Ann. Inst. Statist. Math.
Vol. 49, No. 4, 693–710 (1997)

TESTING HYPOTHESES ABOUT THE POWER LAW PROCESS UNDER FAILURE TRUNCATION USING INTRINSIC BAYES FACTORS

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(Received February 7, 1996; revised October 3, 1996)

Abstract. Conventional Bayes factors for hypotheses testing cannot typically accommodate the use of standard noninformative priors, as such priors are defined only up to arbitrary constants which affect the values of the Bayes factors. To circumvent this problem, Berger and Pericchi (1996, *J. Amer. Statist. Assoc.*, **91**, 109–122) introduced a new criterion called the Intrinsic Bayes Factor (IBF). In this paper, we use their methodology to test several hypotheses regarding the shape parameter of the power law process. Assuming that we have data from the process according to the failure-truncation sampling scheme, we derive the arithmetic and geometric IBF's using the reference priors. We deduce a set of intrinsic priors that correspond to these IBF's, as the observed number of failures tends to infinity. We then use these results to analyze an actual data set on the failures of an aircraft generator.

Key words and phrases: Automatic Bayes factor, Intrinsic Bayes factors, (non-homogeneous) Poisson process, reference prior, repairable systems, tests of hypotheses, Weibull process.

1. Introduction

The power law process is a non-homogeneous Poisson process $\{X(t), t \geq 0\}$ with intensity function $\lambda(t) = \beta t^{\beta-1}/\alpha^\beta$, $\alpha > 0$, $\beta > 0$, and mean-value function $\Lambda(t) = E(X(t)) = (t/\alpha)^\beta$. This process has been widely used to model reliability growth (Crow (1982)), software reliability, and repairable systems (Ascher and Feinhold (1984) and Rigdon and Basu (1989)). In view of the arguments of Ascher (1981), we prefer the phrase power law process to the alternative name, Weibull process, available in the literature to describe the $X(t)$ process.

Statistical hypothesis testing for the parameters α, β have been considered in the literature from a frequentist standpoint. See Rigdon and Basu (1989) for a review in this regard. Hypotheses about the shape parameter β often have interpretations for the system that is modeled by $X(t)$ process. When $\beta = 1$, the power law process becomes a homogeneous Poisson process and the frequency of failures is time-independent. For $\beta > 1$, the frequency of failures increases with time, while for $\beta < 1$, the failure frequency decreases with time. Thus, from the view point of system reliability, the hypotheses $H_1 : \beta = 1$, $H_2 : \beta > 1$ and $H_3 : \beta < 1$ respectively mean that the system is experiencing no change over time, degradation over time and improvement over time. Frequentist testing of point null hypotheses of the type H_1 have been derived under two sampling schemes, namely failure truncation and time truncation. In the former protocol, a predetermined number, n, of successive failure times of the $X(t)$ process are obtained for inference regarding α and β. For example, failure times of a "complex type of aircraft generator", stopped after the thirteenth failure, due to Duane (1964), and discussed by Rigdon and Basu (1989) and by Bar Lev *et al.* (1992) is given in Table 1:

Table 1. Failure times in hours for the aircraft generator.

i = Failure number	y_i = Failure time
1	55
2	166
3	205
4	341
5	488
6	567
7	731
8	1308
9	2050
10	2453
11	3115
12	4017
13	4596

In time-truncation, the observation of the failures is restricted to a pre-fixed interval $[0, t_0]$, and the number of failures, $X(t_0)$, along with the failure times during this interval, are recorded. Here, we are concerned only with testing for β under failure truncation. Our results in the case of time-truncation will be reported elsewhere.

In this paper, we take a Bayesian approach to the problem of testing H_i, $i = 1, 2, 3$ using reference priors for (α, β) and the new criterion, called the Intrinsic Bayes Factor (IBF), due to Berger and Pericchi (1996), where problems of model selection and hypotheses testing are addressed. In the next section, we describe briefly the IBF methodology. In Section 3, towards implementation of the method to solve our problem, expressions for IBF's are derived and the asymptotic behavior of the IBF is established. Finally, computations of the IBF for, and our conclusions from, the data in Table 1, are presented in Section 4.

2. The IBF methodology

In the context of general hypotheses testing, the IBF approach of Berger and Pericchi (1996) can be summarized as follows.

Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be an observable with $\boldsymbol{Y} \sim f(\boldsymbol{y} \mid \boldsymbol{\theta})$, where $\boldsymbol{\theta} \epsilon \Theta$ is a finite-dimensional parameter. Suppose we wish to test the hypotheses $H_i : \boldsymbol{\theta} \epsilon \Theta_i$, where $\Theta_i \subset \Theta$, $i = 1, 2, \ldots, q$. Bayesian hypotheses testing proceeds by selecting a prior distribution $\pi_i(\boldsymbol{\theta})$ for $\boldsymbol{\theta}$ under H_i, together with the prior probability p_i of H_i being true, $i = 1, 2, \ldots, q$. The posterior probability that H_i is true is then

$$P(H_i \mid \boldsymbol{y}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} B_{ji} \right)^{-1} \tag{2.1}$$

where, B_{ji}, the Bayes factor of H_j to H_i, is defined by

$$B_{ji} = \frac{m_j(\boldsymbol{y})}{m_i(\boldsymbol{y})} = \frac{\int_{\Theta_j} f(\boldsymbol{y} \mid \boldsymbol{\theta}) \pi_j(\boldsymbol{\theta}) d\boldsymbol{\theta}}{\int_{\Theta_i} f(\boldsymbol{y} \mid \boldsymbol{\theta}) \pi_i(\boldsymbol{\theta}) d\boldsymbol{\theta}}, \tag{2.2}$$

$m_i(\boldsymbol{y})$ being the marginal or predictive density of $\boldsymbol{Y}$ under H_i. The posterior probabilities in (2.1) are then used to select the most plausible hypothesis.

If one were to use some non-informative priors $\pi_i^N(\boldsymbol{\theta})$, (2.2) becomes

$$B_{ji}^N = \frac{m_j^N(\boldsymbol{y})}{m_i^N(\boldsymbol{y})} = \frac{\int_{\Theta_j} f(\boldsymbol{y} \mid \boldsymbol{\theta}) \pi_j^N(\boldsymbol{\theta}) d\boldsymbol{\theta}}{\int_{\Theta_i} f(\boldsymbol{y} \mid \boldsymbol{\theta}) \pi_i^N(\boldsymbol{\theta}) d\boldsymbol{\theta}}. \tag{2.3}$$

Being usually improper, π_i^N are defined only up to arbitrary constants.

Hence, the corresponding Bayes factor, B_{ji}^N, is indeterminate. One solution to this indeterminancy problem, due to Berger and Pericchi (1996), begins with the assumption that we can split the data vector $\boldsymbol{y}$ into $\boldsymbol{y}(\ell)$, the so-called *training sample*, and the remainder of the data, $\boldsymbol{y}(-\ell)$, such that

$$0 < m_i^N(\boldsymbol{y}(\ell)) < \infty, \qquad \forall i = 1, 2, \ldots, q. \tag{2.4}$$

In view of (2.4), the posteriors $\pi_i^N(\boldsymbol{\theta} \mid \boldsymbol{y}(\ell))$ are well-defined. Now, consider the Bayes factors, $B_{ji}(\ell)$, for the rest of the data $\boldsymbol{y}(-\ell)$, using $\pi_i^N(\boldsymbol{\theta} \mid \boldsymbol{y}(\ell))$ as the priors:

$$\begin{aligned} B_{ji}(\ell) &= \frac{\int_{\Theta_j} f(\boldsymbol{y}(-\ell) \mid \boldsymbol{\theta}, \boldsymbol{y}(\ell)) \pi_j^N(\boldsymbol{\theta} \mid \boldsymbol{y}(\ell)) d\boldsymbol{\theta}}{\int_{\Theta_i} f(\boldsymbol{y}(-\ell) \mid \boldsymbol{\theta}, \boldsymbol{y}(\ell)) \pi_i^N(\boldsymbol{\theta} \mid \boldsymbol{y}(\ell)) d\boldsymbol{\theta}} \\ &= B_{ji}^N \times B_{ij}^N(\boldsymbol{y}(\ell)) \end{aligned} \tag{2.5}$$

where B_{ji}^N is given by (2.3) and

$$B_{ij}^N(\boldsymbol{y}(\ell)) = m_i^N(\boldsymbol{y}(\ell)) / m_j^N(\boldsymbol{y}(\ell)). \tag{2.6}$$

In (2.5), any arbitrary ratio, c_j/c_i say, that multiples B_{ji}^N would be cancelled by the ratio c_i/c_j forming the multiplicand in $B_{ij}^N(\boldsymbol{y}(\ell))$. Also, while the expression for $B_{ji}(\ell)$ apparently requires the conditional distribution of $\boldsymbol{y}(-\ell)$ given $\boldsymbol{y}(\ell)$, (2.6) renders $B_{ji}(\ell)$ in terms of the simpler marginal densities of $\boldsymbol{y}(\ell)$.

As training samples play a fundamental role in our testing H_i, $i = 1, 2, \ldots, q$, we will need

Definition 2.1. A training sample $\boldsymbol{y}(\ell)$, will be called *proper* if (2.4) holds and *minimal* (MTS) if it is proper and none of its subsets is proper.

Berger and Pericchi (1996) advocated various summaries based on $B_{ji}(\ell)$'s in (2.5) from many training samples to test H_i, $i = 1, 2, \ldots, q$. Generically termed the Intrinsic Bayes Factor (IBF), two summaries are given by

Definition 2.2. The Arithmetic Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{AI} = \frac{1}{L}\sum_{\ell=1}^{L} B_{ji}(\ell). \tag{2.7a}$$

The Geometric Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{GI} = \left(\prod_{\ell=1}^{L} B_{ji}(\ell)\right)^{1/L} \tag{2.7b}$$

where L is the number of all possible minimal training samples (the typical dependence of L on the sample size n, here suppressed, will be indicated by L_n).

Introducing Correction Factors, CFA_{ij} and CFG_{ij}, through (2.6) by

$$CFA_{ij} \equiv CFA_{ij}(L_n) \stackrel{\text{def}}{=} \frac{1}{L_n}\sum_{\ell=1}^{L_n} B_{ij}^N(\boldsymbol{y}(\ell)), \tag{2.8a}$$

$$CFG_{ij} \equiv CFG_{ij}(L_n) \stackrel{\text{def}}{=} \left(\prod_{\ell=1}^{L_n} B_{ij}^N(\boldsymbol{y}(\ell))\right)^{1/L_n} \tag{2.8b}$$

we obtain from (2.5), (2.7a), (2.7b), (2.8a) and (2.8b)

$$B_{ji}^{AI} = B_{ji}^N \times CFA_{ij}, \tag{2.9a}$$

$$B_{ji}^{GI} = B_{ji}^N \times CFG_{ij}. \tag{2.9b}$$

This re-write of the IBF's will be used in the rest of the paper.

Remarks 2.1. (i) One can calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{AI} from (2.9a) and (2.9b).

(ii) Though other versions of the IBF are available, see Berger and Pericchi (1996), we will discuss, in this paper, only the arithmetic and the geometric IBF's given by (2.9a) and (2.9b).

The IBF methodology provides a new approach to model selection and hypotheses testing. Berger and Pericchi (1996) have demonstrated that it can be implemented even in many non-standard situations. It is fully automatic in the sense of requiring only default or standard noninformative priors for its computation. Most importantly, it seems to correspond to very reasonable actual Bayes factors (asymptotically, as $n \to \infty$), provided the correction factor $CFA_{ij}(L_n)$ in (2.9a) and (2.9b) almost surely converges, as $n \to \infty$, to a non-zero value. In order for this convergence to hold, the hypothesis H_j must typically, in some sense, be "more complex" than the hypothesis H_i. Unfortunately, in many problems, it is often not clear which hypothesis (if either) is more complex. For example, in the context of the Power Law process, there is no natural way to pick one of the hypotheses $H_1 : \beta = 1$, $H_2 : \beta > 1$ and $H_3 : \beta < 1$ as being more complex than any of the remaining two. This is typically the case when hypotheses are, as in this example, not nested. Following Berger and Pericchi (1996), we therefore introduce the *encompassing hypothesis* $H_0 : \beta > 0$. The idea here is to formulate a hypothesis which is (minimally) more complex than the hypotheses being tested. It should be noted here that, if one were to test hypothesis using the Geometric Bayes factors, then complications regarding the choice of a more complex hypothesis do not arise and an encompassing approach is not needed. See Berger and Pericchi (1996) for details regarding the encompassing approach to model selection and hypotheses testing.

Returning to the general set up introduced earlier in this section, suppose that H_i, $i = 1, 2, \ldots, q$ are nonnested hypotheses. Suppose also that H_i is nested in a certain hypothesis, H_0 say, $\forall i = 1, 2, \ldots, q$, so that H_0 is the encompassing hypothesis. Assume further that $\pi_0(\boldsymbol{\theta})$ is the prior for $\boldsymbol{\theta}$ under H_0. Then, we have

DEFINITION 2.3. (Berger and Pericchi (1996)) The encompassing arithmetic IBF of H_j to H_i is given by

$$B_{ji}^{0AI} \stackrel{\text{def}}{=} \frac{B_{0i}^{AI}}{B_{0j}^{AI}} = B_{ji}^{N}\left(\frac{CFA_{i0}}{CFA_{j0}}\right), \qquad 1 \le i,\ j \le q, \tag{2.10}$$

where B_{ji}^N and CFA_{i0} are given respectively by (2.3) and (2.8a) and (2.8b).

Remarks 2.2. (i) In implementing (2.10) in any given situation, minimal training samples (see Definition 2.1) have to be defined relative to all the $(q+1)$ hypotheses H_i, $i = 0, 1, 2, \ldots, q$.

(ii) The posterior probabilities of H_i, $i = 1, 2, \ldots, q$, given by (2.1), would become, in this encompassing approach,

$$P(H_i \mid \boldsymbol{y}) = \left(\sum_{j-1}^{q} \frac{p_j}{p_i} B_{ji}^0\right)^{-1}. \tag{2.11}$$

Using the posterior probabilities, which are computed through (2.11), we can test H_i, $i = 1, 2, \ldots, q$.

3. Intrinsic Bayes Factors

This section is organized as follows. In Subsections 3.1 to 3.5, we discuss various aspects of testing H_i, $i = 1, 2, 3$ from the point of view of arithmetic Bayes factors in (2.7a). In Subsection 3.1, we determine the MTS, given the data $\boldsymbol{y} = (y_1, \ldots, y_n)$. In Subsection 3.2, we derive expressions for the IBF's given by (2.9a) and (2.9b) for two types of MTS. In Subsection 3.3, we establish the L_1 and almost-sure convergence of the correction factor in (2.8a) and (2.8b) relating to one type of MTS. Expected Intrinsic Bayes Factors are presented in Subsection 3.4. In Subsection 3.5, we derive the intrinsic priors under H_i, $i = 1, 2, 3$. Finally, in Subsection 3.6, we summarize our results for testing H_i, $i = 1, 2, 3$. using the geometric Bayes factors in (2.7b).

3.1 *Minimal training Samples*

The goal here is to determine the set of all possible MTS's for the data $\boldsymbol{y}$. To this end, we use Definition 2.1 and the *reference priors* $\pi_i^N(\alpha, \beta)$, $i = 0, 1, 2, 3$, say, corresponding respectively to $H_0 : \beta > 0$, $H_1 : \beta = 1$, $H_2 : \beta > 1$ and $H_3 : \beta < 1$. For an excellent review of, and algorithms to derive, reference priors in general, see Berger and Bernardo (1992). In view of the advantages of using reference priors in deriving the IBF that have been demonstrated by Berger and Pericchi (1996) we focus, in this paper, solely on the reference priors. It can be shown (see (A.2) and its proof in Appendix) that the reference priors for H_i, $i = 0, \ldots, 3$ are respectively given by

$$\pi_0^N(\alpha, \beta) = \frac{1}{\alpha\beta} I(\alpha > 0, \beta > 0) \tag{3.1}$$

$$\pi_1^N(\alpha) = \frac{1}{\alpha} I(\alpha > 0) \tag{3.2}$$

$$\pi_2^N(\alpha, \beta) = \frac{1}{\alpha\beta} I(\alpha > 0, \beta > 1) \tag{3.3}$$

$$\pi_3^N(\alpha, \beta) = \frac{1}{\alpha\beta} I(\alpha > 0, 0 < \beta < 1), \tag{3.4}$$

where, for any set A, $I(A)$ means the indicator function of A. It should be noted here that (3.1) corresponds to one of the priors used by Bar-Lev *et al.* (1992) (see equation (2.2), with $\gamma = 1$, of their paper), who were concerned with estimation and prediction problems relating to the power law process.

We now derive the marginals with respect to the reference priors given by (3.1) to (3.4). For this, we first observe that the joint pdf of $(Y_1, \ldots, Y_n)$ is given by

$$f(y_1, \ldots, y_n \mid \alpha, \beta) = \left(\frac{\beta}{\alpha}\right)^n \left(\prod_{i=1}^n \frac{y_i}{\alpha}\right)^{\beta - 1} \exp\left\{-\left(\frac{y_n}{\alpha}\right)^\beta\right\}, \quad 0 < y_1 < y_2 < \cdots < y_n. \tag{3.5}$$

Moreover, the marginal pdf of Y_k, $1 \le k \le n$, is given by

$$f(y_k \mid \alpha, \beta) = \beta \frac{\alpha^{-\beta k}}{\Gamma(k)} y_k^{\beta k - 1} \exp\left\{ -\left(\frac{y_k}{\alpha}\right)^{\beta} \right\} \tag{3.6}$$

and the joint pdf of (Y_k, Y_ℓ), $1 \le k < \ell \le n$ is

$$\begin{aligned} f(y_k, y_\ell \mid \alpha, \beta) = {} & \frac{\beta^2 \alpha^{-2\beta}}{\Gamma(k)\Gamma(\ell - k)} (y_k y_\ell)^{\beta - 1} \\ & \times \left(\frac{y_k}{\alpha}\right)^{\beta(k-1)} \left[\left(\frac{y_\ell}{\alpha}\right)^{\beta} - \left(\frac{y_k}{\alpha}\right)^{\beta} \right]^{\ell - k - 1} \exp\left\{ -\left(\frac{y_\ell}{\alpha}\right)^{\beta} \right\}, \\ & \qquad 0 < y_k < y_\ell < \infty. \end{aligned}$$

Now, we introduce some notation for the marginals that we will use. For $i = 0, 1, 2, 3$ and $1 \le k < \ell \le n$, let $m_i(y_k)$, $m_i(y_k, y_\ell)$ and $m_i(\boldsymbol{y})$ be respectively the marginal densities of Y_k, (Y_k, Y_ℓ) and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ under the hypothesis H_i. Thus, for example,

$$m_i(\boldsymbol{y}) = \int_0^\infty \int_0^\infty f(\boldsymbol{y} \mid \alpha, \beta) \pi_i^N(\alpha, \beta) d\alpha d\beta.$$

It is easy to verify that $m_0(y_k)$ is infinite for any $1 \le k \le n$. The expressions for the other marginals can be written in simple closed forms. In the following lemma, we give the marginals for any two observations. Proof is routine and hence is omitted.

LEMMA 3.1. *For $n \ge 2$ and $1 \le k < \ell \le n$, we have*

$$m_0(y_k, y_\ell) = \frac{1}{y_k y_\ell \log\left(\dfrac{y_\ell}{y_k}\right)},$$

$$m_1(y_k, y_\ell) = \frac{k/\ell}{y_k y_\ell} f_B\left(\frac{y_k}{y_\ell}; k+1, \ell - k\right),$$

$$m_2(y_k, y_\ell) = \frac{1}{y_k y_\ell \log\left(\dfrac{y_\ell}{y_k}\right)} F_B\left(\frac{y_k}{y_\ell}; k, \ell - k\right),$$

and

$$m_3(y_k, y_\ell) = \frac{1}{y_k y_\ell \log\left(\dfrac{y_\ell}{y_k}\right)} \left[1 - F_B\left(\frac{y_k}{y_\ell}; k, \ell - k\right)\right]$$

where $f_B(x; a, b)$ and $F_B(x; a, b)$ are, respectively, the pdf and the cdf of the Beta (a, b) distribution, evaluated at x.

It is clear from the above that the marginal density of (Y_k, Y_ℓ) is finite for all $1 \le k < \ell \le n$ under each hypothesis, and hence we conclude that any training sample of size two is an MTS.

3.2 *Arithmetic Bayes factors*

The marginals corresponding to the whole data $\boldsymbol{y}$ can also be expressed in closed forms. We give these in the following lemma. Again, the proof involves routine calculations and is omitted.

LEMMA 3.2.

$$m_0(\boldsymbol{y}) = \Gamma(n)\Gamma(n-1) y_n^{-n} Q^{-(n-1)} e^Q,$$
$$m_1(\boldsymbol{y}) = \Gamma(n) y_n^{-n},$$
$$m_2(\boldsymbol{y}) = m_0(\boldsymbol{y})(1 - F_{\chi^2}(2Q, 2(n-1))),$$

and

$$m_3(\boldsymbol{y}) = m_0(\boldsymbol{y}) - m_2(\boldsymbol{y}),$$

where $Q = \sum_{k=1}^{n} \log(\frac{y_n}{y_k})$ *and* $F_{\chi^2}(x,\nu)$ *is the cdf of the* χ^2_ν *random variable evaluated at* x.

Now, we give the expressions for the various Bayes factors. In line with the notation in Section 2, we let $B^N_{ji}(k,\ell)$ and B^N_{ji} represent the Bayes factors computed using the MTS $\{y_k, y_\ell\}$ and the full data, respectively. Thus, we get the following from Lemmas 3.1 and 3.2.

THEOREM 3.1. *For* $n \geq 2$ *and* $1 \leq k < \ell \leq n$, *we have*

$$B^N_{10}(k,\ell) = \frac{k}{\ell} \log\left(\frac{y_\ell}{y_k}\right) f_B\left(\frac{y_k}{y_\ell}; k+1, \ell-k\right)$$
$$B^N_{20}(k,\ell) = F_B\left(\frac{y_k}{y_\ell}; k, \ell-k\right)$$
$$B^N_{30}(k,\ell) = 1 - B^N_{20}(k,\ell)$$
$$B^N_{01} = \Gamma(n-1) e^Q Q^{-(n-1)}$$
$$B^N_{02} = 1/[1 - F_{\chi^2}(2Q, 2(n-1))]$$
$$B^N_{03} = B^N_{02}/(B^N_{02} - 1).$$

Now, the arithmetic intrinsic Bayes factors B^I_{0i}, $(i = 1,2,3)$, and B^{0I}_{ji}, $(1 \leq i,j \leq 3)$, are given by (see (2.7a), (2.7b) and (2.10))

$$B^I_{0i} = B^N_{0i} \times CFA_{i0}, \tag{3.7}$$

and

$$B^{0I}_{ji} = B^N_{ji} \times \frac{CFA_{i0}}{CFA_{j0}}, \tag{3.8}$$

where CFA_{i0} is the correction factor (see (2.8a) and (2.8b)) given by

$$CFA_{i0} = \frac{1}{\binom{n}{2}} \sum_{1 \leq k < \ell \leq n} \sum B^N_{i0}(k,\ell). \tag{3.9}$$

3.3 *Choice of CFA and the asymptotics*

As we have seen, the set of MTS's include all training samples of size 2. It is, however, not necessary to use the entire set of MTS's in the calculation of the correction factor. An interesting alternative is to use only the MTS's which consist of (two) consecutive observations. Then the corresponding correction factor and the IBF's are given, respectively, by

$$CFA^*_{i0} = \frac{1}{n-1}\sum_{k=1}^{n-1} B^N_{i0}(k, k+1), \tag{3.10a}$$

and

$$B^{0I}_{ji} = B^N_{ji} \times \frac{CFA^*_{i0}}{CFA^*_{j0}}. \tag{3.10b}$$

Then the IBF in (3.8) can be computed by using CFA^*_{i0} in place of CFA_{i0}. The motivation for the use of CFA^*_{i0} is the fact that the quantities $B^N_{i0}(k, k+1)$ are, as will be shown, i.i.d. and hence CFA^*_{i0} is an average of $(n-1)$ i.i.d. terms. In fact, letting $v_k = (\frac{y_k}{y_{k+1}})^k$ for $k \geq 1$, we can write CFA^*_{i0}, $(i = 1, 2, 3)$, using the common expression (see Theorem 3.1),

$$CFA^*_{i0} = \frac{1}{n-1}\sum_{k=1}^{n-1} g_i(v_k), \tag{3.11}$$

where $g_1(x) = -x \log x$, $g_2(x) = x$ and $g_3(x) = 1 - x$. That CFA^*_{i0} is the average of $(n-1)$ i.i.d. terms is now established by the following.

THEOREM 3.2. *Let $< Y_i,\ i \geq 1 >$ be the occurrence times of the power law process with mean function $\Lambda(x) = (\frac{x}{\alpha})^\beta$. Then, the sequence $< V_i,\ i \geq 1 >$ of random variables are independent, with a common Beta $(\beta, 1)$ distribution.*

PROOF. Let $n \geq 2$ be a fixed but otherwise arbitrary integer. Then, it suffices to show that $V_1, V_2, \ldots, V_{n-1}$ are independent, with a common Beta $(\beta, 1)$ distribution. To this end, note that the joint p.d.f. of $(Y_1, \ldots, Y_n)$ is given by (3.5). Consider the transformation of variables from $(Y_1, \ldots, Y_n)$ to $(V_1, \ldots, V_{n-1}, Y_n)$, whose Jacobian is given by $y_n^{n-1}/(n-1)!$. Thus, the joint p.d.f. of $(V_1, \ldots, V_{n-1}, Y_n)$, after some algebra, is

$$\left(\prod_{i=1}^{n-1} \beta v_i^{(\beta-1)}\right) \cdot \beta \frac{y_n^{n\beta-1} e^{-(y_n/\alpha)^\beta}}{\alpha^{n\beta}(n-1)!} \tag{3.12}$$
$$0 < v_i < 1,\ i = 1, \ldots, n-1, \quad y_n > 0.$$

The desired conclusion, and the theorem, follow immediately upon integrating y_n out of the expression in (3.12). □

As Berger and Pericchi (1996) pointed out, for the intrinsic Bayes factors B^{AI}_{ji}'s to correspond, asymptotically, to actual Bayes factors, it is important that the correction factors converge, as $n \to \infty$. That this requirement is indeed satisfied by CFA^*_{i0}'s is contained in the following.

THEOREM 3.3. (a) *For $k \geq 1$, we have*

$$E(B^N_{10}(k, k+1)) = \beta/(\beta+1)^2$$
$$E(B^N_{20}(k, k+1)) = \beta/(\beta+1)$$
$$E(B^N_{30}(k, k+1)) = 1/(\beta+1)$$

(b) *For $i = 1, 2,$ and 3, CFA^*_{i0} in (3.10a) converges in L_1 and almost surely to $\frac{\beta}{(\beta+1)^2}$, $\frac{\beta}{(\beta+1)}$ and $\frac{1}{(\beta+1)}$, respectively.*

PROOF. Let U be a Beta $(\beta, 1)$ random variable. Then, for $i = 1, 2, 3$, $E(B^N_{i0}(k, k+1)) = E(g_i(U))$. Part (a) readily follows from this equation. From (3.11) and Theorem 3.2, it is now clear that CFA^*_{i0}, $i = 1, 2, 3$, converge in L_1 and almost surely to the stated limits. Now the result follows upon calculating the expected values of $g_i(X)$, $i = 1, 2, 3$. □

We have not been able to verify an analogous result for the corrections factor CFA_{i0} in (3.9).

3.4 *The expected intrinsic Bayes factors*

When the sample size is small, the correction factors can have high variability (as statistics), and that may result in possible instability in the IBF's. As a way of overcoming this problem, Berger and Pericchi (1996) recommend replacing the correction factors by their expectations, evaluated at the MLE's of any parameters that may be present. In the present context, using CFA^*_{i0} from (3.10a) in place of CFA_{i0} in (3.7), the expected arithmetic intrinsic Bayes factors are defined by

$$B^{EAI}_{0i} = B^N_{0i} \cdot \frac{1}{n-1} \sum_{k=1}^{n-1} E^{H_0}_{\hat{\beta}_0}[B^N_{i0}(k, k+1)],$$

where $\hat{\beta}_0$ is given by

$$\hat{\beta}_0 = n / \sum_{i=1}^{n-1} \log(Y_n/Y_i) \tag{3.13}$$

and is the MLE of β under H_0 (see p. 254, Rigdon and Basu (1989)). The expected intrinsic Bayes factors B^{0EI}_{ji}, $1 \leq i, j \leq 3$, are then defined by

$$B^{0EAI}_{ji} = B^{EAI}_{0i} / B^{EAI}_{0j}. \tag{3.14}$$

Thus, we get, using part (a) of Theorem 3.3 and the above definitions,

$$B^{0EAI}_{12} = B^N_{12}(\hat{\beta}_0 + 1), \tag{3.15a}$$

$$B^{0EAI}_{13} = B^N_{13}(\hat{\beta}_0 + 1)/\hat{\beta}_0 \tag{3.15b}$$

and

$$B^{0EAI}_{23} = B^N_{23}/\hat{\beta}_0. \tag{3.15c}$$

3.5 *Intrinsic priors*

The main motivation for the use of IBF is the fact that, for large samples, it typically corresponds to actual Bayes factors with respect to some (reasonable) priors. Such priors, when they exist, are called intrinsic priors. These priors are neither unique nor necessarily proper. See Berger and Pericchi (1996) for further motivation and details. We now turn to the derivation of one set of intrinsic priors, π_i^I, $i = 1, 2, 3$, say, for our problem. To this end, we denote the Bayes factor with respect to a prior π_i under H_i and a prior π_j under H_j, $1 \leq i, j \leq 3$, by B_{ji}. Then, for $j = 2, 3$,

$$B_{j1} = B_{j1}^N \frac{\pi_j(\hat{\alpha}_j, \hat{\beta}_j)\pi_1^N(\hat{\alpha}_1)}{\pi_j^N(\hat{\alpha}_j, \hat{\beta}_j)\pi_1(\hat{\alpha}_1)}(1 + o_p(1)), \tag{3.16}$$

where $\hat{\alpha}_1 = Y_n/n$ is the MLE of α under H_1 and $(\hat{\alpha}_j, \hat{\beta}_j)$ is the joint MLE of (α, β) under H_j, $j = 2, 3$. It may be noted that, with $\hat{\beta}_0$ given by (3.13),

$$\hat{\beta}_2 = \max\{1, \hat{\beta}_0\}$$
$$\hat{\beta}_3 = \min\{1, \hat{\beta}_0\}$$

and $\hat{\alpha}_j = Y_n/(n^{1/\hat{\beta}_j})$ (see p. 254, Rigdon and Basu (1989)).

Now comparing the expression for B_{j1} in (3.16) with the one provided by (2.10), we find that the intrinsic priors will be the solutions of the equations

$$\frac{\pi_j(\hat{\alpha}_j, \hat{\beta}_j)\pi_1^N(\hat{\alpha}_1)}{\pi_j^N(\hat{\alpha}_j, \hat{\beta}_j)\pi_1(\hat{\alpha}_1)}(1 + o_p(1)) = \frac{CFA_{10}^*}{CFA_{j0}^*}, \quad j = 2, 3. \tag{3.17}$$

We let $\pi_1(\cdot) \equiv \pi_1^N(\cdot)$ in (3.17) and seek π_j to satisfy

$$\frac{\pi_j(\hat{\alpha}_j, \hat{\beta}_j)}{\pi_j^N(\hat{\alpha}_j, \hat{\alpha}_j)}(1 + o_p(1)) = \frac{CFA_{10}^*}{CFA_{j0}^*}, \quad j = 2, 3. \tag{3.18}$$

Passing to the almost sure limit (as $n \to \infty$) under H_j in (3.18), and letting, for $j = 2, 3$

$$\Delta_j(\alpha, \beta) = \lim_{n\to\infty} \frac{CFA_{10}^*}{CFA_{j0}^*},$$

we get $\forall(\alpha, \beta)\epsilon\Theta_j$,

$$\frac{\pi_j(\alpha, \beta)}{\pi_j^N(\alpha, \beta)} = \Delta_j(\alpha, \beta), \quad j = 2, 3. \tag{3.19}$$

Now, using part (b) of Theorem 3.3, we obtain

$$\Delta_2(\alpha, \beta) = \frac{1}{(\beta + 1)} I(\beta > 1) \tag{3.20}$$

$$\Delta_3(\alpha, \beta) = \frac{\beta}{\beta + 1} I(0 < \beta < 1). \tag{3.21}$$

To find a solution to (3.19) to (3.21), that is to find intrinsic priors π_i^I under H_i, $i = 2, 3$, we use the fact that $\pi_1(\alpha) = \pi_1^N(\alpha)$ and write (for $j = 2, 3$).

$$\pi_j(\alpha, \beta) = \pi_j^1(\beta \mid \alpha)\pi_j^2(\alpha),$$

and

$$\pi_j^2(\alpha) = c_j \pi_1^N(\alpha).$$

In the above, we have chosen $\pi_j^2(\alpha)$ to be of the same form as $\pi_1^N(\alpha)$, except for a constant multiple c_j (allowed for a possible need for calibration). Thus, we confine our choice of (search of solutions for) π_j to be $\pi_1(\alpha) = \pi_1^N(\alpha)$, and $\pi_j(\alpha, \beta) = c_j \pi_j^1(\beta \mid \alpha) \cdot \pi_1^N(\alpha)$, for $j = 2, 3$. Now, using (3.19), (3.1)–(3.4) and Theorem 3.3, we get, for $j = 2, 3$,

$$\pi_j^1(\beta \mid \alpha) = \frac{\Delta_j(\alpha, \beta)}{c_j \beta}.$$

Thus, in view of (3.20) and (3.21),

$$\pi_2^1(\beta \mid \alpha) = \frac{1}{c_2 \beta(\beta + 1)} I(\beta > 1)$$

and

$$\pi_3^1(\beta \mid \alpha) = \frac{1}{c_3(\beta + 1)} I(0 < \beta < 1).$$

Now, choosing $c_2 = c_3 = \log 2$ gives the following intrinsic priors:

$$\pi_1^I(\alpha) = \frac{1}{\alpha} I(\alpha > 0),$$
$$\pi_2^I(\alpha) = \frac{\log 2}{\alpha} I(\alpha > 0), \qquad \pi_2^I(\beta \mid \alpha) = \frac{1}{(\log 2)\beta(\beta + 1)} I(\beta > 1),$$

and

$$\pi_3^I(\alpha) = \frac{\log 2}{\alpha} I(\alpha > 0), \qquad \pi_3^I(\beta \mid \alpha) = \frac{1}{(\log 2)(\beta + 1)} I(0 < \beta < 1).$$

The intrinsic priors given above are equivalent to a proper prior for β (given α), and to an improper prior of the form c/α for the (common parameter) α. Existence of such priors is certainly reassuring as it means that the IBF's in (3.10b) correspond, asymptotically, to the use of these priors, and hence to actual Bayes factors. Use of these IBF's, however, do not correspond to a properly calibrated Bayesian analysis, due to the presence of log 2 in the (intrinsic) prior for α for H_2 and H_3. Since α is a scale parameter, the properly calibrated prior for α ought to be $1/\alpha$, rather than $\log 2/\alpha$. Indeed, these IBF's can be easily adjusted, for instance, by multiplying B_{12}^{0I} and B_{13}^{0I} by $\log 2$, to obtain Bayes factors that do correspond to a properly calibrated Bayesian analysis. Such an easy adjustment to the IBF's is possible here due to the simple form of the intrinsic priors, and to the fact that α is a scale parameter.

3.6 *Geometric intrinsic Bayes factors*

We now summarize our results regarding the test of H_i, $i = 1, 2, 3$, using the Geometric IBF's defined in (2.7b) and the reference priors in (3.2) to (3.4). Recalling from Subsection 3.1 that a typical MTS under H_i, $i = 1, 2, 3$ is $\{y_k, y_\ell\}$, where $1 \le k < \ell \le n$, we get from (2.9b)

$$B_{ji}^{GI} = B_{ji}^{N} \times CFG_{ij} \tag{3.22}$$

where

$$CFG_{ij} = \left[\prod_{k=1}^{n-1} \prod_{\ell=k+1}^{n} B_{ij}^{N}(k,\ell)\right]^{1/(n(n-1)/2)} . \tag{3.23}$$

By restricting to MTS's of the type $\{y_k, y_{k+1}\}$, the correction factor in (3.23) collapses to

$$CFG_{ij}^{*} = \left[\prod_{i=1}^{n-1} B_{ij}^{N}(k,k+1)\right]^{1/(n-1)} . \tag{3.24}$$

Although the encompassing approach is unnecessary in implementing the geometric Bayes factors, we shall find it convenient to compute $B_{ij}^N(k,\ell)$ through the relation

$$B_{ij}^{N}(k,\ell) = B_{i0}^{N}(k,\ell)/B_{j0}^{N}(k,\ell), \tag{3.25}$$

where $B_{i0}(k,\ell)$ are given by Theorem 3.1. It follows from (3.25) and the relations among $B_{i0}^N(k,k+1)$ and $V_k = (Y_k/Y_{k+1})^k$ implicit in (3.10a) and (3.11) that

$$B_{12}^{N}(k,k+1) = \frac{g_1(V_k)}{g_2(V_k)} = -\log V_k \tag{3.26a}$$

$$B_{13}^{N}(k,k+1) = \frac{g_1(V_k)}{g_3(V_k)} = \frac{V_k \log V_k}{1-V_k} \tag{3.26b}$$

$$B_{23}^{N}(k,k+1) = \frac{g_2(V_k)}{g_3(V_k)} = \frac{V_k}{1-V_k}. \tag{3.26c}$$

We now turn to the derivation of intrinsic priors under the geometric approach. Beginning with the simplier version of the correction factor, CFG_{ij}^*, in (3.24), we note that

$$\begin{aligned} CFG_{ij}^{*} &= \exp\left\{\frac{1}{n}\sum_{k=1}^{n-1} \log B_{ij}^{N}(k,k+1)\right\} \\ &= \exp\left\{\frac{1}{n}\sum_{k=1}^{n-1} \log\left[\frac{g_i(V_k)}{g_i(V_k)}\right]\right\}. \end{aligned} \tag{3.27}$$

Since V_k's were shown in Theorem 3.2 to be i.i.d. with a common Beta $(\beta, 1)$ distribution, we obtain by applying the classical SLLN to (3.27) that, as $n \to \infty$,

$$CFG^*_{ij} \overset{H_\ell}{\to} \exp\left\{E^{H_\ell}\left(\log\left[\frac{g_i(V_1)}{g_j(V_1)}\right]\right)\right\} \quad \text{a.s.,} \tag{3.28}$$

for $\ell = 1, 2, 3$. It follows, after some simplification, from (3.26a), (3.26b), (3.26c) and (3.28) that, as $n \to \infty$,

$$CFG^*_{12} \overset{H_2}{\to} \frac{c_2}{\beta}, \tag{3.29}$$

where $\log c_2 = \int_0^\infty (\log t)e^{-t}dt$,

$$CFG^*_{13} \overset{H_3}{\to} \frac{c_2}{\beta} \exp\left\{h(\beta) - \frac{1}{\beta}\right\}, \tag{3.30}$$

where $h(\beta) \overset{\text{def}}{=} \sum_{k=1}^\infty \frac{\beta}{k(k+\beta)}$.

Now, the intrinsic priors via the geometric approach can be shown to obey the equations

$$\frac{\pi_j(\hat{\alpha}_j, \hat{\beta}_j)\pi_1^N(\hat{\alpha}_1)}{\pi_j^N(\hat{\alpha}_j, \hat{\beta}_j)\pi_1(\hat{\alpha}_1)}(1 + o_p(1)) = CFG^*_{ij}. \tag{3.31}$$

The following solution to (3.31) can now be obtained using (3.29) and (3.30) and arguments similar to those used in Subsection 3.5. We omit the details.

$$\pi_1^I(\alpha) = \frac{1}{\alpha}I(\alpha > 0)$$
$$\pi_2^I(\alpha) = \frac{c_2}{\alpha}I(\alpha > 0), \qquad \pi_2^I(\beta \mid \alpha) = \frac{1}{\beta^2}I(\beta > 1)$$
$$\pi_3^I(\alpha) = \frac{c_3}{\alpha}I(\alpha > 0), \qquad \pi_3^I(\beta \mid \alpha) = \frac{1}{c_3\beta^2}\exp\left\{h(\beta) - \frac{1}{\beta}\right\}I(0 < \beta < 1),$$

where $c_2 = 0.561$ and $c_3 = 0.741$. Note that these intrinsic priors are very similar to the ones derived via the arithmetic approach, for large values of β, i.e., for $\beta > 1$. This similarity does not, however, extend to the range $0 < \beta < 1$, where the two priors exhibit somewhat different shapes. Although it is not very clear whether this could result in substantially different values for IBF's, we suspect that they would not, since the range where the discrepancy occurs, viz: $0 < \beta < 1$, is small, and both priors are bounded. In particular, we find that these IBF values are fairly close for the example in Section 4 (see Tables 3 and 4).

4. An example: aircraft generator data

We shall now summarize our findings regarding the test of $H_1 : \beta = 1$, $H_2 : \beta > 1$ and $H_3 : \beta < 1$, assuming that the data in Table 1 come from a power law process. We first consider the case of arithmetic IBF's using the reference priors in (3.1) to (3.4).

In Tables 2 and 3, we provide the values of IBF's B_{0i}^{AI}, B_{12}^{0AI}, B_{13}^{0AI} and B_{23}^{0AI} for the two versions (3.9) and (3.10a) of the correction factor, namely CFA_{i0} and CFA_{i0}^*; other B_{ij}^{0AI}'s can be computed using Table 3 (see (3.7) and (3.8)). It should be noted that the numbers of MTS's used in computing CFA_{i0} and CFA_{i0}^* are $L = 78$ and $L = 12$ respectively.

It is clear from Tables 2 and 3 that, even though the number of MTS's involved in computing CFA_{i0} is about six times the number of MTS's used in CFA_{i0}^*, the values of the IBF's are comparable between these two types of correction factors. In fact, the values of B_{12}^{0AI} and B_{13}^{0AI} would get even closer, if they were adjusted by multiplying by $\log 2$, as indicated at the end of Subsection 3.5. In addition to the asymptotic properties of CFA_{i0}^* (see Theorem 3.3), this comparison gives additional credence to using the simpler correction factor CFA_{i0}^* instead of CFA_{i0} in testing H_i, $i = 1, 2, 3$. It follows from Table 2 that $H_3 : \beta < 1$ is the most plausible hypothesis. We therefore conclude that the aircraft generator being monitored is experiencing reliability improvement.

If we were to test H_1, H_2, H_3 by assigning prior probabilities p_1, p_2, p_3 respectively to these hypotheses, then, using B_{ij}^{0AI}'s from Table 3 and equation (3.8), we could compute the posterior probabilities of H_i, $i = 1, 2, 3$. For instance, if we employ the IBF's from Table 3 that correspond to reference priors and CFA_{i0}^*, and stipulate that $p_1 = p_2 = p_3 = 1/3$, then we get the following posterior probabilities:

$$P(H_1 \mid data) = 0.14$$
$$P(H_2 \mid data) = 0.02$$
$$P(H_3 \mid data) = 0.84$$

As before, this illustration suggests that $H_3 : \beta < 1$ is the most plausible hypothesis.

Finally, in situations like the one involving the use of CFA_{i0}^* where the number of MTS's, L, is small, the use of the expected IBF's (see (3.15a) and (3.15b)) is desirable. For the example here, these values, under the reference priors, are $B_{12}^{0EAI} = 19.73$, $B_{13}^{0EAI} = 0.167$ and $B_{23}^{0EAI} = 0.008$. It follows that $H_3 : \beta < 1$ is the most plausible hypothesis. It is also noteworthy that the values of the expected IBF's are remarkably close to the corresponding values in Table 3.

Remarks 4.1. It is easy to see that the Jeffreys priors, $\pi_i^J(\alpha, \beta)$, say, under H_i, $i = 0, 1, 2, 3$ are given by

$$\pi_0^J(\alpha, \beta) = \frac{1}{\alpha} I(\alpha > 0, \beta > 0)$$

Table 2. Arithmetic IBF's of the encompassing H_0 to H_1, H_2 and H_3 under the reference priors.

	B_{0i}^{AI} using	
Hypothesis	CFA_{i0}^*	CFA_{i0}
H_1	4.403	5.18
H_2	47.178	29.027
H_3	0.776	0.864

Table 3. Arithmetic IBF's for H_1, H_2 and H_3 under the reference priors.

Correction factor	B_{12}^{0AI}	B_{13}^{0AI}	B_{23}^{0AI}
CFA_{i0}^*	10.713	0.176	0.016
CFA_{i0}	5.599	0.166	0.029

Table 4. Geometric IBF's for H_1, H_2 and H_3.

Choice of correction factor	B_{12}^{GI}	B_{13}^{GI}	B_{23}^{GI}
CFG	3.26	0.25	0.078
CFG^*	7.82	0.19	0.024

$$\pi_1^J(\alpha) = \frac{1}{\alpha} I(\alpha > 0)$$
$$\pi_2^J(\alpha, \beta) = \frac{1}{\alpha} I(\alpha > 0, \beta > 1)$$

and

$$\pi_3^J(\alpha, \beta) = \frac{1}{\alpha} I(\alpha > 0, 0 < \beta < 1).$$

These priors readily follow from the Fisher-Information matrix $I(\alpha, \beta)$ in (A.1) of the Appendix. Also, it should be noted here that these priors coincide with those employed by Bar-lev *et al.* (1992); put $\gamma = 0$ in equation (2.2) of their paper. The expressions of the various IBF's under the Jeffreys priors, and their values for the aircraft data, are given in Lingham and Sivaganesan(1996).

We now conclude this section by tabulating the geometric IBF's under the reference prior and using the two corrections factors in (3.23) and (3.24).

As before, we conclude from Table 4 that $H_3 : \beta < 1$ is the most plausible hypothesis.

Acknowledgements

The authors would like to express their sincere gratitude to Professors James Berger and Louis Pericchi for the insight and comments they provided on an earlier version of this paper. We would also like to thank a referee for many valuable comments.

Appendix. Derivation of the reference prior

There has been much development recently on the derivation of reference priors, beginning with the paper by Berger and Bernardo (1989). There, and in the subsequent papers, the authors presented an algorithm to construct reference priors when the parameters are classified into ordered groups according to their inferential importance. Here, we derive the reference prior for (α, β) using their algorithm, and treating β as the parameter of interest, and α as the nuisance parameter.

Using the likelihood obtained from (3.5), we get the Fisher information matrix

$$I(\alpha, \beta) = \begin{bmatrix} \frac{n + c_2}{\beta^2} & -\frac{c_1}{\alpha} \\ \frac{c_1}{\alpha} & \frac{n\beta^2}{\alpha^2} \end{bmatrix} \tag{A.1}$$

where $c_1 = E(U \log U)$ and $c_2 = E[U(\log U)^2]$, with U having the Gamma $(n, 1)$ distribution. Thus, we write

$$h_2(\alpha, \beta) = |I_{22}| = \frac{n\beta^2}{\alpha^2}$$
$$h_1(\alpha, \beta) = |I|/|I_{22}| = \frac{c_3}{n\beta^2}$$

where $c_3 = n(n + c_2) - C_1^2$. Now,

$$\pi_2^\ell(\alpha \mid \beta) = \frac{\sqrt{h_2}}{\psi_2^\ell(\beta)} = \frac{1}{\psi_2^\ell(\beta)} \frac{\sqrt{n}\beta}{\alpha} I_{(a_\ell, b_\ell)}(\alpha)$$

where $0 < a_\ell < b_\ell < \infty$, and

$$\psi_2^\ell(\beta) = \int_{a_\ell}^{b_\ell} \frac{\sqrt{n}\beta}{\alpha} dx = \sqrt{n}\beta \log\left(\frac{b_\ell}{a_\ell}\right).$$

Thus,

$$\pi_2^\ell(\alpha, \beta) = \frac{1}{\alpha\left(\log \frac{b_\ell}{a_\ell}\right)} I_{(a_\ell, b_\ell)}(\alpha).$$

Next, we compute

$$E^{\pi_2^\ell(\alpha|\beta)}(\log |h_1(\alpha, \beta)|) = \log\left(\frac{c_3}{n\beta^2}\right),$$

and write, for $0 < c_\ell < d_\ell < \infty$,

$$\pi^\ell(\alpha,\beta) \propto \pi_2^\ell(\alpha,\beta)\exp\left\{\frac{1}{2}\log\frac{c_3}{n\beta^2}\right\} I_{(a_\ell,b_\ell)}(\alpha) I_{(c_\ell,d_\ell)}(\beta)$$
$$= \frac{1}{\alpha\beta}\cdot\frac{1}{(\log b_\ell/a_\ell)} I_{(a_\ell,b_\ell)}(\alpha) I_{(c_\ell,d_\ell)}(\beta).$$

Finally, the reference prior $\pi(\alpha,\beta)$ is given by

(A.2)
$$\pi(\alpha,\beta) = \lim_{\ell\to\infty}\frac{\pi^\ell(\alpha,\beta)}{\pi^\ell(1,1)} = \frac{1}{\alpha\beta},$$

where we assume that $a_\ell < 1 < b_\ell$, $c_\ell < 1 < d_\ell$ and a_ℓ, $c_\ell \to 0$ and b_ℓ, $d_\ell \to \infty$ as $\ell \to \infty$.

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