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QUANTUM NONLINEAR OSCILLATOR WITH TWO DEGREES OF FREEDOM IN A LASER FIELD

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Abstract

The semiclassical dynamics of a quantum nonlinear oscillator with two degrees of freedom and anharmonicity of the fourth order in a periodic laser field is studied both analytically and numerically. In the absence of external excitation and dissipation, the equations of motion for the mean values of the coordinate and momentum operators of both degrees of freedom reduce to the equation of a one-dimensional nonlinear pendulum. The general solution of this equation is written in terms of the Jacobian elliptic functions. As can be expected, the energy of the free oscillator is redistributed periodically between degrees of freedom. The periodic excitation of the nonlinear oscillator may substantially change its motion pattern. Using as an example an oscillator with two coupled vibrational degrees of freedom, it is numerically shown that the amount of laser photons absorbed depending on the parameter values and initial conditions may vary with time in a rather complex manner, including chaotic oscillations. A nonlinear oscillator is capable of manifesting bistable behavior with allowance for dissipation. The analytical condition for the origination of bistability is found. Examples of the bistable dependence of the number of quanta in the oscillator vibrational mode on the level of laser excitation are presented.

1. Introduction

A nonstationary quantum harmonic oscillator with two degrees of freedom is a system with a quadratic Hamiltonian. The exact solutions of the equation of motion of such a system can be obtained in a variety of ways [1, 2]. A Hamiltonian with higher-order terms with respect to the coordinate and momentum operators, which describes the anharmonic quantum oscillator, gives rise, in general, to nonintegrable equations of motion. The model of the anharmonic quantum oscillator with periodic excitation and single degree of freedom is extensively used in various domains of physics. In molecular laser physics, this model describes in general terms the excitation of oscillations in molecules under exposure to laser radiation (see [3] and references therein). Multiphoton absorption of infrared laser radiation by polyatomic molecules may result in their dissociation and stimulate chemical reactions [4]. From the classical standpoint, the dynamic (deterministic) chaos and diffusion in the phase space are the main mechanisms of the multiphoton absorption and dissociation [5, 6]. The model mentioned is used in quantum optics to describe light propagation through a Kerr-type medium, for example, in problems of fiber optics.

The nonlinear quantum oscillator with two degrees of freedom is an apparent generalization of the one-dimensional model. The laser selectively excites the chosen vibrational mode of a molecule, generally speaking, of the anharmonic molecule. Furthermore, this mode is coupled with the other vibrational modes of the molecule, whose set is modeled by the second degree of freedom. The self-radiation field of the medium is the second degree of freedom in the problem of light propagation through a resonator with a nonlinear Kerr-type medium. For the nonlinear quantum oscillator with two degrees of freedom, solution of the equations

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of motion is possible using numerical methods. One can show examples of the special selection of the type of nonlinearity when an analytical solution is possible. The dynamical theory of Fermi resonance was constructed in [7]. This phenomenon consists in coincidence of the fundamental-tone frequency of one of the molecular vibrations with that of the first overtone of another type of vibrations. The corresponding Hamiltonian describes two harmonic oscillators, which are coupled in a nonlinear manner. The solution for such an oscillator was obtained in [7] in terms of elliptical functions.

In this paper, we study both analytically and numerically the dynamics of the nonlinear quantum oscillator with two degrees of freedom in a periodic external field with a Hamiltonian of the type

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}(t). \tag{1}$$

The steady-state Hamiltonian in (1)

$$\hat{H}_0 = \hbar\omega_a \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) + \hbar\omega_b \left(\hat{b}^\dagger \hat{b} + \frac{1}{2}\right) + \hbar Q \hat{b}^\dagger \hat{b}^\dagger \hat{b}\hat{b} + \hbar G \,(\hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b}) \tag{2}$$

describes the system, where one degree of freedom, denoted by "a" index, is modeled by the harmonic oscillator and the second degree of freedom "b" is modeled by the anharmonic oscillator with the nonlinearity factor Q. The nonlinear term in the Hamiltonian (2) arises as a result of the rotary wave approximation applied to the conventional anharmonicity operator $(\hat{b} + \hat{b}^\dagger)^4$. The coupling of the degrees of freedom is assumed to be linear with the coefficient G and is described by the last term in the Hamiltonian $\hat{H}_0$. The operator

$$\hat{H}_{\text{int}}(t) = \hbar\Omega_0 \left(\hat{b} e^{i\omega_l t} + \hat{b}^\dagger e^{-i\omega_l t}\right) \tag{3}$$

describes excitation of the mode by the external laser field. The latter is assumed to be periodic, classical, and single-mode.

Depending on the relationship between the parameters of Hamiltonians (1)–(3) and relaxation times, the Hamiltonian and dissipative dynamics as well as the quantum and semiclassical ones should be distinguished. In the present paper, within the framework of the unified semiclassical approach based on the Heisenberg equations of motion for the expected values of operators, the following investigations were carried out:

1. The general analytical solution of the equations of motion of the free nonlinear oscillator with the Hamiltonian $\hat{H}_0$ was found.

2. The system of equations was obtained and the Hamiltonian dynamics of the periodically excited nonlinear oscillator with the Hamiltonian $\hat{H}_0 + \hat{H}_{\text{int}}(t)$ was numerically studied.

3. The bistability of a forced nonlinear oscillator with dissipation was studied both analytically and numerically.

2. General Solution of the Equations of Motion of a Free Quantum Nonlinear Oscillator with Two Degrees of Freedom

The Hamiltonian of the free quantum nonlinear oscillator with two degrees of freedom without regard for dissipation has the form of (2). The Heisenberg equations for operators of quantum annihilation in the corresponding modes take the form

$$\frac{d}{dt}\,\hat{a} = -i\,(\omega_a \hat{a} + G\hat{b}), \quad \frac{d}{dt}\,\hat{b} = -i\,(\omega_b \hat{b} + 2Q\hat{b}^\dagger \hat{b}\hat{b} + G\hat{a}). \tag{4}$$

The equations of motion for $\hat{a}^\dagger$ and $\hat{b}^\dagger$ are derived by the Hermitian conjugation of Eqs. (4). Let us pass on from the operators to the mean values in Eqs. (4) and introduce the new variables x_a, y_a, x_b, and y_b in the following way:

$$< \hat{a} >= x_a + iy_a, \quad < \hat{b} >= x_b + iy_b. \tag{5}$$

In this case, the mean value of the product of operators will be considered to be equal to the product of their mean values. Substituting expressions (5) into Eqs. (4), we obtain

$$\begin{aligned} \dot{x}_a + i\dot{y}_a &= -i\,(\omega_a x_a + i\omega_a y_a + Gx_b + iGy_b), \\ \dot{x}_b + i\dot{y}_b &= -i\,(\omega_b x_b + i\omega_b y_b + 2Q\,(x_b^2 + y_b^2)(x_b + iy_b) + Gx_a + iGy_a). \end{aligned} \tag{6}$$

Equating the imaginary and real parts, we obtain the following system:

$$\begin{aligned} \dot{x}_a &= \omega_a y_a + Gy_b, \\ \dot{y}_a &= -\omega_a x_a - Gx_b, \\ \dot{x}_b &= \omega_b y_b + 2Q\left(x_b^2 + y_b^2\right) y_b + Gy_a, \\ \dot{y}_b &= -\omega_b x_b - 2Q\left(x_b^2 + y_b^2\right) x_b - Gx_a. \end{aligned} \tag{7}$$

The variables can be represented in the form

$$\begin{aligned} x_a &= \sqrt{n_a}\,\cos\phi_a, \qquad y_a = \sqrt{n_a}\,\sin\phi_a, \\ x_b &= \sqrt{n_b}\,\cos\phi_b, \qquad y_b = \sqrt{n_b}\,\sin\phi_b, \end{aligned} \tag{8}$$

and, correspondingly, $< \hat{a} >= \sqrt{n_a}\,e^{i\phi_a}$ and $< \hat{b} >= \sqrt{n_b}\,e^{i\phi_b}$. After substitution of Eqs. (8) into Eqs. (7) a little manipulation yields

$$\begin{aligned} \dot{n}_a &= -2G\,\sqrt{n_a n_b}\,\sin(\phi_a - \phi_b), \\ \dot{n}_b &= 2G\,\sqrt{n_a n_b}\,\sin(\phi_a - \phi_b), \\ \dot{\phi}_a &= -\omega_a - G\,\sqrt{\frac{n_b}{n_a}}\,\cos(\phi_a - \phi_b), \\ \dot{\phi}_b &= -\omega_b - 2Qn_b - G\,\sqrt{\frac{n_a}{n_b}}\,\cos(\phi_a - \phi_b). \end{aligned} \tag{9}$$

From the first two equations of (9) the conservation law follows:

$$N = n_a + n_b. \tag{10}$$

We introduce the new designations

$$\phi = \phi_a - \phi_b, \quad n' = n_a - n_b, \quad n_a = \frac{N + n'}{2}, \quad n_b = \frac{N - n'}{2} \tag{11}$$

and subtract the second equation of (9) from the first one. We obtain

$$\begin{aligned} \dot{n}' &= -2G\,\sqrt{N^2 - n'^2}\,\sin\phi, \\ \dot{\phi} &= \omega_b - \omega_a + Q\,(N - n') + \frac{2Gn'\cos\phi}{\sqrt{N^2 - n'^2}}\,. \end{aligned} \tag{12}$$

Let us introduce the new variable $n = n'/N$ which satisfies the following properties:

$$\begin{aligned} -1 \le n \le 1; \quad & n = 1,\ n_a = N,\ n_b = 0; \\ & n = -1,\ n_a = 0,\ n_b = N; \\ & n = 0,\ n_a = n_b = N/2. \end{aligned} \tag{13}$$

Furthermore, we designate

$$\omega = \omega_a - \omega_b, \quad M = NQ, \quad R = M - \omega. \tag{14}$$

The system of equations (12) takes the form

$$\begin{aligned} \dot{n} &= -2G\sqrt{1-n^2}\,\sin\phi, \\ \dot{\phi} &= -\omega + M(1-n) + \frac{2Gn\cos\phi}{\sqrt{1-n^2}}\,. \end{aligned} \tag{15}$$

Differentiating the first equation of the system and substituting expressions for $\dot{n}$ and $\dot{\phi}$, we obtain the following equation:

$$\ddot{n} = -4G^2 n - 2G\sqrt{1-n^2}\,(R - Mn)\cos\phi \tag{16}$$

with the initial conditions

$$n(0) = n_0, \quad \dot{n}(0) = n_1, \quad \phi(0) = \phi_0. \tag{17}$$

From the first equation of system (15) we obtain

$$\sin\phi_0 = -\frac{n_1}{2G\sqrt{1-n_0^2}}. \tag{18}$$

Since the system discussed is the Hamiltonian one, its energy is a conserved quantity. However, it is convenient to use another conserved quantity obtained from the energy by dropping the constant factors and terms. Such a conservation law can be written as

$$E = \omega n + \frac{M}{2}\,n\,(n-2) + 2G\sqrt{1-n^2}\cos\phi. \tag{19}$$

Using this conservation law, one can express $\cos\phi$ in terms of n as follows:

$$\cos\phi = \frac{E - \omega n - \frac{M}{2}\,n\,(n-2)}{2G\sqrt{1-n^2}}\,. \tag{20}$$

Substituting the initial conditions into conservation law (19) and taking into account that

$$\cos\phi_0 = s_0\sqrt{1-\sin^2\phi_0} = s_0\sqrt{1 - \frac{n_1^2}{4G^2\,(1-n_0^2)}} \tag{21}$$

we have

$$E = \omega n_0 + \frac{M}{2}\,n_0\,(n_0 - 2) + S, \quad S = s_0 G\sqrt{4 - 4n_0^2 - \frac{n_1^2}{G^2}}, \tag{22}$$

where $s_0 = \pm 1$ depending on the quadrant in which the phase ϕ_0 is situated. Substitution of expression (20) into Eq. (16) and use of relationship (22) yield the main result of Sec. 2, namely, the differential equation for the normalized difference n of the occupation numbers in the modes

$$\ddot{n} = a_3 n^3 + a_2 n^2 + a_1 n + a_0. \tag{23}$$

The coefficients in Eq. (23) are independent of time and n and are expressed in terms of the system parameters and initial conditions in the following way:

$$\begin{aligned} a_3 &= -\frac{M^2}{2}, \quad a_2 = 3\,\frac{MR}{2}, \\ a_1 &= -4G^2 - R^2 + \frac{M^2}{2}\,n_0^2 - MRn_0 + MS, \\ a_0 &= -\frac{R}{2}\left(Mn_0^2 - 2Rn_0 + 2S\right). \end{aligned} \tag{24}$$

The second-order equation (23) is equivalent to the following first-order equation:

$$\dot{n}^2 = an^4 + bn^3 + cn^2 + dn + e = -a\,P_4(n), \tag{25}$$

where

$$\begin{aligned} a &= \frac{a_3}{2} = -\frac{M^2}{4}, \quad b = \frac{2a_2}{3} = MR, \\ c &= a_1 = -4G^2 - R^2 + \frac{M^2}{2}\,n_0^2 - MRn_0 + MS, \\ d &= 2a_0 = -R\left(Mn_0^2 - 2Rn_0 + 2S\right), \\ e &= n_1^2 - (an_0^4 + bn_0^3 + cn_0^2 + dn_0) = \\ &= n_1^2 + 2RSn_0 - (R^2 + MS - 4G^2)\,n_0^2 + MRn_0^3 - \frac{M^2}{4}\,n_0^4. \end{aligned} \tag{26}$$

Integration of (25) yields

$$\frac{s_1}{\sqrt{-a}} \int\limits_{n_0}^{n} \frac{dn'}{\sqrt{P_4(n')}} = \int\limits_0^t dt', \tag{27}$$

where $s_1 = \pm 1$ depending on the initial conditions. As is evident from the first equation of system (15), the sign of s_1 is determined by the quadrant in which ϕ_0 is situated.

Let us denote the roots of the polynomial $P_4(x)$ by α_i $(i = 1, 2, 3, 4)$. The integral on the left-hand side of equality (27) reduces to an elliptic integral by the substitution $n' = f(\phi')$

$$\int\limits_{n_0}^{n} \frac{dn'}{\sqrt{P_4(n')}} = \mu \int\limits_{\phi_0}^{\phi} \frac{d\phi'}{\sqrt{1 - k^2 \sin^2 \phi'}}\,. \tag{28}$$

In this case, the function $f(\phi')$ and coefficients μ and k depend both on α_i in themselves and their relationship with n and n_0. Two different cases are possible:

1. α_i are real $(\forall i)$. Let us assume that $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. There are two possibilities again in this case:

   (a) $\alpha_4 \le n, n_0 \le \alpha_3$. In this situation,

$$\begin{aligned} \phi' &= f_{1a}^{-1}(n') = \arcsin \sqrt{\frac{(\alpha_1 - \alpha_3)(n' - \alpha_4)}{(\alpha_3 - \alpha_4)(\alpha_1 - n')}}\,, \\ k_1 &= \sqrt{\frac{(\alpha_3 - \alpha_4)(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}, \quad \mu_1 = \frac{2}{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}\,. \end{aligned} \tag{29}$$

   The solution for the difference of occupation numbers looks like

$$\begin{aligned} n(t) &= \frac{\alpha_1 r_{1a} \mathrm{sn}^2(m_1 t + \psi_{1a},\, k_1) + \alpha_4}{r_{1a} \mathrm{sn}^2(m_1 t + \psi_{1a},\, k_1) + 1}, \\ r_{1a} &= \frac{\alpha_3 - \alpha_4}{\alpha_1 - \alpha_3}, \quad m_1 = s_1\,\frac{\sqrt{-a}}{\mu_1}, \quad \psi_{1a} = \mathrm{F}\big(f_{1a}^{-1}(n_0),\, k_1\big), \end{aligned} \tag{30}$$

   where $\mathrm{F}(\theta,\, k)$ is an elliptic integral of the first kind

$$\mathrm{F}(\theta,\, k) = \int\limits_0^{\theta} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}\,. \tag{31}$$

(b) $\alpha_2 \le n, n_0 \le \alpha_1$. In this case,

$$\phi' = f_{1b}^{-1}(n') = \arcsin\sqrt{\frac{(\alpha_1-\alpha_3)(n'-\alpha_2)}{(\alpha_1-\alpha_2)(n'-\alpha_3)}}\,, \tag{32}$$

and the solution has the form

$$\begin{gathered} n(t) = \frac{\alpha_2 - \alpha_3 r_{1b}\mathrm{sn}^2(m_1 t + \psi_{1b},\, k_1)}{1 - r_{1b}\mathrm{sn}^2(m_1 t + \psi_{1b},\, k_1)}\,, \\ r_{1b} = \frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}, \quad \psi_{1b} = \mathrm{F}\big(f_{1b}^{-1}(n_0),\, k_1\big). \end{gathered} \tag{33}$$

2. $\alpha_1 > \alpha_2$ are real roots and $\alpha_3 = \beta + i\gamma$, $\alpha_4 = \beta - i\gamma$, $\gamma > 0$. In this instance,

$$\begin{gathered} \phi' = f_2^{-1}(n') = 2\arctan\sqrt{\frac{\cos\theta_1}{\cos\theta_2}\,\frac{\alpha_1-n'}{n'-\alpha_2}}\,, \\ k_2 = \sin\frac{\theta_1-\theta_2}{2}, \quad \mu_2 = -\frac{\sqrt{\cos\theta_1\,\cos\theta_2}}{\gamma}\,, \\ \theta_1 = \arctan\frac{\alpha_1-\beta}{\gamma}\,, \quad \theta_2 = \arctan\frac{\alpha_2-\beta}{\gamma}\,, \end{gathered} \tag{34}$$

and the solution has the form

$$\begin{gathered} n(t) = \frac{\alpha_1 + \alpha_2 r_2 + (\alpha_1 - \alpha_2 r_2)\mathrm{cn}(m_2 t + \psi_2,\, k_2)}{1 + r_2 + (1-r_2)\mathrm{cn}(m_2 t + \psi_2,\, k_2)}, \\ r_2 = \frac{\cos\theta_2}{\cos\theta_1}, \quad m_2 = s_1\,\frac{\sqrt{-a}}{\mu_2}, \quad \psi_2 = \mathrm{F}\big(f_2^{-1}(n_0),\, k_2\big). \end{gathered} \tag{35}$$

Thus, the semiclassical equations of motion of a free quantum nonlinear oscillator with two degrees of freedom reduce to a single differential second-degree equation in difference of the occupation numbers. The exact analytical solution in terms of the elliptic functions is obtained for this equation. The form of the solution depends on the system parameters and initial conditions.

The phase portraits in the plane n–ϕ provide a pictorial representation of the behavior of the vibrations of a free nonlinear oscillator with two degrees of freedom. These phase portraits are shown in Fig. 1 for various relationships between the system parameters. Let us normalize Eqs. (15) to the coefficient of mode–mode coupling G. As a result we obtain two control parameters, namely, the dimensionless frequency mismatch $\omega/G \equiv (\omega_a - \omega_b)/G$ and the multiplicative nonlinearity factor $M/G \equiv NQ/G$, where $N = n_a + n_b$ is the conserved quantity of the total number of quanta.

Figure 1a shows the phase portrait in the case where the nonlinearity equals zero ($M/G = 0$) and the normalized mismatch is equal to unity ($\omega/G = 1$). Figure 1b shows the phase portrait for the stronger mismatch ($\omega/G = 5$) and multiplicative nonlinearity factor $M/G = 2$. The phase portrait in Fig. 1c relates to the case of small mismatch ($\omega/G = 0.2$) and weak nonlinearity ($M/G = 0.5$). The phase portrait in Fig. 1d is appropriate for the following parameters: $\omega/G = 3.1$ and $M/G = 3$.

3. Dynamics of Excitation of the Nonlinear Quantum Oscillator by the Laser Field

The Hamiltonian of the quantum nonlinear oscillator with two degrees of freedom is a nonstationary operator in the periodic external field. This operator has form (1)–(3) without regard for dissipation. To

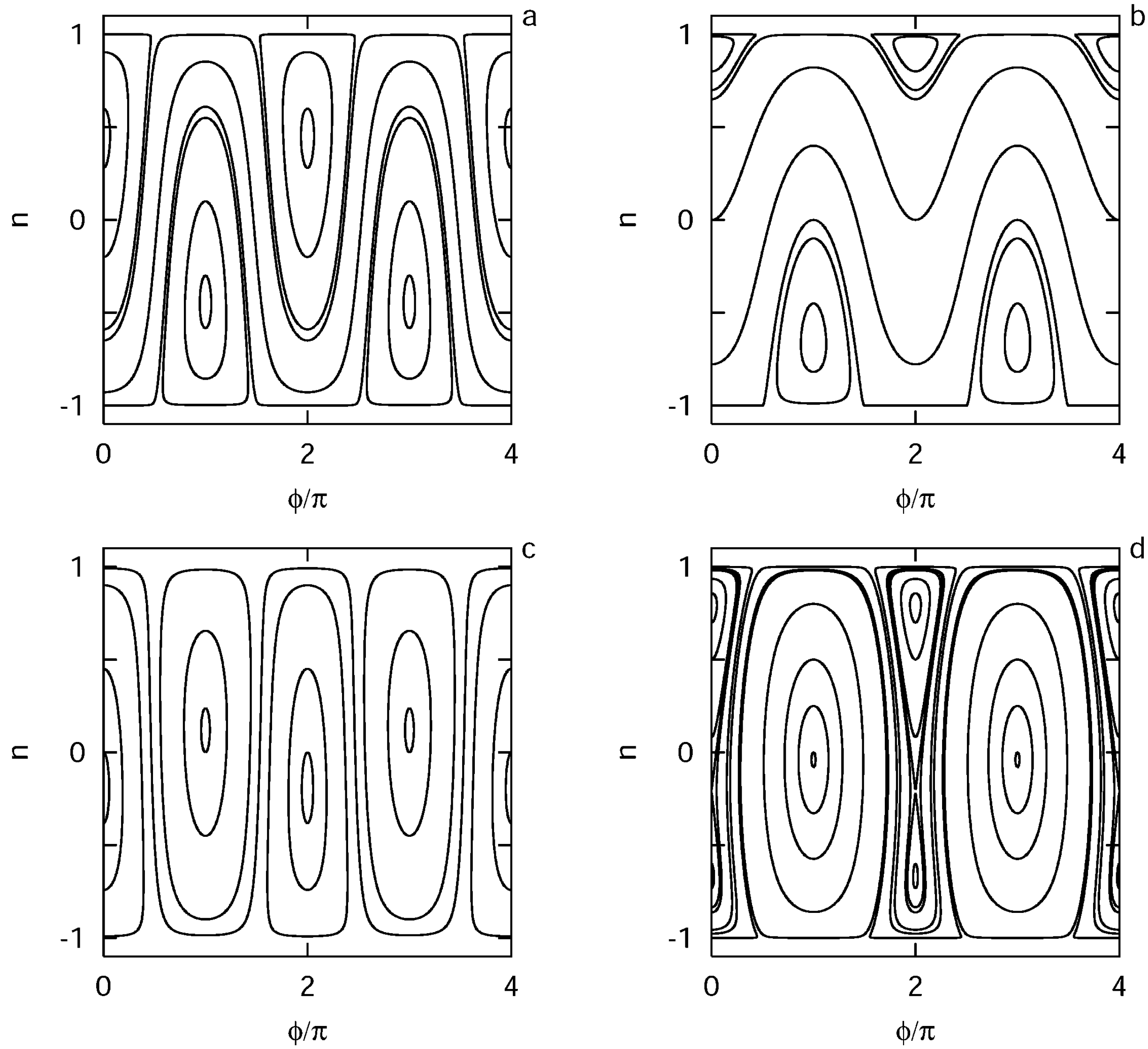

Fig. 1. Phase portraits in polar coordinates of the nonlinear oscillator with two degrees of freedom and without the external force and relaxation: $\omega/G = 1$, $M/G = 0$ (a); $\omega/G = 5$, $M/G = 2$ (b); $\omega/G = 0.2$, $M/G = 0.5$ (c); and $\omega/G = 3.1$, $M/G = 3$ (d).

exclude the time dependence in this Hamiltonian, we change to a rotating coordinate system using the following unitary transformation:

$$\hat{U} = \exp\Big(-i\omega_l t\,(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + 1)\Big). \tag{36}$$

Then the rearranged Hamiltonian has the form

$$\begin{aligned}\hat{H}_{\text{eff}} &= \hbar\,(\omega_a - \omega_l)\,(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar\,(\omega_b - \omega_l)\,(\hat{b}^\dagger\hat{b} + \frac{1}{2}) \\ &+ \hbar Q\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b} + \hbar G\,(\hat{b}^\dagger\hat{a} + \hat{a}^\dagger\hat{b}) + \hbar\Omega_0\,(\hat{b} + \hat{b}^\dagger).\end{aligned} \tag{37}$$

The dimensionless equations of motion for mean values (5) in the rotating coordinate system are derived using the Hamiltonian $H_{\rm eff}$

$$\begin{aligned} \frac{d}{d\tau}\,x_a &= \Delta_a y_a + y_b, \\ \frac{d}{d\tau}\,y_a &= -\Delta_a x_a - x_b, \\ \frac{d}{d\tau}\,x_b &= \Delta_b y_b + q\,(x_b^2 + y_b^2)\,y_b + y_a, \\ \frac{d}{d\tau}\,y_b &= -\Delta_b x_b - q\,(x_b^2 + y_b^2)\,x_b - x_a - \Omega, \end{aligned} \tag{38}$$

where $\tau = Gt$, $\Delta_a = (\omega_a - \omega_l)/G$, $\Delta_b = (\omega_b - \omega_l)/G$, $q = 2Q/G$, and $\Omega = \Omega_0/G$. This is a Hamiltonian system with two degrees of freedom and having the integral of motion due to the conservation of energy in the Hamiltonian $H_{\rm eff}$. As distinct from the free nonlinear oscillator from the previous section, the total number of quanta of the forced oscillator $N(\tau) \equiv n_a(\tau) + n_b(\tau) = x_a^2 + y_a^2 + x_b^2 + y_b^2$ is not conserved but satisfies the following equation:

$$\frac{dN(\tau)}{d\tau} = -2\Omega y_b. \tag{39}$$

To reveal the peculiarities of the redistribution of energy of the laser field in degrees of freedom, one should calculate not only the total number of excitations $N(\tau)$ but also the number of quanta in these modes $n_a(\tau)$ and $n_b(\tau)$. The accuracy of the calculation of these parameters was controlled by means of the integral of motion of system (38)

$$< H_{\rm eff} >= \Delta_a\,(x_a^2 + y_a^2) + \Delta_b\,(x_b^2 + y_b^2) + \frac{q}{2}\,(x_b^2 + y_b^2)^2 + 2\,(x_a x_b + y_a y_b) + 2\Omega x_b. \tag{40}$$

The evolution of the total number of excitation quanta in both modes $N(\tau)$ was calculated for the following initial condition:

$$x_a(0) = y_a(0) = x_b(0) = y_b(0) = 1. \tag{41}$$

Depending on the relationship between the control parameters of system (38), the signal $N(\tau)$ can be regular or chaotic (in terms of the exponential sensitivity to the small variations of the initial conditions). The presence or absence of Hamiltonian chaos in the system was established by calculations of the maximum Lyapunov exponent λ. The mismatches were chosen to be $\Delta_a = 10$ and $\Delta_b = -2.5$. Then the remaining control parameters of system (38), namely, the normalized (to G) nonlinearity factor q and normalized Rabi frequency Ω of the laser field, were varied.

Figure 2 shows the dependence $N(\tau)$ for $q = 0.5$ and $\Omega = 1$. The signal presented in Fig. 1 not only looks regular, but its regularity is confirmed by the calculated Lyapunov exponent, $\lambda = 0$. The signal with $q = 2$ and $\Omega = 3$ shown in Fig. 3 has the maximum Lyapunov exponent $\lambda \simeq 1$ and represents the case of a chaotic signal. Still further chaos with $\lambda \simeq 1.3$ arises in the system with $q = 5$ and $\Omega = 2.5$ (see Fig. 4).

4. Dissipative Dynamics: Bistability

In this section, we consider the dissipative dynamics of the forced nonlinear oscillator with the Hamiltonian $H_{\rm eff}$ of type (37). The relaxation is phenomenologically introduced to the Heisenberg equations (38), which

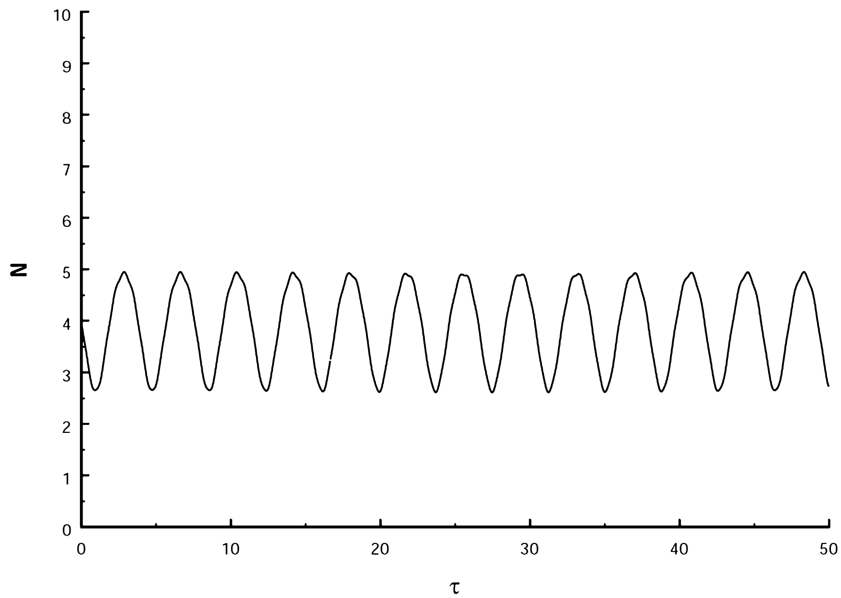

Fig. 2. Regular time dependence of the total number of excitations N of the nonlinear oscillator in the laser field ($q = 0.5$, $\Omega = 1$, $\lambda = 0$).

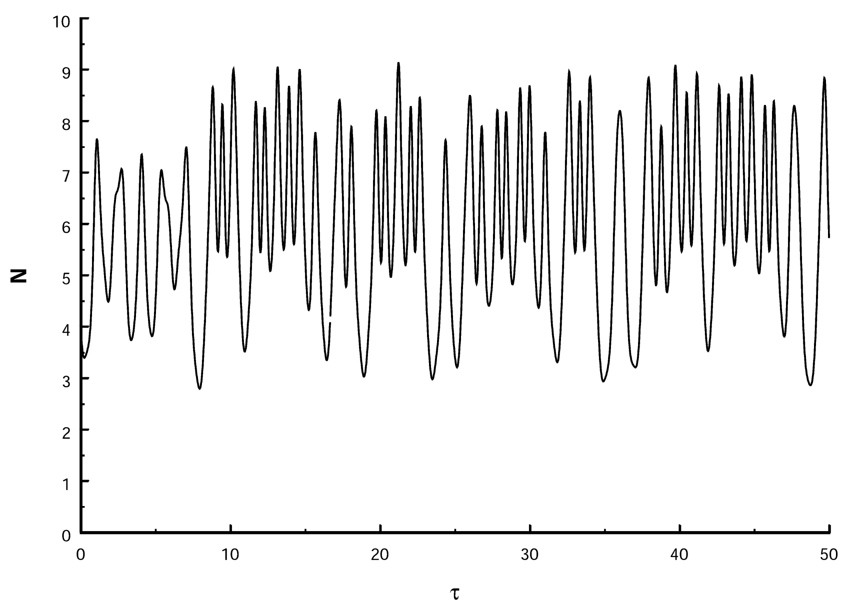

Fig. 3. Oscillations $N(\tau)$ with moderate Hamiltonian chaos ($q = 2$, $\Omega = 3$, $\lambda \simeq 1$).

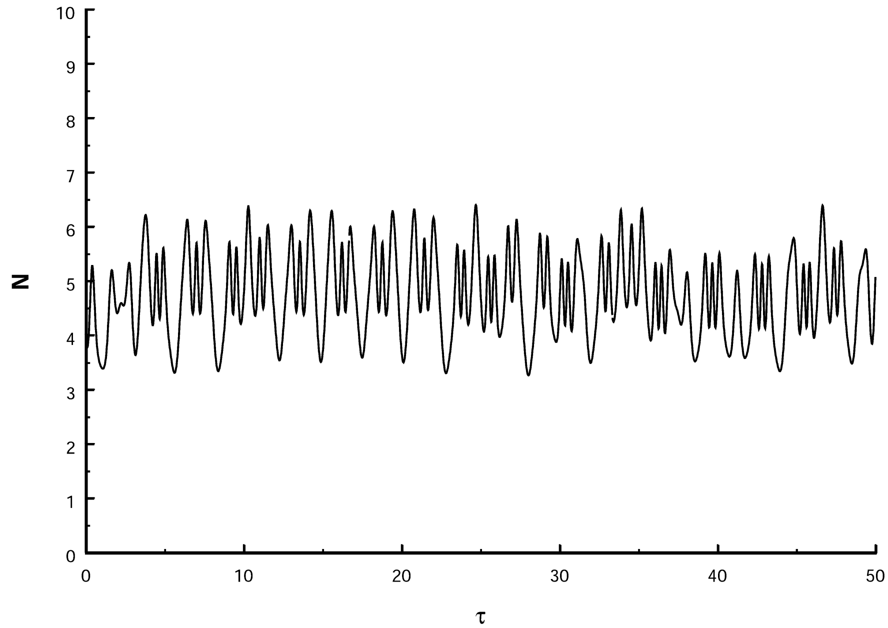

Fig. 4. Strong Hamiltonian chaos of the excited nonlinear oscillator with two degrees of freedom ($q = 5$, $\Omega = 2.5$, $\lambda \simeq 1.3$.

now have the following form:

$$\begin{aligned} \frac{d}{d\tau}\,x_a &= \Delta_a y_a + y_b - \gamma_a x_a, \\ \frac{d}{d\tau}\,y_a &= -\Delta_a x_a - x_b - \gamma_a y_a, \\ \frac{d}{d\tau}\,x_b &= \Delta_b y_b + q\,(x_b^2 + y_b^2)\,y_b + y_a - \gamma_b x_b, \\ \frac{d}{d\tau}\,x_b &= -\Delta_b x_b - q\,(x_b^2 + y_b^2)\,x_b - x_a - \gamma_b y_b - \Omega, \end{aligned} \tag{42}$$

where $\gamma_{a,b}$ are the relaxation rates of excitations in the corresponding degrees of freedom normalized to the value of G. The dynamics of the photons absorbed is determined by the expression

$$\frac{dN(\tau)}{d\tau} = -2\,(\Omega y_b + \gamma_a n_a + \gamma_b n_b). \tag{43}$$

Since system (42) supposedly does not possess an analytical solution, we restrict our consideration to the study of the stationary points (time-independent solutions). For this purpose, the left-hand sides of Eqs. (42) are set equal to zero

$$\begin{aligned} &\Delta_a y_a + y_b - \gamma_a x_a = 0, \\ &\Delta_a x_a + x_b + \gamma_a y_a = 0, \\ &\Delta_b y_b + q\left(x_b^2 + y_b^2\right) y_b + y_a - \gamma_b x_b = 0, \\ &\Delta_b x_b + q\left(x_b^2 + y_b^2\right) x_b + x_a + \gamma_b y_b + \Omega = 0. \end{aligned} \tag{44}$$

In this case, Eq. (43) takes the form

$$\gamma_a\,(x_a^2 + y_a^2) + \gamma_b\,(x_b^2 + y_b^2) + \Omega y_b = 0. \tag{45}$$

Using the first two equations of system (44), we express x_a and y_a in terms of x_b and y_b:

$$x_a = -\frac{1}{K^2}(\Delta_a x_b - \gamma_a y_b), \quad y_a = -\frac{1}{K^2}(\Delta_a y_b - \gamma_a x_b), \quad K^2 = \Delta_a^2 + \gamma_a^2. \tag{46}$$

Substituting this result into equality (45), we arrive at

$$r n_b + \Omega y_b = 0, \quad r = \gamma_b + \gamma_a \frac{1}{K^2}, \quad n_b = x_b^2 + y_b^2, \tag{47}$$

whence it follows that

$$y_b = -\frac{r}{\Omega} n_b. \tag{48}$$

The remaining two equations of system (44) on substitution (46) takes the form

$$\begin{aligned} z y_b + q n_b y_b - r x_b &= 0, \\ z x_b + q n_b x_b + r y_b + \Omega &= 0, \end{aligned} \tag{49}$$

where $z = \Delta_b - \Delta_a/K^2$. We obtain from the first equation

$$x_b = -\frac{q}{\Omega} n_b^2 - \frac{z}{\Omega} n_b. \tag{50}$$

The net result can be obtained by substituting expressions (48) and (50) either into the second equation of system (49) or into the definition $n_b = x_b^2 + y_b^2$. In any case, we obtain the following equation with respect to n_b:

$$q^2 n_b^3 + 2qz n_b^2 + (z^2 + r^2)\, n_b - \Omega^2 \equiv P_3(n_b) = 0. \tag{51}$$

Depending on the system parameters, this equation can possess different numbers of positive roots (it is evident that negative roots are physically meaningless). Let us consider this dependence in greater detail.

1. $z \geq 0$. Resorting to Euclid's rule, it is easy to verify that there is only one positive root in this case.

2. $z < 0$. In this case, according to the Euclid's rule, the equation possesses either a single or three positive roots. To verify the number of them, one should consider the derivative of the left-hand side of (51)

$$P_2(n_b) = P_3'(n_b) = 3q^2 n_b^2 + 4qz n_b + z^2 + r^2. \tag{52}$$

   There are two possibilities in this case:

   (a) $z^2 - 3r^2 < 0$. Under this condition, the discriminant of the equation $P_2(n_b) = 0$ becomes negative, and consequently the function $P_3(n_b)$ does not have extrema. Thus, Eq. (51) possesses one positive root.

   (b) $z^2 - 3r^2 > 0$. In this case, the function $P_3(n_b)$ has minimum and maximum respectively at the points

$$n_{\max} = \frac{-2z - \sqrt{z^2 - 3r^2}}{3q}, \quad n_{\min} = \frac{-2z + \sqrt{z^2 - 3r^2}}{3q}. \tag{53}$$

   It is easy to see that $n_{\min}$ and $n_{\max}$ are positive. The last conditions for the existence of three roots are the inequalities $P_3(n_{\max}) > 0$ and $P_3(n_{\min}) < 0$. Rearranging them, we can obtain

$$-(z^2 - 3r^2)^{3/2} < z^3 + 9zr^2 + 13.5q\Omega^2 < (z^2 - 3r^2)^{3/2}. \tag{54}$$

   The inequalities $z < 0$ and $z^2 - 3r^2 > 0$ are the prerequisites to bistability in the response of the anharmonic oscillator to the external laser action. Namely, given the fulfillment of these conditions, bistability equation (51) possesses three positive roots. Thus, the dependence $\Omega^2(n_b)$ has a maximum and a minimum. As a consequence, the sigmoid function $n_b(\Omega^2)$ has three branches. Hence, there is a certain interval of Ω^2 where the system is bistable [8].

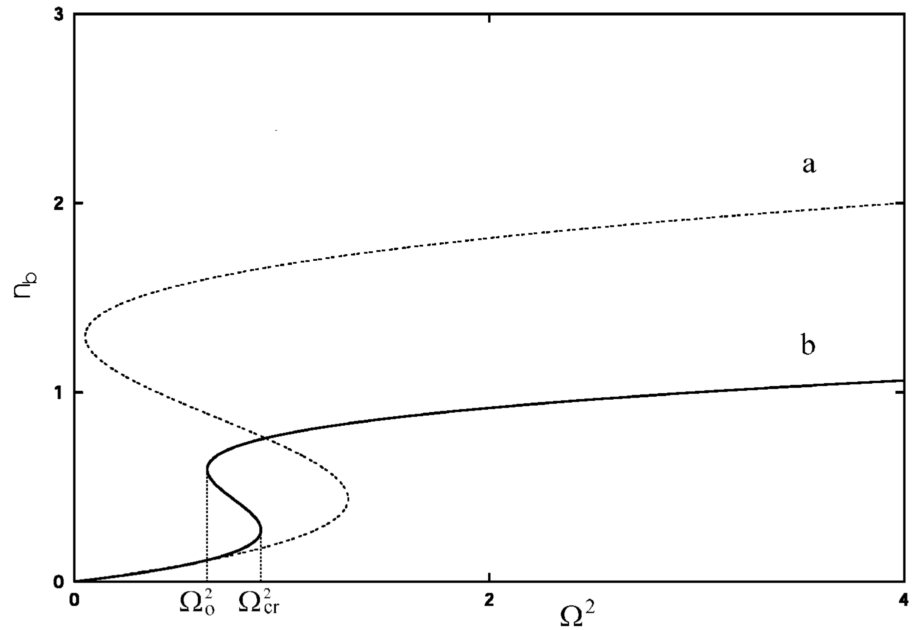

Fig. 5. Curves of bistable behavior of the excited dissipative nonlinear oscillator with two degrees of freedom: $\Delta_a = 10$, $\Delta_b = -2.5$, $q = 2$, $\gamma_a = 0.1$, $\gamma_b = 0.2$ (a) and $\Delta_a = 10$, $\Delta_b = -2.5$, $q = 4$, $\gamma_a = 1.5$, $\gamma_b = 1$ (b).

Figure 5 shows the bistability curves for the same values of the normalized mismatches ($\Delta_a = 10$, $\Delta_b = -2.5$) as in the previous section. Curve a refers to the case with the nonlinearity factor $q = 2$ and rates of damping of excitations in the modes $\gamma_a = 0.1$ and $\gamma_b = 0.2$. Different values of these control parameters ($q = 4$, $\gamma_a = 1.5$, $\gamma_b = 1$) were taken when calculating the bistability curve b. The excitation of the anharmonic oscillator gradually increases and follows the lower branch of the sigmoid bistability curve with the square of the dimensionless Rabi frequency of the laser field Ω^2. When some critical value of $\Omega^2_{\rm cr}$ is reached, the mean number of quanta n_b increases abruptly and tends to increase gradually with Ω^2 following the upper branch of the curve. With the decrease in Ω^2, abrupt transition from the upper to the lower branch occurs as another critical value Ω^2_c is reached. Thus, a hysteresis loop arises with low and high levels of excitation of the anharmonic oscillator.

The bistable behavior of the system is due to both the anharmonicity in one of the degrees of freedom Q and the coupling between the degrees of freedom G. This coupling, at certain values of parameters, will result in the formation of portions on the sigmoid curve with the differential gain $dn_b/d\Omega^2 > 1$ (i.e., the slope of the dependence of n_b on Ω^2 can be greater than unity). For the quantum-optical implementation of the model with the Kerr-type medium, this situation bears a close analogy to the well-known phenomenon of the differential amplification of the intensity of a light beam passing through a cavity with a two-level medium as compared to that of an incident beam [9]. For the molecular implementation of the model, such a differential gain implies the acceleration of excitation of the vibrational mode. This phenomenon can be of interest, for example, for the acceleration of chemical reactions with the use of laser excitation.

5. Conclusions

We have studied the dynamics of the quantum nonlinear oscillator with two degrees of freedom in the case of anharmonicity of the simplest type. The general exact analytical solution of the Heisenberg equations of motion for a free oscillator has been found in the semiclassical approximation. This solution describes the periodic energy exchange between the degrees of freedom in terms of the elliptic functions.

Upon the turning on the periodic external force, the problem becomes nonintegrable even though damping is ignored. We have numerically shown that the oscillator vibrations can be regular or chaotic depending on the value of the anharmonicity coefficient and strength of the external excitation. An external difference in the regular and chaotic signals has been demonstrated in calculations of the total number of oscillator excitations or the number of absorbed photons in the case of laser excitation. The calculation of the maximum Lyapunov exponent serves as evidence of this difference. The mechanism of the origination of Hamiltonian chaos in a periodically excited quantum nonlinear oscillator with two degrees of freedom is presumably of the homoclinic nature. In other words, it is associated with the transversal intersection of the stable and unstable manifolds of the hyperbolic singularity. However, this assumption requires analytical and numerical proof, which will be the subject of our subsequent study.

In Sec. 4, we have shown that a periodically excited nonlinear quantum oscillator with two degrees of freedom and relaxation exhibits the bistability effect in certain situations. The prerequisite to the origination of bistability has been analytically obtained. The characteristic sigmoid curves of the bistable dependence of the number of quanta in the anharmonic mode on the square of the laser Rabi frequency have been constructed.

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