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Characterization of Stability for Cone Increasing Constraint Mappings

RENÉ HENRION
Weierstrass Institute for Applied Analysis and Stochastics, 10117 Berlin, Germany
e-mail: henrion@wias-berlin.de

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Abstract. We investigate stability (in terms of metric regularity) for the specific class of cone increasing constraint mappings. This class is of interest in problems with additional knowledge on some nondecreasing behavior of the constraints (e.g. in chance constraints, where the occurring distribution function of some probability measure is automatically nondecreasing). It is demonstrated, how this extra information may lead to sharper characterizations. In the first part, general cone increasing constraint mappings are studied by exploiting criteria for metric regularity, as recently developed by Mordukhovich. The second part focusses on genericity investigations for global metric regularity (i.e. metric regularity at all feasible points) of nondecreasing constraints in finite dimensions. Applications to chance constraints are given.

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1. Introduction

The concept of metric regularity as introduced by Robinson [27] is fundamental for deriving stability results in parametric programming. It is closely related to several other well-known conditions in stability analysis. Recall, for instance, the equivalence between metric regularity and pseudo-Lipschitzness (see [1] and [29]) of multifunctions which was established by Borwein and Zhuang [4] and Penot [26]. For many different areas of optimization theory (smooth, convex, nonsmooth, finite-, infinite-dimensional, semi-infinite, etc.) characterizations of metric regularity in terms of constraint qualifications have been found (e.g. [2, 3, 9, 10, 15, 25, 27–29, 32]). Significant progress in the nonsmooth setting was made by Mordukhovich who established a condition for his coderivative of multifunctions which is an equivalent criterion of metric regularity in finite dimensions [22] and, under additional hypotheses, is at least sufficient in infinite dimensions [24]. For closely related investigations involving Ioffe's approximate coderivative [13], which is the topological counterpart of Mordukhovich's sequentially defined coderivative, we refer to Jourani and Thibault [16, 18].

The purpose of this paper is to demonstrate how the characterization of metric regularity of constraint systems may be improved in case where the constraint mapping has the additional property of being cone increasing. By this, we mean a mapping $f\colon X \to Y$ together with cones $K_x \subseteq X$ and $K_y \subseteq Y$ such that $x_1 - x_2 \in K_x$ implies $f(x_1) - f(x_2) \in K_y$. The motivation for this investigation came from stability analysis of chance constraints [11]. To give a simplified idea, assume that $h\colon \mathbb{R}^n \to \mathbb{R}^m$ is a mapping which indicates the production $h_j(x)$ $(j = 1, \ldots, m)$ of a certain good (e.g. energy) as a function of n decision variables x_i at m different times. Of course, decisions have to be taken in such a way that the production meets the demand ξ_j for this good at all times, so $h(x) \geqslant \xi$ is a natural requirement. Unfortunately, in general, the demand is a random variable which can be observed only after decisions have been taken. Therefore, it is not reasonable to model the constraint in the deterministic way above but rather to replace it by a stochastic formulation like $\mu(h(x) \geqslant \xi) \geqslant p^0$ where μ is a probability measure for the m-dimensional random variable ξ, and p^0 is some fixed probability level. So, the constraint has to be fulfilled with a certain probability at least, i.e. it is a chance constraint. In addition, some nonstochastic constraints (e.g. capacity constraints for the decision variables) may enter the model in the form $x \in C$ where $C \subseteq \mathbb{R}^n$ is some closed subset. It is convenient to reformulate this chance constraint by introducing the distribution function F_μ corresponding to μ which is defined for $y \in \mathbb{R}^m$ as $F_\mu(y) = \mu(\xi \leqslant y)$:

$$(F_\mu \circ h)(x) \geqslant p^0, \quad x \in C. \tag{1}$$

Since the true underlying measure of ξ is not generally known, one usually replaces it by empirical measures which are based on observations of ξ and which may be understood as perturbations of μ. Then, the question of (Lipschitzian) stability of optimal values and local minimizers with respect to such pertubations arises in a problem with a corresponding cost function. As a key result in this direction, Römisch and Schultz [31] showed that the question of stability of the chance constraint w.r.t. perturbations of μ may be reduced to metric regularity of the constraint mapping $(F_\mu \circ h)(x)$ w.r.t. perturbations of the right-hand side probability level (in [31] an equivalent formulation in terms of pseudo-Lipschitzness was used). The study of metric regularity of (1) in a nonsmooth context (note that F_μ is only upper semicontinuous in general and also h might be nonsmooth) has several specific features. First, the constraint mapping has the structure of a composite function, hence nonsmooth chain rules are of interest. Second, as a distribution function, F_μ is automatically nondecreasing, i.e. in the above terminology, it is $(\mathbb{R}^m_+, \mathbb{R}_+)$-cone increasing. In [11], these particular properties were combined with Mordukhovich's coderivative criterion to arrive at verifiable criteria of metric regularity, namely conditions for the density of μ and constraint qualifications for h.

In the first part of this paper, certain ideas of [11] are generalized to a partially infinite-dimensional setting, this means to a finite number of inequality constraints in an infinite-dimensional space. In particular, the information on nondecreasing behavior is used to get a more precise constraint qualification ensuring metric regularity for composite mappings or to characterize metric regularity w.r.t. some unperturbed, fixed set. It is also shown that, for certain cone increasing constraint mappings, the verification of metric regularity via Mordukhovich's coderivative is equivalent to the corresponding condition using Clarke's coderivative, which might be easier to handle. Of course, both criteria differ significantly in general. For other papers, also considering nondecreasing mappings in the context of subdifferentiation, we refer to [7, 8, 21].

The second part of the paper re-addresses the finite-dimensional situation, but from a different viewpoint: Usually, metric regularity of a feasible set mapping is required to hold at the local minimizers of some optimization problem. Since this condition is hard to verify, one could substitute it by a global version, namely metric regularity at all feasible points. This requirement seems to be strong. On the other hand, it is known, that such global metric regularity is a generic property of smooth constraint functions, i.e. in some sense it is typically fulfilled. This follows from the well-known equivalence of metric regularity with the Mangasarian–Fromovitz Constraint Qualification and the fact, that even the stronger Linear Independence Constraint Qualification holds at all feasible points for a generic set of smooth constraint functions (see [14]). A similar result does not hold in the locally Lipschitzian case (see Example 3.12 below). On the other hand, for the particular class of nondecreasing, locally Lipschitzian constraint mappings, genericity properties may be derived again. It turns out that the results are sensitive to the structure of some possible additional fixed constraint set (not subject to perturbations), usually reflecting simple capacity limitations. Referring back to the application in chance constraints of the type (1), special attention is devoted to the subclass of distribution functions.

2. Preliminaries

In this section, some basic concepts from multivalued analysis will be recalled. Let X, Y be arbitrary sets. For a multifunction $\Phi\colon\ X \rightrightarrows Y$ put

$$\begin{aligned}
&\operatorname{Ker}\Phi = \{x \in X \mid 0 \in \Phi(x)\},\\
&\operatorname{Im}\Phi = \{y \in Y \mid y \in \Phi(x), x \in X\},\\
&\operatorname{Gph}\Phi = \{(x,y) \in X \times Y \mid y \in \Phi(x)\},\\
&\Phi^{-1}(y) = \{x \in X \mid y \in \Phi(x)\}.
\end{aligned}$$

Now let X, Y be two normed spaces. A multifunction $\Phi\colon\ X \rightrightarrows Y$ is called *metrically regular* at some point $(x^0, y^0) \in \operatorname{Gph}\Phi$ if there are constants $a > 0$ and $\varepsilon > 0$ such that

$$\operatorname{dist}(x, \Phi^{-1}(y)) \leqslant a \cdot \operatorname{dist}(y, \Phi(x)) \quad \forall (x,y) \in B_\varepsilon(x^0) \times B_\varepsilon(y^0).$$

The abstract form of constraint sets writes as $C \cap F^{-1}(K)$, where $C \subseteq X$ and $K \subseteq Y$ are closed subsets of the respective spaces (K usually being a closed convex cone) and $F\colon X \to Y$ is the constraint function. Then, F is said to be metrically regular with respect to C at some feasible point $x^0 \in C \cap F^{-1}(K)$, if the associated multifunction

$$\Phi(x) = \begin{cases} -F(x) + K, & \text{for } x \in C, \\ \emptyset, & \text{else,} \end{cases}$$

is metrically regular at $(x^0, 0)$. It is easily seen that this is equivalent to the conventional definition of metric regularity for constrained systems

$$\exists \varepsilon > 0\ \exists a > 0\ \forall (x, y) \in (C \cap B_\varepsilon(x^0)) \times B_\varepsilon(0),$$
$$\text{dist}\,(x, C \cap F^{-1}(K - y)) \leqslant a \cdot \text{dist}\,(F(x), K - y).$$

F is simply called metrically regular in the case $C = X$.

Given two cones $K_x \subseteq X$ and $K_y \subseteq Y$, a mapping $f\colon X \to Y$ is called (K_x, K_y)-increasing at some point $\bar{x} \in X$ if there exists some $\varepsilon > 0$ such that

$$x_1 - x_2 \in K_x \Longrightarrow f(x_1) - f(x_2) \in K_y \quad \forall x_1, x_2 \in B(\bar{x}, \varepsilon).$$

For a Banach space X with dual X^* and a multifunction $\Phi\colon X \rightrightarrows X^*$ denote by

$$\limsup_{x \to \bar{x}} \Phi(x) = \{x^* \in X^* \mid \exists x_n \to \bar{x}, \exists x_n^* \overset{*}{\rightharpoonup} x^*, x_n^* \in \Phi(x_n)\}$$

the sequential Kuratowski–Painlevé upper limit with respect to the norm topology in X and the weak-star topology in X^*. To a cone $K \subseteq X$ its polar cone $K^0 \subseteq X^*$ is assigned by $K^0 = \{x^* \in X^* \mid \langle x^*, x \rangle \leqslant 0\ \forall x \in K\}$.

Next, we introduce Mordukhovich's normal cone which is based on the set of Fréchet ε-normals:

DEFINITION 2.1. Let $C \subseteq X$ be a nonempty subset of a Banach space X and $\varepsilon \geqslant 0$.

(1) The set of Fréchet ε-normals ($\varepsilon \geqslant 0$) to C at some $x \in \overline{C}$ is

$$\widehat{N}_\varepsilon(C; x) = \left\{ x^* \in X^* \mid \limsup_{\substack{u \in C \\ u \to x}} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leqslant \varepsilon \right\}.$$

(2) The (Mordukhovich-) normal cone to C at some $\bar{x} \in \overline{C}$ is

$$N(C; x) = \limsup_{\substack{x \to \bar{x} \\ x \in C \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(C; x).$$

In [23] it is shown that for Asplund spaces (i.e. those Banach spaces on which every continuous convex function is Fréchet differentiable at a dense set of points) one can let $\varepsilon = 0$ in the definition of the normal cone. It is noted, that in infinite dimensions, this normal cone lacks the property of being weak star closed unless a normal compactness assumption introduced by Loewen [20] is made for the set C:

DEFINITION 2.2. A closed set $C \subseteq X$ is said to be normally compact around $\bar{x} \in C$ if there exist $\gamma, \sigma > 0$ and a compact set $S \subseteq X$ such that

$$\sigma \|x^*\| \leqslant \max_{s \in S} \langle x^*, s \rangle \quad \forall x^* \in \widehat{N}_0(C; x)\ \forall x \in B(\bar{x}, \gamma) \cap C.$$

In [20] Loewen showed that the multifunction $x \mapsto N(C; x)$ is closed near $\bar{x} \in C$ in the norm x weak star topology of $X \times X^*$ provided that C is normally compact around $\bar{x}$ and that X is a reflexive Banach space. In particular, $N(C; \bar{x})$ is a weak star closed set then.

We also make use of Clarke's tangent cone (see [6]) to a set C at some point $x \in C$:

$$\begin{aligned} T_c(C; x) = \{h \in X \mid \forall x_n \to x(\{x_n\} \subseteq C) \\ \forall t_n \downarrow 0\ \exists h_n \to h : x_n + t_n h_n \in C\} \end{aligned}$$

and of its polar, the Clarke's normal cone $N_c(C; x) = T_c^0(C; x)$. In any Banach space, one has $N(C; x) \subseteq N_c(C; \bar{x})$, while in Asplund spaces, the two introduced normal cones are related by (see [23]) $N_c(C; x) = \overline{\text{co}}^* N(C; x)$, where $\overline{\text{co}}^*$ denotes the weak star closed, convex hull.

With a multifunction Φ: $X \rightrightarrows Y$ one may associate a multifunction $D^*\Phi(\bar{x}, \bar{y})$: $Y^* \rightrightarrows X^*$ at some point $(\bar{x}, \bar{y}) \in \text{Gph}\, \Phi$ which is called the coderivative of Φ and is defined by

$$D^*\Phi(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N(\text{Gph}\, \Phi; (\bar{x}, \bar{y}))\}.$$

The just defined coderivative relates to Mordukhovich's normal cone. If, instead, it relates to Clarke's normal cone N_c, then we shall use the symbol D_c^* for distinction. From the inclusions for the corresponding normal cones it follows that

$$\text{Im}\, D^*\Phi(\bar{x}, \bar{y}) \subseteq \text{Im}\, D_c^*\Phi(\bar{x}, \bar{y}) \quad \text{and} \quad \text{Ker}\, D^*\Phi(\bar{x}, \bar{y}) \subseteq \text{Ker}\, D_c^*\Phi(\bar{x}, \bar{y}).$$

For special multifunctions $\Phi(x) = f(x) + \mathbb{R}^+ = \text{epi}\, f$, where f: $X \to \mathbb{R}$ is lower semicontinuous and 'epi' denotes the epigraph, one gets the corresponding Mordukhovich's and Clarke's subdifferentials

$$D^*\Phi(x, f(x))(1) = \partial f(x) \quad \text{and} \quad D_c^*\Phi(x, f(x))(1) = \partial_c f(x).$$

Moreover, such epigraphical multifunctions satisfy

$$\begin{aligned} &\text{Ker}\, D^*\Phi(\bar{x}, f(\bar{x})) = \{0\} \Longleftrightarrow 0 \notin \partial f(\bar{x}) \\ &\text{and} \\ &\text{Ker}\, D_c^*\Phi(\bar{x}, f(\bar{x})) = \{0\} \Longleftrightarrow 0 \notin \partial_c f(\bar{x}). \end{aligned} \tag{2}$$

The following facts are collected from Mordukhovich [22–24]. The statement of the first theorem is a finite-dimensional reduction of the original result. For a corresponding version obtained by Jourani and Thibault in terms of Ioffes topological coderivative (which we do not consider here), we refer to [16, 17].

THEOREM 2.3. *Let X be an Asplund space and $\Phi\colon X \rightrightarrows \mathbb{R}^m$ a multifunction with closed graph such that $(\bar{x}, \bar{y}) \in \mathrm{Gph}\,\Phi$. Then, the condition*

$$\mathrm{Ker}\, D^*\Phi(\bar{x}, \bar{y}) = \{0\}$$

is sufficient to imply metric regularity of Φ at $(\bar{x}, \bar{y})$. If, moreover, X is finite dimensional, then it is both necessary and sufficient for metric regularity.

THEOREM 2.4. *Let C_1, C_2 be two closed subsets of an Asplund space X such that $\bar{x} \in C_1 \cap C_2$. If one of these sets is normally compact in the sense of Definition* 2.2 *and if the condition*

$$N(C_1; \bar{x}) \cap -N(C_2; \bar{x}) = \{0\}$$

holds, then one has $N(C_1 \cap C_2; \bar{x}) \subseteq N(C_1; \bar{x}) + N(C_2; \bar{x})$.

THEOREM 2.5. *Let $F\colon X \to Y$ be a continuous function between Asplund spaces and $f\colon Y \to \mathbb{R}$ a locally Lipschitzian function. Then, at any fixed $\bar{x} \in X$, one has*

$$\partial(f \circ F)(\bar{x}) \subseteq \bigcup_{y^* \in \partial f(F(\bar{x}))} D^*F(\bar{x}, F(\bar{x}))(y^*).$$

3. Results

3.1. METRIC REGULARITY FOR CONE INCREASING CONSTRAINT MAPPINGS

In this section, we deal with constraint mappings modelling a finite number of inequalities in an infinite-dimensional space with additional cone increasing behaviour. The following simple observation is basic for introducing this information into the characterization of metric regularity:

PROPOSITION 3.1. *Let X, Y be Banach spaces, $K_x \subseteq X$ a closed cone, $K_y \subseteq Y$ a closed, convex cone and $f\colon X \to Y$ a (K_x, K_y)-increasing function around $\bar{x} \in X$. Then, the associated multifunction $\Phi\colon X \rightrightarrows Y$ defined by $\Phi(x) := -f(x) + K_y$ satisfies:*

$$\mathrm{Im}\, D^*\Phi(\bar{x}, \bar{y}) \subseteq \mathrm{Im}\, D^*_c\Phi(\bar{x}, \bar{y}) \subseteq K_x^0 \quad \forall \bar{y} \in \Phi(\bar{x}).$$

Proof. Only the second inclusion has to be shown. Assume that $x^* \in \mathrm{Im}\, D^*_c \Phi(\bar{x}, \bar{y})$, that is, there exists some $y^* \in Y^*$ such that $(x^*, -y^*) \in N_c(\mathrm{Gph}\,\Phi; (\bar{x}, \bar{y}))$. We show that $(h, 0) \in T_c(\mathrm{Gph}\,\Phi; (\bar{x}, \bar{y}))$ for all $h \in K_x$. For any $(x, y) \in \mathrm{Gph}\,\Phi$ in a small neighborhood of $(\bar{x}, \bar{y})$ we have $f(x + h) - f(x) \in K_y$ and $f(x) + y \in K_y$, hence, by convexity of K_y it holds $f(x + h) + y \in K_y$. Therefore, $(x + h, y) \in \mathrm{Gph}\,\Phi$. Now consider arbitrary sequences $(x_n, y_n) \to (\bar{x}, \bar{y})$ ($(x_n, y_n) \in \mathrm{Gph}\,\Phi$) and $t_n \downarrow 0$. Then $(x_n, y_n) + t_n(h, 0) = (x_n + t_n h, y_n) \in \mathrm{Gph}\,\Phi$ (since $t_n h \in K_x$ for all $n \in \mathbb{N}$), so $(h, 0) \in T_c(\mathrm{Gph}\,\Phi; (\bar{x}, \bar{y}))$ and we conclude that $\langle x^*, h\rangle = \langle (x^*, -y^*), (h, 0)\rangle \leqslant 0$ for all $h \in K_x$. Therefore, $x^* \in K_x^0$ as was to be proved. □

COROLLARY 3.2. *Let* X *be a Banach space,* $K_x \subseteq X$ *a closed cone and* $f\colon X \to \mathbb{R}$ *a* $(K_x, \mathbb{R}_+)$*-increasing function around* $\bar{x} \in X$*. Then* $\partial f(\bar{x}) \subseteq \partial_c f(\bar{x}) \subseteq -K_x^0$*. In particular, for* $X = \mathbb{R}^n$ *and* $K_x = \mathbb{R}^n_+$*, one has* $\partial f(\bar{x}) \subseteq \partial_c f(\bar{x}) \subseteq \mathbb{R}^n_+$*.*

Proof. Since $-f$ is $(-K_x, \mathbb{R}_+)$- increasing around $\bar{x}$, it follows from Proposition 3.1 that

$$\partial f(\bar{x}) = D^* f^{\text{epi}}(\bar{x}, f(\bar{x}))(1) \subseteq D_c^* f^{\text{epi}}(\bar{x}, f(\bar{x}))(1) = \partial_c f(\bar{x}) \subseteq -K_x^0,$$

where $f^{\text{epi}}(x) = f(x) + \mathbb{R}_+$. □

The next lemma deals with a constraint mapping having the structure of a composite function with the outer function being cone increasing. This structure is motivated by the chance constraint (1) discussed in the introductory section (recall, that in (1) F_μ as a distribution function is $(\mathbb{R}^m_+, \mathbb{R}_+)$-increasing).

LEMMA 3.3. *Let* $F\colon X \to Y$ *be a continuous function between Asplund spaces,* $K_y \subseteq Y$ *a closed cone, and* $f\colon Y \to \mathbb{R}$ *a locally Lipschitzian function which is* $(K_y, \mathbb{R}_+)$*-increasing. Then, the constraint* $(f \circ F)(x) \geqslant 0$ *is metrically regular at some feasible point* $\bar{x}$ *if* $f(F(\bar{x})) > 0$ *or if, in the binding case, the following two conditions are satisfied:*

(1) $0 \notin \partial(-f)(F(\bar{x}))$.
(2) $0 \notin D^*F(\bar{x}, F(\bar{x}))(y^*) \quad \forall y^* \in K_y^0 \setminus \{0\}$.

Proof. According to the definitions, we have to verify metric regularity of the multifunction $\Phi(x) = -(f \circ F)(x) + \mathbb{R}_+$ at $(\bar{x}, 0) \in \operatorname{Gph} \Phi$. This is clear in the nonbinding case $f(F(\bar{x})) > 0$ where, due to continuity, both distances occurring in the definition of metric regularity equal zero locally. For the binding case we apply Theorem 2.3. The sufficient criterion $\operatorname{Ker} D^*\Phi(\bar{x}, 0) = \{0\}$ for metric regularity is equivalent in the present context to $0 \notin \partial(-(f \circ F))(\bar{x})$. Now, condition (1) above along with Corollary 3.2 (applied to $-f$) give $\partial(-f)(F(\bar{x})) \subseteq K_y^0 \setminus \{0\}$. Therefore, $0 \notin \{D^*F(\bar{x}, F(\bar{x}))(y^*) \mid y^* \in \partial(-f)(F(\bar{x}))\}$ due to condition (2) above, and Theorem 2.5 yields $0 \notin \partial(-(f \circ F))(\bar{x})$, which was to be shown. □

Note, that Lemma 3.3 provides separate constraint qualifications for the two functions in the composition. Condition (2) is substantially improved by introducing additional information on cone increasing behavior. In order to illustrate this fact, assume for a moment, that $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and $K_y = \mathbb{R}^m_+$. Without exploiting the nondecreasing behavior of f, condition (2) would have to be required to hold for all $y^* \in \mathbb{R}^m$, which means the linear independence of the gradients $\nabla F_i(\bar{x})$. But restricting condition (2) to $y^* \in K_y^0 \setminus \{0\} = \mathbb{R}^m_- \setminus \{0\}$ only, amounts to the negative (or, equivalently, positive) linear independence of these gradients. In some sense, one may compare this difference with the one between the Linear Independence and the Mangasarian–Fromovitz Constraint Qualification for inequality systems.

The following proposition is an auxiliary result, and similar versions of it are proved in [12] or [15].

PROPOSITION 3.4. *Let X be a Banach space. With some locally Lipschitzian mapping f: $X \to \mathbb{R}^m$ associate the multifunction Φ: $X \rightrightarrows \mathbb{R}^m$ defined by $\Phi(x) := -f(x) + \mathbb{R}^m_+$. Then, at any $(\bar{x}, \bar{y}) \in \operatorname{Gph}\Phi$, it holds that*

(1) $\bar{y}^* \in \mathbb{R}^m_-$ *and* $\|\bar{x}\|^* \leqslant \eta\|\bar{y}^*\|$ *for all* $(\bar{x}^*, \bar{y}^*) \in N(\operatorname{Gph}\Phi; (\bar{x}, \bar{y}))$ *and for some* $\eta > 0$.
(2) $\operatorname{Gph}\Phi$ *is normally compact around* $(\bar{x}, \bar{y})$.

Proof. Let $L, \varepsilon > 0$ be such that L is a Lipschitz modulus of f in $B(\bar{x}, \varepsilon)$. Consider any $(x, y) \in \operatorname{Gph}\Phi \cap (B(\bar{x}, \varepsilon/2) \times B(\bar{y}, \varepsilon/2))$. Choose an arbitrary $h \in X \setminus \{0\}$. Then, for $0 < t < \varepsilon/(2\|h\|)$, one has

$$f(x + th) + y \geqslant f(x + th) - f(x) \geqslant -Lt\|h\|\mathbf{1} \quad (\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^m).$$

Consequently,

$$(x + th, y + Lt\|h\|\mathbf{1}) \in \operatorname{Gph}\Phi$$

and for any $(x^*, y^*) \in \widehat{N}_\delta(\operatorname{Gph}\Phi; (x, y))$ (with arbitrary $\delta \geqslant 0$) it follows that

$$\begin{aligned}
&\langle x^*, h\rangle + L\|h\|\langle y^*, \mathbf{1}\rangle \\
&\quad = \|(h, L\|h\|\mathbf{1})\| \lim_{t\downarrow 0} \left\langle (x^*, y^*), \frac{(x + th, y + Lt\|h\|\mathbf{1}) - (x, y)}{t\|(h, L\|h\|\mathbf{1})\|} \right\rangle \\
&\quad \leqslant \|(h, L\|h\|\mathbf{1})\| \limsup_{\substack{(x', y') \to (x, y) \\ (x', y') \in \operatorname{Gph}\Phi}} \left\langle (x^*, y^*), \frac{(x', y') - (x, y)}{\|(x', y') - (x, y)\|} \right\rangle \\
&\quad \leqslant \delta\|(h, L\|h\|\mathbf{1})\|.
\end{aligned} \tag{3}$$

Next, fix any index i with $1 \leqslant i \leqslant m$ and observe that $(x, y + te_i) \in \operatorname{Gph}\Phi$, where e_i denotes the ith standard unit vector in $\mathbb{R}^m$ (the remaining variables fixed as above). It follows for any $(x^*, y^*) \in \widehat{N}_\delta(\operatorname{Gph}\Phi; (x, y))$ (where $\delta \geqslant 0$ is arbitrary and brackets refer to the components):

$$y^*[i] = \lim_{t\downarrow 0} \langle (x^*, y^*), t^{-1}((x, y + te_i) - (x, y))\rangle \leqslant \delta \tag{4}$$

with the same argumentation as in (3). Now, corresponding to some $(\bar{x}^*, \bar{y}^*) \in N(\operatorname{Gph}\Phi; (\bar{x}, \bar{y}))$ there exist sequences $(x_n, y_n) \to (\bar{x}, \bar{y}), (x_n^*, y_n^*) \xrightarrow{*} (\bar{x}^*, \bar{y}^*)$ and $\delta_n \downarrow 0$ such that $(x_n, y_n) \in \operatorname{Gph}\Phi$ and $(x_n^*, y_n^*) \in \widehat{N}_{\delta_n}(\operatorname{Gph}\Phi; (x_n, y_n))$. Consequently, for all $h \in X$ (the excluded case $h = 0$ follows trivially), one gets by (3)

$$\begin{aligned}
\langle \bar{x}^*, h\rangle &= \lim_n \langle x_n^*, h\rangle \leqslant \lim_n \{\delta_n\|(h, L\|h\|\mathbf{1})\| - L\|h\|\langle y_n^*, \mathbf{1}\rangle\} \\
&= -L\|h\|\langle \bar{y}^*, \mathbf{1}\rangle
\end{aligned}$$

and by (4): $\bar{y}^*[i] = \lim_n y_n^*[i] \leqslant \lim_n \delta_n = 0$, hence $\bar{y}^* \in \mathbb{R}^m_-$. Finally, interchanging h and $-h$ provides

$$|\langle \bar{x}^*, h\rangle| \leqslant L\|h\|\|\bar{y}^*\|_1 \leqslant L\rho\|h\|\|\bar{y}^*\| \quad \forall h \in X,$$

where $\|\cdot\|_1$ refers to the sum norm and ρ is some modulus of norm equivalence in $\mathbb{R}^m$. Putting $\eta := L\rho$, one arrives at $\|\bar{x}^*\| \leqslant \eta\|\bar{y}^*\|$. It remains to check the last assertion of the proposition. If we reconsider (3) and (4) but with $\delta = 0$, then the same reasoning as in the lines before gives that

$$\begin{aligned} &y^* \leqslant 0 \quad \text{and} \quad \|x^*\| \leqslant \eta\|y^*\|,\\ &\forall (x^*, y^*) \in \widehat{N}_0(\operatorname{Gph}\Phi; (x, y)),\\ &\forall (x, y) \in \operatorname{Gph}\Phi \cap (B(\bar{x}, \varepsilon/2) \times B(\bar{y}, \varepsilon/2)). \end{aligned}$$

Therefore, $\|(x^*, y^*)\| = \|x^*\| + \|y^*\| \leqslant (1+\eta)\|y^*\| \leqslant \tau(1+\eta)\langle -\mathbf{1}, y^*\rangle$, where τ is another modulus of norm equivalence in $\mathbb{R}^m$. Now, normal compactness of $\operatorname{Gph}\Phi$ around $(\bar{x}, \bar{y})$ follows according to Definition 2.2 with $\sigma := \tau^{-1}(1+\eta)^{-1}, \gamma := \varepsilon/2, S := \{(0, -\mathbf{1})\}$. □

The subsequent theorem relates the conditions for Mordukhovich and Clarke's coderivative in the case of cone increasing constraint mappings. It is known that, in general, the criterion $\operatorname{Ker} D_c^*\Phi(\bar{x}, 0) = \{0\}$ is too strong for the characterization of metric regularity. Take, for instance, the one-dimensional multifunction $\Phi(x) = -|x| + \mathbb{R}^+$ which is metrically regular at $(0, 0)$ but where $\operatorname{Ker} D_c^*\Phi(0, 0) = \mathbb{R}_+$ (note, however, that $\operatorname{Ker} D^*\Phi(0, 0) = \{0\}$). On the other hand, the theorem will show that, for certain cone increasing constraints (modelling a finite number of inequalities in an infinite-dimensional space), both conditions are equivalent in order to check metric regularity of the associated multifunction.

THEOREM 3.5. *Let X be a reflexive Banach space, $K_x \subseteq X$ a closed cone with the property*

$$\exists \hat{x} \in X\colon\ \langle x^*, \hat{x}\rangle > 0\ \forall x^* \in K_x^0 \setminus \{0\}, \tag{5}$$

and $f\colon X \to \mathbb{R}^m$ a $(K_x, \mathbb{R}^m_+)$-increasing, locally Lipschitzian mapping around $\bar{x} \in X$. Then, the multifunction $\Phi\colon X \rightrightarrows \mathbb{R}^m$ defined by $\Phi(x) := -f(x) + \mathbb{R}^m_+$ satisfies

$$\operatorname{Ker} D_c^*\Phi(\bar{x}, \bar{y}) = \{0\} \Longleftrightarrow \operatorname{Ker} D^*\Phi(\bar{x}, \bar{y}) = \{0\} \quad \forall \bar{y} \in \Phi(\bar{x})$$

(*note that Φ is the multifunction associated to the constraint system $f_i(x) \geqslant 0$ $(i = 1, \ldots, m)$, in the context of verifying metric regularity*).

Proof. Due to $\operatorname{Ker} D^*\Phi(\bar{x}, \bar{y}) \subseteq \operatorname{Ker} D_c^*\Phi(\bar{x}, \bar{y})$ one has to show the direction '⇐', so assume that $\operatorname{Ker} D^*\Phi(\bar{x}, \bar{y}) = \{0\}$. This is equivalent to $0 \notin$

$D^*\Phi(\bar{x},\bar{y})[S^{m-1}]$ where $S^{m-1} = \{y \in \mathbb{R}^m \mid \|y\|_1 = 1\}$ and $\|\cdot\|_1$ refers to the sum norm in $\mathbb{R}^m$. First note, that $D^*\Phi(\bar{x},\bar{y})[S^{m-1}]$ is weak*-compact. In fact, from Proposition 3.4 we derive that $D^*\Phi(\bar{x},\bar{y})[S^{m-1}] \subseteq B(0,\gamma)$ for some $\gamma > 0$, hence it is bounded. It remains to show the weak*-closedness. Let $x^*_\alpha \xrightarrow{*} x^*$ with $x^*_\alpha \in D^*\Phi(\bar{x},\bar{y})[S^{m-1}]$ be a convergent net. By definition, there exists a net $y^*_\alpha \in S^{m-1}$ such that $(x^*_\alpha, -y^*_\alpha) \in N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y}))$. By the compactness of S^{m-1}, there is a convergent subnet $y^*_{\alpha'} \to y^* \in S^{m-1}$, so $(x^*_{\alpha'}, -y^*_{\alpha'}) \xrightarrow{*} (x^*, -y^*)$ with $(x^*_{\alpha'}, -y^*_{\alpha'}) \in N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y}))$. According to Proposition 3.4, $\mathrm{Gph}\,\Phi$ is normally compact around $(\bar{x},\bar{y})$ (compare Definition 2.2), hence $N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y}))$ is weak star closed. It follows that $(x^*, -y^*) \in N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y}))$, so $x^* \in D^*\Phi(\bar{x},\bar{y})[S^{m-1}]$, which was to be shown.

As a consequence of the weak*-compactness, there is some $\hat{x}^* \in D^*\Phi(\bar{x},\bar{y})[S^{m-1}]$ with $\langle \hat{x}^*, \hat{x}\rangle = \min\{\langle x^*, \hat{x}\rangle \mid x^* \in D^*\Phi(\bar{x},\bar{y})[S^{m-1}]\}$, where $\hat{x}$ refers to (5). Proposition 3.1 provides $D^*\Phi(\bar{x},\bar{y})[S^{m-1}] \subseteq K^0_x$, hence, by assumption, $\hat{x}^* \in K^0_x \setminus \{0\}$. Now (5) yields $\langle \hat{x}^*, \hat{x}\rangle > 0$. Thus, $D^*\Phi(\bar{x},\bar{y})[S^{m-1}] \subseteq H^* := \{x^* \in X^* \mid \langle x^*, \hat{x}\rangle \geqslant \langle \hat{x}^*, \hat{x}\rangle\}$ and $0 \notin H^*$. We are done if we can show that

$$D^*_c\Phi(\bar{x},\bar{y})[S^{m-1}] \subseteq \overline{\mathrm{co}}^* D^*\Phi(\bar{x},\bar{y})[S^{m-1}] \tag{6}$$

since then $D^*_c\Phi(\bar{x},\bar{y})[S^{m-1}] \subseteq H^*$, due to the convexity and the weak*-closedness of H^*. In particular, $0 \notin D^*_c\Phi(\bar{x},\bar{y})[S^{m-1}]$ from which the desired relation $\mathrm{Ker}\, D^*_c\Phi(\bar{x},\bar{y}) = \{0\}$ follows. Before proving (6), the following implication is shown:

$$\begin{aligned} &(x^*, -y^*) \in \mathrm{co}\, N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y})),\\ &y^* \in S^{m-1} \Longrightarrow x^* \in \mathrm{co}\, D^*\Phi(\bar{x},\bar{y})[S^{m-1}]. \end{aligned} \tag{7}$$

To see this, consider any $(x^*, -y^*) \in \mathrm{co}\, N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y}))$ with $y^* \in S^{m-1}$. This means the existence of some $\gamma_i \geqslant 0$ $(i = 1,\dots,k)$ and of $(x^*_i, -y^*_i) \in N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y}))$ such that $\sum_{i=1}^k \gamma_i = 1$ and $(x^*, -y^*) = \sum_{i=1}^k \gamma_i (x^*_i, -y^*_i)$. We may assume that $y^*_i \neq 0$, since otherwise Proposition 3.4 implies $x^*_i = 0$ and the term $(x^*_i, -y^*_i)$ may then be removed from the sum. Furthermore, we know from Proposition 3.4, that $-y^*_i \in \mathbb{R}^m_-$ $(i = 1,\dots,k)$, so $y^* = \sum_{i=1}^k \gamma_i y^*_i$ implies $\|y^*\|_1 = \sum_{i=1}^k \gamma_i \|y^*_i\|_1$. By the cone property of N one has

$$([\|y^*_i\|_1]^{-1} x^*_i, -[\|y^*_i\|_1]^{-1} y^*_i) \in N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y})).$$

Therefore, $w^*_i := [\|y^*_i\|_1]^{-1} x^*_i \in D^*\Phi(\bar{x},\bar{y})[S^{m-1}]$. It results

$$x^* = \sum_{i=1}^k \gamma_i x^*_i = \sum_{i=1}^k \gamma_i \|y^*_i\|_1 [\|y^*_i\|_1]^{-1} x^*_i = \sum_{i=1}^k \delta_i w^*_i,$$

where $\delta_i \geqslant 0$ and $\sum_{i=1}^k \delta_i = \|y^*\|_1 = 1$. Consequently, $x^* \in \mathrm{co}\, D^*\Phi(\bar{x},\bar{y})[S^{m-1}]$.

In order to verify (6), consider $\bar{x}^* \in D_c^*\Phi(\bar{x},\bar{y})(\bar{y}^*)$ with $\bar{y}^* \in S^{m-1}$. Then,

$$(\bar{x}^*, -\bar{y}^*) \in N_c(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y})) = \overline{\mathrm{co}}^* N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y})),$$

so there is a net $(x_\alpha^*, -y_\alpha^*) \xrightarrow{*} (\bar{x}^*, -\bar{y}^*)$ with $(x_\alpha^*, -y_\alpha^*) \in \mathrm{co}\,N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y}))$. Hence, with $(v_\alpha^*, -r_\alpha^*) := ([\|y_\alpha^*\|_1]^{-1}x_\alpha^*, -[\|y_\alpha^*\|_1]^{-1}y_\alpha^*)$, one also has (due to $[\|y_\alpha^*\|_1 \to 1)$

$$(v_\alpha^*, -r_\alpha^*) \in \mathrm{co}\,N(\mathrm{Gph}\,\Phi; (\bar{x},\bar{y})) \quad \text{and} \quad (v_\alpha^*, -r_\alpha^*) \xrightarrow{*} (\bar{x}^*, -\bar{y}^*),$$

but $r_\alpha^* \in S^{m-1}$. From (7), it follows that $v_\alpha^* \in \mathrm{co}\,D^*\Phi(\bar{x},\bar{y})[S^{m-1}]$. Consequently, $\bar{x}^* \in \overline{\mathrm{co}}^* D^*\Phi(\bar{x},\bar{y})[S^{m-1}]$, which finishes the proof. □

COROLLARY 3.6. *If $m = 1$ in the setting of Theorem 3.5, then*

$$0 \in \partial f(\bar{x}) \Longleftrightarrow 0 \in \partial_c f(\bar{x}) \Longleftrightarrow 0 \in \partial_c(-f)(\bar{x}) \Longleftrightarrow 0 \in \partial(-f)(\bar{x}).$$

Proof. Apply Theorem 3.5 along with (2) to see that $0 \in \partial_c(-f)(\bar{x}) \Leftrightarrow 0 \in \partial(-f)(\bar{x})$. Now, f being $(K_x, \mathbb{R}_+)$-increasing, it follows that $-f$ is $(-K_x, \mathbb{R}_+)$-increasing. But, since K_x satisfies (5), the same must hold true for $-K_x$ (take $-\hat{x}$ and note that $(-K_x)^0 = -K_x^0$), hence Theorem 3.5 may also be applied to $-f$, which yields $0 \in \partial f(\bar{x}) \Leftrightarrow 0 \in \partial_c f(\bar{x})$. The remaining equivalence is obvious by $\partial_c(-f)(\bar{x}) = -\partial_c f(\bar{x})$ (cf. [6]). □

COROLLARY 3.7. *Let f: $\mathbb{R}^n \to \mathbb{R}^m$ be a $(\mathbb{R}^n_+, \mathbb{R}^m_+)$-increasing, locally Lipschitzian mapping defining the constraint $f(x) \geqslant 0$. Then f is metrically regular at some feasible point $\bar{x} \in \mathbb{R}^n$, if and only if* $\mathrm{Ker}\,D_c^*\Phi(\bar{x}, 0) = \{0\}$ *for $\Phi(x) = -f(x) + \mathbb{R}^m_+$. In particular, for $m = 1$, one has*

$$f \text{ is metrically regular at } \bar{x} \Longleftrightarrow f(\bar{x}) > 0 \text{ or } 0 \notin \partial_c f(\bar{x}).$$

Proof. By Theorem 2.3, f is metrically regular at $\bar{x}$ if and only if $\mathrm{Ker}\,D^*\Phi(\bar{x}, 0) = \{0\}$. Apply Theorem 3.5. The first equivalence in the last statement follows from $\mathrm{Ker}\,D_c^*\Phi(\bar{x}, 0) = \{0\}$ if $f(\bar{x}) > 0$ and from (2). □

It is clear, that in Theorem 3.5 some cone property has to be required for K_x. Otherwise, one could take the example $f(x) = |x|$ discussed before the statement of the theorem. Here, f is trivially $(0, \mathbb{R}_+)$-increasing, but the equivalence between the two coderivative conditions does not hold for the associated multifunction $\Phi(x) = -|x| + \mathbb{R}^+$. Of course, $K_x = \{0\}$ violates (5). On the other hand, the required cone property is not too restrictive. It holds, in particular for the usual positivity cones $\mathbb{R}^n_+$ or l^p_+, L^p_+ with $2 \leqslant p < \infty$, so it is not necessary – although sufficient – to have nonempty interior.

Theorem 3.5 allows the following characterization of the coderivative of (single-valued) cone increasing functions in terms of subdifferentials of their components:

LEMMA 3.8. *Let X be a reflexive Banach space and F: $X \to \mathbb{R}^m$ a locally Lipschitzian, $(K_x, \mathbb{R}^m_+)$-increasing function with the closed cone K_x satisfying* (5). *Then, the following equivalences hold at any $\bar{x} \in X$* (*with F understood as a single-valued multifunction and $D^*F(\bar{x}) := D^*F(\bar{x}, F(\bar{x}))$*):

$$\begin{aligned}
&\operatorname{Ker} D^*F(\bar{x}) \cap (\mathbb{R}^m_+ \cup \mathbb{R}^m_-)\\
&\quad = \{0\} \Longleftrightarrow 0 \notin \partial F_i(\bar{x}) \ (i = 1, \ldots, m)\\
&\qquad\qquad \Longleftrightarrow 0 \notin \partial(-F_i)(\bar{x}) \ (i = 1, \ldots, m)\\
&\qquad\qquad \Longleftrightarrow \operatorname{Ker} D^*(-F)(\bar{x}) \cap (\mathbb{R}^m_+ \cup \mathbb{R}^m_-) = \{0\}.
\end{aligned}$$

Proof. The last equivalence will follow from the first one, since $-F$ is $(-K_x, \mathbb{R}^m_+)$-increasing and $-K_x$ satisfies (5) if K_x does so. The second equivalence follows from Corollary 3.6, since each F_i: $X \to \mathbb{R}$ is $(K_x, \mathbb{R})$-increasing. To verify the first equivalence, we apply the so-called scalarization formula (cf. [23]) which allows to write the coderivative of a single-valued, locally Lipschitzian mapping with finite-dimensional image space in terms of subdifferentials of linear combinations of the components:

$$\begin{aligned}
D^*F(\bar{x})(y^*) \ &= \ \partial\langle y^* \circ F\rangle(\bar{x}) \subseteq \sum_{i=1}^m \partial(y_i^* \cdot F_i)(\bar{x})\\
&= \begin{cases} \sum_{i=1}^m y_i^* \partial F_i(\bar{x}), & y^* \in \mathbb{R}^m_+,\\ \sum_{i=1}^m (-y_i^*)\partial(-F_i)(\bar{x}), & y^* \in \mathbb{R}^m_-. \end{cases}
\end{aligned}$$

Here, the scalarization is followed by an inclusion resulting from the sum rule and a case distinction which is necessary since, in contrast to Clarke's subdifferential, only positive scalars may be pulled out from the approximate subdifferential. Now the direction '⇒' of the first equivalence is obtained from the scalarization (equation above) along with $0 \notin D^*F(\bar{x})(e_i)$, where e_i denotes the ith standard unit vector of $\mathbb{R}^m$. The reverse direction would follow from the relation $0 \notin \operatorname{co}\{\bigcup_{i=1}^m \partial_a F_i(\bar{x})\} \cup \operatorname{co}\{\bigcup_{i=1}^m \partial_a(-F_i)(\bar{x})\}$. Indeed, in this case, 0 cannot belong to either of the two sums in the case distinction above, provided that $y^* \in \mathbb{R}^m_+ \setminus \{0\}$ or $y^* \in \mathbb{R}^m_- \setminus \{0\}$. Consequently, $0 \notin D^*F(\bar{x})(y^*)$ for all such y^*, which entails the left-hand side of the first equivalence. In order to verify the indicated relation, assume the right-hand side of the first equivalence. The reflexivity of X implies that the subdifferentials $\partial F_i(\bar{x})$ of the locally Lipschitzian functions F_i are weak star compact. Consequently, there exist a $\hat{x}_i^* \in \partial F_i(\bar{x})$ such that

$$\gamma_i = \max\{\langle x^*, \hat{x}\rangle \mid x^* \in \partial F_i(\bar{x})\} = \langle \hat{x}_i^*, \hat{x}\rangle,$$

where $\hat{x}$ refers to (5). On the other hand, each component F_i is $(K_x, \mathbb{R}_+)$-increasing, so Corollary 3.2 along with the assumption $0 \notin \partial F_i(\bar{x})$ gives $-\hat{x}_i^* \in K_x^0 \setminus \{0\}$. Then, (5) provides $\gamma_i < 0$. Setting $\gamma := \max \gamma_i < 0$, we obtain

$$\partial F_i(\bar{x}) \subseteq \{x^* \in X^* \mid \langle x^*, \hat{x} \rangle \leqslant \gamma\} =: H, \quad i = 1, \ldots, m,$$

where H is a convex set not containing zero. Therefore, $0 \notin \text{conv}\,\{\bigcup_{i=1}^m \partial F_i(\bar{x})\}$. At the same time, one has $0 \notin \partial(-F_i)(\bar{x})$ from the already proved second equivalence in the statement of the lemma. Then, the $-F_i$ are $(-K_x, \mathbb{R}^+)$-increasing, such that the analogous argumentation as before – but taking the γ_i as corresponding minima rather than maxima – leads to $0 \notin \text{co}\,\{\bigcup_{i=1}^m \partial(-F_i)(\bar{x})\}$, which finishes the proof. □

The lemma allows to re-address the chain rule of Lemma 3.3, now in a more specific setting, which allows completely to resolve metric regularity of the composite function in terms of subdifferentials.

COROLLARY 3.9. *Let X be a reflexive Banach space and $F\colon X \to \mathbb{R}^m$, $f\colon \mathbb{R}^m \to \mathbb{R}$ locally Lipschitzian functions that are $(K_x, \mathbb{R}_+^m)$- and $(\mathbb{R}_+^m, \mathbb{R}_+)$-increasing, respectively, where K_x satisfies* (5). *Then, the constraint $(f \circ F)(x) \geqslant 0$ is metrically regular at some feasible $\bar{x}$ if $f(F(\bar{x})) > 0$ or if, in the binding case, the conditions*

$$0 \notin \partial(-f)(F(\bar{x})) \quad \textit{and} \quad 0 \notin \partial F_i(\bar{x}) \quad (i = 1, \ldots, m)$$

are fulfilled.

The proof directly results from combining Lemma 3.8 with Lemma 3.3 and noting that $(\mathbb{R}_+^m)^0 = \mathbb{R}_-^m$. By the way, the same conclusion holds true if in the definition of the increasing behavior of f and F, respectively, $\mathbb{R}_+^m$ is replaced by $\mathbb{R}_-^m$ in one or both cases.

Specifying the previous results to the chance constraint (1), where the outer function F_μ is automatically $(\mathbb{R}_+^m, \mathbb{R}_+)$-increasing, one obtains

COROLLARY 3.10. *In the chance constraint* (1) *let $C = \mathbb{R}^n$ and F_μ, h be locally Lipschitzian. Then, this constraint is metrically regular at some feasible point $\bar{x}$ if $F_\mu(h(\bar{x})) > p^0$ or if, in the binding case, the conditions*

$$0 \notin \partial_{(c)}(-F_\mu)(h(\bar{x})) \quad \textit{and} \quad 0 \notin \sum_{i=1}^m \lambda_i \partial h_i(\bar{x}) \ \forall \lambda \in \mathbb{R}_-^m \setminus \{0\}$$

are satisfied. Here, the symbol $\partial_{(c)}$ indicates that both subdifferentials ∂ and ∂_c could be used equivalently. If, moreover, h is $(\mathbb{R}_+^n, \mathbb{R}_+^m)$-increasing (*which might be a reasonable assumption since h measures a kind of production*), *then the second condition is implied by*

$$0 \notin \partial_{(c)} h_i(\bar{x}) \ (i = 1, \ldots, m).$$

The proof follows from Lemma 3.3, Corollary 3.6, Corollary 3.9 and from the scalarization formula (cf. proof of Lemma 3.8).

Now, we turn to a characterization of metric regularity of constraint systems including unperturbed constraint sets. It is clear that for this issue the behavior of the constraint function along the unperturbed set is crucial. We shall call a closed subset $C \subseteq X$ *tangentially generating* at some $\bar{x} \in C$ if Clarke's tangent cone is a generating cone there, i.e. $T_c(C;\bar{x}) - T_c(C;\bar{x}) = X$. This may be understood as a kind of constraint qualification for the set C.

LEMMA 3.11. *Let X be an Asplund space, $f\colon X \to \mathbb{R}^m$ a locally Lipschitzian mapping and $C \subseteq X$ a closed subset that is tangentially generating at some point $\bar{x} \in C$ fulfilling $f(\bar{x}) \geqslant 0$. Then, the constraint $f(x) \geqslant 0$ is metrically regular at $\bar{x}$ with respect to C if*

(1) Ker $D^*\Phi(\bar{x}, 0) = \{0\}$, *where* $\Phi(x) := -f(x) + \mathbb{R}^m_+$.
(2) f *is* $(T_c(C;\bar{x}), \mathbb{R}^m_+)$*-increasing around* $\bar{x}$.

Proof. According to Section 2, we have to show metric regularity of the multifunction

$$\Phi_1(x) := \begin{cases} -f(x) + \mathbb{R}^m_+, & \text{if } x \in C,\\ \emptyset, & \text{else}\end{cases}$$

at the point $(\bar{x}, 0) \in \operatorname{Gph}\Phi_1$. By Theorem 2.3 it remains to check that $\operatorname{Ker} D^*\Phi_1(\bar{x}, 0) = \{0\}$, so choose any $y^* \in \mathbb{R}^m$ with $(0, -y^*) \in N(\operatorname{Gph}\Phi_1; (\bar{x}, 0))$. We have to show that $y^* = 0$. Obviously, we can write $\operatorname{Gph}\Phi_1 = \operatorname{Gph}\Phi \cap (C \times \mathbb{R}^m)$ with Φ as introduced in the statement of the lemma. Now, for arbitrary $(x^*, z^*) \in N(\operatorname{Gph}\Phi; (\bar{x}, 0)) \cap -N(C \times \mathbb{R}^m; (\bar{x}, 0))$ one has $z^* = 0$ due to $N(C \times \mathbb{R}^m; (\bar{x}, 0)) = N(C;\bar{x}) \times \{0\}$. But then, since $(\bar{x}, 0) \in \operatorname{Gph}\Phi$, Proposition 3.4(1) provides $x^* = 0$, hence $N(\operatorname{Gph}\Phi; (\bar{x}, 0)) \cap -N(C \times \mathbb{R}^m; (\bar{x}, 0)) = \{0\}$. Also from Proposition 3.4, we know that $\operatorname{Gph}\Phi$ is normally compact around $(\bar{x}, 0)$. Therefore, Proposition 2.4 yields $(0, -y^*) \in N(\operatorname{Gph}\Phi; (\bar{x}, 0)) + [N(C;\bar{x}) \times \{0\}]$. This means the existence of some $x^* \in -N(C;\bar{x})$ such that $(x^*, -y^*) \in N(\operatorname{Gph}\Phi; (\bar{x}, 0))$, i.e. $x^* \in D^*\Phi(\bar{x}, 0)(y^*)$. By assumption (2) of this lemma and by Proposition 3.1 we know that $x^* \in (T_c(C;\bar{x}))^0 = N_c(C;\bar{x})$. On the other hand, $x^* \in -N(C;\bar{x}) \subseteq -N_c(C;\bar{x})$. Now, $T_c(C;\bar{x})$ is a generating cone, since C is tangentially generating at $\bar{x}$, so its polar $N_c(C;\bar{x})$ must be a pointed cone. Hence, $x^* = 0$ and $y^* \in \operatorname{Ker} D^*\Phi(\bar{x}, 0)$. Now, assumption (1) of this lemma gives $y^* = 0$, which completes the proof. □

Lemma 3.11 decomposes metric regularity of f with respect to C into usual metric regularity of f without regard to C and cone increasing behavior along C. This might facilitate the verification as compared to directly checking the sufficient criterion of Theorem 2.3 for the multifunction Φ_1 defined in the proof of the lemma.

3.2. GLOBAL METRIC REGULARITY OF FINITE-DIMENSIONAL, NONDECREASING CONSTRAINT MAPPINGS

In this section, we study global metric regularity of finite-dimensional, nondecreasing (i.e. $(\mathbb{R}^n_+, \mathbb{R}^m_+)$-increasing) constraint mappings. More precisely, we mean metric regularity w.r.t. C at *all* feasible points of the constraint

$$M = \{x \in \mathbb{R}^n \mid f(x) \geqslant 0, \text{ and } x \in C\}, \tag{8}$$

where $C \subseteq \mathbb{R}^n$ is closed, $f \in \mathcal{C}^{0,1}(\mathbb{R}^n, \mathbb{R}^m)$ and f satisfies $x \geqslant y \Rightarrow f(x) \geqslant f(y)$ with the partial orders of $\mathbb{R}^n, \mathbb{R}^m$, respectively. By $\mathcal{C}^{0,1}(\mathbb{R}^n, \mathbb{R}^m)$ we denote the space of locally Lipschitzian mappings f: $\mathbb{R}^n \to \mathbb{R}^m$. In the case $m = 1$, the symbol $\mathcal{C}^{0,1}(\mathbb{R}^n)$ will be used. We note that the same concept of 'global metric regularity' was used in [22] in a different sense and should not be mixed up with its meaning here.

As mentioned in the introductory section, global metric regularity is a typical or generic property of continuously differentiable constraint mappings. Here, 'generic' refers to the fact, that it is fulfilled for a dense G_δ-set (a countable intersection of open sets) in the space of continuously differentiable mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ endowed with a suitable topology. As we shall see from an example below, a similar statement does not hold true for locally Lipschitzian constraint mappings in a natural topology.

First, we endow $\mathcal{C}^{0,1}(\mathbb{R}^n)$ with a metric. For $f \in \mathcal{C}^{0,1}(\mathbb{R}^n)$ define the function $\psi_f(x) = \max\{\|y\| \mid y \in \partial_c f(x)\}$. Obviously, ψ_f is nonnegative and it is uppersemicontinuous due to the uppersemicontinuity of the set-valued mapping $\partial_c f(\cdot)$. Furthermore, it has the following properties (for arbitrary $f, g \in \mathcal{C}^{0,1}(\mathbb{R}^n)$ and arbitrary $x \in \mathbb{R}^n$):

$$\psi_{f+g}(x) \leqslant \psi_f(x) + \psi_g(x), \tag{9}$$

$$d^H(\partial_c f(x), \partial_c g(x)) \leqslant \psi_{f-g}(x). \tag{10}$$

Here d^H refers to the Hausdorff distance of closed subsets of $\mathbb{R}^n$. Relation (9) is based on the sum rule for Clarke's subdifferential. To see (10), recall the representation of the Hausdorff distance between compact, convex sets by means of their support functionals, which for Clarke's subdifferential is the generalized directional derivative d^0 (cf. [6]). Since d^0 fulfills a triangular inequality w.r.t. f, g for fixed point and direction, we have

$$\begin{aligned} & d^H(\partial_c f(x), \partial_c g(x)) \\ & \quad = \sup_{\|h\| \leqslant 1} |d^0 f(x; h) - d^0 g(x; h)| \\ & \quad \leqslant \sup_{\|h\| \leqslant 1} \max\{d^0(f-g)(x; h), d^0(g-f)(x; h)\} \\ & \quad = \sup_{\|h\| \leqslant 1} \max\{\max\{y(h) \mid y \in \partial_c(f-g)(x)\}, \end{aligned}$$

$$
\begin{aligned}
&\max\{y(h) \mid y \in \partial_c(g-f)(x)\}\} \\
&= \max\{\max\{\sup_{\|h\|\leqslant 1} y(h) \mid y \in \partial_c(f-g)(x)\}, \\
&\qquad \max\{\sup_{\|h\|\leqslant 1} y(h) \mid y \in -\partial_c(f-g)(x)\}\} \\
&= \max\{\max\{\|y\| \mid y \in \partial_c(f-g)(x)\}, \\
&\qquad \max\{\|y\| \mid -y \in \partial_c(f-g)(x)\}\} = \psi_{f-g}(x)
\end{aligned}
$$

which is (10).

Now, put $d_i(f,g) = \max\{\max\{|f(x)-g(x)|, \psi_{f-g}(x)\} \mid x \in B(0,i)\}$ where $B(0,i)$ denotes the closed ball around 0 with radius $i \in \mathbb{N}$. Note that the 'max'-sign is justified by uppersemicontinuity of $\psi_{f-g}(x)$. It is easy to verify, that d_i defines a metric for locally Lipschitzian functions restricted to $B(0,i)$. In fact, reflexivity of d_i follows from the symmetry $-\partial_c f = \partial_c(-f)$, while the triangular inequality basically relies on (9). This metric is reasonable in the sense that the entity $\max\{\psi_f(x) \mid x \in B(0,i)\}$ defines the smallest Lipschitz constant of f on $B(0,i)$. Obviously,

$$d(f,g) = \sum_{i=1}^{\infty} 2^{-i} d_i(f,g)/(1+d_i(f,g)) \tag{11}$$

is a metric on $\mathcal{C}^{0,1}(\mathbb{R}^n)$ and $\sum_{i=1}^m d(f_i,g_i)$ is a metric on $\mathcal{C}^{0,1}(\mathbb{R}^n,\mathbb{R}^m)$ which is compatible with the product topology induced by the metric on the single factors. $(\mathcal{C}^{0,1}(\mathbb{R}^n,\mathbb{R}^m), d)$ is a complete metric space. Furthermore, one easily verifies the property

$$d(f,g) < 2^{1-i}(i \in \mathbb{N}) \Longrightarrow d_i(f,g) \leqslant d(f,g)/(2^{1-i} - d(f,g)) \tag{12}$$

for $f,g \in \mathcal{C}^{0,1}(\mathbb{R}^n)$. The following example shows, that there exists a nonempty open set of functions $f \in \mathcal{C}^{0,1}(\mathbb{R})$ with the property that global metric regularity is violated for the constraint $f(x) \geqslant 0$. This means, for all of these f there is at least one feasible point violating metric regularity. Consequently, the set of functions satisfying global metric regularity cannot contain a dense G_δ (since the complement contains a nonempty open subset).

EXAMPLE 3.12. Define $\hat{f} \in \mathcal{C}^{0,1}(\mathbb{R})$ to meet the property $\partial_c \hat{f}(x) = [-1,1]\ \forall x \in \mathbb{R}$ (see Rockafellar [30] for the construction of such a function). Then, $\hat{f}$ is obiously not constant, so there are $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and, without loss of generality $\hat{f}(x_1) < \hat{f}(x_2)$. Take some $\bar{x} \in (x_1,x_2)$ with $\hat{f}(x_1) < \hat{f}(\bar{x}) < \hat{f}(x_2)$ and define $\bar{f}(x) := \hat{f}(x) - \hat{f}(\bar{x})$. Then, $\bar{f} \in \mathcal{C}^{0,1}(\mathbb{R}), \partial_c \bar{f}(x) = [-1,1]\ \forall x \in \mathbb{R}$ and $\bar{f}(x_1) < \bar{f}(\bar{x}) = 0 < \bar{f}(x_2)$. Choose $i \in \mathbb{N}$ such that $x_1, x_2 \in [-i,i]$ and set

$$V := \{f \in \mathcal{C}^{0,1}(\mathbb{R}) \mid d(f,\bar{f}) < \varepsilon\},$$

where $\varepsilon := 2^{1-i} \min\{1/3, |\bar{f}(x_1)|/(2 + |\bar{f}(x_1)|), |\bar{f}(x_2)|/(2 + |\bar{f}(x_2)|)\} > 0$. V is a nonempty and open subset of $\mathcal{C}^{0,1}(\mathbb{R})$. Suppose for a moment that, for each $f \in V$ there exists some $x \in \mathbb{R}$ such that $f(x) = 0$ and $0 \in \partial(-f)(x)$ (Mordukhovich's subdifferential here). Then (compare Section 2),

$$0 \in \partial(-f)(x) = D^*\text{epi}\,(-f)(x, 0)(1),$$

hence Ker $D^*\text{epi}\,(-f)(x,0) \neq \{0\}$ and the multifunction $\Phi(z) = -f(z) + \mathbb{R}_+$ is not metrically regular at $(x, 0)$ (see the statement of equivalence in Theorem 2.3) or, in other words, for all f from an open set V the constraint $f(x) \geqslant 0$ is not metrically regular at some feasible point x (depending on f). This is what had to be shown by the example.

In order to verify the property used above, choose any $f \in V$. By $\varepsilon < 2^{1-i}$ one may apply (12) to $f, \bar{f}$ and continue the estimation using the definition of ε to arrive at $d_i(f, \bar{f}) < |\bar{f}(x_k)|/2$ $(k = 1, 2)$. In particular, $f(x_1) < 0 < f(x_2)$, so there is some $x^* \in (x_1, x_2)$ with $f(x^*) = 0$. Similarly, (10) along with the definitions of d_i and of ε provide

$$\begin{aligned} d^H(\partial_c f(x), \partial_c \bar{f}(x)) &\leqslant d_i(f, \bar{f}) < d(f, \bar{f})/(2^{1-i} - d(f, \bar{f})) \\ &< \frac{1}{3} 2^{1-i} \Big/ \frac{2}{3} 2^{1-i} = \frac{1}{2} \quad \forall x \in [-i, i]. \end{aligned}$$

Since $\partial_c \bar{f}(x) = [-1, 1]$ $\forall x \in [-i, i]$ and $\partial_c f(x)$ is a closed interval, one derives $[-1/2, 1/2] \subseteq \partial_c f(x)$ $\forall x \in [-i, i]$. Due to $\partial_c(-f)(x) = -\partial_c f(x)$ it follows $[-1/2, 1/2] \subseteq \partial_c(-f)(x)$ $\forall x \in [-i, i]$. Now, from a theorem of Katriel ([19], Theorem 1) it is known, that in the one-dimensional case $\partial_c(-f)$ and $\partial(-f)$ agree on a dense subset of $D \subseteq \mathbb{R}$, hence $[-1/2, 1/2] \subseteq \partial(-f)(x)$ $\forall x \in D \cap [-i, i]$. But the graph of the multifunction $x \mapsto \partial(-f)(x)$ being closed and $D \cap [-i, i]$ being dense in $[-i, i]$, one arrives at $[-1/2, 1/2] \subseteq \partial(-f)(x)$ $\forall x \in [-i, i]$. In particular, $0 \in \partial(-f)(x^*)$. □

This example demonstrates by the way the potential of Mordukhovich's subdifferential as a theoretical tool for characterizing stability (or non-stability). The decisive argument in the example was to exploit the coderivative condition from Theorem 2.3 as a *necessary* criterion for metric regularity in finite dimensions.

Although the example proves global metric regularity not to be a typical property for general, locally Lipschitzian constraint functions, we shall see in the following, that genericity considerations are not in vain for the specific subclass of nondecreasing functions. As a preparatory result, we show

LEMMA 3.13. *For arbitrary compact subsets $K \subseteq \mathbb{R}^n$ it holds that $V^K = \{f \in \mathcal{C}^{0,1}(\mathbb{R}^n) \mid \partial_c f(x) \subseteq \text{int}\,\mathbb{R}^n_+ \ \forall x \in K\}$ is an open subspace of $(\mathcal{C}^{0,1}(\mathbb{R}^n), d)$.*

Proof. Choose any $f \in V^K$. First we show that the function

$$\delta(x) = \min\{\|y - z\| \mid y \in \partial_c f(x), z \in \partial\mathbb{R}^n_+\},$$

where $\partial\mathbb{R}^n_+$ refers to the boundary of the positive orthant of $\mathbb{R}^n$, is lowersemicontinuous (the 'min'-sign is justified by compactness of $\partial_c f(x)$). Now, for any converging sequence $x_k \to \bar{x}$ one has $\delta(x_k) = \|y_k - z_k\|$ with $y_k \in \partial_c f(x_k), z_k \in \partial\mathbb{R}^n_+$. Obviously, $\{y_k\}$ is a bounded sequence due to the upper semicontinuity of $\partial_c f$. On the other hand, $\|z_k\| \leqslant \|z_k - y_k\| + \|y_k\| \leqslant 2\|y_k\|$ (since $0 \in \partial\mathbb{R}^n_+$), hence $\{z_k\}$ is a bounded sequence, too. Consequently, for some subsequences we may assume $y_{k_l} \to \bar{y}, z_{k_l} \to \bar{z}$, where $\bar{y} \in \partial_c f(\bar{x})$ (by closedness of the set-valued mapping $\partial_c f$) and $\bar{z} \in \partial\mathbb{R}^n_+$. Consequently, $\delta(x_{k_l}) = \|y_{k_l} - z_{k_l}\| \to \|\bar{y} - \bar{z}\| \geqslant \delta(\bar{x})$, so δ is lowersemicontinuous.

Next, denote by $\bar{\delta}$ the minimum of δ over the compact set K. Then $f \in V^K$ implies $\bar{\delta} > 0$. Let $l \in \mathbb{N}$ such that $K \subseteq B(0, l)$. We claim that $g \in V^K$ for all $g \in \mathcal{C}^{0,1}(\mathbb{R}^n)$ satisfying $d(f, g) < 2^{1-l}\bar{\delta}/(2 + \bar{\delta})$. In fact, from (10) and (11) one gets for all these g:

$$d^H(\partial_c f(x), \partial_c g(x)) \leqslant \psi_{f-g}(x) \leqslant d_l(f, g) < \bar{\delta}/2 \quad \forall x \in K. \tag{13}$$

On the other hand $g \notin V^K$ would imply the existence of some $y^* \in \partial_c g(x) \setminus \operatorname{int}\mathbb{R}^n_+$ for some $x \in K$. Now, for any $y \in \partial_c f(x)$ we have $y \in \operatorname{int}\mathbb{R}^n_+$, hence there is some $\tau \in [0, 1]$ such that $y^\tau = y^* + \tau(y - y^*) \in \partial\mathbb{R}^n_+$. Then

$$\bar{\delta} \leqslant \delta(x) \leqslant \|y^\tau - y\| = (1 - \tau)\|y^* - y\| \leqslant \|y^* - y\|.$$

Since $y \in \partial_c f(x)$ was arbitrary it results the contradiction to (13)

$$\bar{\delta} \leqslant \min\{\|y^* - y\| \mid y \in \partial_c f(x)\} \leqslant d^H(\partial_c f(x), \partial_c g(x)) < \bar{\delta}/2.$$

Therefore $g \in V^K$ for all $g \in \mathcal{C}^{0,1}(\mathbb{R}^n)$ from the indicated neighborhood of f. □

Next, we introduce the following subsets of $\mathcal{C}^{0,1}(\mathbb{R}^n, \mathbb{R}^m)$:

$$\begin{aligned}
\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m) &= \{f \in \mathcal{C}^{0,1}(\mathbb{R}^n, \mathbb{R}^m) \mid \forall x, y \in \mathbb{R}^n\colon\ x \leqslant y \Rightarrow f(x) \leqslant f(y)\},\\
\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m) &= \{f \in \mathcal{C}^{0,1}(\mathbb{R}^n, \mathbb{R}^m) \mid f_i \text{ is the distribution function}\\
&\qquad \text{of some probability measure on } \mathbb{R}^n \text{ for } i = 1, \ldots, m\},\\
\mathcal{D}(\mathbb{R}^n, \mathbb{R}^m) &= \{f \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m) \mid f_i \text{ has a density for } i = 1, \ldots, m\},\\
V &= \{f \in \mathcal{C}^{0,1}(\mathbb{R}^n) \mid \partial_c f(x) \subseteq \operatorname{int}\mathbb{R}^n_+ \ \forall x \in \mathbb{R}^n\}.
\end{aligned}$$

Along with general nondecreasing functions from $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ we consider the sets $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^m)$ for application to chance constraints. Obviously, with the identification $f = (f_1, \ldots, f_m)$, one may write $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m) = \mathcal{M}(\mathbb{R}^n) \times \cdots \times \mathcal{M}(\mathbb{R}^n)$, and similarly for $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^m)$. Furthermore, $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$, where each of the inclusions may be strict. To see this for the first inclusion, consider the uniform distribution over the line segment $\overline{\{0, 0\}, \{1, 1\}}$ in $\mathbb{R}^2$; then, the distribution function becomes

$$f(x, y) = \min\{1, \max\{0, \min\{x, y\}\}\} \in \mathcal{F}(\mathbb{R}^2, \mathbb{R}),$$

but it does not have a density since the Lebesgue measure of the line segment is zero.

Finally, it holds that

$$V = \bigcap_{i \in \mathbb{N}} V^{B(0,i)}. \tag{14}$$

In the following we consider all these sets as metric subspaces of $(\mathcal{C}^{0,1}(\mathbb{R}^n, \mathbb{R}^m), d)$.

LEMMA 3.14. *$V \cap \mathcal{M}(\mathbb{R}^n), V \cap \mathcal{F}(\mathbb{R}^n)$ and $V \cap \mathcal{D}(\mathbb{R}^n)$, respectively are dense subsets of $\mathcal{M}(\mathbb{R}^n), \mathcal{F}(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$, respectively.*

Proof. Let any $f \in \mathcal{M}(\mathbb{R}^n), \mathcal{F}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)$, respectively, and any $\varepsilon > 0$ be given. It has to be shown, that there exists $g \in \mathcal{M}(\mathbb{R}^n), \mathcal{F}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)$, respectively, such that $g \in V$ and $d(g, f) < \varepsilon$. Define

$$\Phi(x) = (2\pi)^{-n/2} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \mathrm{e}^{-\|y\|^2/2}\, \mathrm{d}y_n \cdots \mathrm{d}y_1.$$

Being the distribution function of some multivariate normal distribution, Φ simultaneously belongs to $\mathcal{D}(\mathbb{R}^n), \mathcal{F}(\mathbb{R}^n)$ and $\mathcal{M}(\mathbb{R}^n)$. Putting

$$g = \frac{f + \gamma\Phi}{1 + \gamma}, \quad \text{where} \quad \gamma > 0, \tag{15}$$

we see that, by convexity of the sets $\mathcal{D}(\mathbb{R}^n), \mathcal{F}(\mathbb{R}^n)$ and $\mathcal{M}(\mathbb{R}^n)$, it holds $g \in \mathcal{M}(\mathbb{R}^n)$, $\mathcal{F}(\mathbb{R}^n)$, $\mathcal{D}(\mathbb{R}^n)$, respectively, whenever $f \in \mathcal{M}(\mathbb{R}^n)$, $\mathcal{F}(\mathbb{R}^n)$, $\mathcal{D}(\mathbb{R}^n)$, respectively. From the equalities

$$|g(x) - f(x)| = \gamma(1+\gamma)^{-1}|f(x) - \Phi(x)| \quad \text{and}$$
$$\psi_{g-f}(x) = \gamma(1+\gamma)^{-1}\psi_{f-\Phi}(x)\ \forall x \in \mathbb{R}^n$$

one derives $d(g, f) < \varepsilon$ as soon as $\gamma > 0$ becomes sufficiently small. It remains to show that $g \in V$. By the sum rule for Clarke's subdifferential, one has for any $x \in \mathbb{R}^n$ and any $\xi \in \partial_c g(x)$, that $\xi = (1+\gamma)^{-1}\xi' + \gamma(1+\gamma)^{-1}\nabla\Phi(x)$ with $\xi' \in \partial_c f(x)$. But $\partial_c f(x) \subseteq \mathbb{R}^n_+$ due to f being $(\mathbb{R}^n_+, \mathbb{R}_+)$-increasing (compare Corollary 3.2) and $\nabla\Phi(x) \in \operatorname{int}\mathbb{R}^n_+$, hence $\xi \in \operatorname{int}\mathbb{R}^n_+$ and $g \in V$ (recall that $\gamma > 0$). □

Before stating the first genericity result, we need a sufficient criterion for metric regularity of (8) which, in contrast to the general coderivative condition of Theorem 2.3, is formulated in terms of subdifferentials of the components f_i of f and of the normal cone to the fixed set C. Using Theorem 2.4, it is easily shown, that the coderivative condition is implied by the relation

$$\sum_{i \in I(\bar{x})} \lambda_i \partial(-f_i)(\bar{x}) \cap N(C; \bar{x}) = \emptyset \quad \forall \lambda_i \leqslant 0\ (i \in I(\bar{x})), \sum_{i \in I(\bar{x})} \lambda_i < 0,$$

where $I(\bar{x}) = \{i \in \{1, \ldots, m\} \mid f_i(\bar{x}) = 0\}$. However, with respect to the metric for locally Lipschitzian functions as introduced above in terms of Clarke's subdifferential, it is more reasonable to replace ∂ and N by ∂_c and N_c in the last relation. Although this would result in a much stronger condition in general, no essential information is lost in the subsequent results. Explicitly, we consider the constraint qualification

$$\sum_{i \in I(\bar{x})} \lambda_i \partial_c f_i(\bar{x}) \cap N_c(C; \bar{x}) = \emptyset \quad \forall \lambda_i \geqslant 0 \ (i \in I(\bar{x})), \sum_{i \in I(\bar{x})} \lambda_i > 0. \tag{16}$$

THEOREM 3.15. *It holds*

(1) $V^K \cap \mathcal{M}(\mathbb{R}^n)$, $V^K \cap \mathcal{F}(\mathbb{R}^n)$, $V^K \cap \mathcal{D}(\mathbb{R}^n)$, *respectively, are open and dense in* $\mathcal{M}(\mathbb{R}^n)$, $\mathcal{F}(\mathbb{R}^n)$, $\mathcal{D}(\mathbb{R}^n)$, *respectively, for all compact sets* $K \subseteq \mathbb{R}^n$.

(2) $V \cap \mathcal{M}(\mathbb{R}^n)$, $V \cap \mathcal{F}(\mathbb{R}^n)$, $V \cap \mathcal{D}(\mathbb{R}^n)$, *respectively, is a dense* G_δ *in* $\mathcal{M}(\mathbb{R}^n)$, $\mathcal{F}(\mathbb{R}^n)$, $\mathcal{D}(\mathbb{R}^n)$, *respectively.*

(3) *The set of functions* $f \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ *for which the constraint set* $f(x) \geqslant 0$ *is globally metrically regular, contains a dense* G_δ *in* $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$. *Similarly, the set of functions* $f \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)(\mathcal{D}(\mathbb{R}^n, \mathbb{R}^m))$ *for which the constraint set* $f(x) \geqslant p$ *is globally metrically regular for all* $p \in \mathbb{R}^m$, *contains a dense* G_δ *in* $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)(\mathcal{D}(\mathbb{R}^n, \mathbb{R}^m))$.

Proof. (1) follows from Lemma 3.13, Lemma 3.14 and $V \subseteq V^K$. (2) follows from (1), (14) and Lemma 3.14. To verify (3) note first that by (2) the set $V' = (V \times \cdots \times V) \cap \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ is a dense G_δ in $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ such that for $f \in V'$ one has $\partial_c f_i(x) \subseteq \operatorname{int} \mathbb{R}^n_+ \ \forall x \in \mathbb{R}^n, \ i = 1, \ldots, m$. On the other hand, there is no additional fixed constraint set C, hence $C = \mathbb{R}^n, N_c(C; x) = 0$ and these functions satisfy:

$$\sum_{i=1}^{m} \lambda_i \partial_c f_i(x) \cap \{0\} = \emptyset \quad \forall x \in \mathbb{R}^n \ \forall \lambda_i \geqslant 0 \ (i = 1, \ldots, m), \sum_{i=1}^{m} \lambda_i > 0. \tag{17}$$

This follows from the fact that a nontrivial positive linear combination of subsets of $\operatorname{int} \mathbb{R}^n_+$ again yields a subset of $\operatorname{int} \mathbb{R}^n_+$ which in particular excludes the origin. But then, by (16), metric regularity of f holds at all feasible points. The same argumentation (using the statements in parantheses of (2)) provides the corresponding assertion when replacing $\mathcal{M}$ by $\mathcal{F}$ or $\mathcal{D}$, respectively. Note, that (17) holds for all $i \in \{1, \ldots, m\}$ (not just for $i \in I(x)$ as required in (16)). Therefore, the actual value of the right-hand side $p \in \mathbb{R}^m$ is irrelevant in the constraint $f(x) \geqslant p$. □

The reason for the slightly different formulations in the third statement of the theorem is that, for $f \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ or $f \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^m)$, it does not make sense to restrict constraints to the form $f(x) \geqslant 0$. Rather, one would consider the inequality $f(x) \geqslant p$ with $0 \leqslant p \leqslant 1$ being some probability level. Formally writing this constraint in zero form is not reasonable since then $f(x) - p$ is no longer a distribution function in general, which has to be nonnegative.

The following example demonstrates that, even in the case $n = m = 1$ the property of global metric regularity for the constraint $f(x) \geqslant 0$ is not open in $\mathcal{D}(\mathbb{R})$. Much less it is open in the bigger sets $\mathcal{F}(\mathbb{R}), \mathcal{M}(\mathbb{R})$ or $\mathcal{C}^{0,1}(\mathbb{R})$.

EXAMPLE 3.16. Let $\phi(x) = (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}$ be the density for the one-dimensional standard normal distribution $\Phi(x) = \int_{-\infty}^{x} \phi(y)\,\mathrm{d}y$ and for $k \in \mathbb{N}$ define

$$\phi_k(x) = \begin{cases} 0 & \text{if } x \leqslant -k, \\ \phi(x)(1 - \Phi(-k))^{-1} & \text{if } x > -k. \end{cases}$$

For $\Phi_k(x) = \int_{-\infty}^{x} \phi_k(y)\,\mathrm{d}y$ it follows

$$\Phi_k(x) = \begin{cases} 0 & \text{if } x \leqslant -k, \\ (\Phi(x) - \Phi(-k))(1 - \Phi(-k))^{-1} & \text{if } x > -k. \end{cases}$$

Therefore $\Phi, \Phi_k \in \mathcal{D}(\mathbb{R})$ $(k \in \mathbb{N})$. Now, some elementary calculation shows that

$$|\Phi(x) - \Phi_k(x)| \leqslant \Phi(-k), \quad \text{and}$$
$$\psi_{\Phi - \Phi_k}(x) \leqslant \max\{\phi(-k), (2\pi)^{-1/2}\Phi(-k)(1 - \Phi(-k))^{-1}\}$$

for all $x \in \mathbb{R}$ and all $k \in \mathbb{N}$. But $\lim_{k\to\infty} \Phi(-k) = \lim_{k\to\infty} \phi(-k) = 0$, therefore we get $\lim_{k\to\infty} d(\Phi_k, \Phi) = 0$. On the other hand, the constraint $\Phi(x) \geqslant 0$ is trivially metrically regular at all feasible points (no binding occurs due to strict positivity of Φ), while all the constraints $\Phi_k(x) \geqslant 0$ fail to be metrically regular at all points $x \leqslant -k$, which are feasible by definition. In fact, the whole interval $(-\infty, -k]$ becomes infeasible after a small right-hand side perturbation of the constraint. Consequently, global metric regularity of the constraint $f(x) \geqslant 0$ cannot be open in $\mathcal{D}(\mathbb{R})$ as far as the metric d is used. □

The phenomenon encountered in the example is also known even for smooth constraint functions. In the smooth case, it is possible to avoid such ‘asymptotic failure’ of global metric regularity by introducing the so-called Whitney topology (see [14]). The introduction of an analogous topology in $\mathcal{C}^{0,1}(\mathbb{R}^n, \mathbb{R}^m)$ in order to arrive at a similar ’open and dense’ result in the present context of nondecreasing functions, seems not to be successful apart from the trivial case $n = m = 1$.

On the other hand, genericity of global metric regularity in terms of an ‘open and dense’ statement can be shown in the presence of additional fixed constraints C (see (8)) with appropriate structure, namely compact box-constraints, which are quite typical for the unperturbed part of feasibility. Note that now, metric regularity w.r.t. a closed subset comes into play.

LEMMA 3.17. *Let $[a, b]$ be a rectangle in R^n with $a \leqslant b$. Then, the set of functions $f \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ for which the constraint $f(x) \geqslant 0$ is globally metrically regular with respect to $[a, b]$ contains an open and dense subset of $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$.*

Proof. Define $W = V^{[a,b]} \cap \{f \in \mathcal{M}(\mathbb{R}^n) \mid f(b) \neq 0\}$. $V^{[a,b]} \cap \mathcal{M}(\mathbb{R}^n)$ is open and dense in $\mathcal{M}$ by item (1) in Theorem 3.15. But $\{f \in \mathcal{M}(\mathbb{R}^n) \mid f(b) \neq 0\}$ is clearly open and dense in $\mathcal{M}(\mathbb{R}^n)$ too (to verify density, add to f a small constant, which of course provides a function still in $\mathcal{M}(\mathbb{R}^n)$). So, W is representable as the intersection of two open and dense subsets of $\mathcal{M}(\mathbb{R}^n)$, hence is itself open and dense in $\mathcal{M}(\mathbb{R}^n)$. Therefore, $W' = W \times \cdots \times W$ is open and dense in $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$. It remains to show, that for all functions $f \in W'$ the constraint $f(x) \geqslant 0$ is metrically regular with respect to $[a, b]$ at all feasible points. Again, as in the proof of Theorem 3.15, one has

$$\sum_{i=1}^{m} \lambda_i \partial_c f_i(x) \subseteq \operatorname{int} \mathbb{R}^n_+, \quad \forall \lambda_i \geqslant 0 \ (i = 1, \ldots, m), \sum_{i=1}^{m} \lambda_i > 0, \forall x \in [a, b].$$

But, due to the simple structure of a rectangle, it holds $N_c([a, b]; x) \cap \operatorname{int} \mathbb{R}^n_+ = \emptyset \ \forall x \in [a, b] \setminus \{b\}$. Hence, (16) holds for all $x \in [a, b] \setminus \{b\}$. Finally, at $x = b$ all components of $f \in W'$ are unequal to zero by definition, so (16) is trivially satisfied by emptyness of the active index set $I(x)$. □

This lemma allows the following generalization to noncompact box constraints (e.g. nonnegativity constraints):

COROLLARY 3.18. *Let* J_1, J_2 *be two not necessarily disjoint subsets of* $\{1, \ldots, n\}$ *and assume*

$$C = \{x \in \mathbb{R}^n \mid x_i \geqslant a_i \ (i \in J_1), \ x_i \leqslant b_i \ (i \in J_2)\}.$$

Then the set of functions $f \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ *for which the constraint* $f(x) \geqslant 0$ *is globally metrically regular with respect to* C *contains a dense* G_δ *in* $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$.

Proof. Put $C_j = \{x \in C \mid x_i \geqslant -j \ (i \notin J_1), x_i \leqslant j \ (i \notin J_2)\}$. Then C_j is a rectangle in $\mathbb{R}^n$ and from the proof of Lemma 3.17 there follows existence of an open and dense set $W_j \subseteq \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$, such that (16) with C replaced by C_j (!) is fulfilled for all $f \in W_j$ at all $x \in C_j$ with $f(x) \geqslant 0$. Now, set $W = \bigcap_{j \in \mathbb{N}} W_j$ and consider an arbitrary function $f \in W$ and an arbitrary $x \in C$ with $f(x) \geqslant 0$. Then, for some $j \in \mathbb{N}$, we have $x \in C_j (\subseteq C)$, hence $N_c(C; x) \subseteq N_c(C_j; x)$. Therefore (16) holds for f at x because of $W \subseteq W_j$. On the other hand, W is a countable intersection of open and dense subsets of $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$. In particular, W is a G_δ-subset. In order to verify density of W note, that the following characterization of nondecreasing functions is valid:

$$f \in \mathcal{M}(\mathbb{R}^n) \Longleftrightarrow \partial_c f(x) \subseteq \mathbb{R}^n_+ \ \forall x \in \mathbb{R}^n. \tag{18}$$

This simple observation is based on Corollary 3.2 and on the mean-value theorem for Clarke's subdifferential (see [6]). From here, it is easily seen that $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ is a closed subspace of $\mathcal{C}^{0,1}(\mathbb{R}^n, \mathbb{R}^m)$ (recall the definition of the metric d in (11) to see from the just given characterization of $\mathcal{M}(\mathbb{R}^n)$, that $f \in \mathcal{M}(\mathbb{R}^n)$ provided that $d(f_n, f) \to 0$ and $f_n \in \mathcal{M}(\mathbb{R}^n)$). In particular, $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ is a Baire space. Therefore, the countable intersection of open and dense subsets is dense itself. □

At this point one could ask in how far the box structure of C in Lemma 3.17 and Corollary 3.18 is necessary to arrive at genericity results for global metric regularity. While this question is not completely clear, the result is negative even for very simple sets C, as far as the use of the sufficient condition (16) for metric regularity is concerned. Although not a proof for nongenericity of global metric regularity with respect to certain closed subsets C, it is a strong indicator at least. More precisely, the following example (where C is a closed halfspace) shows that there may exist a nonempty, open subset of nondecreasing functions such that (16) is violated at least at one feasible point.

EXAMPLE 3.19. As it was shown by Borwein, Moors and Xianfu in [5], for any polytope $P \subseteq \mathbb{R}^n$ there exists a Lipschitzian function h: $\mathbb{R}^n \to \mathbb{R}$, such that $\partial_c h \equiv P$. In particular, we may choose h: $\mathbb{R}^2 \to \mathbb{R}$, such that $\partial_c h \equiv \overline{(2,1);(1,2)}$ where the line on top refers to the corresponding line segment. For the fixed direction $d = (-1, 1)$ Clarke's directional derivative of h is computed at the origin as

$$\limsup_{\substack{y \to 0 \\ t \downarrow 0}} t^{-1}(h(y + td) - h(y)) = \max\{\langle v, d\rangle \mid v \in \partial_c h(0)\} = 1.$$

Consequently, there exist points y_i $(i = 0, 1, 2)$ and numbers $t_2 > t_1 > 0$ with

$$y_i = y_0 + t_i d \ (i = 1, 2) \quad \text{and} \quad h(y_2) > h(y_1) > h(y_0).$$

Now define $f(x) = h(x) - h(y_1)$. Then

$$\begin{aligned} &f(y_2) > f(y_1) = 0 > f(y_0), \quad \text{and} \\ &\partial_c f(x) = \partial_c h(x) = \overline{(2,1);(1,2)} \ \forall x \in \mathbb{R}^2. \end{aligned} \tag{19}$$

By (18) one has $f \in \mathcal{M}(\mathbb{R}^2)$. Take $C = \{x \in \mathbb{R}^2 \mid (1,1) \cdot (x - y_1) \leqslant 0\}$ as an unperturbed set and consider the constraint $f(x) \geqslant 0, x \in C$. Since f is active at y_1, condition (16) at this feasible point means $\partial_c f(y_1) \cap N_c(C; y_1) = \emptyset$ (N_c being a cone). By defintion, however, $N_c(C; y_1)$ is the positive span of $(1,1)$, which of course meets $\partial_c f(y_1)$. Therefore (16) is violated at y_1. Next consider a small (in the sense of the metric (11)) perturbation g of f. On the one hand the functional values of g will be close to those of f. Consequently, since $[y_0; y_2] \subseteq C$, a continuity argument from (19) provides existence of some point $y_3 \in [y_0; y_2]$ such that $g(y_3) = 0$. This means that y_3 is a feasible point of the constraint $g(x) \geqslant 0, x \in C$ and g is active at y_3. On the other hand the deviation of $\partial_c g(y_3)$ from $\partial_c f(y_3) = \overline{(2,1);(1,2)}$ will be small, too (in the sense of Hausdorff distance) so, keeping in mind that Clarke's subdifferential is always convex, the condition $\partial_c g(y_3) \cap N_c(C; y_3) = \emptyset$ – which is condition (16) for g –

is violated at y_3. Note that $N_c(C; y_3)$ is the positive span of $(1, 1)$ again, since $y_3 - y_1$ is a multiple of $d = (-1, 1)$. Summarizing, validity of (16) at all feasible points of the constraint $g(x) \geqslant 0, x \in C$ is violated for locally Lipschitzian g from a whole open neighborhood of the nondecreasing function f. □

The results obtained so far suggest that genericity of global metric regularity of nondecreasing constraint functions hinges on presence or absence as well as the structure of additional unperturbed constraints. In the remainder, we re-address the chance constraint (1) in terms of global metric regularity. Now, both F_μ and h are assumed to be locally Lipschitzian. Let us first disregard the subset C. By Corollary 3.10, global metric regularity of (1) holds, provided that

(1) $0 \notin \partial_c F_\mu(y)$ for all $y \in \mathbb{R}^m$ with $F_\mu(y) = p^0$.
(2) $0 \notin \sum_{i=1}^{m} \lambda_i \partial h_i(x) \quad \forall \lambda \leqslant 0 \ (\lambda \neq 0)$ for all $x \in \mathbb{R}^n$ with $F_\mu(h(x)) = p^0$

(note, that the condition $0 \notin \partial_c(-F_\mu)(h(\bar{x}))$ in Corollary 3.10 is equivalent to $0 \notin \partial_c F_\mu(h(\bar{x}))$ due to the rule $\partial_c(-f) = -\partial_c f$ of Clarke's subdifferential). From Theorem 3.15(2) we see, that the first condition is generic in the class $\mathcal{F}(\mathbb{R}^m)$ of locally Lipschitzian distribution functions over $\mathbb{R}^m$ as well as in the subclass $\mathcal{D}(\mathbb{R}^m)$ of those distribution functions having a density. The separate genericity of the second condition in terms of h (with a given F_μ) cannot be derived in general. Indeed, it is easy to construct a counterexample on the basis of Example 3.12. However, under the restriction $h \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$, which might be reasonable since h is a kind of production function, the second condition is implied by $0 \notin \partial_c h_i(x),\ i = 1, \ldots, m,\ \forall x \in \mathbb{R}^n$ (cf. Corollary 3.10), which, again by Theorem 3.15(2), is a generic property (see proof of (17)).

Now, we include an additional deterministic constraint C into the considerations. For the case of C having a simple box structure, Lemma 3.17 has shown that global metric regularity is satisfied for an open and dense subset of nondecreasing mappings. But note that, in the concrete context of the chance constraint (1), the constraint mapping has a composite structure, so genericity has to be formulated in terms of F_μ and h. This is done in the next theorem.

THEOREM 3.20. *In the chance constraint* (1) *assume that* $C = [a, b]$ $(a, b \in \mathbb{R}^n, a \leqslant b)$. *Denote by* P *the set of functions* $(F_\mu, h) \in \mathcal{F}(\mathbb{R}^m) \times \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ (*or* $\in \mathcal{D}(\mathbb{R}^m) \times \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$, *respectively*), *such that the constraint* $F_\mu(h(x)) \geqslant p$ *is globally metrically regular with respect to* C. *Then,* P *contains a subset* W *which is open and dense in* $\mathcal{F}(\mathbb{R}^m) \times \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ (*or in* $\mathcal{D}(\mathbb{R}^m) \times \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$, *respectively*) *with the topology induced by the metric* (11).

Proof. The proof for the case $F_\mu \in \mathcal{D}(\mathbb{R}^m)$ runs exactly along the same lines as for $F_\mu \in \mathcal{F}(\mathbb{R}^m)$, hence we restrict considerations to the latter case. Define the set

$$W = \{(F, g) \in \mathcal{F}(\mathbb{R}^m) \times \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m) \mid \begin{array}{l} 1.\ \partial_c F(y) \subseteq \operatorname{int} \mathbb{R}^m_+ \ \forall y \in g([a,b]), \\ 2.\ \partial_c g_i(x) \subseteq \operatorname{int} \mathbb{R}^n_+ \ \forall x \in [a,b] \\ \quad (i = 1, \ldots, m), \\ 3.\ F(g(b)) \neq p \}. \end{array}$$

First we show that W is open and dense in $\mathcal{F}(\mathbb{R}^m) \times \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$. To verify openness, let $(\overline{F}, \bar{g}) \in W$ be arbitrarily given. Due to the compactness of $\bar{g}([a, b])$ and because of the upper semicontinuity of $\partial_c \overline{F}$, there exists a compact neighborhood $\mathcal{O}$ of $\bar{g}([a, b])$, such that $\partial_c \overline{F}(y) \subseteq \operatorname{int} \mathbb{R}^m_+ \ \forall y \in \mathcal{O}$. Denote by T an open neighborhood of $\bar{g}$ in $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ such that $g([a, b]) \subseteq \mathcal{O} \ \forall g \in T$. Introducing the sets

$$\begin{aligned} V &= \{g \in T \mid \partial_c g_i(x) \subseteq \operatorname{int} \mathbb{R}^n_+ \ \forall x \in [a,b] \ (i = 1, \ldots, m)\} \\ U &= \{F \in \mathcal{F}(\mathbb{R}^m) \mid \partial_c F(y) \subseteq \operatorname{int} \mathbb{R}^m_+ \ \forall y \in \mathcal{O}\} \\ S &= \{(F, g) \in \mathcal{F}(\mathbb{R}^m) \times \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m) \mid F(g(b)) \neq p\}, \end{aligned}$$

we see that $(\overline{F}, \bar{g}) \in (U \times V) \cap S$. From Lemma 3.13 we know that U is open in $\mathcal{F}(\mathbb{R}^m)$ and V is open in $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ and, obviously, S is open in $\mathcal{F}(\mathbb{R}^m) \times \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$. From the very definitions it follows that $(U \times V) \cap S \subseteq W$, hence $(\overline{F}, \bar{g})$ is an interior point of W, which means that W is open.

To check the density of W, consider an arbitrary pair $(\overline{F}, \bar{g}) \in \mathcal{F}(\mathbb{R}^m) \times \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$. By Theorem 3.15(1), a small perturbation of $\bar{g}$ in $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ suffices to provide some $g^1 \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ with property (2) in the definition of W. Again, Theorem 3.15(1) yields the existence of some $F^1 \in \mathcal{F}(\mathbb{R}^m)$ which is arbitrarily close to $\overline{F}$ and satisfies $\partial_c F^1(y) \subseteq \operatorname{int} \mathbb{R}^m_+ \ \forall y \in g^1([a, b])$, which is property (1) for the pair (F^1, g^1) in the definition of W.

Now, suppose that $F^1(g^1(b)) = p$. For some $c \in \operatorname{int} \mathbb{R}^m_+$, put $g^2(x) = g^1(x) + c \ \forall x \in \mathbb{R}^n$. Obviously, $g^2 \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$ and g^2 is close to g^1 if $\|c\|$ is chosen to be small. Furthermore, for small $\|c\|$, the pair (F^1, g^2) satisfies the first two properties of the definition of W (the openness of these two properties was implicitly shown already in the verification of the openness of W itself). Finally, exploiting the upper semicontinuity of $\partial_c F^1$ and the first property of W, let $\|c\|$ be small enough to meet

$$0 \notin \partial_c F^1(g^1(b) + tc) \ \forall t \in [0, 1].$$

On the other hand, $\partial_c F^1(y) \subseteq \mathbb{R}^m_+ \ \forall y \in \mathbb{R}^m$ (see Corollary 3.2) due to F^1 being nondecreasing. But then, the mean value theorem for Clarke's subdifferential provides (for some $t' \in [0, 1]$):

$$\begin{aligned} &F^1(g^2(b)) - p \\ &\quad = F^1(g^2(b)) - F^1(g^1(b)) \in \langle \partial_c F^1(g^1(b) + t'c), c \rangle \in \mathbb{R}_+ \setminus \{0\}, \end{aligned}$$

hence $F^1(g^2(b)) > p$. Summarizing, $(F^1, g^2) \in W$, and (F^1, g^2) is arbitrarily close to $(\overline{F}, \bar{g})$, which means the density of W.

To finish the proof, we have to show that W is contained in the set P defined in the theorem, so let some pair $(\overline{F}, \bar{g}) \in W$ be arbitrarily given. From the chain rule of Clarke's subdifferential along with the definition of W, it follows that

$$\begin{aligned}&\partial_c(\overline{F} \circ \bar{g} - p)(x) \\ &\quad = \partial_c(\overline{F} \circ \bar{g})(x) \subseteq \overline{\text{conv}}\left\{ \sum_{i=1}^{m} \lambda_i \partial_c \bar{g}_i(x) \mid \lambda \in \partial_c \overline{F}(\bar{g}(x)) \right\} \\ &\qquad\qquad \subseteq \text{int}\, \mathbb{R}^n_+ \quad \forall x \in [a, b].\end{aligned}$$

The last inclusion follows from the fact that the occurring subdifferentials are compact, that $\lambda_i > 0$ (first property of W) and that $\partial_c \bar{g}_i(x) \subseteq \text{int}\, \mathbb{R}^n_+$ (second property of W). Now, the specific structure of box constraints gives $N_c([a, b]; x) \cap \text{int}\, \mathbb{R}^n_+ = \emptyset \ \forall x \in [a, b] \setminus \{b\}$, and, exploiting $\overline{F}(\bar{g}(b)) \neq p$ (third property of W), we see that the sufficient criterion (16) for metric regularity is satisfied, thus $(\bar{F}, \bar{g}) \in P$. □

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