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Estimation and testing procedures for the reliability functions of generalized half logistic distribution

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ABSTRACT

Two measures of reliability are considered, $R(t) = P(X > t)$ and $P = P(X > Y)$. Estimation and testing procedures are developed for $R(t)$ and P under Type II cesoring and a sampling scheme of Bartholomew (1963). Two types of point estimators are considered (i) uniformly minimum variance unbiased estimators (UMVUEs) and (ii) maximum likelihood estimators (MLEs). A new technique of obtaining these estimators is introduced. A comparative study of different methods of estimation is done. Testing procedures are developed for the hypotheses related to different parametric functions.

1. Introduction and preliminaries

The reliability function $R(t)$ is defined as the probability of failure-free operation until time t. Thus, if the random variable (rv) X denotes the lifetime of an item or system, then $R(t) = P(X > t)$. Another measure of reliability under stress–strength set-up is the probability $P = P(X > Y)$, which represents the reliability of an item or system of random strength X subject to random stress Y. A lot of work has been done in the literature for the point and interval estimation and testing for $R(t)$ and P under censorings and complete sample case. For a brief review, one may refer to Bartholomew (1957, 1963), Basu (1964), Chao (1982), Chaturvedi and Pathak (2012, 2013, 2014), Chaturvedi and Rani (1997, 1998), Chaturvedi and Singh (2006, 2008), Chaturvedi and Surinder (1999), Chaturvedi and Tomer (2002, 2003), Constantine, Karson, and Tse (1986), Johnson (1975), Kelley, Kelley, and Schucany (1976), Pugh (1963), Sathe and Shah (1981), Tong (1974, 1975), Tyagi and Bhattacharya (1989), and others.

Half logistic model, obtained as the distribution of the absolute standard logistic variate, is probability model considered by Balakrishnan (1985). Balakrsihnan and Hossain (2007) considered generalized (Type II) version of logistic distribution and derived some interesting properties of the distribution. Ramakrsihnan (2008) considered two generalized versions of HLD namely Type I and Type II along with point estimation of scale parameters and estimation of stress–strength reliability based on complete sample. Arora, Bhimani, and Patel (2010) obtained the MLE of the shape parameter in a GHLD based on Type I progressive censoring with varying failure rates. Kim, Kang, and Seo (2011) proposed Bayes estimators of the

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shape parameter and the reliability function for the GHLD based on progressively Type II censored data under various loss functions. Seo, Lee, and Kang (2012) developed an entropy estimation method for upper record values from the GHLD. Azimi (2013) derived the Bayes estimators of the shape parameter and the reliability function for the GHLD based on Type II doubly censored samples. Seo and Kang (2014) derived the entropy of a GHLD by using Bayes estimators of an unknown parameter in the GHLD based on Type II censored samples. They also compared these estimators in terms of the mean squared error and the bias.

Let the life X of an item have the GHLD, then cumulative distribution function (cdf) and probability density function (pdf) of the (rv) X are, respectively

$$F(x;\lambda)=1-\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\lambda},\quad x>0,\ \lambda>0$$

and

$$f(x;\lambda)=\frac{\lambda}{1+e^{-x}}\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\lambda},\quad x>0,\ \lambda>0. \tag{1.1}$$

Here, it should be noted that λ is the shape parameter and, for $\lambda=1$, it comes out to be the half-logistic distribution.

The purpose of the present paper is many-fold. We develop point estimation procedures under Type II censoring and a sampling scheme proposed by Bartholomew (1963). Testing procedures are also proposed. As far as point estimation is considered, we derive UMVUEs and MLEs. A new technique of obtaining UMVUEs and MLEs is developed. For obtaining UMVUEs, the major role is played by the estimators of the powers of the parameter. With the help of estimators of the powers of the parameter, we obtain the estimators of pdf at a specified point, which is subsequently used to obtain the UMVUEs of $R(t)$ and P. The MLEs of the parameter is derived. Utilizing the invariance property of the MLEs, the MLE of the pdf at a specified point is obtained, which is subsequently used to obtain the MLEs of $R(t)$ and P. Thus, we have established an interrelationship between various estimation problems and functional forms of the parametric functions to be estimated are not needed.

In Sections 2 and 3, respectively, we provide point estimators under Type II cesoring and a sampling scheme proposed by Bartholomew (1963). In Section 4, we developed test procedures. In Section 5, we present numerical findings. Finally, in Section 6, discussions are made and conclusions are presented.

2. Point estimators under Type II censoring

Suppose n items are put on a test and the test is terminated after the first r ordered observations are recorded. Let $0<X_{(1)}\le X_{(2)}\cdots\le X_{(r)}$, $0<r<n$, be the lifetimes of first r ordered observations. Obviously, $(n-r)$ items survived until $X_{(r)}$.

Lemma 1. *Let $S_r=\sum_{i=1}^{r}\ln\left\{\frac{1}{2}\left(e^{x_{(i)}}+1\right)\right\}+(n-r)\ln\left\{\frac{1}{2}\left(e^{x_{(i)}}+1\right)\right\}$. Then, S_r is complete and sufficient for the distribution given at (1.1). Moreover, the pdf of S_r is*

$$g(s_r;\lambda)=\frac{\lambda^r s_r^{r-1}}{\Gamma(r)}\exp(-\lambda s_r),\quad s_r>0. \tag{2.1}$$

Proof. (1.1) can be written as

$$f(x;\lambda)=\frac{\lambda}{1+e^{-x}}\exp\left(-\lambda\ln\left\{\frac{1}{2}\left(e^{x}+1\right)\right\}\right),\quad x>0,\ \lambda>0. \tag{2.2}$$

From (2.2), the joint pdf of $X_{(1)}\le X_{(2)}\le\cdots X_{(n)}$ is

$$f^*(x_{(1)},x_{(2)},\ldots,x_{(n)};\lambda)=n!\lambda^n\prod_{i=1}^{n}\left(\frac{1}{1+e^{-x_{(i)}}}\right)\exp\left(-\lambda\sum_{i=1}^{n}\ln\left\{\frac{1}{2}\left(e^{x_{(i)}}+1\right)\right\}\right). \tag{2.3}$$

Integrating out $x_{(r+1)},x_{(r+2)},\ldots,x_{(n)}$ from (2.3) over the region $x_{(r)}\le x_{(r+1)}\le\cdots\le x_{(n)}$, the joint pdf of $X_{(1)}\le X_{(2)}\le\cdots X_{(r)}$ comes out to be

$$h(x_{(1)},x_{(2)},\ldots,x_{(r)};\lambda)=\lambda^r n(n-1)\ldots(n-r+1)\prod_{i=1}^{r}\left(\frac{1}{1+e^{-x_{(1)}}}\right)\exp(-\lambda s_r). \tag{2.4}$$

It follows easily from (2.2) that the rv $U=\lambda\ln\left\{\frac{1}{2}\left(e^{x}+1\right)\right\}$ has exponential distribution with mean life $1/\lambda$. Moreover, if we consider the transformation $Z_i=(n-i+1)\{U_{(i)}-U_{(i-1)}\},\ i=1,2,\ldots,r;\ U_0=0$, then Z_i's are independent and identically distributed (i.i.d.) rv's, each having exponential distribution with mean life $1/\lambda$. It is easy to see that $\sum_{i=1}^{r}Z_i=S_r$. Result (2.1) now follows from the additive property of gamma distribution (see Johnson & Kotz, 1970, p. 170). It follows from (2.4) that S_r is sufficient for the distribution given at (1.1). Since the distribution of S_r belongs to exponential family of distributions, it is also complete (see Rohatgi, 1976, p. 347).

The following theorem provides the UMVUEs of the powers of λ.

Theorem 1. *For $q \in (-\infty, \infty)$, the UMVUE of λ^q is*

$$\hat{\lambda}_{\mathrm{II}}^q = \begin{cases} \dfrac{\Gamma(r)}{\Gamma(r-q)} S_r^{-q}, & r-q>0 \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. From (2.1),

$$\begin{aligned} E(S_r^{-q}) &= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty s_r^{r-q-1} \exp(-\lambda s_r) ds_r \\ &= \frac{\Gamma(r-q)}{\Gamma(r)} \lambda^q \end{aligned}$$

and the theorem follows from Lehmann–Scheffé theorem (see Rohatgi, 1976, p. 357). □

In the following lemma, we provide the UMVUE of the sampled pdf (1.1) at a specified point x.

Lemma 2. *The UMVUE of $f(x;\lambda)$ at a specified point x is*

$$\hat{f}_{\mathrm{II}}(x;\lambda) = \begin{cases} (r-1)\left(1+e^{-x}\right)^{-1} S_r^{-1} \left(1 - S_r^{-1} \ln\left\{\frac{1}{2}(e^x+1)\right\}\right)^{r-2}, & \ln\left\{\frac{1}{2}(e^x+1)\right\} < S_r \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. We can write (2.2) as

$$f(x;\lambda) = \left(1+e^{-x}\right)^{-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\ln\left\{\frac{1}{2}(e^x+1)\right\}\right)^i \lambda^{i+1}. \tag{2.5}$$

Using lemma 1 of Chaturvedi and Tomer (2002) and Theorem 1, from (2.5), the UMVUE of $f(x;\lambda)$ at a specified point x is

$$\begin{aligned} \hat{f}_{\mathrm{II}}(x;\lambda) &= \left(1+e^{-x}\right)^{-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\ln\left\{\frac{1}{2}(e^x+1)\right\}\right)^i \lambda_{\mathrm{II}}^{i+1} \\ &= (r-1)\left(1+e^{-x}\right)^{-1} S_r^{-1} \sum_{i=0}^{r-2} (-1)^i \binom{r-2}{i} \left(S_r^{-1} \ln\left\{\frac{1}{2}(e^x+1)\right\}\right)^i, \quad \ln\left\{\frac{1}{2}(e^x+1)\right\} < S_r \end{aligned}$$

and the lemma holds. □

In the following theorem, we obtain the UMVUE of $R(t)$.

Theorem 2. *The UMVUE of $R(t)$ is given by*

$$\hat{R}_{\mathrm{II}}(t) = \begin{cases} \left(1 - S_r^{-1} \ln\left\{\frac{1}{2}(e^t+1)\right\}\right)^{r-1}, & \ln\left\{\frac{1}{2}(e^x+1)\right\} < S_r \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. Since $F(x, s_r) = f(x;\lambda) g(s_r;\lambda)$ is a continuous function of (X, S_r) on the rectangle $[t,\infty) \times [0,\infty)$, the conditions of Fubini's theorem (see Bilodeau, Thie, & Keough, 2010, p. 207) are satisfied for the change of order of integration. Let us consider the expected value of the integral $\int_t^\infty \hat{f}_{II}(x;\lambda)dx$ with respect to S_r, i.e.

$$\begin{aligned} \int_0^\infty \left\{\int_t^\infty \hat{f}_{\mathrm{II}}(x;\lambda)dx\right\} g(s_r;\lambda) ds_r &= \int_t^\infty \left[E s_r \left\{\hat{f}_{\mathrm{II}}(x;\lambda)\right\}\right] dx \\ &= \int_t^\infty \hat{f}_{\mathrm{II}}(x;\lambda) dx \\ &= R(t). \end{aligned} \tag{2.6}$$

We conclude from (2.6) that the UMVUE of $R(t)$ can be obtained simply by integrating $\hat{f}_{II}(x;\lambda)$ from t to ∞. Thus, from Lemma 2,

$$\hat{R}_{II}(t) = (r-1)S_r^{-1}\int_t^{\infty}\left(1+e^{-x}\right)^{-1}\left(1-S_r^{-1}\ln\left\{\frac{1}{2}\left(e^x+1\right)\right\}\right)^{r-2}dx$$
$$= (r-1)\int_{S_r^{-1}\ln\left\{\frac{1}{2}(e^t+1)\right\}}^{1}(1-y)^{r-2}dy$$

and the theorem follows.

Let X and Y be two independent rv's following the classes of distributions $f_1(x;\lambda_1)$ and $f_2(y;\lambda_2)$ respectively, where

$$f_1(x;\lambda_1) = \frac{\lambda_1}{1+e^{-x}}\exp\left(-\lambda_1\ln\left\{\frac{1}{2}\left(e^x+1\right)\right\}\right), \quad x>0,\ \lambda_1>0$$

and

$$f_2(y;\lambda_2) = \frac{\lambda_2}{1+e^{-y}}\exp\left(-\lambda_2\ln\left\{\frac{1}{2}\left(e^y+1\right)\right\}\right), \quad y>0,\ \lambda_2>0.$$

Let n items on X and m items on Y are put on a life test and the truncation numbers for X and Y are r_1 and r_2, respectively. Let us denote by

$$S_{r_1} = \sum_{i=1}^{r_1}\ln\left\{\frac{1}{2}\left(e^{x_{(i)}}+1\right)\right\} + (n-r_1)\ln\left\{\frac{1}{2}\left(e^{x_{(r_1)}}+1\right)\right\}$$

and

$$T_{r_2} = \sum_{j=1}^{r_2}\ln\left\{\frac{1}{2}\left(e^{y_{(j)}}+1\right)\right\} + (m-r_2)\ln\left\{\frac{1}{2}\left(e^{y_{(r_2)}}+1\right)\right\}. \quad \square$$

In what follows, we obtain the UMVUE of P.

Theorem 3. *The UMVUE of P is given by*

$$\hat{P}_{II} = \begin{cases} (r_2-1)\sum_{i=0}^{r_2-2}(-1)^i\binom{r_2-2}{i}\left(\frac{S_{r_1}}{T_{r_2}}\right)^{i+1}B(i+1,r_1), & S_{r_1}<T_{r_2} \\ (r_2-1)\sum_{j=0}^{r_1-1}(-1)^j\binom{r_1-1}{j}\left(\frac{T_{r_2}}{S_{r_1}}\right)^{j}B(j+1,r_2-1), & S_{r_1}>T_{r_2}. \end{cases}$$

Proof. It follows from Lemma 3 that the UMVUES of $f_1(x;\lambda_1)$ and $f_2(y;\lambda_2)$ at specified points x and y, respectively, are

$$\hat{f}_{1II}(x;\lambda_1) = \begin{cases} (r_1-1)\left(1+e^{-x}\right)^{-1}S_{r_1}^{-1}\left(1-S_{r_1}^{-1}\ln\left\{\frac{1}{2}\left(e^x+1\right)\right\}\right)^{r_1-2}, & \ln\left\{\frac{1}{2}\left(e^x+1\right)\right\}<S_{r_1} \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

and

$$\hat{f}_{2II}(y;\lambda_2) = \begin{cases} (r_2-1)\left(1+e^{-y}\right)^{-1}T_{r_2}^{-1}\left(1-T_{r_2}^{-1}\ln\left\{\frac{1}{2}\left(e^y+1\right)\right\}\right)^{r_2-2}, & \ln\left\{\frac{1}{2}\left(e^y+1\right)\right\}<T_{r_2} \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

From the arguments similar to those adopted in proving Theorem 2, it can be shown that the UMVUE of P is given by

$$\hat{P}_{II} = \int_{y=0}^{\infty}\int_{x=y}^{\infty}\hat{f}_{1II}(x;\lambda_1)\hat{f}_{2II}(y;\lambda_2)dxdy$$
$$= \int_{y=0}^{\infty}\hat{R}_{1II}(y;\lambda_1)\hat{f}_{2II}(y;\lambda_2)dy,$$

which on using Theorem 2 and (2.8) gives that

$$\hat{P}_{\mathrm{II}} = (r_2-1)\int_{y=0}^{\infty}(1+e^{-y})^{-1}T_{r_2}^{-1}\left(1-S_{r_1}^{-1}\ln\left\{\frac{1}{2}(e^y+1)\right\}\right)^{r_1-1}\left(1-T_{r_2}^{-1}\ln\left\{\frac{1}{2}(e^y+1)\right\}\right)^{r_2-2}dy,$$

$$\ln\left\{\frac{1}{2}(e^y+1)\right\} < S_{r_1},\ \ln\left\{\frac{1}{2}(e^y+1)\right\} < T_{r_2}$$

$$= (r_2-1)\int_{y=0}^{\min\{S_{r_1},T_{r_2}\}}(1+e^{-y})^{-1}T_{r_2}^{-1}\left(1-S_{r_2}^{-1}\ln\left\{\frac{1}{2}(e^y+1)\right\}\right)^{r_1-1}\times\left(1-T_{r_2}^{-1}\ln\left\{\frac{1}{2}(e^y+1)\right\}\right)^{r_2-2}dy. \quad (2.9)$$

Now, from (2.9), for $S_{r_1} < T_{r_2}$,

$$\hat{P}_{\mathrm{II}} = (r_2-1)\int_{1-\frac{S_{r_1}}{T_{r_2}}}^{1} Z^{r_2-2}\left\{1-(1-z)\frac{T_{r_2}}{S_{r_1}}\right\}^{r_1-1}dz$$

$$= (r_2-1)\int_0^1 (1-u)^{r_1-1}\left\{1-\frac{S_{r_1}}{T_{r_2}}u\right\}^{r_2-1}\frac{S_{r_1}}{T_{r_2}}du$$

$$= (r_2-1)\sum_{i=0}^{r_2-2}(-1)^i\binom{r_2-2}{i}\left(\frac{S_{r_1}}{T_{r_2}}\right)^{i+1}\int_0^1 u^i(1-u)^{r_1-1}du$$

and the first assertion follows. Furthermore, for $S_{r_1} > T_{r_2}$,

$$\hat{P}_{\mathrm{II}} = (r_2-1)\sum_{j=0}^{r_1-1}(-1)^j\binom{r_1-1}{j}\left(\frac{T_{r_2}}{S_{r_1}}\right)^j\int_0^1 u^{r_2-2}(1-u)^j du$$

and the second assertion follows. □

Since the likelihood function is of the same form as (2.3), it can be easily seen that the MLE of λ under Type II censoring is

$$\widetilde{\lambda}_{\mathrm{II}} = \frac{r}{S_r}. \quad (2.10)$$

From (2.10) and one-to-one property of the MLEs, the MLE of $f(x;\lambda)$ at a specified point x is

$$\widetilde{f}_{\mathrm{II}}(x;\lambda) = \frac{r}{S_r}\left(\frac{1}{1+e^{-x}}\right)\exp\left(-\frac{r}{S_r}\ln\left\{\frac{1}{2}(e^x+1)\right\}\right). \quad (2.11)$$

In the following theorem we obtain the MLE of $R(t)$.

Theorem 4. *The MLE of $R(t)$ is given by*

$$\widetilde{R}_{\mathrm{II}}(t) = \exp\left(-\frac{r}{S_r}\ln\left\{\frac{1}{2}(e^t+1)\right\}\right).$$

Proof. We know that,

$$\widetilde{R}_{\mathrm{II}}(t) = \int_t^{\infty}\widetilde{f}_{\mathrm{II}}(x;\lambda)dx,$$

which, on using (2.11), gives us

$$\widetilde{R}_{\mathrm{II}}(t) = \frac{r}{S_r}\int_t^{\infty}\left(\frac{1}{1+e^{-x}}\right)\exp\left(-\frac{r}{S_r}\ln\left\{\frac{1}{2}(e^x+1)\right\}\right)dx$$

$$= \int_{\frac{r}{S_r}\ln\left\{\frac{1}{2}(e^t+1)\right\}}^{\infty} e^{-y}dy$$

and the theorem follows. □

In the following theorem, we obtain the MLE of P.

Theorem 5. *The MLE of P is given by*

$$\widetilde{P}_{\mathrm{II}} = r_2 S_{r_1}\left(r_2 S_{r_1} + r_1 T_{r_2}\right)^{-1}.$$

Proof. From one-to-one property of the MLEs,

$$\widetilde{P}_{\rm II} = \int_{y=0}^{\infty}\int_{x=y}^{\infty}\widetilde{f}_{1\rm II}(x;\lambda_1)\widetilde{f}_{2\rm II}(y;\lambda_2)dxdy$$
$$= \int_{y=0}^{\infty}\widetilde{R}_{1\rm II}(y;\lambda_1)\widetilde{f}_{2\rm II}(y;\lambda_2)dy,$$

which, on using Lemma 3 and Theorem 4, gives that

$$\widetilde{P}_{\rm II} = \frac{r_2}{T_{r_2}}\int_{y=0}^{\infty}\exp\left(-\frac{r_1}{S_{r_1}}\ln\left\{\frac{1}{2}(e^y+1)\right\}\right)\left(\frac{1}{1+e^{-y}}\right)\exp\left(-\frac{r_2}{T_{r_2}}\ln\left\{\frac{1}{2}(e^y+1)\right\}\right)dy$$
$$= \int_{y=0}^{\infty}\exp\left(-\frac{r_1T_{r_2}}{r_2S_{r_1}}v\right)e^{-v}dv$$
$$= \int_{y=0}^{\infty}\exp\left\{-\left(1+\frac{r_1T_{r_2}}{r_2S_{r_1}}\right)v\right\}dv$$

and the theorem follows. □

3. Point estimators under the sampling scheme of Bartholomew (1963)

Let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the failure times of n items under test from (1.1). The test begins at time $X_{(0)} = 0$ and the system operates till $X_{(1)} = x_{(1)}$ when the first failure occurs. The failed item is replaced by a new one and the system operates till the second failure occurs at time $X_{(2)} = x_{(2)}$, and so on. The experiment is terminated at time t_0. Here, $X_{(i)}$ is the time until ith failure measured from time 0.

Lemma 3. *if $N(t_0)$ be the number of failures during the interval $[0, t_0]$, then*

$$P[N(t_0) = r|t_0] = \frac{\left(n\lambda\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^r}{r!}\exp\left(-n\lambda\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right).$$

Proof. Let us make the transformations

$$W_1 = \ln\left\{\frac{1}{2}(e^{X_{(1)}}+1)\right\},\qquad W_2 = \ln\left\{\frac{1}{2}(e^{X_{(2)}}+1)\right\} - \ln\left\{\frac{1}{2}(e^{X_{(1)}}+1)\right\},\ldots,$$
$$W_n = \ln\left\{\frac{1}{2}(e^{X_{(n)}}+1)\right\} - \ln\left\{\frac{1}{2}(e^{X_{(n-1)}}+1)\right\}.$$

The pdf of W_1 is

$$h(w_1) = n\lambda\exp(-n\lambda w_1).$$

Moreover, $W_2, \ldots, W_n$, are independent and identically distributed as W_1. Using the monotonicity property of $\ln\left\{\frac{1}{2}(e^x+1)\right\}$,

$$P[N(t_0) = r|t_0] = P[X_{(r)} \le t_0] - P[X_{(r+1)} \le t_0]$$
$$= P\left[\ln\left\{\frac{1}{2}(e^{X_{(r)}}+1)\right\} \le \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right] - P\left[\ln\left\{\frac{1}{2}(e^{X_{(r+1)}}+1)\right\} \le \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right]$$
$$= P\left[W_1+W_2+\cdots W_r \le \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right] - P\left[W_1+W_2+\cdots W_{r+1} \le \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right]. \quad (3.1)$$

From the additive property of exponentially distributed rv's (see Johnson & Kotz, 1970, p. 170), $U = n\lambda\sum_{i=1}^{r}W_i$ follows gamma distribution with pdf

$$h(u) = \frac{1}{\Gamma(r)}u^{r-1}e^{-1},\quad u > 0. \quad (3.2)$$

Using (3.2) and a result of Patel, Kapadia, and Owen (1976, p. 244), we obtain from (3.1) that

$$P[N(t_0)=r|t_0] = \frac{1}{\Gamma(r+1)}\int_{\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}^{n\lambda}}^{\infty} e^{-u}u^r du - \frac{1}{\Gamma(r)}\int_{\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}^{n\lambda}}^{\infty} e^{-u}u^{r-1}du$$

$$= \exp\left\{-n\lambda\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right\}\left(\sum_{j=0}^{r}\frac{\left\{n\lambda\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right\}^j}{j!} - \sum_{j=0}^{r-1}\frac{\left\{n\lambda\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right\}^j}{j!}\right)$$

and the lemma follows. □

In the following theorem, we derive the UMVUE of λ^q, where q is a positive integer.

Theorem 6. *For q to be a positive integer, the UMVUE of λ^q is given by*

$$\hat{\lambda}_I^q = \begin{cases} \frac{r!}{(r-q)!}\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-q}, & r-q\ge 0\\ 0, & \textit{otherwise}. \end{cases}$$

Proof. It follows from Lemma 3 and Fisher–Neyman factorization theorem (see Rohatgi, 1976, p. 341) that r is sufficient for estimating λ. Moreover, since the distribution of r belongs to exponential family, it is also complete (see Rohatgi, 1976, p. 347). The theorem now follows from the result that the qth factorial moment of distribution of r is given by

$$E\left\{r(r-1)\dots(r-q+1)\right\} = \left(n\lambda\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^q. \quad \square$$

In the following lemma, we obtain the UMVUE of the sampled pdf (1.1) at a specified point x.

Lemma 4. *The UMVUE of $f(x;\lambda)$ at a specified point x is*

$$\hat{f}_I(x;\lambda) = \begin{cases} r\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}(1+e^{-x})\right)^{-1} \\ \quad\times\left[1-\ln\left\{\frac{1}{2}(e^x+1)\right\}\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}\right]^{r-1}, & \ln\left\{\frac{1}{2}(e^x+1)\right\} < n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\\ 0, & \textit{otherwise}. \end{cases}$$

Proof. Using lemma 1 of Chaturvedi and Tomer (2002) and Theorem 6, from (2.5), the UMVUE of $f(x;\lambda)$ at a specified point x is

$$\hat{f}_I(x;\lambda) = (1+e^{-x})^{-1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\ln\left\{\frac{1}{2}(e^x+1)\right\}\right)^i\lambda_I^{(i+1)}$$

$$= (1+e^{-x})^{-1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\ln\left\{\frac{1}{2}(e^x+1)\right\}\right)^i\left\{\frac{r!}{(r-i-1)!}\right\}\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-(i+1)}$$

$$= r\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}(1+e^{-x})\right)^{-1}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\left(\ln\left\{\frac{1}{2}(e^x+1)\right\}\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}\right)^i,$$

$$\ln\left\{\frac{1}{2}(e^x+1)\right\} < n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}$$

and the lemma follows. □

In the following theorem, we derive the UMVUE of $R(t)$.

Theorem 7. *The UMVUE of $R(t)$ is given by*

$$\hat{R}_I(t) = \begin{cases} \left[1-\ln\left\{\frac{1}{2}(e^t+1)\right\}\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}\right]^r, & \ln\left\{\frac{1}{2}(e^t+1)\right\} < n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\\ 0, & \textit{otherwise}. \end{cases}$$

Proof. From the arguments similar to those adopted in the proof of Theorem 2, using Lemma 4,

$$\hat{R}_I(t) = \int_t^{\infty} \hat{f}_I(x;\lambda)dx$$

$$= \frac{r}{n\ln\left\{\frac{1}{2}\left(e^{t_0}+1\right)\right\}} \int_t^{\infty} (1+e^{-x})^{-1}\left[1-\ln\left\{\frac{1}{2}(e^x+1)\right\}\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}\right]^{r-1},$$

$$\ln\left\{\frac{1}{2}(e^x+1)\right\} < n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}$$

$$= r\int_{\ln\left\{\frac{1}{2}\left(e^x+1\right)\right\}\left(n\ln\left\{\frac{1}{2}\left(e^{t_0}+1\right)\right\}\right)^{-1}}^{1} (1-y)^{r-1}dy$$

and the theorem follows. □

In what follows, we obtain UMVUE of P. Suppose n items on X and m items on Y are put through a life test and t_0 and t_{00} are their truncation times, respectively. Let r_1 items on X and r_2 items on Y fail before times t_0 and t_{00}, respectively.

Theorem 8. *The UMVUE of P is given by*

$$\hat{P}_I = \begin{cases} r_2\int_0^1\left[1-m\ln\left\{\frac{1}{2}\left(e^{t_{00}}+1\right)\right\}\left(n\ln\left\{\frac{1}{2}\left(e^{t_0}+1\right)\right\}\right)^{-1}z\right]^{r_1}(1-z)^{r_2-1}dz, \\ \quad n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\} > m\ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\} \\ r_2\int_0^{n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\left(m\ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\}\right)^{-1}}\left[1-m\ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\}\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}z\right]^{r_1} \\ \quad\times(1-z)^{r_2-1}dz, \quad n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\} > m\ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\}. \end{cases}$$

Proof. Using the arguments similar to those applied in the proofs of Theorems 2 and 5, we get

$$\hat{P}_I = \int_{y=0}^{\infty}\int_{x=y}^{\infty}\hat{f}_{1I}(x;\lambda_1)f_{2I}(y;\lambda_2)dxdy$$

$$= \int_{y=0}^{\infty}\hat{R}_{1I}(y;\lambda_1)\hat{f}_{2I}(y;\lambda_2)dy$$

$$= r_2\left(m\ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\}\right)^{-1}\int_{y=0}^{\infty}(1+e^{-y})^{-1}\left[1-\ln\left\{\frac{1}{2}(e^y+1)\right\}\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}\right]^{r_1}$$

$$\times\left[1-\ln\left\{\frac{1}{2}(e^y+1)\right\}\left(m\ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\}\right)^{-1}\right]^{r_2-1}dy,$$

$$\ln\left\{\frac{1}{2}(e^y+1)\right\} < n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}, \qquad \ln\left\{\frac{1}{2}(e^y+1)\right\} < m\ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\}$$

$$= r_2\left(m\ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\}\right)^{-1}\int_{y=0}^{\min\{n\ln(e^{t_0}+1),m\ln(e^{t_{00}}+1)\}}\left(1+e^{-y}\right)^{-1}$$

$$\times\left[1-\ln\left\{\frac{1}{2}(e^y+1)\right\}\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}\right]^{r_1}$$

$$\times\left[1-\ln\left\{\frac{1}{2}(e^y+1)\right\}\left(m\ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\}\right)^{-1}\right]^{r_2-1}dy. \tag{3.3}$$

The theorem now follows from (3.3). □

Corollary 1. *In the case when $t_0 = t_{00}$, but $\lambda_1 \neq \lambda_2$,*

$$\hat{P}_I = \begin{cases} r_2\sum_{i=0}^{r_1}(-1)^i\binom{r_1}{i}\left(\frac{m}{n}\right)^i B(i+1,r_2), & m < n \\ r_2\sum_{i=0}^{r_2}(-1)^i\binom{r_2-1}{j}\left(\frac{n}{m}\right)^{j+1}B(j+1,r_1+1), & m > n. \end{cases}$$

Proof. From Theorem 8, for $m < n$,

$$\hat{P}_I = r_2 \int_0^1 \left(1 - \frac{m}{n} z\right)^{r_1} (1-z)^{r_2-1} dz$$

$$= r_2 \sum_{i=0}^{r_1} (-1)^i \binom{r_1}{i} \left(\frac{m}{n}\right)^i \int_0^1 z^i (1-z)^{r_2-1} dz$$

and the first assertion follows. Again from Theorem 8, for $m > n$,

$$\hat{P}_I = r_2 \int_0^{n/m} \left(1 - \frac{m}{n} z\right)^{r_1} (1-z)^{r_2-1} dz$$

$$= r_2 \left(\frac{n}{m}\right) \int_0^1 (1-u)^{r_1} \left(1 - \frac{n}{m} u\right)^{r_2-1} du$$

$$= r_2 \sum_{i=0}^{r_2-1} (-1)^j \binom{r_2-1}{j} \left(\frac{n}{m}\right)^{j+1} \int_0^1 u^j (1-u)^{r_1} du$$

and the second assertion follows. □

It follows from Lemma 3, that the MLE of λ under the sampling scheme of Bartholomew (1963) is

$$\widetilde{\lambda}_I = r\left(n \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}. \tag{3.4}$$

From (3.4) and one-to-one property of the MLEs, the MLE of $f(x; \lambda)$ at a specified point x is

$$\widetilde{f}_I(x;\lambda) = r\left(n \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}(e^{-x}+1)\right)^{-1} \exp\left[-r \ln\left\{\frac{1}{2}(e^{x}+1)\right\}\left(n \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}\right]. \tag{3.5}$$

Theorem 9. *The MLE of* $R(t)$ *is given by*

$$\widetilde{R}_I(t) = \exp\left[-r \ln\left\{\frac{1}{2}(e^{t}+1)\right\}\left(n \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}\right].$$

Proof. From (3.5),

$$\widetilde{R}_I(t) = \int_t^\infty \widetilde{f}_I(x;\lambda) dx$$

$$= r\left(n \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1} \int_t^\infty (e^{-x}+1)^{-1} \exp\left[-r \ln\left\{\frac{1}{2}(e^{x}+1)\right\}\left(n \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}\right] dx$$

$$= \int_{r \ln\left\{\frac{1}{2}(e^t+1)\right\}\left(n \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1}}^\infty e^{-y} dy$$

and the theorem follows. □

Theorem 10. *The MLE of* P *is given by*

$$\widetilde{P}_I = \int_0^\infty \exp\left[-r_1 m \ln\left\{\frac{1}{2}(e^{t_{00}}+1)\right\}\left(r_2 n \ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1} v\right] e^{-v} dv.$$

Proof. From Theorem 9 and (3.5),

$$\widetilde{P}_I = \int_{y=0}^\infty \int_{x=y}^\infty \widetilde{f}_{1I}(x;\lambda_1) \widetilde{f}_{2I}(y;\lambda_2) dx dy$$

$$= \int_0^\infty \widetilde{R}_{1I}(x;\lambda_1) \widetilde{f}_{2I}(y;\lambda_2) dy$$

$$= r_2\left(m\ln\left\{\frac{1}{2}\left(e^{t_{00}}+1\right)\right\}\right)^{-1}\int_0^\infty\left(1+e^{-y}\right)^{-1}\exp\left[-r_1\ln\left\{\frac{1}{2}\left(e^{y}+1\right)\right\}\left(n\ln\left\{\frac{1}{2}\left(e^{t_0}+1\right)\right\}\right)^{-1}\right]$$
$$\times \exp\left[-r_2\ln\left\{\frac{1}{2}\left(e^{y}+1\right)\right\}\left(m\ln\left\{\frac{1}{2}\left(e^{t_{00}}+1\right)\right\}\right)^{-1}\right]dy$$

and the theorem follows on putting $Z = r_2\ln\left\{\frac{1}{2}\left(e^{y}+1\right)\right\}\left(m\ln\left\{\frac{1}{2}\left(e^{t_{00}}+1\right)\right\}\right)^{-1}$. □

Corollary 2. *In the case when $t_0 = t_{00}$, but, $\lambda_1 \neq \lambda_2$,*
$\widetilde{P}_I = r_2 n(r_2 n + r_1 m)^{-1}$.

Remarks. (i) In the literature, researchers have dealt with the estimation of $R(t)$ and P, separately. If we look at the proofs of Theorems 2–5 and 7–10, we observe that the UMVUE(S) / MLE(S) of power(s) of parameter is used to obtain UMVUE(S) / MLE(S) of the sampled pdf(s), which is (are) subsequently used to estimate $R(t)$ and P. Thus, for both the estimation problems, the basic role is played by the estimator(s) of power(s) of parameter. In this way, we have justified estimation of power(s) of parameter.
(ii) We have established interrelationship between the two estimation problems.
(iii) In the present approaches of obtaining UMVUES and MLES, one does not need the expressions of $R(t)$ and P.
(iv) It follows from Theorem 1 that $V(\hat{\lambda}_{II}) = \lambda^2(r-2)^{-1} \to 0$ as $r \to \infty$. Moreover, from (2.10), $E(\widetilde{\lambda}_{II}) = r\lambda(r-1)^{-1} \to \lambda$ as $r \to \infty$ and $V(\widetilde{\lambda}_{II}) = r^2\lambda^2(r-1)^{-2}(r-2)^{-1} \to 0$ as $r \to \infty$. Thus, $\hat{\lambda}_{II}$ and $\widetilde{\lambda}_{II}$ are consistent estimators of λ. Since $\hat{f}_{II}(x;\lambda)$, $\widetilde{f}_{II}(x;\lambda)$, $\hat{R}_{II}(t)$, $\widetilde{R}_{II}(t)$, $\hat{P}_{II}$ and $\widetilde{P}_{II}$ are continuous functions of consistent estimators, they are also consistent estimators.
(v) Similarly, it follows from Theorem 6 that $V(\hat{\lambda}_I) = \lambda\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1} \to 0$ as $n \to \infty$. Moreover, from (3.4), $E(\widetilde{\lambda}_I) = \lambda$ and $V(\widetilde{\lambda}_I) = \lambda\left(n\ln\left\{\frac{1}{2}(e^{t_0}+1)\right\}\right)^{-1} \to 0$ as $n \to \infty$. Thus, $\hat{\lambda}_I$ and $\widetilde{\lambda}_I$ are also consistent estimators of λ. Since, $\hat{f}_I(x;\lambda)$, $\widetilde{f}_I(x;\lambda)$, $\hat{R}_I(t)$, $\widetilde{R}_I(t)$, $\hat{P}_I$ and $\widetilde{P}_I$ are continuous functions of consistent estimators, they are also consistent estimators.

4. Test procedures for various hypotheses

An important hypothesis in life-testing experiments is $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$. It follows from (2.4) that, the likelihood function for observing λ is given by

$$L(\lambda|\mathbf{x}) = n(n-1)\dots(n-r+1)\lambda^r\left\{\prod_{i=1}^{r}\left(\frac{1}{1+e^{-x_{(i)}}}\right)\right\}\exp(-\lambda S_r). \tag{4.1}$$

Now, under H_0

$$\sup_{\Theta_0} L(\lambda|\mathbf{x}) = n(n-1)\dots(n-r+1)\lambda_0^r\left\{\prod_{i=1}^{r}\left(\frac{1}{1+e^{-x_{(i)}}}\right)\right\}\exp(-\lambda_0 S_r), \quad \Theta_0 = \{\lambda : \lambda = \lambda_0\}$$

and

$$\sup_{\Theta} L(\lambda|\mathbf{x}) = n(n-1)\dots(n-r+1)\left\{\frac{r}{S_r}\right\}^r\left\{\prod_{i=1}^{r}\left(\frac{1}{1+e^{-x_{(i)}}}\right)\right\}\exp(-r), \quad \Theta = \{\lambda : \lambda > 0\}.$$

Therefore, the likelihood ratio (LR) is given by

$$\phi(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\lambda|\mathbf{x})}{\sup_{\Theta} L(\lambda|\mathbf{x})} = \frac{(\lambda_0 S_r)^r}{r^r}\exp(-\lambda_0 S_r + r). \tag{4.2}$$

We note that the first term on the right hand side of (4.2) is an increasing function of S_r. Denoting by $\chi^2_{(2r)}(.)$, the Chi-square statistic with $2r$ degrees of freedom and using the fact that $2\lambda_0 S_r \sim \chi^2_{2r}$ the critical region is given by

$\{0 < S_r < k_0\} \cup \{k_0' < S_r < \infty\}$,

where, k_0 and k_0' are obtained such that $P\left[\chi^2_{(2r)} < 2\lambda_0 k_0 \text{ or } 2\lambda_0 k_0' < \chi^2_{(2r)}\right] = \alpha$.

Similarly, using Lemma 4, it can be shown that, under the sampling scheme of Bartholomew (1963), the critical region for testing $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$ is given by

$$\{r < k_1 \text{ or } r > k_1'\}, \qquad r \sim Poisson\left(n\lambda \ln\left\{\frac{1}{2}\left(e^{t_0}+1\right)\right\}\right).$$

Another important hypothesis in life-testing experiments is $H_0 : \lambda \leq \lambda_0$ against $H_1 : \lambda > \lambda_0$ It follows from (2.4) that, for $\lambda_1 < \lambda_2$,

$$\frac{h(x_{(1)}, x_{(2)}, \ldots, x_{(r)}; \lambda_1)}{h(x_{(1)}, x_{(2)}, \ldots, x_{(r)}; \lambda_2)} = \left(\frac{\lambda_1}{\lambda_2}\right)^r \exp\left\{(\lambda_2 - \lambda_1)S_r\right\}. \tag{4.3}$$

It follows from (4.3) that $h(x_{(1)}, x_{(2)}, \ldots, x_{(r)}; \lambda)$ has monotone likelihood ratio in S_r Thus, the uniformly most powerful critical region (UMPCR) for testing H_0 against H_1 is given by (see Lehmann, 1959, p. 88)

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if } S_r \leq k_0'' \\ 0, & \text{otherwise,} \end{cases}$$

where, k_0'' is obtained such that $P\left[\chi^2_{(2r)} \leq 2\lambda_0 k_0''\right] = \alpha$.

Similarly, using Lemma 4, it can be shown that, under the sampling scheme of Bartholomew (1963), the UMPCR for testing $H_0 : \lambda \leq \lambda_0$ against $H_1 : \lambda \geq \lambda_0$ is given by

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if } r \leq k_1'' \\ 0, & \text{otherwise,} \end{cases}$$

where, k_1'' is obtained such that $P\left[r \leq k_1''\right] = \alpha$.

It can be shown that $P = \lambda_1(\lambda_1 + \lambda_2)^{-1}$. Suppose, we want to test $H_0 : P = P_0$ against $H_1 : P \neq P_0$. It follows that H_0 is equivalent to $\lambda_1 = K\lambda_2$, where $K = P_0(1 - P_0)^{-1}$. Thus, $H_0 : \lambda_1 = K\lambda_2$, and $H_1 : \lambda_1 \neq K\lambda_2$. It can be shown that, under H_0,

$$\widetilde{\lambda}_1 = (r_1 + r_2)\left(S_{r_1} + K^{-1}T_{r_2}\right)^{-1}$$

and

$$\widetilde{\lambda}_2 = (r_1 + r_2)\left(KS_{r_1} + T_{r_2}\right)^{-1}.$$

For a generic constant k, the likelihood of observing $x_{(1)}, x_{(2)}, \ldots, x_{(r_1)}$ and $y_{(1)}, y_{(2)}, \ldots, y_{(r_2)}$ is

$$L\left(\lambda_1, \lambda_2 | x_{(1)}, x_{(2)}, \ldots, x_{(r_1)}, y_{(1)}, y_{(2)}, \ldots, y_{(r_2)}\right) = k\lambda_1^{r_1}\lambda_2^{r_2}\exp\left(-\lambda_1 S_{r_1} - \lambda_2 T_{r_2}\right).$$

Thus,

$$\sup_{\Theta_0} L\left(\lambda_1, \lambda_2 | x_{(1)}, x_{(2)}, \ldots, x_{(r_1)}, y_{(1)}, y_{(2)}, \ldots, y_{(r_2)}\right) = k\left(KS_{r_1} + T_{r_2}\right)^{-(r_1+r_2)}\exp\left\{-(r_1+r_2)\right\} \tag{4.4}$$

and

$$\sup_{\Theta} L\left(\lambda_1, \lambda_2 | x_{(1)}, x_{(2)}, \ldots, x_{(r_1)}, y_{(1)}, y_{(2)}, \ldots, y_{(r_2)}\right) = k(S_{r_1})^{-r_1}(T_{r_2})^{-r_2}\exp\left\{-(r_1+r_2)\right\}. \tag{4.5}$$

From (4.4) and (4.5), the likelihood ratio criterion is

$$\phi(x_{(1)}, x_{(2)}, \ldots, x_{(r_1)}, y_{(1)}, y_{(2)}, \ldots, y_{(r_2)}) = k\left(\frac{S_{r_1}}{T_{r_2}}\right)^{r_1}\left(1 + K\frac{S_{r_1}}{T_{r_2}}\right)^{-(r_1+r_2)}.$$

Denoting by $F(a, b)$ the F-statistic with (a, b) degrees of freedom and using the fact that $\frac{S_{r_1}}{T_{r_2}} \sim \frac{r_1\lambda_2}{r_2\lambda_1}F(2r_1, 2r_2)$, the critical region is given by

$$\left\{\frac{S_{r_1}}{T_{r_2}} < k_2 \text{ or } \frac{S_{r_1}}{T_{r_2}} > k_2'\right\},$$

where, k_2 and k_2' are obtained such that $P\left[F(2r_1, 2r_2) < \frac{Kr_2}{r_1}k_2 \text{ or } \frac{Kr_2}{r_1}k_2' < F(2r_1, 2r_2)\right] = \alpha$.

5. Simulation studies and real life data analysis

In order to investigate the performances of estimators obtained under Type II censoring, we have generated the following sample of size $n = 50$ from (1.1) with ($\lambda = 2, r = 35$).

Sample 1. **0.03557**, **0.04390**, **0.05849**, **0.07581**, **0.08980**, **0.09512**, **0.15476**, **0.15672**, **0.15883**, **0.18973**, **0.19693**, **0.19744**, **0.25391**, **0.29171**, **0.29456**, **0.29595**, **0.30608**, **0.35567**, **0.35874**, **0.40511**, **0.51386**, **0.53380**, **0.53397**, **0.53851**, **0.54353**, **0.64291**, **0.69078**, **0.74089**, **0.77141**, **0.78130**, **0.85495**, **0.88166**, **0.88959**, **0.93080**, **0.96709**, 0.99934, 1.01677, 1.04662, 1.14260, 1.16183, 1.31862, 1.40385, 1.67627, 1.68607, 1.73305, 1.80235, 1.96938, 2.31859, 2.60060, 3.16414.

Here, $R(1.5) = 0.1331163$ and for $q = 2$, we have $\lambda^2 = 4.0$, $\hat{\lambda}^2_{\mathrm{II}} = 3.6706660$ and $\widetilde{\lambda}^2_{\mathrm{II}} = 4.0076340$. For the same sample we also observe that $S_{35} = 17.48332$, $\hat{R}_{\mathrm{II}}(1.5) = 0.1327089$ and $\widetilde{R}_{\mathrm{II}}(1.5) = 0.1328605$.

In order to obtain the estimates of P, we have generated two samples of sizes $n = 40$ and $m = 50$ from X and Y populations with ($\lambda_1 = 1.5, r_1 = 30$) and ($\lambda_2 = 2.5, r_2 = 35$) respectively.

Sample 2. **0.01390**, **0.01999**, **0.02407**, **0.10932**, **0.14757**, **0.22437**, **0.23892**, **0.24690**, **0.31548**, **0.31598**, **0.33807**, **0.37595**, **0.41559**, **0.43652**, **0.51984**, **0.56983**, **0.58488**, **0.58512**, **0.60319**, **0.64334**, **0.68058**, **0.75704**, **0.90135**, **0.94744**, **0.95669**, **0.97402**, **0.98917**, **1.05542**, **1.12621**, **1.15212**, 1.23539, 1.24687, 1.33331, 1.43316, 1.54729, 1.59163, 2.05774, 2.19345, 2.47098, 2.63974.

Sample 3. **0.00561**, **0.01720**, **0.03715**, **0.05352**, **0.06566**, **0.07054**, **0.07495**, **0.14592**, **0.18657**, **0.18920**, **0.25015**, **0.26663**, **0.27329**, **0.27555**, **0.27924**, **0.28733**, **0.34623**, **0.34653**, **0.35539**, **0.36918**, **0.38105**, **0.38257**, **0.41951**, **0.44008**, **0.44338**, **0.44734**, **0.45479**, **0.45757**, **0.47485**, **0.48859**, **0.49504**, **0.52409**, **0.60360**, **0.63920**, **0.65156**, 0.68071, 0.71227, 0.72568, 0.82186, 0.86101, 0.91877 0.92865, 0.98415, 1.00283, 1.19383, 1.21007, 1.32116, 1.45646, 2.30930, 2.96547.

For these two samples, we have $S_{30} = 16.96735$, $T_{35} = 11.85217$, $P = 0.6250000$, $\hat{P}_{\mathrm{II}} = 0.6279613$ and $\widetilde{P}_{\mathrm{II}} = 0.6254931$.

In order to compare the performances of estimators obtained under the sampling scheme of Bartholomew (1963), first we have generated the following sample of size $n = 50$ from (1.1) with ($\lambda = 1.5, t_0 = 0.5$).

Sample 4. **0.01482**, **0.01501**, **0.02427**, **0.03059**, **0.04498**, **0.04616**, **0.04883**, **0.07409**, **0.12092**, **0.12726**, **0.14244**, **0.20716**, **0.22625**, **0.26113**, **0.26478**, **0.26734**, **0.28106**, **0.30782**, **0.32811**, **0.38861**, **0.39100**, **0.45515**, 0.51289, 0.59869, 0.69617, 0.71387, 0.76371, 0.79266, 0.8222, 0.84549 0.85455 0.92352, 0.93130, 0.99186, 1.01629, 1.16967, 1.17996, 1.18517, 1.25318, 1.52630, 1.52848 1.55159, 1.55272, 1.67781, 1.77616, 1.84483, 1.97855, 2.01709, 2.45541, 2.59038.

Here, we observe that $r = 22$. Again, for $q = 2$, we obtained $\lambda^2 = 2.2500000$, $\hat{\lambda}^2_{\mathrm{I}} = 2.3415660$ and $\widetilde{\lambda}^2_{\mathrm{I}} = 2.4530690$. For the same sample we have $R(0.8) = 0.4882488$, $\hat{R}_{\mathrm{I}}(0.8) = 0.4669112$ and $\widetilde{R}_{\mathrm{I}}(0.8) = 0.5851892$.

In order to obtain the estimates of P, we have generated samples of size $n = 50$ and $m = 30$ from X and Y populations, respectively, for ($\lambda_1 = 1.5, t_0 = 1.0$) and ($\lambda_2 = 2.5, t_{00} = 1.0$).

Sample 5. **0.04369**, **0.05539**, **0.06486**, **0.06557**, **0.17431**, **0.20020**, **0.20764**, **0.21177**, **0.29985**, **0.30308**, **0.37475**, **0.38439**, **0.40060**, **0.43005**, **0.44059**, **0.47568**, **0.51463**, **0.56283**, **0.60545**, **0.61437**, **0.62927**, **0.70739**, **0.79281**, **0.85155**, **0.85884**, **0.90207**, **0.97803**, **0.99725**, 1.12290, 1.13807, 1.15342, 1.20459, 1.22793, 1.29950, 1.38227, 1.43545, 1.64240, 1.64661, 1.75696, 1.79753, 1.85814, 1.97025, 2.04686, 2.26356, 2.28556, 2.28863, 2.42240, 2.55530, 2.93035, 3.11324.

Sample 6. **0.05832**, **0.06896**, **0.6935**, **0.09352**, **0.10416**, **0.10787**, **0.12729**, **0.15383**, **0.16423**, **0.18712**, **0.27414**, **0.31500**, **0.34236**, **0.35265**, **0.39001**, **0.39691**, **0.43116**, **0.45483**, **0.53379**, **0.56427**, **0.66435**, **0.67791**, **0.79948**, **0.82972**, **0.83108**, **0.92230**, **0.95735**, **0.98113**, 1.43755, 1.52208.

Table 1
Failure log times to breakdown of an insulating fluid testing experiment.

0.270027	1.02245	1.15057	1.42311	1.54116	1.57898	1.8718	1.9947
2.08069	2.11263	2.48989	3.45789	3.48186	3.52371	3.60305	4.28895

Table 2
Estimates of powers of λ based on Type II censoring.

r	4	8	12	16
$\hat{\lambda}_{II}$	0.21725	0.37317	0.41954	0.55071
$\hat{\lambda}^2_{II}$	0.03146	0.11936	0.16001	0.28306
$\tilde{\lambda}_{II}$	0.28966	0.42648	0.45768	0.58742
$\tilde{\lambda}^2_{II}$	0.08391	0.18189	0.20947	0.34506

Table 3
Estimates of $R(t)$ based on Type II censoring.

t		r			
		4	8	12	16
0.5	$\hat{R}_{II}(t)$	0.9402	0.89976	0.88825	0.85598
	$\tilde{R}_{II}(t)$	0.92185	0.88709	0.87935	0.84787
0.8	$\hat{R}_{II}(t)$	0.89972	0.83471	0.81679	0.76678
	$\tilde{R}_{II}(t)$	0.87071	0.81559	0.80352	0.75521
1	$\hat{R}_{II}(t)$	0.87124	0.79032	0.76852	0.7079
	$\tilde{R}_{II}(t)$	0.83558	0.76761	0.75291	0.6947
1.2	$\hat{R}_{II}(t)$	0.84185	0.74568	0.7204	0.65036
	$\tilde{R}_{II}(t)$	0.80005	0.72004	0.70294	0.6361
1.5	$\hat{R}_{II}(t)$	0.79656	0.67926	0.64963	0.56791
	$\tilde{R}_{II}(t)$	0.74672	0.6505	0.63036	0.55307

Table 4
Estimates of powers of λ based on the sampling scheme of Bartholomew (1963).

t_0 r	1.5 4	2 8	3 11	4.3 16
$\hat{\lambda}_I$	0.24795	0.34873	0.29188	0.27622
$\hat{\lambda}^2_I$	0.04611	0.10641	0.07745	0.07153
$\tilde{\lambda}_I$	0.24795	0.34873	0.29188	0.27622
$\tilde{\lambda}^2_I$	0.06148	0.12161	0.08519	0.0763

Here, we observe that $r_1 = 28, r_2 = 28, P = 0.6250000, \hat{P}_I = 0.6271232$ and $\tilde{P}_I = 0.6250000$.

For the theory developed in Section 4, for testing the hypothesis $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$ under Type II sampling scheme, we have considered Sample 1. Now with the help of chi-square table at $\alpha = 5\%$ level of significance (LOS), we obtained $k_0 = 12.18939$ and $k_0' = 23.7558$. Hence, in this case we may accept H_0 at 5% LOS as $S_{35} = 17.48332$.

Again, for testing $H_0 : \lambda \leq \lambda_0$ against $H_1 : \lambda > \lambda_0$ we have considered Sample 1. Now at 5% LOS, we obtained $k_0'' = 12.93482$ and hence in this case we may accept H_0 as $S_{35} = 17.48332$.

In order to test $H_0 : P = P_0$ against $H_1 : P \neq P_0$ under Type II sampling scheme, we have considered Samples 2 and 3. For these two samples, we obtain $S_{30}/T_{35} = 1.431582$. Now, with the help of F-table at 5% LOS, we obtain $k_2 = 0.869331$ and $k_2' = 2.325576$. Hence, in this case we may accept H_0 at 5% LOS.

For testing the hypothesis $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$ under the sampling scheme of Bartholomew (1963), we have considered Sample 4. Using the fact that $r \sim \text{Poisson}\left(n\lambda \ln\left\{\frac{1}{2}(e^{t_0} + 1)\right\}\right)$, with help of Poisson table at 5% LOS, we obtain $k_1 = 11$ and $k_1' = 28$. Hence, in this case we may accept H_0 at 5% LOS as $r = 22$.

Again, for testing $H_0 : \lambda \leq \lambda_0$ against $H_1 : \lambda > \lambda_0$ at 5% LOS, we obtained $k_1'' = 12$. Hence in this case we may accept H_0 at 5% LOS as $r = 22$ (corresponding to Sample 4).

Table 1 gives the failure log times to breakdown of an insulating fluid experiment (see Nelson, 1982). Seo, Kim, and Kang (2013), applying Kolmogorov test, showed that the data follow GHLD.

In what follows, we compute estimators of powers of λ and $R(t)$ based on different types of censorings (see Tables 2–5).

Table 5
Estimates of $R(t)$ based on the sampling scheme of Bartholomew (1963).

t	t_0 / r	1.5 / 4	2 / 8	3 / 11	4.3 / 16
0.5	$\hat{R}_I(t)$	0.93214	0.90613	0.92099	0.92516
	$\widetilde{R}_I(t)$	0.88621	0.85396	0.91726	0.96516
0.8	$\hat{R}_I(t)$	0.88665	0.84498	0.86901	0.87584
	$\widetilde{R}_I(t)$	0.8632	0.8251	0.90017	0.95774
1	$\hat{R}_I(t)$	0.85488	0.80314	0.83318	0.8418
	$\widetilde{R}_I(t)$	0.84402	0.80122	0.88583	0.95144
1.2	$\hat{R}_I(t)$	0.82229	0.76096	0.79683	0.80723
	$\widetilde{R}_I(t)$	0.82117	0.77299	0.86861	0.94381
1.5	$\hat{R}_I(t)$	0.77248	0.69798	0.74208	0.75506
	$\widetilde{R}_I(t)$	0.7788	0.72129	0.83633	0.92924

6. Discussions and conclusions

If we look at the simulation results under Type II censoring scheme, it is clear that estimated values of MLES are better than estimated values of UMVUES. It is also clear that, under the sampling scheme of Bartholomew (1963), estimated values of UMVUEs of λ and R(t) are better than their corresponding estimated values of MLEs. But, under the sampling scheme of Bartholomew (1963), estimated value of MLE of P is more efficient than estimated value of UMVUE of P.

Thus, the problems of estimating $R(t)$ and P are considered. UMVUES and MLES are derived. A comparative study of the two methods of estimation is done. By estimating the sampled pdf to obtain the estimators of $R(t)$ and P, an interrelationship between the two estimation problems is established. Simulation study is performed.

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