Journal of Mathematical Imaging and Vision 7, 85–102 (1997)

Locality and Adjacency Stability Constraints for Morphological Connected Operators

JOSÉ CRESPO
Departamento de Lenguajes y Sistemas Informáticos e Ingeniería de Software, Facultad de Informática, Universidad Politécnica de Madrid, 28660 Boadilla del Monte (Madrid), Spain

RONALD W. SCHAFER
School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.

Abstract. This paper investigates two constraints for the connected operator class. For binary images, connected operators are those that treat grains and pores of the input in an all or nothing way, and therefore they do not introduce discontinuities. The first constraint, called connected-component (c.c.) locality, constrains the part of the input that can be used for computing the output of each grain and pore. The second, called adjacency stability, establishes an adjacency constraint between connected components of the input set and those of the output set. Among increasing operators, usual morphological filters can satisfy both requirements. On the other hand, some (non-idempotent) morphological operators such as the median cannot have the adjacency stability property. When these two requirements are applied to connected and idempotent morphological operators, we are lead to a new approach to the class of filters by reconstruction. The important case of translation invariant operators and the relationships between translation invariance and connectivity are studied in detail. Concepts are developed within the binary (or set) framework; however, conclusions apply as well to flat non-binary (gray-level) operators.

Keywords: connectivity, mathematical morphology, connected operator, connected-component locality, adjacency stability

Connected operators do not introduce discontinuities. For binary images (or sets), they treat the connected components of the input and its complement in an all or nothing way. The relationship between the general class of connected operators and morphological connected filters will be investigated in this paper. This will be done by presenting two constraints called *connected-component* (*c.c.*) *locality* and *adjacency stability*.

Connectivity plays an important role in this paper, and we are going to be interested in those openings (or respectively closings) that exclusively remove grains (respectively fill pores) of the input set. These are all connected operators (but they are not, of course, the only ones). An important group of connected filters is the class of filters by reconstruction [2, 4, 26, 27]. In this work (as in [4]), filters by reconstruction are those combinations of openings $\tilde{\gamma}$ and closings $\tilde{\varphi}$ by reconstruction that are idempotent. The *actions* of the openings $\tilde{\gamma}$ and closings $\tilde{\varphi}$ by reconstruction are:

(a) removing grains using $\tilde{\gamma}$;
(b) filling pores using $\tilde{\varphi}$; and
(c) both removing grains and filling pores using $\tilde{\varphi}\tilde{\gamma}$, $\tilde{\gamma}\tilde{\varphi}$, $\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$ [5], etc., where $\tilde{\gamma}_i$ and $\tilde{\varphi}_i$ belong respectively to a granulometry $\{\tilde{\gamma}_i\}$ and an anti-granulometry $\{\tilde{\varphi}_i\}$ by reconstruction [16, 26].

We might think that by combining (as in group (c) above) the basic openings $\tilde{\gamma}$ and closings $\tilde{\varphi}$, it would be possible to remove grains and fill pores in such a way that any possible connected operation (i.e., an operation that does not introduce discontinuities) can be implemented. However, this is not true. Figure 1 shows a case in which it is not possible to obtain, for

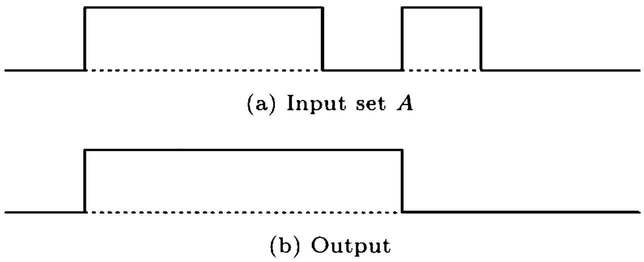


Figure 1. Connected operation: one-dimensional example. For the input shown in part (a), no combination of openings and closings can compute the output (b).

the input in Fig. 1(a), the output displayed in Fig. 1(b) by means of any combination of openings and closings. Figure 2 gives a more complete insight into the problem by showing all possible outputs of connected operators acting on a simple study case: two grains and one pore in a connected one-dimensional space. There are $2^3 = 8$ possibilities; there are three flat zones and there exists two possibilities for each, either being in the output set or in its complement. The following questions, not treated in the mathematical morphology literature, arise then:

- Why is not possible to compute the output shown in Fig. 1 by means of openings and closings by reconstruction?
- Is there any relation between Fig. 2 and morphological connected filters? Which outputs can be computed using $\tilde{\gamma}$ and $\tilde{\varphi}$?
- Is there some reason why all classes of "usual" morphological filters are combinations of extensive and anti-extensive operations?

In this paper, we will address these questions by studying some properties that are satisfied by the usual morphological filters. Our study will be simplified by focusing on the so-called c.c. local operators. Then, adjacency relationships between grains and pores of the input set and the output of a connected operator will be studied.

This paper extends part of the thesis work by Crespo in [2]. The concepts treated in this paper were introduced by Crespo, Serra and Schafer in [5]. The c.c. locality and adjacency stability constraints treated in this paper are most meaningful when applied to connected operators. Nevertheless, they can also be used for non-connected operators. These requirements will allow us to obtain some interesting properties of connected operators that satisfy one or both of them. An

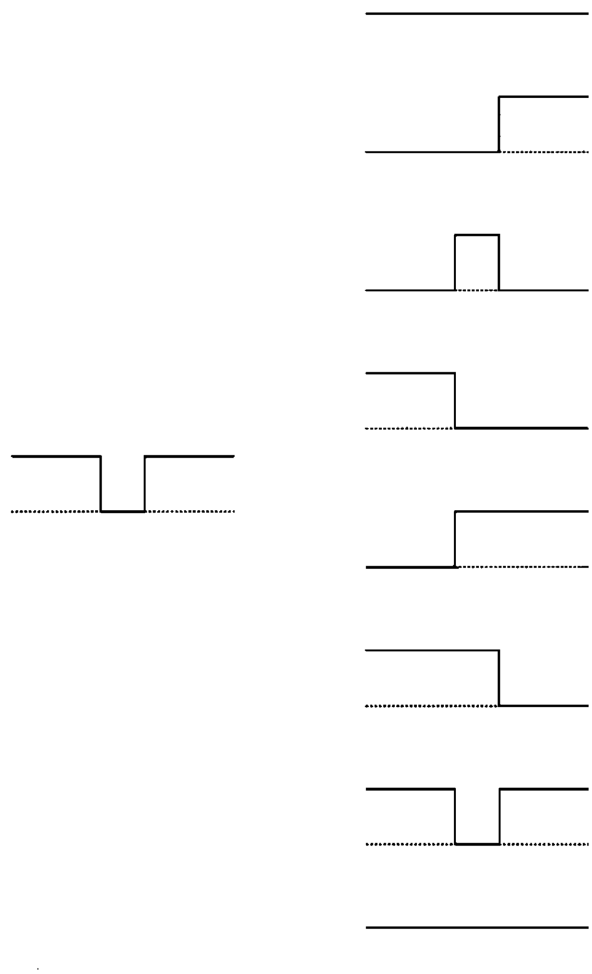


Figure 2. Stability and connected operators: one-dimensional example. For the input set on the left (a non-connected set), there exists eight possible outputs of connected operators. Which outputs cannot be computed using some combination of openings and closings?

important result is the discovery that it is not possible to compute any arbitrary connected operator by means of openings and closings by reconstruction.

This result also applies to the more general case of non-connected operators: it is not possible to compute any arbitrary (connected or not) operator solely by means of (connected or not) openings and closings. When these constraints are applied to morphological filters, it will be observed that openings and closings arise naturally as the building blocks for morphological filtering. If the restriction of translation invariant operators to subspaces (more precisely, their re-definition) is required to be *well behaved* (in the sense that the re-defined operator operates in the same way in subspaces), we find the class of *openings* and *closings by reconstruction* as the only group of connected filters that are both c.c. local and adjacency stable. Thus, in this work we approach the class of morphological filters in an alternative way to the axiomatic way (by means of their definitions) that is normally used in mathematical morphology. The relationship between translation invariance and connectivity is examined in this paper.

The outline of the paper is as follows. Section 1 gives some background on mathematical morphology and on connected operators. Section 2 introduces and investigates the c.c. locality and adjacency stability conditions. The translation invariant operator case is treated in Section 3, which includes a study of the relationships between translation invariance and connectivity. Proofs are included in the paper.

1. Background on Mathematical Morphology

Mathematical morphology is concerned with the application of set theory to image analysis. Morphological signal and image processing rests on a framework established by Matheron and Serra [6, 12–16, 22, 24]. This section offers some background on morphological filtering and on connected operators.

Connected operators [2, 26] are those that do not introduce discontinuities. When they are applied to binary images, for example, either connected components of the foreground (*grains*) are removed or those of the background (*pores*) are filled. They are called morphological when they are *increasing*. Morphological connected filters are those morphological connected operators that are *idempotent*.

Filters by reconstruction [2, 4, 26, 27] are a class of connected filters that are composed of *openings* and *closings by reconstruction*, denoted in the following by $\tilde{\gamma}$ and $\tilde{\varphi}$. When applied to a binary image (a concept equivalent to that of a set), the openings and closings by reconstruction treat each grain or pore independently from the rest of grains or pores.

1.1. Morphological Filtering

This background section reviews some concepts regarding morphological filtering [9, 17–20, 22–24]. Morphological operators operate on an algebraic structure called a *complete lattice* [1, 24], which is the minimal structure required.

Definition. A set T is a complete lattice if: (a) there exists a partial ordering $\leq$ over T; and (b) for any family $\{A_i\}$ of elements in T, there exists: a smallest majorant $\bigvee_i A_i$ called the "sup" (for supremum), and a greatest minorant $\bigwedge_i A_i$ called the "inf" (for infimum).

In all theoretical expressions in this paper, we will be working on the lattice $\mathcal{P}(E)$, where E is a given set of points called *space* and $\mathcal{P}(E)$ denotes the set of all subsets of E (i.e., $\mathcal{P}(E) = \{A : A \subseteq E\}$). In other words, inputs and outputs will be supposed to be sets or, equivalently, binary functions. In this lattice, the sup $\bigvee$ and the inf $\bigwedge$ operations are the set union $\bigcup$ and the set intersection $\bigcap$ operations, while the order relation is the set inclusion relation $\subseteq$. Even though we will work on the lattice $\mathcal{P}(E)$, results are extendable for gray-level functions by means of the so called flat operators [7, 11, 12, 22, 25].

Mathematical morphology deals with *increasing* mappings. A mapping (or transformation) ψ is increasing if it preserves ordering, i.e., if two inputs are ordered then their outputs are likewise ordered. For an increasing set operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$, $A \leq B \Rightarrow \psi(A) \leq \psi(B)$, where $A, B \in \mathcal{P}(E)$. The sup, the inf and the sequential composition of increasing operators is increasing.

Two elementary morphological operations are *erosions* and *dilations*, denoted respectively by ε and δ.

Definition. Let E be any space. The mapping $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ that commute with the inf (or respectively the sup) are called **erosions** ε (respectively **dilations** δ). That is, for all $A_i \in \mathcal{P}(E)$, $\varepsilon(\bigwedge_i A_i) = \bigwedge_i \varepsilon(A_i)$ (respectively $\delta(\bigvee_i A_i) = \bigvee_i \delta(A_i)$).

Before defining what a morphological filter is, let us establish the idempotence concept. A transformation ψ is *idempotent* if when ψ is applied twice it leaves the first output unchanged. Mathematically, this can be expressed as $\psi\psi(A) = \psi(A)$, $\forall A \in \mathcal{P}(E)$, or as $\psi\psi = \psi$.

Definition. A mapping ψ is a **morphological filter** if and only if ψ is increasing and idempotent.

In general when we refer to *filters* we will have the meaning of the previous definition.

An operator ψ is *anti-extensive* (or respectively *extensive*) if $\psi \leq I$ (respectively $\psi \geq I$), where I represents the identity operator (for all $A \in \mathcal{P}(E)$, $I(A) = A$). Notice that, for two operators ψ_1 and ψ_2 defined 2from $\mathcal{P}(E)$ to $\mathcal{P}(E)$, the order relation $\psi_1 \leq \psi_2$ (or respectively $\psi_1 \geq \psi_2$) means that $\psi_1(A) \leq \psi_2(A)$ (respectively $\psi_1(A) \geq \psi_2(A)$) for all $A \in \mathcal{P}(E)$.

Definition. An **opening** γ (or respectively a **closing** φ) is an antiextensive (respectively extensive) filter.

The alternating compositions of an opening and a closing $\varphi\gamma$ and $\gamma\varphi$ are idempotent; i.e., they are filters, called *alternating filters*.

Each morphological operation has a *dual* operation. Two operators ψ_1 and ψ_2 are the dual of each other if $\psi_1 = I^c\psi_2 I^c$, where I^c is the complementation operator, and vice-versa. (For all $A \in \mathcal{P}(E)$, $I^c(A) = A^c$, where A^c is the complement of A.)

Some morphological filters show a robustness property called the strong property [20]. A filter ψ is strong if it is both an $\bigwedge$-filter (i.e., if $\psi = \psi(I \bigwedge \psi)$) and a $\bigvee$-filter (i.e., if $\psi = \psi(I \bigvee \psi)$). That is, ψ is a strong filter if

$$\psi = \psi\left(I \bigwedge \psi\right) = \psi\left(I \bigvee \psi\right). \tag{1}$$

1.2. Connectivity in Mathematical Morphology

Connectivity is introduced in mathematical morphology by the operation that extracts the *connected components* of a set. As will be seen in this section, those operators that do not break the connected components of either the foreground or the background of an image are called *connected operators*.

The Point Opening γ_x. Connectivity is established in [24] by means of the *connected class* concept. A connected class $\mathcal{C}$ in $\mathcal{P}(E)$ is a subset of $\mathcal{P}(E)$ such that (a) $\emptyset \in \mathcal{C}$ and for all $x \in E$, $\{x\} \in \mathcal{C}$; and (b) for each family C_i in $\mathcal{C}$, $\bigwedge_i C_i \neq \emptyset$ implies $\bigvee_i C_i \in \mathcal{C}$. No definition of neighborhood relationships (i.e., no particular topology) has been assumed for E in the definition of the connected class $\mathcal{C}$.

The subclass $\mathcal{C}_x$ that has all members of $\mathcal{C}$ that contain x (i.e., $\mathcal{C}_x = \{C : x \in C \in \mathcal{C}\}$) defines an opening called a *point opening* [21]. The point opening of a point x, denoted by γ_x, has as invariant class (i.e., the class formed by those sets that are left unchanged by γ_x) $\mathcal{C}_x \cup \{\emptyset\}$. For all $x \in E$, $A \in \mathcal{P}(E)$

$$\gamma_x(A) = \bigvee\{C : C \in \mathcal{C}_x, C \leq A\}. \tag{2}$$

The operation γ_x is therefore idempotent (i.e., $\gamma_x(\gamma_x(A)) = \gamma_x(A)$ or, equivalently, $\gamma_x\gamma_x = \gamma_x$) and antiextensive (i.e., $\gamma_x(A) \leq A$ or, equivalently, $\gamma_x \leq I$). Properties satisfied by $\gamma_x(A)$ are:

(a) $\forall x \in E, \gamma_x(\{x\}) = \{x\}$.
(b) $\forall A \in \mathcal{P}(E)$, $\forall x, y \in E$, $\gamma_x(A)$ and $\gamma_y(A)$ are equal or disjoint.
(c) $\forall A \in \mathcal{P}(E)$, $x \notin A$ implies $\gamma_x(A) = \emptyset$.

When we associate, for example, the operation γ_x with the usual connectivity in $\mathbf{Z}^2$, the opening $\gamma_x(A)$, $A \in \mathcal{P}(\mathbf{Z}^2)$, can be defined as the union of all paths that contain x and that are included in A. Figure 3 shows an example of $\gamma_x(A)$ where x belongs to A. It can be seen that the point opening γ_x simply has the effect of selecting the connected component of A to which x belongs. A simple way to implement the $\gamma_x(A)$ operation is by iterating the *geodesic dilation* of the set $\{x\}$ inside A until idempotence [8, 10, 22].

The dual operation of γ_x is the closing φ_x, which is equal to $E \backslash \gamma_x I^c(A)$, for all $A \in \mathcal{P}(E)$, where $\backslash$ denotes set difference. Figure 4 shows a one-dimensional example of both dual operations γ_x and φ_x, along with the pore extraction operation.

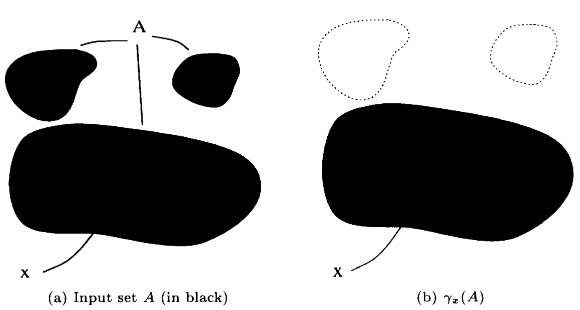


Figure 3. Connected component extraction. The opening $\gamma_x(A)$ extracts the connected component of A to which x belongs.

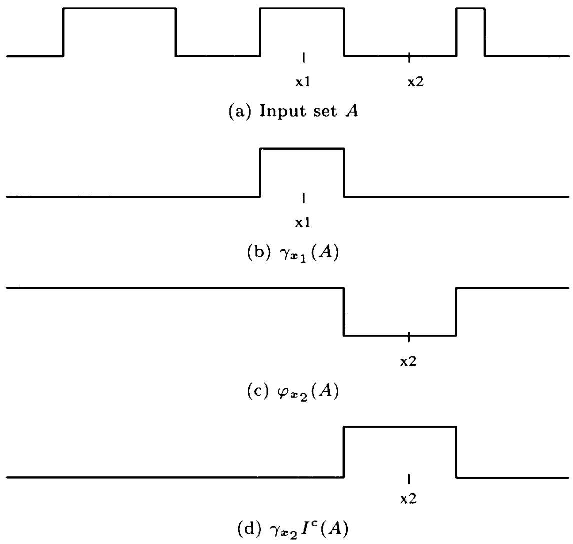


Figure 4. γ_x, φ_x, $\gamma_x I^c$: one-dimensional example. Notice that $\varphi_{x_2}(A)$ (part (c)) is equal to $E \backslash \gamma_{x_2} I^c(A)$.

The operation that extracts the pore to which a point x of the space E belongs is not the dual operation of the grain extraction operation. Figure 4(d) shows a pore extraction operation. For a point x of E, two equivalent ways to extract the pore to which x belongs are $\gamma_x I^c$ or $I^c \varphi_x$. In the following, the first way $\gamma_x I^c$ has been (arbitrarily) chosen.

Connected Operators. Connected operators belong to a class of operators that *consider* the connectivity of an input set A, $A \in \mathcal{P}(E)$. If two points x, y in E are connected in A or in A^c (foreground and background are regarded symmetrically), then for a connected operator the pair x, y will be connected either in the output set or in the complement of the output set. This forces connected operators to process grains and pores in an all-or-nothing way. If a grain is removed (i.e., the grain is modified) then all its component points will be removed. Similarly for pores: either they are filled or they are left unchanged. On the other hand, non-connected operators process sets without any restriction on changes of connectivity from the input set to the output set. In particular, a morphological non-connected operator must only be increasing.

The following definition of connected filter is due to Serra and Matheron. Let us define first the concept of flat zone which is defined more generally for functions rather than for sets [26]. Figure 5 gives an illustrative example.

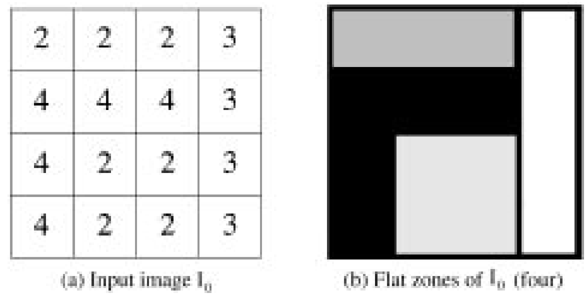


Figure 5. Flat zones example. For an input gray-level image (a), part (b) shows its four flat zones, i.e., those regions with a same function value. Notice that there are *two* flat zones with intensity value 2 (and not one) because pixels with value 2 form two separated regions.

Definition. Let E be a space equipped with γ_x and T a complete lattice. The flat zones of a function $f : E \rightarrow T$ are defined as the largest connected components of points $x \in E$ with the same function value.

Notice that the flat zone of point x in set A, is $F_x(A) = \gamma_x(A) \bigvee \gamma_x I^c(A)$, $A \in \mathcal{P}(E)$ [2].

Definition. An operator ψ is **connected** if and only if it extends the flat zones for its input function.

For the binary case, an equivalent definition of connected operator is that in [26], which applies only to binary morphology: an operator $\psi : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is said to be connected if and only if both set subtractions $A \backslash \psi(A)$ and $\psi(A) \backslash A$ are formed exclusively by connected components of A or of its complement A^c. Figure 6 shows an example.

The previous definition of connected operator, which applies both to binary and gray-level morphology, does not establish how each intensity level of an input function is processed. In addition, notice that growth is not considered in the definition.

Clearly, the class of connected operators is closed under the sup, the inf and the composition of connected operators [26]. Figure 7 shows that discontinuities can be introduced by non-connected operators and that they modify the shape of the preserved connected components.

1.3. *Filters by Reconstruction*

This section discusses an important group of connected filters, the so called *filters by reconstruction*. Filters by

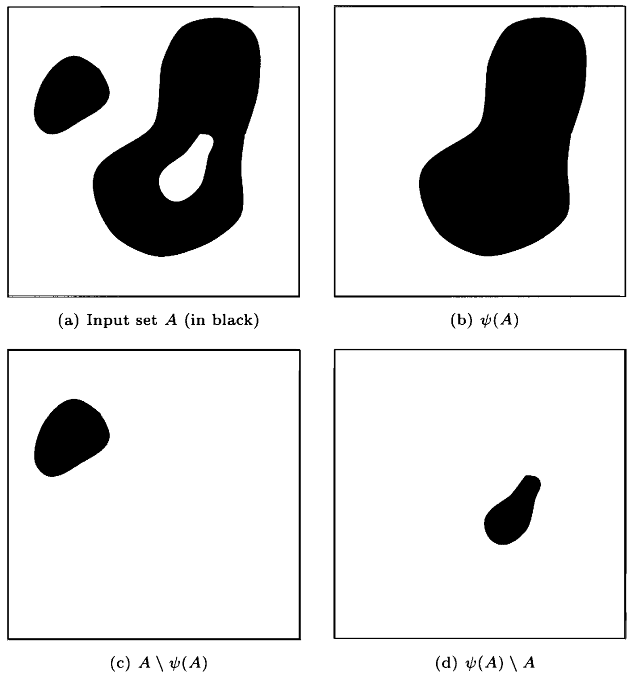


Figure 6. Connected operator example Part (a) shows an input set A and part (b) displays the output $\psi(A)$, where ψ is a connected operator. Both set differences $A \backslash \psi(A)$ (part (c)) and $\psi \backslash A$ (part (d)) are composed only of grains and pores of the input set A.

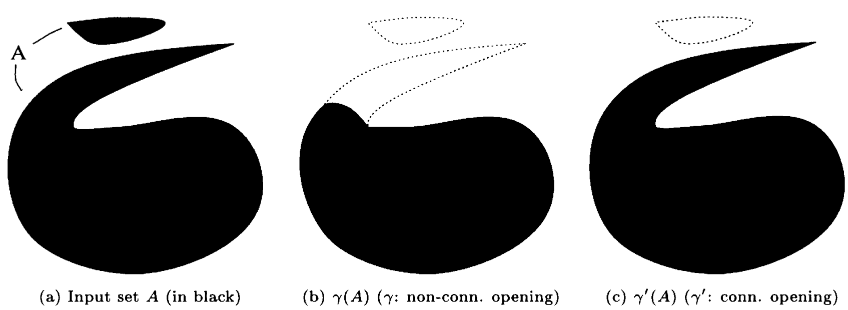


Figure 7. Differences between a non-connected and a connected opening. In this example, one of the two grains of (a) has been *broken* in (b). Notice that image (b) shows a discontinuity that does not exist in (a).

reconstruction are defined by means of the concepts of *trivial opening* $\gamma_\circ$ and *trivial closing* $\varphi_\circ$, which appeared in [21].

Definition. Let E be any space.

(1) An opening $\gamma_\circ : \mathcal{P}(E) \to \mathcal{P}(E)$ is a **trivial opening** if for all $A \in \mathcal{P}(E)$

$$\gamma_\circ(A) = \begin{cases} A, & \text{if A satisfies an increasing criterion} \\ \emptyset, & \text{if A does not satisfy the incr. crit.} \end{cases}$$

(2) A closing $\varphi_\circ : \mathcal{P}(E) \to \mathcal{P}(E)$ is a **trivial closing** if for all $A \in \mathcal{P}(E)$

$$\varphi_\circ(A) = \begin{cases} E, & \text{if A satisfies an increasing criterion} \\ A, & \text{if A does not satisfy the incr. crit.} \end{cases}$$

Increasing criteria often used to build a trivial opening $\gamma_\circ$ or a trivial closing $\varphi_\circ$ are: the area (or number of pixels, when the space of points is a grid of points), the length of the projection in a certain direction, a Minkowski operation (when the space is equipped with translation), etc.

Definition. Let E be a space equipped with γ_x. An opening $\tilde{\gamma}:\mathcal{P}(E)\to\mathcal{P}(E)$ (or respectively a closing $\tilde{\varphi}:\mathcal{P}(E)\to\mathcal{P}(E)$) is an **opening by reconstruction** (respectively **closing by reconstruction**) if and only if

$$\tilde{\gamma}=\bigvee_{x\in E}\gamma_\circ\gamma_x\left(resp.\ \tilde{\varphi}=\bigwedge_{x\in E}\varphi_\circ\varphi_x\right),$$

where $\gamma_\circ$ is a trivial opening (respectively $\varphi_\circ$ is a trivial closing).

Thus, the output of an opening by reconstruction $\tilde{\gamma}$ performed on an input set A is the set formed by all connected components of A (grains of A) that satisfy the increasing criterion of the trivial opening $\gamma_\circ$ that is associated with $\tilde{\gamma}$. The processing preformed by a closing by reconstruction $\tilde{\varphi}$ can be regarded in a similar way; for each pore P of an input set A, the increasing criterion of $\varphi_\circ$ is applied to $E\backslash P$.

Whenever the action of $\tilde{\gamma}$ or of $\tilde{\varphi}$ on a particular flat zone (grain or pore) of a point x using different input sets must be studied, only the grain (for $\tilde{\gamma}$) or pore (for $\tilde{\varphi}$) of x matters. In [4], *filters by reconstruction are those combinations of openings $\tilde{\gamma}$ and closings $\tilde{\varphi}$ by reconstruction that are idempotent.*

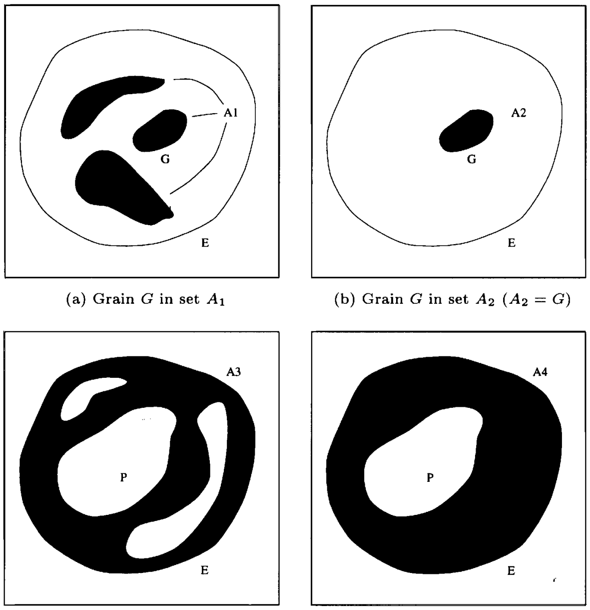
2. Connected-Component Locality and Adjacency Stability

The two constraints presented in this paper are introduced in this section. The first one, called connected-component (c.c.) locality, establishes a limit on which *part* of the input can be used for computing the output of a grain or pore. The second constraint, adjacency stability, restrains in some way the behavior of adjacent flat zones, in particular the switch from grain to pore and vice-versa.

2.1. Connected-Component Locality

The concept of *connected-component (c.c.) local operator*, which is defined next, embraces both increasing and non-increasing operators that treat each grain and pore independently of the rest of the input.

Definition. Let E be a space equipped with γ_x. An operator $\psi:\mathcal{P}(E)\to\mathcal{P}(E)$ is said to be **connected-component local** (or **c.c. local**) if and only if, $\forall A\in\mathcal{P}(E),\forall x\in E$



(a) Grain G in set A_1 (b) Grain G in set A_2 ($A_2=G$)

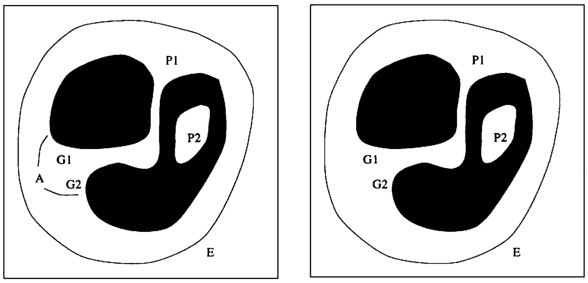
(c) Pore P in set A_3 (d) Pore P in set A_4 ($A_4=E\setminus P$)

Figure 8. Connected-component (c.c.) local operator. A c.c. local operator processes each grain independently of the rest of the input set (sets appear in black, and the space E is shown). Therefore a c.c. local operator preserves (or respectively removes) the grain G in A_1 (part (a)) if and only if it preserves (respectively removes) G in A_2 (part (b)). Similarly for pores. A c.c. local operator preserves (or respectively fills) the pore P in A_3 (part (c)) if and only if this operator preserves (respectively fills) P in A_4 (part (d)).

(a) $\gamma_x(A)\neq\emptyset,\gamma_x\psi(A)=\emptyset\Rightarrow\forall B\in\mathcal{P}(E),\gamma_x(A)=\gamma_x(B):\gamma_x\psi(B)=\emptyset.$
(b) $\gamma_x(A)=\emptyset,\gamma_x\psi(A)\neq\emptyset\Rightarrow\forall B\in\mathcal{P}(E),\gamma_xI^c(A)=\gamma_xI^c(B):\gamma_x\psi(B)\neq\emptyset.$

That is, a connected operator ψ is c.c. local if, for all $x\in E$ and for all $A\in\mathcal{P}(E)$, the fact whether or not $\gamma_x\psi(A)$ is empty or not (i.e., the fact whether or not x belongs to $\psi(A)$) depends exclusively on $\gamma_x(A)$ or on $\gamma_xI^c(A)$. Figure 8 illustrates the c.c. locality concept. Furthermore, if different input sets possess an identical grain G, a c.c. local operator will preserve or remove G in all cases. The same applies to pores. Notice that ψ can be increasing or not in the c.c. locality definition and in Proposition 2.

For a c.c. local operator ψ, we can deduce, from the definition, that: (a) $\gamma_x(A)\neq\emptyset,\gamma_x\psi(A)=\emptyset\Rightarrow\gamma_x\psi(A)=\psi\gamma_x(A)=\emptyset$ (part (a) of the definition when $B=\gamma_x(A)$); and (b) $\gamma_x(A)=\emptyset,\gamma_x\psi(A)\neq\emptyset\Rightarrow\varphi_x\psi(A)=\psi\varphi_x(A)=E$ (part (b) of the definition when $B=\varphi_x(A)$). Thus, a c.c. local connected



(a) Input set A (in black) (b) $\psi(A)$

Figure 9. Invariant class of a c.c. local filter. If a set A is invariant under ψ as shown in the figure (i.e., $A = \psi(A)$), then its grains are also invariant under ψ. The same applies to the set difference of the space E and each pore. Thus, in this case it is known that $\psi(G_1) = G_1$, $\psi(G_2) = G_2$, $\psi(E \backslash P_1) = E \backslash P_1$, and $\psi(E \backslash P_2) = E \backslash P_2$.

operator is one that

(1) Fills grains and/or remove pores.
(2) Treats each grain or pore independently from the rest of grains and pores.

Notice that, because of item (2), if a set B is invariant under a c.c. local operator ψ (i.e., $\psi(B) = B$), then each grain of B is also invariant under ψ and, for each pore P of B, the set formed by $E \backslash P$ is also invariant. Figure 9 shows an example. It is clear that in the c.c. locality definition that grains and pores are treated symmetrically (see also items (1) and (2) above). Therefore, *the dual of a c.c. local operator is dual*, i.e., if ψ is c.c. local, then $I^c \psi I^c$ is also c.c. local.

The following proposition is a direct consequence of the definitions of $\tilde{\gamma}$ and $\tilde{\varphi}$.

Proposition 1. *The opening $\tilde{\gamma}$ and the closing $\tilde{\varphi}$ by reconstruction are c.c. local filters.*

(The proof of Proposition 1 is obvious from the definitions of $\tilde{\gamma}$ and $\tilde{\varphi}$.)

The following proposition states when a c.c. local connected operator can commute with the filters φ_x and γ_x.

Proposition 2. *Let E be a space equipped with γ_x. If $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is a c.c. local connected operator, then*

(a) *If ψ is extensive*: $\psi \varphi_x = \varphi_x \psi$.
(b) *If ψ is antiextensive*: $\psi \gamma_x = \gamma_x \psi$.

Proof: Let us prove part (a) (proof of part (b) is similar). Let $A \in \mathcal{P}(E)$.

(i) Case $x \in A$. We have that: (a) $\varphi_x(A) = E$, and therefore $\psi \varphi_x(A) = E$; (b) $x \in \psi(A)$, and $\varphi_x \psi(A) = E$.
(ii) Case $x \notin A$. Then, $\varphi_x(A) = E \backslash \gamma_x I^c(A)$. Since $\gamma_x I^c(A) = \gamma_x I^c(E \backslash \gamma_x I^c)(A)$, then ψ (extensive) fills or leaves unchanged the pores (which are identical) of x in A and in $E \backslash \gamma_x I^c(A)$ (definition of c.c. local operator). If the pores are filled, then $\psi \varphi_x(A) = E = \varphi_x \psi(A)$. Otherwise, $\psi \varphi_x(A) = E \backslash \gamma_x I^c(A) = \varphi_x \psi(A)$. □

From Proposition 1, together with Proposition 2 we have as particular cases that $\tilde{\gamma} \gamma_x = \gamma_x \tilde{\gamma}$ (presented in [26]) and $\tilde{\varphi} \varphi_x = \varphi_x \tilde{\varphi}$.

The next proposition states when the combination of c.c. local connected operators is c.c. local.

Proposition 3. *The class of c.c. local connected operators is closed under the sup and the inf operations.*

Proof: Let $A \in \mathcal{P}(E)$. Let us prove the proposition for the sup operation (the inf case is analogous). Let $\{\psi_1, \ldots, \psi_n\}$ be a family of c.c. local connected operators.

(i) Case $x \in A$, $x \notin \bigvee_i \psi_i(A)$. If $x \notin \bigvee_i \psi_i(A)$, then $x \notin \psi_i(A), \forall i$. Since all $\psi_i, \forall i$, are local, then $\forall B \in \mathcal{P}(E) : \gamma_x(B) = \gamma_x(A)$, we have $\gamma_x \psi_i(B) = \emptyset, \forall i \Rightarrow \gamma_x(\bigvee_i \psi_i(B)) = \emptyset$.
(ii) Case $x \notin A$, $x \in \bigvee_i \psi_i(A)$. If $x \in \bigvee_i \psi_i(A)$, then $\exists i_0, x \in \psi_{i_0}(A)$. Since ψ_{i_0} is c.c. local, then $\forall B \in \mathcal{P}(E) : \gamma_x I^c(B) = \gamma_x I^c(A)$, we have that $\gamma_x \psi_{i_0}(B) \neq \emptyset \Rightarrow \gamma_x(\bigvee_i \psi_i(B)) \neq \emptyset$. □

However, the sequential composition of c.c. local operators is not c.c. local, in general. Nevertheless, there are some cases in which the sequential composition of c.c. local operators is c.c. local, as stated next.

Proposition 4. *The sequential composition of extensive c.c. local connected operators is c.c. local, as well as the sequential composition of antiextensive c.c. local connected operators.*

Proof: Let us prove the extensive operator case (the anti-extensive operator case is analogous). Let $A \in$

$\mathcal{P}(E)$. Let ψ_1 and ψ_2 be two extensive c.c. local operators, and let us consider the sequential composition $\psi_2\psi_1$ (the proof considering the $\psi_1\psi_2$ operator, and more operators, would be similar).

An extensive operator can only fill pores. Let $x \notin A$, $x \in \psi_2\psi_1(A) \Rightarrow$ either (a) $x \in \psi(A)$, and ψ_1 has filled the pore of x in A; or (b) $x \notin \psi_1(A)$, and ψ_2 has filled the pore of x in $\psi_1(A)$. In both cases, the pore filled is the pore of x in A, because if $x \notin \psi_1(A)$, then $\gamma_x I^c(A) = \gamma_x I^c \psi_1(A)$. Therefore, $\forall B : \gamma_x I^c(B) = \gamma_x I^c(A)$, then $x \in \psi_2\psi_1(B)$. □

The alternating filter by reconstruction $\tilde{\varphi}\tilde{\gamma}$ is not c.c. local (notice that the class of c.c. local operators is not closed under sequential composition). The fact that a grain G of the input set appears in the output is not a consequence only of G but also of its adjacent pores in the case that G has been removed by $\tilde{\gamma}$. An example is shown in Figure 10.

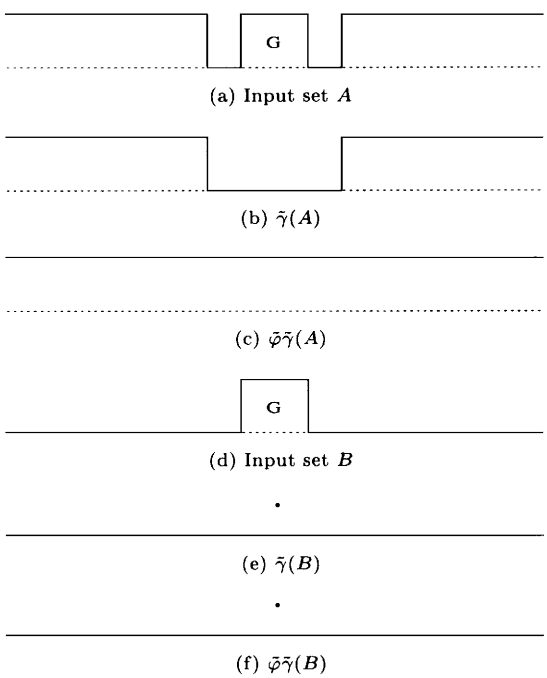


Figure 10. Example of sequential composition of c.c. local operators. This figure shows that the sequential composition of c.c. local operators is not, in general, c.c. local. Consider the alternating filter $\tilde{\varphi}\tilde{\gamma}$. Both $\tilde{\gamma}$ and $\tilde{\varphi}$ are c.c. local as stated in the text. However, $\tilde{\varphi}\tilde{\gamma}$ is not. Imagine that $\tilde{\gamma}$ removes the central grain G in A and B (identical grain). If $\tilde{\varphi}$ fills the resulting pore in $\tilde{\gamma}(A)$ (see part (b)) but not the one in $\tilde{\gamma}(B)$ (see part (e)), we have that G is included in $\tilde{\varphi}\tilde{\gamma}(A)$ (part (c)) but is not in $\tilde{\varphi}\tilde{\gamma}(B)$ (part (f)).

2.2. *Adjacency Stability*

The origin of the adjacency stability concept, which appeared first in [5], was an study of the strong property of the connected operator class. This can be observed by comparing the strong property Eq. (1) with the equation that defines the adjacency stability below. The restriction that the adjacency stability equation poses does not only affect whether or not the strong equation holds, but also has implications regarding the idempotence or non-idempotence (weaker condition than the strong property) of an operator, as will be seen in the next section.

Let us define the concept of adjacency between two sets, which formalizes the intuitive notion of contiguity. Two flat zones $F_x(A)$ and $F_{x'}(A)$ in a space E (equipped with γ_x) are said to be *adjacent* if $F_x(A) \bigvee F_{x'}(A) = \gamma_x(F_x(A) \bigvee F_{x'}(A))$, i.e., if $F_x(A) \bigvee F_{x'}(A)$, for all $A \in \mathcal{P}(E)$, is a connected set. (Notice that $F_x = \gamma_x(A) \bigvee \gamma_x I^c(A)$.) The *adjacent flat zones* of x in an input set A, symbolized by $D_x(A)$, are the pores (if $x \in A$) or the grains (if $x \notin A$) that are adjacent to $F_x(A)$, i.e., $D_x(A) = \bigvee_{x'}\{F_{x'}(A) : x' \in E, F_{x'}(A) \bigvee F_x(A) = \gamma_x(F_{x'}(A) \bigvee F_x(A)\}$. An example of the adjacent flat zones of a point is shown in Fig. 11.

The concept of adjacency stability is established next. This requirement concerns how adjacent grains and pores are treated by an operation.

Definition. Let E be a space equipped with γ_x. An operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is **adjacency stable** if, for all $x \in E$

$$\gamma_x\Big(I \bigvee \psi\Big) = \gamma_x \bigvee \gamma_x \psi. \tag{3}$$

Notice that whereas γ_x does not commute in general under the sup, it commutes always under the inf: $\gamma_x(\bigwedge_i \psi_i) = \bigwedge_i \gamma_x \psi_i$ [2].

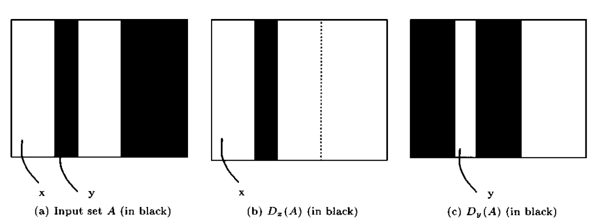


Figure 11. Adjacent flat zones of a point: (a) input set A (set in black); (b) $D_x(A)$: adjacent flat zones of x; and (c) $D_y(A)$: adjacent flat zones of y.

The consequences of adjacency stability on the relationships between the grains of the input and the output are stated in the next proposition.

Proposition 5. *Let E be a space equipped with γ_x. A connected operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is adjacency stable if and only if, for all $A \in \mathcal{P}(E)$, the grains of $\psi(A)$ are a union of the*

(i) *grains of A*
(ii) *pores of A surrounded by grains in (i).*

Proof: Let $A \in \mathcal{P}(E)$. Let us suppose that ψ is an adjacency stable operator and that there exists a grain G of $\psi(A)$ that contains a pore of A that is not surrounded by the grains of A included in $\psi(A)$. Therefore, there exists a grain G' of A that is not included in $\psi(A)$ and that is adjacent to G. Then, $x \in G' \Rightarrow \gamma_x(I \bigvee \psi)(A) = G \bigvee P \neq (\gamma_x \bigvee \gamma_x \psi)(A) = G' \Rightarrow \psi$ is not adjacency stable. A contradiction has been reached. □

Corollary 1. *Let E be a space equipped with γ_x. If a connected operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is adjacency stable then, for all $A \in \mathcal{P}(E)$*

(a) $\gamma_x(A) = \emptyset,\ \gamma_x \psi(A) \neq \emptyset \Rightarrow D_x(A) \leq \psi(A)$.
(b) $\gamma_x(A) \neq \emptyset,\ \gamma_x \psi(A) = \emptyset \Rightarrow D_x(A) \leq I^c \psi(A)$.

The grain-pore relationship is illustrated in Fig. 12.

For the adjacent unstable case displayed in Fig. 12(b), Fig. 13 shows that the adjacency stability equation does not hold for the point marked as x (this point is not the only one). Notice that the fact that an operator behaves as an adjacency stable operator for some inputs does not imply it is adjacency stable. The adjacency stability equation must hold for all $A \in \mathcal{P}(E)$ and for all $x \in E$.

The adjacency stability Eq. (3) treats grains and pores symmetrically. We note this fact because only the grain extraction operation γ_x (and not its dual φ_x) is employed in the definition of adjacency stability. The reason is that what matters is the switch from grain to pore and vice-versa. Therefore either in the input set or in the output set we study only grains and, nevertheless, by doing so we study as well their adjacent pores. This symmetrical treatment can be observed also in Corollary 1. Thus we can state that *the dual of an adjacency stable operator is adjacency stable.*

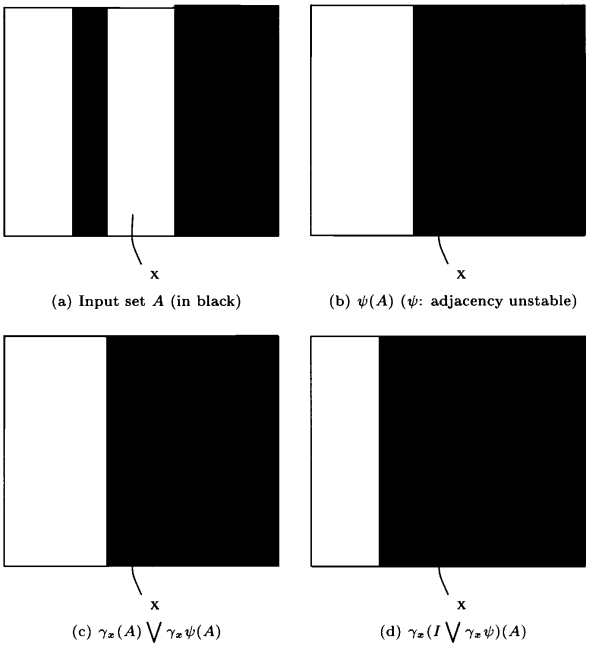
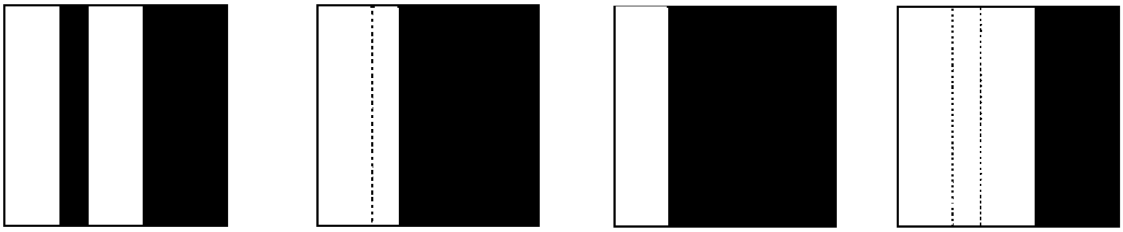


Figure 13. Adjacency stability equation. Parts (a) and (b) display respectively an input set A and the output $\psi(A)$, where ψ is a connected operator. The adjacency stability equation does not hold, since there exists at least one point x (in this case there are clearly more than one: all the points that compose the pore to which x belongs in part (a)) in which $\gamma_x(A) \bigvee \gamma_x\psi(A)$ (part (c)) is not equal to $\gamma_x(I \bigvee \gamma_x \psi)(A)$ (part (d)).



(a) Input set A (in black) (b) Output adj. unst. op. (c) Output adj. st. op. (d) Output adj. st. op.

Figure 12. Adjacency stability. Notice that a pore of the input set A has become a grain in (b) but that it is not surrounded by grains of A that appear in (b). This situation does not happen in cases (b) and (c).

The fact that the class formed by adjacency stable operators is closed under certain operations is stated next in Theorem 1. The following lemma is needed.

Lemma 1. *Let E be a space equipped with γ_x. A connected operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is adjacency stable if and only if, for all $A \in \mathcal{P}(E)$, $\psi(A)$ and $A \backslash \psi(A)$ are not connected to each other.*

Proof:

(i) $\psi(A)$ and $A\backslash\psi(A)$ are not connected to each other:

$$\gamma_x\Big(A \bigvee \psi(A)\Big) = \begin{cases} \gamma_x(A), & x \in A\backslash\psi(A) \\ \gamma_x\psi(A), & x \in \psi(A) \\ \emptyset, & \text{otherwise} \end{cases}$$

That is, $\gamma_x\big(A \bigvee \psi(A)\big) = \gamma_x(A) \bigvee \gamma_x\psi(A), \forall x$.

(ii) ψ is stable: Let us suppose $\psi(A)$ and $A\backslash\psi(A)$ are connected to each other, i.e., there exists at least one grain G of $\psi(A)$ and a grain G' of $A\backslash\psi(A)$ that are adjacent (because ψ is connected, G' is a grain of A). Let $x \in G$. Then, $\gamma_x(G \bigvee G') \geq G \bigvee G' > \gamma_x(G) \bigvee \gamma_x(G') = \gamma_x(G)$, and $\gamma_x(A \bigvee \psi(A)) \geq G \bigvee G' > \gamma_x(A) \bigvee \gamma_x\psi(A) = G$. The operator ψ would be adjacency unstable, and a contradiction has been reached. □

Theorem 1. *The class of adjacency stable connected operators is closed under the sup, the inf and the sequential composition operations.*

Proof: Let us study each case separately. Let $\{\psi_1, \ldots, \psi_n\}$ be a family of adjacency stable operators. The case for the composition will be proved for the two operators case, and by induction this result applies for the composition of an arbitrary number of operators. From Lemma 1, it is known that if a connected operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is adjacency stable, then $A\backslash\psi(A)$ is not connected to $\psi(A)$, $A \in \mathcal{P}(E)$.

(i) $\bigvee_i \psi_i$: Let us suppose that $\bigvee_i \psi_i$ is not adjacency stable. Let $A \in \mathcal{P}(E)$ such that $\exists x : \gamma_x\big(A \bigvee_i \psi_i(A)\big) \neq \gamma_x(A) \bigvee \gamma_x(\bigvee_i \psi_i(A))$. If G is grain of A such that G is not in $\bigvee_i \psi_i(A)$ but G is connected to a grain G' of $\bigvee_i \psi_i(A)$, then clearly G is connected to a certain grain G'' of $\psi_{i_0}(A)$, where $\psi_{i_0} \in \{\psi_i\}$, because the grains of $\bigvee_i \psi_i(A)$ are composed of grains of $\{\psi_i(A)\}$. However, this implies that ψ_{i_0} is adjacency unstable. A contradiction has been reached.

(ii) $\bigwedge_i \psi_i$: Similarly as in (i).

(iii) $\psi_2\psi_1$: Let us suppose that $\psi_2\psi_1$ is not adjacency stable. Let G be a grain of $A\backslash\psi_2\psi_1(A)$ that is connected to a grain of $\psi_2\psi_1(A)$ (notice that G is a grain of A). Because ψ_1 and ψ_2 are connected operators, either $G' \geq G$, where G' is adjacent of $\psi_2\psi_1(A)$, must have appeared in $\psi_1(A)$, and hence ψ_2 is non-stable, or G was not in $\psi_1(A)$ and therefore it was adjacent to a grain of $\psi_1(A)$, in which case ψ_1 is not adjacency stable. In either case, a contradiction has been reached. □

On the other hand, the combination of an adjacency stable and an adjacency unstable operator is in general adjacency unstable. However, there are cases in which the result is adjacency stable: obviously, the inf (or respectively the sup) of an adjacency stable ψ_{st} and an adjacency unstable one ψ_{unst} is stable when $\psi_{\text{st}} \leq \psi_{\text{unst}}$ (respectively $\psi_{\text{st}} \geq \psi_{\text{unst}}$).

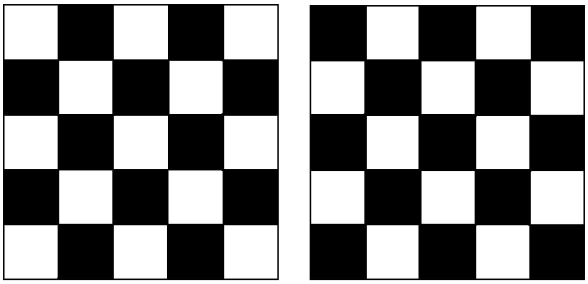
The complementation operator I^c is clearly an adjacency unstable operator. In fact, the operator I^c is the "prototype" of adjacency unstable operator because it switches all grains to pores and vice-versa. Thus, the adjacency stability Eq. (3) does not hold for any point of the space (assuming the space E is a connected set, i.e., that $E = \gamma_x(E)$, $x \in E$). For all $x \in E$ and for all $A \in \mathcal{P}(E)\backslash\{\emptyset, E\}$,

$$\gamma_x\Big(I \bigvee I^c\Big)(A) = A \bigvee A^c = E \neq$$

$$\gamma_x(A) \bigvee \gamma_x I^c(A) = A \text{ or } A^c.$$

Not all adjacency unstable operators are not increasing (as the complementation operator I^c). The median operator is an example of an increasing adjacency unstable operator [2, 5]. Notice that the median operator can be expressed as a sup of erosions [13, 16] (or an inf of dilations) but that some of these erosions are not antiextensive and are adjacency unstable. Therefore the median operator is a sup of erosions but is not a sup of adjacency stable operators (an operation that must be adjacency stable, from Theorem 1). An example is shown in Figure 14.

The next theorem establishes the adjacency stability of all extensive and antiextensive operators, increasing or not. This theorem is followed by a corollary that guarantees the adjacency stability of any combination of openings γ and closings φ, connected or not.



(a) Input set A (in black) (b) Output of median operator

Figure 14. Adjacency unstability of the median operator. For the input displayed in part (a), the 3×3 neighborhood median operator gives part (b) as output. Since (b) is the complement of (a), it is clear that the adjacency stability equation does not hold for any point of the space. (The space is $\mathbf{Z}^2$ and four-connectivity is assumed.)

Theorem 2. *Extensive and anti-extensive mappings are adjacency stable.*

Proof: The proof is obvious from Eq. (3).

(i) ψ is extensive ($\psi \geq I$): the left hand side of Eq. (3) is $\gamma_x(I \bigvee \psi) = \gamma_x\psi$; and the right hand side, $\gamma_x \bigvee \gamma_x\psi = \gamma_x\psi$.
(ii) ψ is antiextensive ($\psi \leq I$): the left hand side of Eq. (3) is $\gamma_x(I \bigvee \psi) = \gamma_x$; and the right hand side, $\gamma_x \bigvee \gamma_x\psi = \gamma_x$. □

A consequence of the previous result (and of Theorem 1) is that *any composition of openings γ and closings φ is adjacency stable*. Clearly, erosions and dilations are also adjacency stable if they are anti-extensive and extensive, respectively.

In Figure 2, only the first, the second, the fourth, the seventh and the eight are outputs of stable filters. In all other cases, Eq. (3) is not true for all x. Thus, in Fig. 2 *all outputs that do not satisfy the stability equation cannot be obtained using openings $\tilde{\gamma}$ and closings $\tilde{\varphi}$ by reconstruction*. We have discovered that openings $\tilde{\gamma}$ and closings $\tilde{\varphi}$ by reconstruction *cannot* compute any connected operation. Notice that this also applies to any kind of openings and closings: It is not possible to compute any operation by means of openings and closings.

The following proposition and theorem state how to build an adjacency stable operator from an unstable one. Proposition 6 gives an obvious result, which is followed by Theorem 3 that establishes the smallest adjacency stable majorant (and minorant) of a connected operator.

Proposition 6. *Let ψ be any operator, adjacency stable or not. The operators*

$$\psi_1 = \psi \bigwedge I \quad \textit{and} \quad \psi_2 = \psi \bigvee I$$

are adjacency stable.

Proof: From Theorem 2, since ψ_1 is antiextensive and ψ_2 is extensive. □

Theorem 3. *Let E be a space equipped with γ_x. Any unstable connected operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ admits a smallest majorant $\hat{\psi}$ and a greatest minorant $\check{\psi}$ that are connected and adjacency stable. Let $A \in \mathcal{P}(E)$. The expressions of, respectively, $\hat{\psi}$ and $\check{\psi}$ are*

$$\hat{\psi}(A) = \bigvee_{x \in \psi(A)} \gamma_x\Big(\psi(A) \bigvee A\Big) \tag{4}$$

$$\check{\psi}(A) = \bigwedge_{x \notin \psi(A)} \varphi_x\Big(\psi(A) \bigwedge A\Big) \tag{5}$$

Proof: We will prove only the majorant $\hat{\psi}$ case (expression (4)). The operator $\hat{\psi}$ is clearly connected. From Lemma 1, it is also adjacency stable because, $\forall A \in \mathcal{P}(E)$, by construction $\hat{\psi}(A)$ is not connected to $A \backslash \hat{\psi}(A)$. In addition, $\hat{\psi}$ is the smallest adjacency stable majorant of $\psi : \hat{\psi}$ adds to $\psi(A)$ all grains of A that did not belong to $\psi(A)$ but that were connected to $\psi(A)$, and each added grain of A is necessary for the adjacency stability of $\hat{\psi}$. □

If ψ is adjacency stable, expressions (4) and (5) give clearly that $\hat{\psi} = \check{\psi} = \psi$. Notice that $\hat{\psi}$ and $\check{\psi}$ are not necessarily extensive nor anti-extensive, unlike ψ_1 and ψ_2 of Proposition 6. This is an expected result since there exist adjacency stable operators that are not extensive nor anti-extensive such as $\tilde{\varphi}\tilde{\gamma}$, $\tilde{\gamma}\tilde{\varphi}$, $\tilde{\gamma}\tilde{\varphi}\tilde{\gamma}$, etc. (which are all stable from Theorem 2).

As an example of how the c.c. locality and adjacency stability concepts can be employed, let us use them to prove, in the following example, the classical theorem that establishes the strong-property of the alternating filters by reconstruction $\tilde{\varphi}\tilde{\gamma}$ and $\tilde{\gamma}\tilde{\varphi}$. This theorem is a simpler version of the theorem by Matheron and Serra that appeared in [21].

Example. Prove that the connected alternating filters $\tilde{\varphi}\tilde{\gamma}$ and $\tilde{\gamma}\tilde{\varphi}$ are strong, where $\tilde{\gamma}$ and $\tilde{\varphi}$ are respectively an opening and a closing by reconstruction.

It is known that $\tilde{\varphi}\tilde{\gamma}$ is an $\bigwedge$-filter: $\tilde{\varphi}\tilde{\gamma} = \tilde{\varphi}\tilde{\gamma}(I \bigwedge \tilde{\varphi}\tilde{\gamma})$ [20]. Let us show that $\tilde{\varphi}\tilde{\gamma}$ is also a $\bigvee$-filter.

Let A be a set. Since $\tilde{\gamma}$ is c.c. local (Proposition 1), we have, from Proposition 2, that $\tilde{\gamma} = \bigvee_x \gamma_x \tilde{\gamma} = \bigvee_x \tilde{\gamma}\gamma_x$. Therefore, $\tilde{\varphi}\tilde{\gamma}(I \bigvee \tilde{\varphi}\tilde{\gamma}) = \tilde{\varphi}(\bigvee_x \tilde{\gamma}\gamma_x)(I \bigvee \tilde{\varphi}\tilde{\gamma})$. From Theorem 2, $\tilde{\varphi}\tilde{\gamma}$ is adjacency stable and (from Lemma 1) $\tilde{\varphi}\tilde{\gamma}(A)$ and $A \backslash \tilde{\varphi}\tilde{\gamma}(A)$ are not connected. Then, $\tilde{\gamma}\gamma_x(I \bigvee \tilde{\varphi}\tilde{\gamma})(A)$ is equal to

$$\begin{cases} \tilde{\gamma}\gamma_x(A) = \emptyset, & x \in A \backslash \tilde{\varphi}\tilde{\gamma}(A). \\ \tilde{\gamma}\gamma_x \tilde{\varphi}\tilde{\gamma}(A), & x \in \tilde{\varphi}\tilde{\gamma}(A). \end{cases}$$

Using again Proposition 2, $\bigvee_x \tilde{\gamma}\gamma_x \tilde{\varphi}\tilde{\gamma} = \bigvee_x \gamma_x \tilde{\gamma}\tilde{\varphi}\tilde{\gamma} = \tilde{\gamma}\tilde{\varphi}\tilde{\gamma}$. Finally, it is known (from [26]) that $\tilde{\gamma}\tilde{\varphi}\tilde{\gamma} = \tilde{\varphi}\tilde{\gamma}$ because $\tilde{\varphi}\tilde{\gamma} \leq \tilde{\gamma}\tilde{\varphi}$.

In the example above the adjacency stability equation arises when studying the strong property of an operator.

3. Translation Invariance Operators

In this section we will discuss translation invariance and study its relationship with connectivity, in particular with connected operators. Notice that c.c. locality, as defined previously, does not assume any translation invariance.

3.1. *Translation Invariance and Connectivity*

Let us denote the translation operator by T_α, where α is the translating vector. We will restrict our discussion to the spaces $\mathbf{R}^2$ or $\mathbf{Z}^2$ (where $\mathbf{R}$ and $\mathbf{Z}$ denote respectively the set of real numbers and the set of integers). The translation of a set A by α is

$$\mathrm{T}_\alpha(A) = \{x : x - \alpha \in A\}.$$

Definition. Let E be $\mathbf{R}^2$ or $\mathbf{Z}^2$. An operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is translation invariant if and only if, for all $A \in \mathcal{P}(E)$

$$\psi \mathrm{T}_\alpha(A) = \mathrm{T}_\alpha \psi(A).$$

We can easily deduce that $\mathrm{T}_\alpha \psi \gamma_x = \psi \mathrm{T}_\alpha \gamma_x$ (from the translation-invariance definition), and that, if $x \notin A$, $\mathrm{T}_\alpha \psi(E \backslash \gamma_x I^c(A)) = \psi(E \bigvee \mathrm{T}_\alpha \gamma_x I^c(A))$ (taking into account that, for all α, $\mathrm{T}_\alpha(E) = E$, where E is $\mathbf{R}^2$ or $\mathbf{Z}^2$).

Proposition 7. *Let E be $\mathbf{R}^2$ or $\mathbf{Z}^2$ equipped with γ_x. If a c.c. local operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is translation invariant then, $\forall A \in \mathcal{P}(E)$, $\forall x \in E$*

(a) $\gamma_x(A) \neq \emptyset, \gamma_x \psi(A) = \emptyset \Rightarrow \forall B \in \mathcal{P}(E), \gamma_{x'} \mathrm{T}_\alpha \gamma_x(A) = \gamma_{x'}(B), x' \in \mathrm{T}_\alpha \gamma_x(A) : \gamma_{x'} \psi(B) = \emptyset.$

(b) $\gamma_x(A) = \emptyset, \gamma_x \psi(A) \neq \emptyset \Rightarrow \forall B \in \mathcal{P}(E), \gamma_{x'} \mathrm{T}_\alpha \gamma_x I^c(A) = \gamma_{x'} I^c(B), x' \in \mathrm{T}_\alpha \gamma_x I^c(A) : \gamma_{x'} \psi(B) \neq \emptyset.$

Proof: Let us prove part (a) (the proof of part (b) is similar). Let $A \in \mathcal{P}(E)$. $\gamma_x(A) \neq \emptyset, \gamma_x \psi(A) = \emptyset \Rightarrow \psi \mathrm{T}_\alpha \gamma_x(A) = \emptyset$, since ψ is translation invariant. Because ψ is c.c. local, then $\psi \mathrm{T}_\alpha \gamma_x(A) = \emptyset \Rightarrow \forall B \in \mathcal{P}(E) : \gamma_{x'}(B) = \gamma_{x'} \mathrm{T}_\alpha \gamma_x(A), x' \in \mathrm{T}_\alpha \gamma_x(A), \Rightarrow \gamma_{x'} \psi(B) = \emptyset.$ □

Clearly, the opening γ_x is not translation invariant: $\gamma_x \mathrm{T}_\alpha \neq \mathrm{T}_\alpha \gamma_x$. The reason is that, obviously, the opening γ_x depends on x. However, an important case arises when, given a certain grain G, the opening γ_x satisfies that, after translating the grain, if x' belongs to the translated grain, $\gamma_{x'}$ recovers exactly the translated grain. If this is the case, we will say that γ_x is *pointwise translation invariant* [3], whose definition is stated next.

Definition. Let E be $\mathbf{R}^2$ or $\mathbf{Z}^2$ equipped with γ_x. The opening γ_x is said to be **pointwise translation invariant** if and only if, $\forall \alpha, x \in E, \forall A \in \mathcal{P}(E)$

$$\mathrm{T}_\alpha \gamma_x(A) = \gamma_{x'} \mathrm{T}_\alpha \gamma_x(A), \forall x' \in \mathrm{T}_\alpha \gamma_x(A). \quad (6)$$

As discussed in [3], if γ_x is not pointwise translation invariant, then *it can be impossible to build certain c.c. local translation invariant operators*. An example of an opening γ_x in $\mathbf{Z}^2$ that is not pointwise translation invariant arises when we employ 8-connectivity at the left side of the space and 4-connectivity at the other side. This is an unusual but possible choice; notice that γ_x is well defined for all points in the space (even for those at the boundary of both sides).

3.2. *Restriction to Subspaces*

The c.c. locality and adjacency stability requirements can greatly restrict what an operator can do. This is particularly true when an operator, defined on a certain space E, is regarded together with its *restriction* to subspaces of E. This is in fact quite common.

We want in general that when a grain has been removed or a pore has been filled in a certain space, the same result could be reproduced for other subspaces that contain the grain or pore. Thus, each $E' \leq E$ defines a "new" operator. We will define precisely, for a c.c. local operator defined on a space E, its restriction to a subspace of E since some theoretical results will be presented regarding it. In order to simplify the problem, only c.c. local translation-invariant operators are treated. Notice that, as discussed in Section 3, it can be impossible to define certain c.c. local and translation invariant operators if γ_x does not satisfy the pointwise translation invariance requirement.

Definition. Let E be a space equipped with γ_x and let $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ be a c.c. local connected operator. The **subspace restricted class** of ψ, denoted by $\mathcal{C}_\psi$, is the set of operators $\psi_i : \mathcal{P}(E'_i) \to \mathcal{P}(E'_i)$, $E'_i \leq E$, that, for all $A \in \mathcal{P}(E'_i)$

(a) $\gamma_x(A) \neq \emptyset \Rightarrow \psi_i \gamma_x(A) = \psi \gamma_x(A) \bigwedge E'_i$.

(b) $\gamma_x(A) = \emptyset \Rightarrow \psi_i(E' \backslash \gamma_x I^c(A)) = \psi(E \backslash \gamma_x I^c(A)) \bigwedge E'_i$.

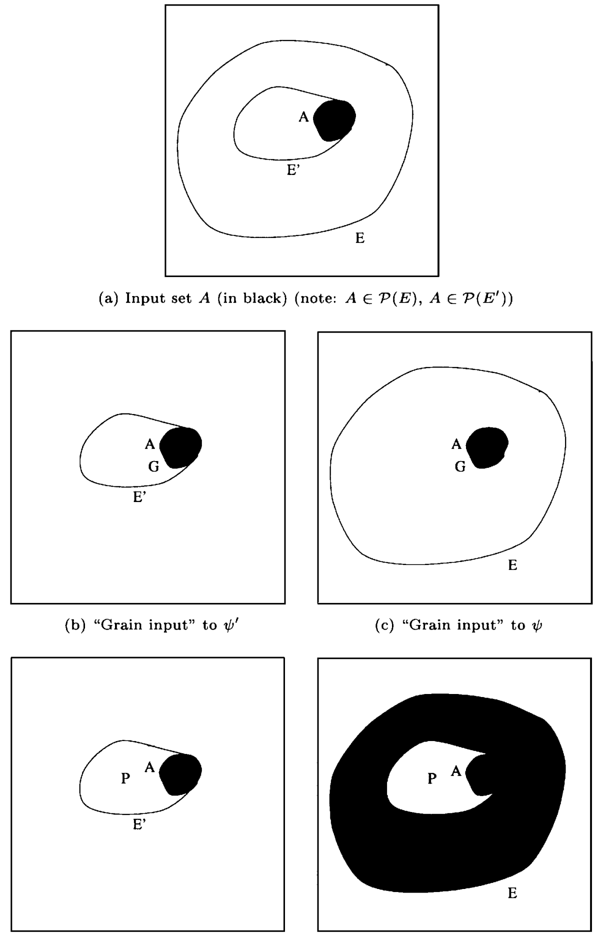
Figure 15 illustrates this concept. Notice that A must be included in E'_i, the domain definition of ψ_i. In addition, it can be noticed that the previous definition applies to both translation-invariant and non translation-invariant operators.

Most often, ψ and any element of $\mathcal{C}_\psi$ are considered usually as the same operator. For example, when we define and implement an erosion the fact that a particular space is *attached*[1] to that operation is usually disregarded. Nevertheless when such an erosion is applied to another space (we employ, for example, an image of different size), the operation is, strictly, different. The distinction is relevant in this paper, and this is the reason why a precise definition of the restriction of an operator to a subspace has been given.

The following proposition relates the adjacency stability of each member of a class $\mathcal{C}_\psi$ with the extensivity and anti-extensivity of ψ.

Proposition 8. *Let E be $\mathbf{R}^2$ or $\mathbf{Z}^2$, and let ψ be a c.c. local connected translation-invariant operator defined on E. Then,*

$\forall \psi_i \in \mathcal{C}_\psi$, ψ_i is adjacency stable $\Leftrightarrow$ ψ is extensive or anti-extensive



(a) Input set A (in black) (note: $A \in \mathcal{P}(E)$, $A \in \mathcal{P}(E')$)

(b) "Grain input" to ψ' (c) "Grain input" to ψ

(d) "Pore input" to ψ' (e) "Pore input" to ψ

Figure 15. Subspace restriction of an operator. In (a) we can see the space E, a subspace E' and an input set A ($A \leq E' \leq E$). The symbol ψ denotes a c.c. local operator defined on E, and ψ' is its restriction to the subspace E'. Parts (b) and (c) show the "inputs" that are considered by ψ' (in (b)) and by ψ (in (c)) when a grain G is processed (in this case $A = G$). The definition of the subspace restriction of an operator, implies that G will be preserved (or respectively removed) by ψ' in E' if G is preserved (or respectively removed) by ψ in E. Parts (d) and (e) show the pore processing case. The pore P will be preserved (or respectively filled) by ψ' in E' if P is preserved (or respectively filled) by ψ in E.

Proof:

(i) The implication in the leftwards sense is true from Theorem 2.

(ii) Let ψ be a non-extensive and a non-antiextensive operator, and let G and P be respectively a grain and a pore that are variant under ψ. Since ψ is translation-invariant (and since E is either $\mathbf{R}^2$ or $\mathbf{Z}^2$), we assume that G and P are chosen to be disjoint and adjacent to each other. Let us call E_{i_0} the subspace of E formed by $G \bigvee P$. Then, the restriction ψ_{i_0} of ψ to E_{i_0} is adjacent unstable: $\gamma_x(I \bigvee \psi')(G) = G \bigvee P = E_{i_0} \neq \gamma_x(G) \bigvee \gamma_x \psi(G) = G$ or P. □

Corollary 2. *If ψ is a c.c. local connected translation-invariant operator, then*

ψ is adjacency unstable $\Leftrightarrow \exists \psi_i \in \mathcal{C}_\psi$ that is not idempotent.

Corollary 3. *If ψ is a c.c. local connected translation-invariant filter, then*

$\forall \psi_i \in \mathcal{C}_\psi$, ψ_i is adjacency stable $\Leftrightarrow \psi$ is an opening $\tilde{\gamma}$ or a closing $\tilde{\varphi}$ by rec.

Corollary 2 and Corollary 3 have important consequences. The first one shows that, if idempotence is desired for all members of $\mathcal{C}_\psi$, where ψ is a c.c. local translation-invariant operator, then ψ *must* be adjacency stable. This result was followed by Corollary 3, in which we find the class of $\tilde{\gamma}$ and $\tilde{\varphi}$ as the only types of connected filters that form classes $\mathcal{C}_\psi$ whose elements can satisfy both the c.c. locality and the adjacency stability conditions.

In the next Corollary 4, the last one of Proposition 8, the self-duality concept is linked to idempotence. An operator ψ is self-dual when $\psi = I^c \psi I^c$.

Corollary 4. *If ψ is a c.c. local connected translation-invariant operator, then,*

ψ is self-dual, $\psi \neq I \Rightarrow \exists \psi_i \in \mathcal{C}_\psi$ that is not idempotent.

Therefore, the c.c. local treatment of grains and pores (as defined in the c.c. locality definition) can be non-compatible with idempotence when both pores and grains are processed symmetrically. However, Corollary 4 does not imply the impossibility of building self-dual morphological filters, besides the trivial identity operator I case, whose subspace restrictions are filters as well. The *morphological center* is a self-dual morphological filter [24] that can satisfy that its subspaces restrictions are all idempotent. Nevertheless, it is not possible, as stated in Corollary 4, that the self-dual morphological center be c.c. local. For example, the operator

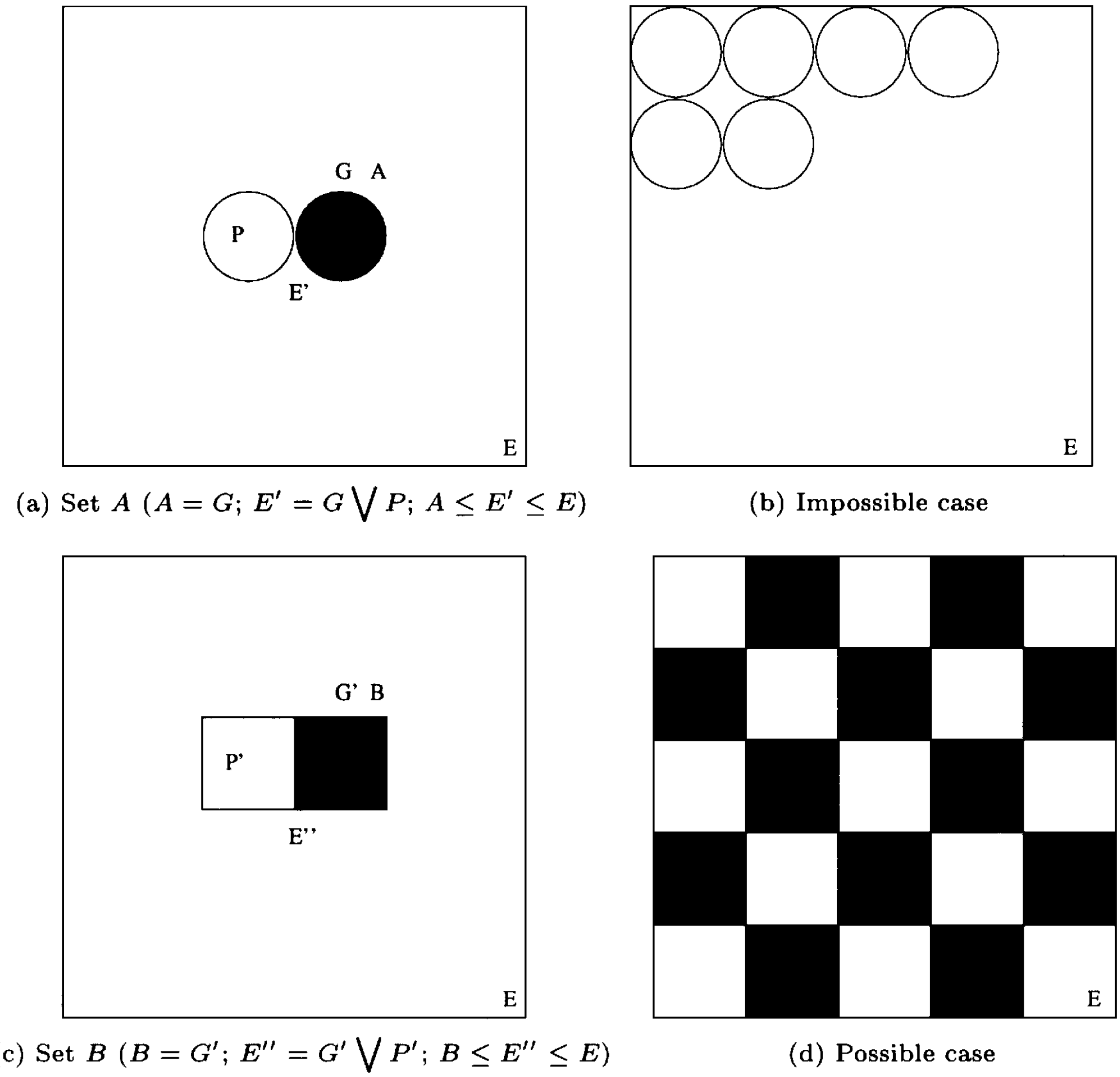
$$\psi = \left(I \bigvee \tilde{\varphi}\tilde{\gamma}\right) \bigwedge \tilde{\gamma}\tilde{\varphi} \qquad (7)$$

is a self-dual filter, and all elements in $\mathcal{C}_\psi$ are also filters. However, ψ is not c.c. local (neither $\tilde{\varphi}\tilde{\gamma}$ nor $\tilde{\gamma}\tilde{\varphi}$ in (7) are c.c. local). Notice that the case $\psi = \left(I \bigvee \tilde{\gamma}\right) \bigwedge \tilde{\varphi}$ (which is self-dual, idempotent and c.c. local) is equal to the identity operator I.

A possible mistake could be to think that if there exists a certain $\psi_i \in \mathcal{C}_\psi$ that is not adjacency stable, then ψ would be adjacency unstable. It is clearly untrue when the operator ψ is not translation invariant, but it is also untrue when it is. An additional condition is that there must exist a way to place at least one variant grain and one variant pore adjacent to each other in the space E of definition of ψ. Figure 16 gives one example in which this is not possible and another one in which it is. Figure 16(a) and Fig. 16(b) refer to the case of a connected translation-invariant operator ψ_{circ} that removes (or respectively fills) grains (respectively pores) that have a circular shape and a certain area. The restriction ψ'_{circ} of the operator to E' (see Fig. 16(a)), where E' is the union of a variant grain G and a variant pore P, is clearly adjacency unstable (in Fig. 16(a), we would have $\psi'_{\text{circ}}(A) = \psi'_{\text{circ}}(G) = I^c(G) = P$, and the adjacency stability equation would not hold for any $x \in E'$). However, the operator ψ_{circ} is not adjacency unstable because a circular grain and a circular pore cannot be placed adjacently in E. Fig. 16(c) and Fig. 16(d) refer to the case of another operator ψ_{sq}, defined on E, that removes (or respectively fills) grains (respectively pores) that are square and that have a certain area. In this case, not only some subspace restrictions are adjacency unstable (in Fig. 16(c), in which the subspace E'' is composed of a variant grain G' and a variant pore P', we would have $\psi'_{\text{sq}}(B) = \psi'_{\text{sq}}(G') = I^c = P$ where ψ'_{sq} is the restriction of ψ_{sq} to E'') but also the operator ψ_{sq} itself. As shown in Fig. 16(d), variant grains and variant pores can be placed in the space E adjacent to each other (in Fig. 16(d), the whole space E has been partitioned into variant grains and pores). The output given by ψ_{sq} when the set in Fig. 16(d) is the input would be its complement.

Similarly to the case concerning adjacency stability, the existence of $\psi_i \in \mathcal{C}_\psi$ that is not idempotent does not imply that ψ is not idempotent. (Notice that, on the other hand, implications in the other sense do not mean anything: if ψ is, for example, idempotent then obviously there exists at least one $\psi_i \in \mathcal{C}_\psi$ that is idempotent because $\psi \in \mathcal{C}_\psi$.)

Previous corollaries apply only to c.c. local operators. However, they have implications as well for non-c.c. local operators since, when a class of operators is established, c.c. locality is normally a characteristic desired for at least some components of the



(a) Set A ($A = G$; $E' = G \bigvee P$; $A \leq E' \leq E$)

(b) Impossible case

(c) Set B ($B = G'$; $E'' = G' \bigvee P'$; $B \leq E'' \leq E$)

(d) Possible case

Figure 16. Variant grains and pores. If ψ_{circ} is a c.c. local connected translation-invariant operator, defined on E, that removes (or respectively fills) grains (respectively pores) that have a circular shape and a certain area, then its restriction to E' is adjacency unstable (see part (a)). We would have $\psi'_{\text{circ}}(A) = \psi_{\text{circ}}(G) = I^c(G) = P$, and the adjacency stability equation would not hold for any $x \in E'$). However, ψ_{circ} would not be adjacent unstable because it is not possible to place any pair of adjacent grains and pores adjacent to each other in E (part (b)). On the other hand, If ψ_{sq} is a c.c. local connected translation-invariant operator, defined on E, that removes (or respectively fills) grains (respectively pores) that are square and that have a certain area, then both its restriction ψ'_{sq} to E'' is adjacency unstable (see part (c)), similarly to the case of ψ'_{circ}, and ψ_{sq}. As shown in part (d), it is possible to place variant grains and pores adjacent to each other in the space E (in (d), the whole space E has been partitioned into variant grains and pores).

class. Notice that, in practice, the c.c. local case is most commonly used. Thus, when we want to build a class of filters (i.e., idempotence is desired for all class members) adjacency stable operators should be used if c.c. locality is desired for some operators of the class.

Regarding the translation-invariant requirement in the previous theoretical results, this condition is sufficient but not necessary (see, for example, the proof of Proposition 8). We have used it in this section because the reasoning with translation-invariant operators is simpler than with the more general case. In part (ii) of Proposition 8, ψ does not need to be translation invariant, rather it must be true that there exists at least one variant (under ψ) grain G and one variant pore P that are adjacent (i.e., $\gamma_x(G \bigvee P) = G \bigvee P$, $x \in G \bigvee P$).

4. Conclusion

Connected operators are those that do not introduce discontinuities. An important class of connected operators is that constituted by filters by reconstruction. We have employed some constraints to approach and to study the class of connected operators. The first constraint, connected component (c.c.) locality, requires a connected operator to depend only on some part (which can be, nevertheless, unbounded) of the input set for computing the output of each grain and pore. The second constraint restricts how adjacency is considered by a connected operator, and is called adjacency stability.

This paper has introduced a way to approach openings and closings (in particular, the classes of openings and closings by reconstruction) that is an alternative to the usual axiomatic way. Openings and closings by reconstruction can form subspace restriction classes of operators that are both c.c. local and adjacency stable. In addition, if translation-invariance is desired, then they are the only classes of morphological connected filters that can satisfy both requirements. A point to be noted is that the c.c. local treatment of grains and pores (as defined in the c.c. locality definition) can be non-compatible with idempotence when both pores and grains are processed symmetrically, i.e., when a self-dual processing is desired. An important result that should be considered when establishing a space connectivity is that it can be impossible to build certain c.c. local and translation invariant operators if the opening that defines the space connectivity (γ_x) does not satisfy the pointwise translation invariance property.

This paper has addressed some questions regarding the connected operator class that have not been previously discussed in the literature. Some of these questions are whether or not openings and closings by reconstruction can compute any connected operation, and why all "usual" classes of filters are composed of extensive and anti-extensive operators. As discussed in our work, it is *not* possible to compute any arbitrary output using openings and closings, and the fact that *only* extensive (closings and certain dilations) and antiextensive mappings (openings and certain erosions) are normally used as building-pieces in morphological processing appear to have some desirable (and possibly unexpected) properties. We notice that, even though we have considered the binary framework, results are extendable for gray-level functions by means of the so called flat operators. The reason is that flat operators process each gray-level independently from the rest.

Acknowledgments

We want to thank the reviewers for suggestions that have greatly improved the paper. We would like to acknowledge the contributions of Dr. Jean Serra at the early stages of this work and the suggestions given by Dr. Christophe Gratin. This work has been supported by the Spanish government under the program *Formación de Postgrado y de Personal Investigador*, by the *École des Mines de Paris* and the *Centre de Morphologie Mathématique*, by the Digital Signal Processing Laboratory at the Georgia Institute of Technology, and by the Joint Services and Electronics Program under contract DAAH-04-93-G-0027.

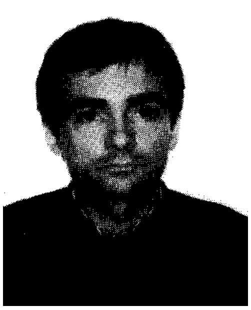
Note

1. An operator is defined on a particular space E, as implicitly indicated in the notation $\psi : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$.

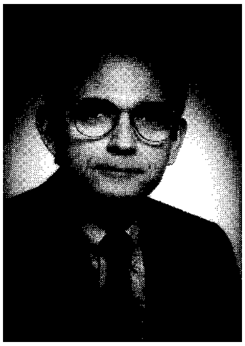
References

1. G. Birkhoff, *Lattice Theory*, American Mathematical Society, Providence, 1984.
2. J. Crespo, "Morphological connected filters and intra-region smoothing for image segmentation," PhD. Thesis, School of Electrical Engineering, Georgia Institute of Technology, Dec. 1993.
3. J. Crespo, "Space connectivity and translation-invariance," in Workshop on Mathematical Morphology and Its *Applications to Image Processing*, Atlanta, May 1996.
4. J. Crespo, J. Serra, and R. Schafer, "Theoretical aspects of morphological filters by reconstruction," *Signal Processing*, Vol. 47, No. 2, pp. 201–225, Nov. 1995.
5. J. Crespo, J. Serra, and R. Schafer, "Image segmentation using connected filters," in *Workshop on Mathematical Morphology*, Barcelona, pp. 52–57, May 1993.
6. C. Giardina and E. Dougherty, *Morphological Methods in Image and Signal Processing*, Prentice-Hall: Englewood Clliffs, 1988.
7. H. Heijmans, "Theoretical aspects of gray-level morphology," *IEEE Trans. Pattern Anal. Machine Intell.*, Vol. 13, pp. 568–582, 1991.
8. H. Heijmans, *Morphological Image Operators (Advances in Electronics and Electron Physics*, P. Hawkes (Ed.), Academic Press: Boston, 1994.
9. H. Heijmans and C. Ronse, "The algebraic basis of mathematical morphology, part I: Dilations and erosions," *Comput. Vision, Graphics and Image Processing*, Vol. 50, pp. 245–295, 1990.
10. C. Lantuéjoul and S. Beucher, "On the use of the geodesic metric in image analysis," *J. Microsc.*, Vol. 121, pp. 39–49, 1981.
11. P. Maragos, "A representation theory for morphological image and signal processing," *IEEE Trans. Pattern Anal. Machine Intell.*, Vol. 11, pp. 586–599, June 1989.
12. P. Maragos and R. Schafer, "Morphological filters—Part I: Their set-theoretic analysis and relations to linear-shift-invariant filters," *IEEE Trans. Acoust. Speech Signal Processing*, Vol. 35, pp. 1153–1169, Aug. 1987.

13. P. Maragos and R. Schafer, “Morphological filters—Part II: Their relations to median, order-statistic, and stack filters,” *IEEE Trans. Acoust. Speech Signal Processing*, Vol. 35, pp. 1170–1184, Aug. 1987.
14. P. Maragos and R. Schafer, “Morphological systems for multidimensional signal processing,” *Proc. of the IEEE*, Apr. 1990, Vol. 78, No. 4, pp. 690–710.
15. G. Matheron, *Éléments pour une Théorie des Milieux Poreux*. Masson: Paris, 1965.
16. G. Matheron, *Random Sets and Integral Geometry*, Wiley: New York, 1975.
17. G. Matheron, *Les Applications Idempotentes*, Report Centre de Géostatistique et de Morphologie Mathématique, E.N.S. des Mines de Paris, 1982.
18. G. Matheron, *Surpotentes et Sous-potentes*, Report Centre de Géostatistique et de Morphologie Mathématique, E.N.S. des Mines de Paris, 1982.
19. G. Matheron, *Filters and Lattices*, Report Centre de Géostatistique et de Morphologie Mathématique, E.N.S. des Mines de Paris, 1983.
20. G. Matheron, “Filters and lattices,” in *Mathematical Morphology Volume II: Theoretical Advances*, J. Serra (Ed.), Academic Press: London, Chap. 6, pp. 115–140, 1988.
21. G. Matheron and J. Serra, “Strong filters and connectivity,” in: *Mathematical Morphology Volume II: Theoretical Advances*, J. Serra (Ed.), Academic Press: London, Chap. 7, pp. 141–157, 1988.
22. J. Serra, *Mathematical Morphology*, Academic Press: London, Vol. I, 1982.
23. J. Serra, *Éléments de Théorie pour l’Optique Morphologique*, Thèse d’État, Université de Paris VI, 1986.
24. J. Serra, *Mathematical Morphology. Volume II: Theoretical Advances*, Academic Press: London, 1988.
25. J. Serra, “Anamorphoses and function lattices,” in *Mathematical Morphology in Image Processing*, E. Dougherty (Ed.), Marcel Dekker: New York, Chap. 13, pp. 483–523, 1993.
26. J. Serra and P. Salembier, “Connected operators and pyramids,” in *Proceedings of SPIE, Non-Linear Algebra and Morphological Image Processing*, San Diego, July 1993, Vol. 2030, pp. 65–76.
27. J. Serra and P. Salembier, “Opérateurs connexes et pyramides,” in *9ème Congrès RFIA, AFCET/AFIA*, Paris, Janvier, Vol. 1, pp. 243–254, 1994.



José Crespo del Arco was born in Madrid on September 13, 1966. At the *Universidad Politécnica de Madrid*, he received the diploma of *Ingeniero de Telecomunicación* in 1989. At the Georgia Institute of Technology, he received a Master of Science in electrical engineering in 1990, a Master of Science in management in 1993, and a Ph.D. in electrical engineering in 1993. He has been a post-doc researcher at Centre de Morphologie Mathématique (E.N.S. des Mines de Paris), an associate professor at Universidad Alfonso X El Sabio, and he is currently an assistant professor at Universidad Politécnica de Madrid (Facultad de Informática). His research interests are image analysis, mathematical morphology, algorithms, videoconference and multimedia. He is a member of IEEE and ACM.



Ronald W. Schafer received the B.S.E.E. and M.S.E.E. degrees from the University of Nebraska, Lincoln, in 1961 and 1962 respectively, and the Ph.D. degree from the Massachusetts Institute of Technology, Cambridge, in 1968.

From 1968 to 1974 he was a member of the Acoustics Research Department, Bell Laboratories, Murray Hill, New Jersey, where he was engaged in research on speech analysis and synthesis, digital signal processing techniques, and digital waveform coding. Since 1974, he has been on the faculty of the Georgia Institute of Technology, where he is now John O. McCarty Professor and Institute Professor of Electrical Engineering. His current research interests include speech and video processing, non-linear signal processing systems, and applications of signal processing in multimedia communication systems. He is coauthor of the widely used textbooks, *Digital Signal Processing*, *Digital Processing of Speech Signals*, and *Discrete-Time Signal Processing*. He has been active in the affairs of the IEEE Acoustics, Speech, and Signal Processing Society, having served as Associate Editor of the Transactions, member of several committees, Vice-President and President of the Society, and Chairman of the 1981 ICASSP.

Dr. Schafer is a Fellow of the IEEE and the Acoustical Society of America and he is a member of Sigma Xi, Eta Kappa Nu, Phi Kappa Phi, and the National Academy of Engineering. He was awarded the Achievement Award and the Society Award of the IEEE ASSP Society in 1979 and 1983 respectively; the 1983 IEEE Region III Outstanding Engineer Award; and he shared the 1980 Emanuel R. Piore Award with L.R. Rabiner. In 1985 he received the Class of 1934 Distinguished Professor Award at Georgia Tech, and he received the 1992 IEEE Education Medal.