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Convex Structure of the Constrained Least Square Problem for Estimating the Forward Rate Sequence

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Abstract. We will show that the constrained least square problem proposed in Konno and Takase [5] for estimating the forward rate sequence by using the market prices of default-free non-callable coupon bonds is in fact a convex minimization problem under more general conditions than those assumed in the subsequent paper by the same authors [6]. Consequently, the constrained least square approach can generate a smooth and accurate forward rate sequence very fast by standard convex minimization algorithms.

Key words: convex minimization problems, forward rate sequence, least square approach, term structure of interest rates

1. Introduction

In a recent article [5], Konno and Takase proposed a constrained least square method for estimating the term structure of interest rates by using the market prices of default-free and non-callable coupon bonds. This method is an improvement on the direct least square method proposed by Carleton and Cooper [1] in 1976, which has long been set aside by practitioners because the calculated forward rate sequence often exhibits a large fluctuation.

Therefore people use a simple spline interpolation method by McCulloch [7] or its extended methods such as those proposed by Vasicek and Fong [8] and Houglet [4]. However, the Vasicek–Fong's method is not entirely free from a large fluctuation of the estimated forward sequence and sometimes it generates negative interest rates. On the other hand, Houglet's method results in a highly nonlinear and nonconvex minimization problem, whose global minimum cannot in general be calculated by the current state-of-the-art mathematical programming methodology.

An alternative approach was recently proposed by Delbaen and Lorimier [2, 3]. This method is intended to minimize the weighted sum of the mispricing and the total amount of jumps of the forward rate sequence. Thus it can generate a relatively smooth forward rate sequence with a good fitting to the market data. Unfortunately however, this method sometimes generates negative forward rates. Also it cannot incorporate the data of coupon bonds because the problem then becomes a nonconvex minimization problem. This is very inconvenient in practice since there are not many discount bonds available in the Japanese market.

Our constrained least square approach, on the other hand, can generate a smooth, nonnegative forward rate sequence very fast by using all available default-free, non-callable discount bonds and coupon bonds. Also, it has been argued in [6] that the calculated locally optimal solution of the nonconvex least square problem is in fact a globally optimal solution under several technical and plausible assumptions. In particular, it has been proved that when we calculate a forward rate sequence for twenty periods by choosing 6 months as one period, the calculated solution is a global minimum if

(i) the coupon rate is less than 3 yen/period and
(ii) there exists at least one maturing bond at the end of each period.

These assumptions are usually valid in the Japanese bond market. However, if we choose 3 months or 1 month as one period, then the sufficient conditions above would not be satisfied and therefore the calculated solution may not be a global minimum. Also, when we calculate a forward rate sequence for more than 20 periods, then some tighter restrictions have to be imposed to guarantee the global optimality of the calculated solution.

The purpose of this short article is to show that the global optimum of the constrained least square problem can be calculated without imposing technical assumptions stated above. We will show that the problem is in fact a convex minimization problem under much more general conditions.

2. Constrained Least Square Problem

Let i_t be the forward rate during period t $(t = 1, \ldots, T)$. Also let z_t be the associated discount rate, i.e.,

$$z_t = [(1 + i_1) \cdots (1 + i_t)]^{-1}, t = 1, \ldots, T. \tag{2.1}$$

Then the theoretical price P_j of the default-free non-callable bond B_j $(j = 1, \ldots, n)$ satisfies the relation:

$$P_j = c_{j1}z_1 + c_{j2}z_2 + \cdots + c_{jT}z_T, \quad j = 1, \ldots, n, \tag{2.2}$$

where c_{jt} is the (fixed) amount of cash flow during period t.

Let ε_j be the pricing error (mispricing) associated with bond B_j. Then the market price p_j can be represented as follows:

$$p_j = c_{j1}z_1 + \cdots + c_{jT}z_T + \varepsilon_j, \quad j = 1, \ldots, n. \tag{2.3}$$

The least square method of Carleton–Cooper can be represented as follows:

$$\left| \begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{n} \left\{ \sum_{t>1}^{T} c_{jt}z_t - p_j \right\}^2 \\ \text{subject to} & 1 \geqslant z_1 \geqslant z_2 \geqslant \cdots \geqslant z_T \geqslant 0. \end{array} \right. \tag{2.4}$$

Constraints on z_t's correspond to the non-negativity of the forward rate i_t's. This problem is a convex quadratic programming problem which can be solved very fast by standard algorithms. Let z_t^* $(t = 1, \ldots, T)$ be an optimal solution of this problem. Then the optimal forward rate i_t^*'s can be recovered by the formula below:

$$i_t^* = z_{t-1}^*/z_t^* - 1, \quad t = 1, \ldots, T. \quad (z_0 = 1) \tag{2.5}$$

Unfortunately, the sequence $(i_1^*, \cdots, i_T^*)$ sometimes suffers a big jump. For example, the computational results reported in [5] show as much as 8 percent jump of forward rate in 6 months, particularly toward the end of the horizon. This is intolerable from the practitioner's point of view since a large fluctuation of interest rate is usually associated with a radical change of economic environment, which cannot be forecasted in advance. This fluctuation may be partly due to the errors associated with market data. Upon closer examination however, we find that it has more to do with the numerical instability associated with the formula (2.5). Since z_{t-1}^*/z_t^* is usually close to 1, small errors in the calculated sequence of discount factors z_t^*'s can lead to a significant error in the calculated forward rate sequence.

This means that we can obtain a smooth forward rate sequence by imposing a smoothness condition on i_t's without deteriorating the least square fitting. The readers are referred to [5] for the validity of this conclusion. This observation led us to the imposition of the smoothness condition:

$$1/(1+\delta) \leqslant (1 + i_{t+1})/(1 + i_t) \leqslant (1 + \delta), \quad t = 1, \ldots, T-1, \tag{2.6}$$

where $\delta > 0$ is a parameter to control the smoothness of the sequence $(i_1, \ldots, i_T)$. Note that the constraint (2.6) can be rewritten as follows:

$$-\delta(1 + i_t) \leqslant i_t - i_{t+1} \leqslant \delta\,(1 + i_{t+1}), \quad t = 1, \ldots, T-1, \tag{2.7}$$

Since i_t is usually less than 0.05 (when one period is less than 6 months), the constraint (2.6) is almost equivalent to the condition

$$|i_t - i_{t+1}| \leqslant 1.05\delta, \quad t = 1, \ldots, T-1. \tag{2.8}$$

The reason why we do not impose a more direct condition $|i_t - i_{t+1}| \leqslant \delta$ instead of (2.6) is that the resulting mathematical programming problem then becomes an intractable nonconvex problem with a non-connected feasible region.

By using the relation $1 + i_t = z_t/z_{t+1}$, the constraint (2.6) can be represented in terms of z_t's as follows:

$$1/(1+\delta) \leqslant z_{t-1}z_{t+1}/z_t^2 \leqslant 1 + \delta, \quad t = 1, \ldots, T-1.$$

The constrained least square problem is thus represented as follows:

$$\left|\begin{array}{ll}\text{minimize} & \sum_{j=1}^{n}\left\{\sum_{t=1}^{T} c_{jt}z_t - p_j\right\}^2 \\ \text{subject to} & 1 \geqslant z_1 \geqslant z_2 \geqslant \cdots \geqslant z_T \geqslant 0 \\ & (1+\delta)^{-1} \leqslant z_{t-1}z_{t+1}/z_t^2 \leqslant 1+\delta, \\ & (z_0 = 1), \quad t = 1,\ldots,T-1.\end{array}\right. \tag{2.9}$$

Let us introduce a new set of variables

$$x_t = -ln z_t, \quad t = 1,\ldots,T. \tag{2.10}$$

Then the problem (2.4) becomes

$$\left|\begin{array}{ll}\text{minimize} & g(x) = \sum_{j=1}^{n}\left\{\sum_{t=1}^{T} c_{jt}\exp(-x_t) - p_j\right\}^2 \\ \text{subject to} & 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_T \\ & -ln(1+\delta) \leqslant x_{t-1} - 2x_t + x_{t+1} \leqslant ln(1+\delta), \\ & t = 1,2,\ldots,T-1.\end{array}\right. \tag{2.11}$$

3. Convexity of the Objective Function

Note that the objection function of (2.11) is not convex in general. We will show below that it is in fact convex in practice.

First let us note that the market price of government bonds whose principal value is 100 yens and whose maturity is less than 10 years is usually greater than 50 yens. Hence we may safely assume that each price has a lower bound which depends on its maturity. Also the market price of each bond will be priced in such a way that the magnitude of mispricing is not more than several yens, which is also assumed below in our domain of optimization.

Let $\alpha_j > 0$ $(j = 1,\ldots,n)$ and let $\alpha = (\alpha_1,\ldots,\alpha_n)$ Also let

$$X_1(\alpha) = \left\{(x_1,\ldots,x_T) \mid \sum_{t=1}^{T} c_{jt}\exp(-x_t) \geqslant \alpha_j, \quad j = 1,\ldots,n\right\} \tag{2.12}$$

$$X_2(\alpha) = \left\{(x_1,\ldots,x_T) \,\middle|\, \left|\sum_{t=1}^{T} c_{jt}\exp(-x_t) - p_j\right| \leqslant \alpha_j, \quad j = 1,\ldots,n\right\} \tag{2.13}$$

and let

$$X(\alpha) = X_1(\alpha) \cap X_2(\alpha). \tag{2.14}$$

THEOREM 1. *The objective function* $g(x)$ *of the problem* (2.11) *is convex on* $X(\alpha)$.

Proof. Let Q be the Hessian matrix of $g(x)$. Then

$$Q = \sum_{j=1}^{n} \xi_j \xi_j^t + \sum_{j=1}^{n} e_j \text{ diag } \xi_j,$$

where

$$e_j = \sum_{t=1}^{T} c_{jt} \exp(-x_t) - p_j, \quad j = 1, \ldots, n,$$

ξ_j is the T-dimensional vector whose tth element ξ_{jt} is given by

$$\xi_{jt} = c_{jt} \exp(-x_t), \quad t = 1, \ldots, T,$$

and diag ξ_i is the diagonal matrix with ξ_{jt}'s as its diagonal elements.

To show that Q is positive semi-definite on $X(\alpha)$, let $(x_1, \ldots, x_T) \in X(\alpha)$ and let $u = (u_1, \ldots, u_T) \neq 0$.

Then

$$\begin{aligned} u^t Q u &= \sum_{j=1}^{n} \left\{ \left(\sum_{t=1}^{T} \xi_{jt} u_t \right)^2 + e_j \sum_{t=1}^{T} \xi_{jt} u_t^2 \right\} \\ &\geqslant \sum_{j=1}^{n} \left\{ \left(\sum_{t=1}^{T} \xi_{jt} u_t \right)^2 - \alpha_j \sum_{t=1}^{T} \xi_{jt} u_t^2 \right\}. \end{aligned}$$

Therefore it suffices to prove that

$$v_j = \left(\sum_{t=1}^{T} \xi_{jt} u_t \right)^2 - \alpha_j \sum_{t=1}^{T} \xi_{jt} u_t^2 \geqslant 0, \quad j = 1, \ldots, n$$

Note that $\Sigma_{t=1}^{T} \xi_{jt} u_t^2 \geqslant 0$ for all j since $\xi_{jt} \geqslant 0, \quad \forall j, t$. Let $\gamma_j > 0$ and let

$$w_j = \min \left\{ \left(\sum_{t=1}^{T} \xi_{jt} u_t \right)^2 \mid \sum_{t=1}^{T} \xi_{jt} u_t^2 = \gamma_j \right\}.$$

Then it is straightforward to show that

$$w_j = \left(\sum_{t=1}^{T} \xi_{jt}\right) \gamma_j .$$

Therefore

$$\begin{aligned} v_j &= \left(\sum_{t=1}^{T} \xi_{jt} u_t\right)^2 - \alpha_j \sum_{t=1}^{T} \xi_{jt} u_t^2 \\ &\geqslant \left(\sum_{t=1}^{T} \xi_{jt} - \alpha_j\right) \sum_{t=1}^{T} \xi_{jt} u_t^2 \geqslant 0. \end{aligned}$$ □

This theorem shows that the objective function of the problem (2.11) is convex if the market price of bonds p_j's are larger than, say 10 yens, and if the maximal amount of mispricing is less than a few yens. The conditions of the theorem are satisfied in practice.

4. Discussions

The constrained least square problem (2.11) can be solved very fast by a quadratic approximation approach according to the numerical experiments reported in [5]. In fact, a global minimum can be obtained in less than a minute by personal computer when $T = 20$ and $n = 100$. It is expected that a larger problem can be solved without much difficulty. This means that we can calculate more accurate forward rate sequence by choosing one month as the length of one period.

An alternative method to obtain a continuous forward rate curve would be to apply a standard spline interpolation method by using the forward rate sequence calculated from the above method with one period being 6 months. Since the forward rate sequence generated by our method is smooth and non-negative, the spline interpolation method would lead to a smooth and non-negative forward rate curve.

As a final remark, for different periods of cash flows, we may modify the constraints of the problem (2.11) in an appropriate way.

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