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Multiplicative bias correction for generalized Birnbaum–Saunders kernel density estimators and application to nonnegative heavy tailed data

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ABSTRACT

In this paper, we show that the multiplicative bias correction (MBC) techniques can be applied for generalized Birnbaum–Saunders (GBS) kernel density estimators. First, some properties of the MBC-GBS kernel density estimators (bias, variance and mean integrated squared error) are shown. Second, the choice of bandwidth is investigated by adopting the popular cross-validation technique. Finally, the performances of the MBC estimators based on GBS kernels are illustrated by a simulation study, followed by a real application for nonnegative heavy tailed (HT) data. In general, in terms of integrated squared bias (ISB) and integrated squared error (ISE), the proposed estimators outperform the standard GBS kernel estimators.

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1. Introduction

Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) continuous random variables with an unknown probability density function (pdf) f on the support $\mathbb{T}$ ($\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = [0, \infty)$). A continuous symmetric or asymmetric kernel estimator $\hat{f}_h(x)$ of $f(x)$ can be defined as follows:

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K_{x,h}(X_i) \tag{1}$$

where $h = h(n) > 0$ is an arbitrary sequence of smoothing parameters (bandwidths) and $K_{x,h}$ is the continuous symmetric or asymmetric kernel with the target x and the bandwidth h. Note that in symmetric case, we have $K_{x,h}(\cdot) = (1/h)K\{(x-\cdot)/h\}$, where $K(\cdot)$ is the kernel function which is a symmetric pdf independent of x and h; see, e.g., Parzen (1962), Rosenblatt (1956) and Silverman (1986). It is well known that the symmetric kernel estimator is inappropriate for estimating densities with support $\mathbb{T} = [0, \infty)$, because it causes boundary bias. Thus, the asymmetric kernels have been proposed as a good solution for avoiding these boundary effects. This simple idea is due to Chen (2000) (gamma and modified gamma kernels), Scaillet (2004) (inverse and reciprocal inverse Gaussian kernels), Jin and Kawczak (2003) (log-normal and Birnbaum–Saunders (BS) kernels) and Marchant, Bertin, Leiva, and Saulo (2013) (generalized Birnbaum–Saunders (GBS) kernels); see also Chen (1999) when the support is $\mathbb{T} = [0, 1]$ and Kokonendji and Senga Kiessé (2011) for discrete case ($\mathbb{T} = \mathbb{N}$).

Assuming that the true density f is twice continuously differentiable, then the bias of (1) is $O(h^2)$ with symmetric kernels and $O(h)$ with asymmetric kernels as $h \to 0$. Jones, Linton, and Nielsen (1995) and Terrell and Scott (1980) proposed the so-called multiplicative bias correction (MBC) techniques, which improve bias from $O(h^2)$ as the bandwidth $h \to 0$, to $O(h^4)$ for kernel density using symmetric second-order kernels, see also Jones and Foster (1993) for the same context. Recently, Hirukawa (2010) and Hirukawa and Sakudo (2014) have demonstrated that these two classes of MBC approaches can be applied to kernel density estimation on the unit interval using beta and modified beta kernels and for density estimation using asymmetric kernels (gamma, modified gamma, inverse Gaussian, reciprocal inverse Gaussian, log-normal and BS kernels), respectively. The authors have shown that the order of magnitude in bias is improved from $O(h)$ to $O(h^2)$.

The main goal of this paper is to extend the application of MBC approaches for GBS kernel density estimation as in Hirukawa (2010) and Hirukawa and Sakudo (2014). These previous studies are motivations of this paper. Our study is also motivated by several points. First, the family of GBS kernels introduced recently by Marchant et al. (2013) has a large number of particular cases such as the BS-classical, BS-power-exponential (BS-PE) and BS-Student-t (BS-t) kernels. Second, the GBS kernels are more appropriate for estimating densities of nonnegative HT data, because of their flexibility and properties; see Jin and Kawczak (2003) and Marchant et al. (2013). As third motivation, some applications of GBS kernel methods can be found in various domains such as in economics, finance, reliability, actuarial and also environmental sciences.

This paper is organized as follows. Section 2 briefly recalls the GBS distribution and standard GBS kernel density estimators. In Section 3 we develop asymptotic properties of MBC-GBS kernel density estimators and adopt the unbiased-cross validation (UCV) procedure of choosing the bandwidth for proposed estimators. Section 4 conducts Monte Carlo simulations to compare sample finite performance of standard GBS and proposed MBC-GBS kernel estimators. Section 5 provide an application on real environmental data and all proofs are given in Section 6. Finally, Section 7 concludes the paper.

2. A short review on GBS kernels

In this section, we present a brief recall on GBS distribution and GBS kernel density estimators.

2.1. GBS distribution

Consider a GBS random variable $T \sim GBS(\alpha, \beta; g)$, where $\alpha > 0$ is the shape parameter, $\beta > 0$ is the scale parameter and g is a real function that generates the density of random variable $Z = \left(\sqrt{T/\beta} - \sqrt{\beta/T}\right)/\alpha$ with standard symmetric distributions; see Marchant et al. (2013) for more details. The probability density function (pdf) of T is given by

$$f_T(t; \alpha, \beta; g) = cg\left(\frac{1}{\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right)\frac{1}{2\alpha}\left(\frac{1}{\sqrt{\beta t}} + \sqrt{\frac{\beta}{t^3}}\right), \quad t > 0; \alpha > 0,\ \beta > 0, \tag{2}$$

Table 1
Constant c, c_{g^2} and density generator g for the indicated distribution.

Distribution	c	c_{g^2}	$g = g(u),\ u > 0$
Normal	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{\pi}}$	$\exp\left(-\frac{1}{2}u\right)$
PE	$\frac{\nu}{2^{\frac{1}{2\nu}}\Gamma(\frac{1}{2\nu})}$	$\frac{\nu}{\Gamma\left(\frac{1}{2\nu}\right)}$	$\exp\left(-\frac{1}{2}u^{\nu}\right), \nu > 0$
t	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{\pi})}$	$\frac{\nu\Gamma(\nu)}{\sqrt{\pi}\Gamma\left(\frac{1+2\nu}{\nu}\right)}$	$\left(1+\frac{u}{\nu}\right)^{-\frac{(\nu+1)}{2}}, \nu > 0$

Table 2
Values of $u_1(g)$ and $u_2(g)$ for the indicated distribution.

Distribution	$u_1(g)$	$u_2(g)$
Normal	1	3
PE	$\frac{2^{\frac{1}{\nu}}\Gamma\left(\frac{3}{2\nu}\right)}{\Gamma\left(\frac{1}{2\nu}\right)}$	$\frac{2^{\frac{2}{\nu}}\Gamma\left(\frac{5}{2\nu}\right)}{\Gamma\left(\frac{1}{2\nu}\right)}$
t	$\frac{\nu}{\nu-2},\ \nu > 2$	$\frac{3\nu^2}{(\nu-2)((\nu-4))},\ \nu > 4$

Table 3
GBS kernels for the indicated distribution.

Distribution	Kernel
Normal	$\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2h}\left(\frac{y}{x}+\frac{x}{y}-2\right)\right)\frac{1}{2\sqrt{h}}\left(\frac{1}{\sqrt{xy}}+\sqrt{\frac{x}{y^3}}\right)$
PE	$\frac{\nu}{2^{\frac{1}{2\nu}}\Gamma(\frac{1}{2\nu})}\exp\left(-\frac{1}{2h^{\nu}}\left(\frac{y}{x}+\frac{x}{y}-2\right)\right)\frac{1}{2\sqrt{h}}\left(\frac{1}{\sqrt{xy}}+\sqrt{\frac{x}{y^3}}\right), \nu > 0$
t	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{\pi})}\left[1+\frac{1}{\nu h}\left(\frac{y}{x}+\frac{x}{y}-2\right)\right]^{-\left(\frac{\nu+1}{2}\right)}\frac{1}{2\sqrt{h}}\left(\frac{1}{\sqrt{xy}}+\sqrt{\frac{x}{y^3}}\right), \nu > 0$

where $c = 1/\int_{-\infty}^{\infty} g(y^2)dy$ is the normalization constant; see Table 1 for some examples of g and c. The kth moment of the GBS distribution is given by

$$\mathbb{E}(T^k) = \beta^k \sum_{i=0}^{k}\binom{2k}{2i}\sum_{j=0}^{i}\binom{i}{j}\mathbb{E}(U^{k+j-i})\left(\frac{\alpha}{2}\right)^{2(k+j-i)}, \quad k = 1, 2, \ldots \tag{3}$$

where $U = Z^2$ follows a generalized chi-square ($G\chi^2$) distribution with one degree of freedom, and denoted $U \sim G\chi^2(g)$; see Fang, Kotz, and Ng (1990) and Marchant et al. (2013).

2.2. GBS kernel estimator

Given a random sample $X_1, \ldots, X_n$, the GBS kernel estimator of an unknown pdf f with nonnegative support is given by Marchant et al. (2013)

$$\begin{aligned}\widehat{f}_{GBS}(x) &= \frac{1}{n}\sum_{i=1}^{n} K_{GBS(h^{1/2},x;g)}(X_i)\\ &= \frac{1}{n}\sum_{i=1}^{n} cg\left(\frac{1}{h}\left(\frac{X_i}{x}+\frac{x}{X_i}-2\right)\right)\frac{1}{2\sqrt{h}}\left(\frac{1}{\sqrt{xX_i}}+\sqrt{\frac{x}{X_i^3}}\right), \quad x > 0,\end{aligned} \tag{4}$$

where $x > 0$ is the target (point where the density is estimated) and $h > 0$ is a bandwidth (or smoothing parameter). The GBS kernel K_{GBS} is obtained with parameters $\alpha = h^{1/2}$, $\beta = x$ and generator g; see Table 3 for some examples of GBS Kernels. Note that the GBS-Normal kernel reduces to the original Birnbaum–Saunders kernel proposed by Jin and Kawczak (2003).

The expressions of the bias and variance for $\widehat{f}_{GBS}(x)$ are derived by Marchant et al. (2013). The asymptotic bias when $h \to 0$ is given by

$$\text{bias}(\widehat{f}_{GBS}(x)) = \frac{hu_1(g)}{2}\left(xf'(x) + x^2 f''(x)\right) + o(h), \tag{5}$$

where $u_1(g) = \mathbb{E}(U)$ is given in Table 2. Similarly, when $n \to \infty$ and $h \to 0$ the asymptotic variance for $x/h \to \infty$ (interior x) is

$$\text{Var}(\widehat{f}_{GBS}(x)) = \frac{c^2}{c_{g^2} nh^{1/2} x} f(x) + o\left(\frac{1}{nh^{1/2}}\right), \tag{6}$$

where $c = 1/\int_{-\infty}^{\infty} g(y^2)dy$ and $c_{g^2} = 1/\int_{-\infty}^{\infty} g^2(y^2)dy$ are given in Table 1 for some generator g. Note that we can easily show that the variance approximation for $x/h \to \kappa$, where $\kappa > 0$ is a constant (boundary x) is (see, e.g., Chen, 2000, Hirukawa & Sakudo, 2014 and Scaillet, 2004)

$$\mathrm{Var}(\widehat{f}_{GBS}(x)) = O\left\{(nh^{3/2})^{-1}\right\}. \tag{7}$$

The mean integrated squared error (MISE) is also given in Marchant et al. (2013) and is expressed as

$$\begin{aligned}\mathrm{MISE}(\widehat{f}_{GBS}) &= \int_0^\infty \mathrm{bias}^2(\widehat{f}_{GBS}(x))dx + \int_0^\infty \mathrm{Var}(\widehat{f}_{GBS}(x))dx \\ &= \left(\frac{hu_1(g)}{2}\right)^2 \int \left(xf'(x) + x^2 f''(x)\right)^2 dx + \left(\frac{c^2}{c_{g^2} nh^{1/2}}\right) \int \frac{1}{x} f(x)dx + o\left(h^2 + \frac{1}{nh^{\frac{1}{2}}}\right).\end{aligned} \tag{8}$$

The bandwidth h that minimizes (8) is given by

$$h_{GBS}^{opt} = \left[\frac{c^2 \int \frac{1}{x} f(x)dx}{c_{g^2 u_1^2(g)} \int (xf'(x) + x^2 f'')^2 dx}\right]^{2/5} n^{-2/5}. \tag{9}$$

It is clear that the bandwidth given by (9) cannot be used directly in practice. For this reason, Marchant et al. (2013) and Ziane, Adjabi, and Zougab (2015) have adapted respectively the biased and unbiased cross-validation (BCV and UCV) procedures to bandwidth selection for GBS kernel estimators (4).

3. MBC for GBS density estimators

In this section, we adapt two classes of MBC techniques for GBS kernel density estimator, originally proposed by Jones et al. (1995) and Terrell and Scott (1980) for symmetric kernel density estimator. Note that these MBC techniques have also extended recently by Hirukawa (2010) and Hirukawa and Sakudo (2014) for kernel density estimation on the unit interval using beta and modified beta kernels and for density estimation using asymmetric kernels (gamma, modified gamma, inverse Gaussian, reciprocal inverse Gaussian, log-Normal and Birnbaum–Saunders kernels), respectively.

3.1. Estimators

Based on the same idea of Hirukawa (2010) and Terrell and Scott (1980), the MBC kernel density estimator using the GBS kernels, which we simply denote as TS-GBS kernel density estimators, can be adapted as follows:

$$\widehat{f}_{TS\text{-}GBS}(x) = \left\{\widehat{f}_{GBS,h}(x)\right\}^{\frac{1}{1-a}} \left\{\widehat{f}_{GBS,h/a}(x)\right\}^{-\frac{a}{1-a}}, \tag{10}$$

where $\widehat{f}_{GBS,h}$ and $\widehat{f}_{GBS,h/a}$ denote the GBS kernel density estimators given by (4) with bandwidths h and h/a, respectively, with $a \in (0, 1)$ is a constant that does not depend on the target x; see, e.g., Hirukawa (2010).

The second class of MBC techniques for symmetric kernel density estimators is attributed to Jones et al. (1995) (see also Hirukawa, 2010 and Hirukawa & Sakudo, 2014 for asymmetric kernel density estimators). The analogue of their estimators using GBS kernels, which we denote as JLN-GBS kernel density estimators is given by

$$\widehat{f}_{JLN\text{-}GBS}(x) = \widehat{f}_{GBS}(x) \left\{\frac{1}{n}\sum_{i=1}^n \frac{K_{GBS(h^{1/2},x;g)}(X_i)}{\widehat{f}_{GBS}(X_i)}\right\}, \tag{11}$$

where $K_{GBS(h^{1/2},x;g)}$ is the GBS kernel.

3.2. Asymptotic properties

The following theorems present the asymptotic bias and variance of the TS-GBS and JLN-GBS kernel estimators. We assume that

A1. f has four continuous and bounded derivatives.
A2. The sequence of bandwidths $h = h(n)$ satisfies $\lim_{n\to\infty} h = 0$ and $\lim_{n\to\infty} nh^{7/2} = \infty$.

Note that these assumptions have been discussed in Hirukawa (2010) and Hirukawa and Sakudo (2014).

Theorem 1. *Let $\widehat{f}_{TS\text{-}GBS}$ be the TS-GBS kernel estimator defined by (10). For a given $x > 0$ and under assumptions* A1 *and* A2*, then:*

(i) *the bias of the TS-GBS kernel estimators admits the following expansion*

$$\mathrm{bias}(\widehat{f}_{TS\text{-}GBS}) = \frac{1}{a}\left[\frac{1}{2}\left\{\frac{l_1^2(x)}{f(x)} - l_2(x)\right\}\right] h^2 + o(h^2),$$

where $l_1(x) = \frac{1}{2}xu_1(g)f' + \frac{1}{2}x^2u_1(g)f''$ *and* $l_2(x) = \frac{1}{4}x^2u_2(g)f'' + \frac{1}{4}x^3u_2(g)f''' + \frac{1}{24}x^4u_2(g)f''''$ *with* $u_k(g) = \mathbb{E}(U^k)$, $k \in \{1, 2\}$ *given in* Table 2.

(ii) *the variance of the TS-GBS kernel estimators is given by*

$$\mathrm{Var}(\widehat{f}_{TS\text{-}GBS}) = \begin{cases} \dfrac{\psi(a)c^2}{c_{g^2}nh^{1/2}x}f(x) + o\left(\dfrac{1}{nh^{1/2}}\right) & \text{for } x/h \to \infty \\ O\left\{(nh^{3/2})^{-1}\right\} & \text{for } x/h \to \kappa, \end{cases}$$

where $\psi(a) = \frac{(1+a^{5/2})(1+a)^{1/2} - 2\sqrt{2}a^{3/2}}{(1+a)^{1/2}(1-a)^2}$, $c = 1/\int_{-\infty}^{\infty} g(y^2)dy$, $c_{g^2} = 1/\int_{-\infty}^{\infty} g^2(y^2)dy$ *and* $\kappa > 0$ *is a constant.*

Proof. The proof is given in Section 6.

Theorem 2. *Let* $\widehat{f}_{JLN\text{-}GBS}$ *be the JLN-GBS kernel estimator defined by* (11). *For a given* $x > 0$ *and under assumptions* A1 *and* A2, *then:*

(i) *the bias of the JLN-GBS kernel estimator is given by*

$$\mathrm{bias}(\widehat{f}_{JLN\text{-}GBS}) = -f(x)\left[\frac{1}{2}xu_1(g)q'(x) + \frac{1}{2}x^2u_1(g)q''(x)\right] h^2 + o(h^2),$$

where $q(x) = l_1(x)/f(x)$ *and* $l_1(x)$ *is the same as given in* Theorem 1.

(ii) *the variance of the JLN-GBS kernel estimators has asymptotic form*

$$\mathrm{Var}(\widehat{f}_{TS\text{-}GBS}) = \begin{cases} \dfrac{c^2}{c_{g^2}nh^{1/2}x}f(x) + o\left(\dfrac{1}{nh^{1/2}}\right) & \text{for } x/h \to \infty \\ O\left\{(nh^{3/2})^{-1}\right\} & \text{for } x/h \to \kappa, \end{cases}$$

where $c = 1/\int_{-\infty}^{\infty} g(y^2)dy$, $c_{g^2} = 1/\int_{-\infty}^{\infty} g^2(y^2)dy$ *and* $\kappa > 0$ *is a constant.*

Proof. The proof is given in Section 6.

3.3. Global property

The criterion to use for the global propriety is the mean integrated squared error (MISE) defined as

$$\mathrm{MISE}(\widehat{f}_{MBC\text{-}GBS}) = \int_0^{\infty} \mathrm{bias}^2(\widehat{f}_{MBC\text{-}GBS}(x))dx + \int_0^{\infty} \mathrm{Var}(\widehat{f}_{MBC\text{-}GBS}(x))dx, \tag{12}$$

where $\widehat{f}_{MBC\text{-}GBS}$ is the TS-GBS or the JLN-GBS kernel density estimators.

The mean integrated squared error (MISE) of the TS-GBS kernel estimators given in (10) is expressed as

$$\mathrm{MISE}(\widehat{f}_{TS\text{-}GBS}) = \frac{h^4}{4a^2}\int_0^{\infty}\left\{\frac{l_1^2(x)}{f(x)} - l_2(x)\right\}^2 dx + \frac{\psi(a)c^2}{c_{g^2}nh^{1/2}}\int_0^{\infty}\frac{f(x)}{x}dx + o\left(\frac{1}{nh^{1/2}} + h^4\right). \tag{13}$$

The optimal bandwidth minimizing the corresponding MISE (13) is such that

$$h_{TS\text{-}GBS}^{opt} = \left\{\frac{c^2a^2\psi(a)\int_0^{\infty}\frac{f(x)}{x}dx}{2c_{g^2}\int_0^{\infty}\left\{\frac{l_1^2(x)}{f(x)} - l_2(x)\right\}^2 dx}\right\}^{2/9} n^{-2/9}. \tag{14}$$

Similarly, the MISE of the JLN-GBS kernel estimators given in (11) is given by

$$\mathrm{MISE}(\widehat{f}_{JLN\text{-}GBS}) = h^4\int_0^{\infty} f^2(x)\left[\frac{1}{2}xu_1(g)q'(x) + \frac{1}{2}x^2u_1(g)q''(x)\right]^2 dx$$
$$+ \frac{c^2}{c_{g^2}nh^{1/2}}\int_0^{\infty}\frac{f(x)}{x}dx + o\left(\frac{1}{nh^{1/2}} + h^4\right). \tag{15}$$

By minimizing (15) in the bandwidth h, we obtain the optimal value

$$h^{opt}_{JLN\text{-}GBS} = \left\{ \frac{c^2 \int_0^\infty \frac{f(x)}{x} dx}{8c_{g^2} \int_0^\infty f^2(x) \left[\frac{1}{2} x u_1(g) q'(x) + \frac{1}{2} x^2 u_1(g) q''(x)\right]^2 dx} \right\}^{2/9} n^{-2/9}. \tag{16}$$

Note that the bandwidths (14) and (16) cannot be employed in practice. Then, the next subsection presents a practical procedure to bandwidth selection.

3.4. Choice of bandwidth for MBC-GBS kernel estimators

Because the optimal bandwidths given by (14) and (16) depend on the unknown density f and on its derivatives f', f'', f''' and f'''', then it cannot be exploited in practice. In this paper, we adopt the popular unbiased cross validation (UCV) method (see Hagmann & Scaillet, 2007 for semi-parametric MBC technique). In the case of the UCV technique, for a given estimator $\widehat{f}_{MBC\text{-}GBS}$, which denotes the TS-GBS or JLN-GBS kernel estimators, the optimal bandwidth h_{ucv} of h is obtained by

$$h_{ucv} = \arg\min_h \mathrm{UCV}(h),$$

where

$$\mathrm{UCV}(h) = \int \widehat{f}^{2}_{MBC\text{-}GBS}(x) dx - \frac{2}{n} \sum_{i=1}^{n} \widehat{f}^{(-i)}_{MBC\text{-}GBS}(X_i),$$

where $\widehat{f}^{(-i)}_{MBC\text{-}GBS}(y)$ is the leave-one-out estimator computed as $\widehat{f}_{MBC\text{-}GBS}(y)$ by excluding the observation X_i. For the TS-GBS kernel estimators, the UCV function is given

$$\begin{aligned} \mathrm{UCV}_{TS\text{-}GBS}(h) = & \int_0^\infty \left\{\widehat{f}_{GBS,h}(x)\right\}^{\frac{2}{1-a}} \left\{\widehat{f}_{GBS,h/a}(x)\right\}^{-\frac{2a}{1-a}} dx - \frac{2}{n(n-1)} \\ & \times \sum_i \left[\left\{ \sum_{j\neq i} K_{GBS(h^{1/2},X_i;g)}(X_j) \right\}^{\frac{1}{1-a}} \left\{ \sum_{j\neq i} K_{GBS((h/a)^{1/2},X_i;g)}(X_j) \right\}^{-\frac{a}{1-a}} \right]. \end{aligned} \tag{17}$$

In the case of JLN-GBS kernel estimators, the expression of UCV is

$$\begin{aligned} \mathrm{UCV}_{JLN\text{-}GBS}(h) = & \frac{1}{n^2} \int_0^\infty \widehat{f}_{GBS}(x)^2 \left\{ \sum_{i=1}^{n} \frac{K_{GBS(h^{1/2},x;g)}(X_i)}{\widehat{f}_{GBS}(X_i)} \right\}^2 dx - \frac{2}{n(n-1)} \\ & \times \sum_i \sum_{j\neq i} K_{GBS(h^{1/2},X_i;g)}(X_j) \frac{\widehat{f}_{GBS}(X_i)}{\widehat{f}_{GBS}(X_j)}. \end{aligned} \tag{18}$$

4. Simulation study

This section investigates the MBC-GBS kernel density estimators considered in the previous section and compares their performances with the GBS kernel estimators. We consider three densities displayed in Fig. 1 and defined as follows (see also Barros, Paula, & Leiva, 2009, Hirukawa & Sakudo, 2014, Jin & Kawczak, 2003 and Marchant et al., 2013):

(a) a Burr(1, 3, 1) density:

$$f(x) = \frac{3x^2}{(1+x^3)^2}, \quad x > 0$$

(b) a Pareto(1, 1) density:

$$f(x) = \frac{1}{(1+x)^2}, \quad x > 0$$

(c) a GBS(1/2, 3/2; $t_{\nu=1}$) density:

$$\frac{2}{3\pi}\left[1 + \frac{8}{3}x + \frac{6}{x} + 7\right]^{-1} \left[\sqrt{\frac{3}{2x}} + \sqrt{\frac{9}{8x^3}}\right], \quad x > 0.$$

Note that these considered densities have heavy tails. The Burr(1, 3, 1) density has been already studied and analyzed by Jin and Kawczak (2003) and Marchant et al. (2013) using standard BS and GBS kernel density estimators respectively.

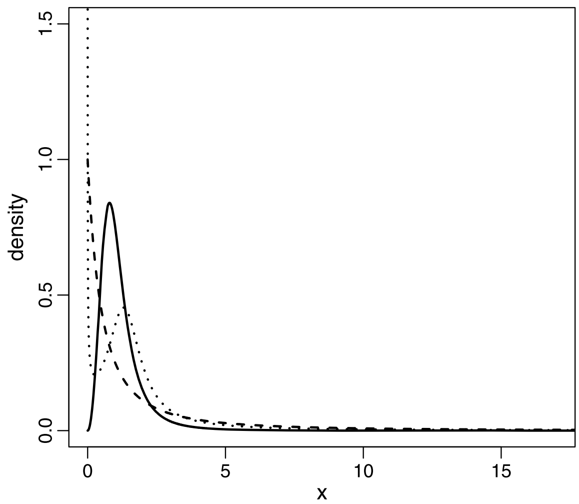

Fig. 1. Distributions in the simulation study. The black line represents the Burr distribution, the dashed line represents the Pareto distribution and the dot line represents the GBS distribution.

For Burr(1, 3, 1), Pareto(1, 1) and GBS(1/2, 3/2; $t_{\nu=1}$) distributions, 1000 replications of sizes $n = 25, 50, 100$ and 200 are generated. We apply the MBC-GBS (TS-GBS and JLN-GBS) kernel estimators using the BS-PE($\nu = 2$) and BS-$t(\nu = 5)$ kernels for estimating the density of nonnegative HT data generated from Burr(1, 3, 1), Pareto(1, 1) and GBS(1/2, 3/2; $t_{\nu=1}$) distributions. Note that for the TS-GBS kernel estimator, the parameter a is fixed as $a = 0.5$. However, the optimal value can be obtained in the sense of mean integrated squared error (MISE), see Hirukawa (2010). For comparison, we also use the standard GBS kernel estimators based on BS-PE($\nu = 2$) and BS-$t(\nu = 5)$ kernels. For bandwidth choice, which is one of the difficulties in nonparametric kernel methods, we used the UCV method developed in the previous section. We examine the performances of the estimators via integrated squared error (ISE) and integrated squared bias (ISB) given respectively by

$$\mathrm{ISE} := \int \left[\widehat{f}(x) - f(x)\right]^2 dx \tag{19}$$

and

$$\mathrm{ISB} := \int \left[\mathbb{E}\{\widehat{f}(x)\} - f(x)\right]^2 dx, \tag{20}$$

where $\widehat{f}$ is the MBC-GBS kernel estimator or the GBS kernel estimator with BS-PE($\nu = 2$) and BS-$t(\nu = 5)$ kernels. Note that the integrated variance (IV) given by $\mathrm{IV} := \int \mathrm{Var}\{\widehat{f}(x)\}dx$ is also computed.

From Tables 4–7, we can observe immediately that:

1. the means of ISE, ISB and IV based on 1000 replications decrease as sample size n increases for the all estimators;
2. the mean of the bandwidths obtained with UCV technique decreases as sample size n increases as for the ISE and ISB;
3. the TS-GBS and JLN-GBS kernel estimators outperform the standard GBS kernel estimators in the sense of ISE and ISB for several combinations of sample size and distribution. However, we can see that the TS-GBS and JLN-GBS kernel estimators underperform the standard GBS kernel estimators in certain cases, in particular for ISE criteria (see for example the cases of Burr ($n = 50$) with BS-$t(\nu = 5)$ kernel, Pareto ($n = 25, 50$) with BS-PE($\nu = 2$) kernel and GBS ($n = 25, 50, 100$ and 200) with BS-PE($\nu = 2$) kernel). This result can be caused by small sample size or by the values of bandwidth obtained with UCV method.
4. in terms of ISE and ISB, the performances of JLN-GBS and TS-GBS kernel estimators are mixed depending on the distribution. For example, in case of Burr, the JLN-GBS kernel estimator based on BS-$t(\nu = 5)$ kernel in general works better than the other competitors in the sense of ISB.

The comparison is also given in Fig. 2 for Burr distribution. The best smoothing quality is obtained by using the MBC-GBS kernel estimators. The standard GBS kernel estimators tend to underestimate the density.

5. Application to air pollution data

This section illustrates the performances of MBC-GBS kernel density estimators for O_3 data. These data concern the study of the daily tropospheric ozone concentrations (in ppb = ppm × 1000) observed in New York during May–September, 1973, provided by the New York State Department of Conservation; see, e.g., Leiva, Vilca, Balakrishnan, and Sanhueza (2010) and Nadarajah (2008). These data have been also analyzed recently by Saulo, Leiva, Ziegelmann, and Marchant (2013). Note

Table 4
Some expected values of ISE and their standard errors between parentheses based on 1000 replications for the Burr, Pareto and GBS distributions.

Density	Kernel	Estimator	$n = 25$	$n = 50$	$n = 100$	$n = 200$
Burr	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.06118(0.03671)	0.02378(0.01947)	0.01445(0.00987)	0.00762(0.00304)
		$\widehat{f}_{TS\text{-}GBS}$	0.05276(0.03913)	0.02170(0.01766)	0.01396(0.01062)	0.00680(0.00463)
		$\widehat{f}_{JLN\text{-}GBS}$	0.05375(0.04523)	0.01980(0.01526)	0.01558(0.01499)	0.00731(0.00562)
Burr	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.07593(0.09846)	0.03714(0.04780)	0.01874(0.01771)	0.00721(0.00214)
		$\widehat{f}_{TS\text{-}GBS}$	0.06524(0.11635)	0.03759(0.05738)	0.01624(0.01303)	0.00603(0.00265)
		$\widehat{f}_{JLN\text{-}GBS}$	0.06976(0.12537)	0.03809(0.06616)	0.01779(0.02890)	0.00726(0.00558)
Pareto	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.04864(0.03882)	0.01612(0.01080)	0.01533(0.01303)	0.00694(0.00294)
		$\widehat{f}_{TS\text{-}GBS}$	0.05510(0.05889)	0.01905(0.01198)	0.01611(0.01492)	0.00717(0.00408)
		$\widehat{f}_{JLN\text{-}GBS}$	0.04665(0.04923)	0.01672(0.01110)	0.01560(0.01366)	0.00567(0.00568)
Pareto	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.07505(0.18984)	0.02071(0.01588)	0.01707(0.01235)	0.00705(0.00486)
		$\widehat{f}_{TS\text{-}GBS}$	0.04623(0.03100)	0.01917(0.01513)	0.01656(0.01425)	0.00610(0.00259)
		$\widehat{f}_{JLN\text{-}GBS}$	0.04742(0.04179)	0.01890(0.02139)	0.01671(0.02004)	0.00557(0.00421)
GBS	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.03453(0.02918)	0.02149(0.01203)	0.01388(0.00719)	0.01107(0.00913)
		$\widehat{f}_{TS\text{-}GBS}$	0.03945(0.01830)	0.03363(0.01886)	0.02865(0.01996)	0.02688(0.01918)
		$\widehat{f}_{JLN\text{-}GBS}$	0.03546(0.03216)	0.02411(0.01300)	0.01272(0.00634)	0.00856(0.00501)
GBS	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.03338(0.02712)	0.02324(0.01915)	0.01481(0.00854)	0.01165(0.00789)
		$\widehat{f}_{TS\text{-}GBS}$	0.03463(0.03027)	0.02158(0.01543)	0.01427(0.01048)	0.01085(0.00693)
		$\widehat{f}_{JLN\text{-}GBS}$	0.03729(0.03351)	0.02297(0.01427)	0.01377(0.00355)	0.00925(0.00521)

Table 5
Empirical ISB values for the Burr, Pareto and GBS distributions based on 1000 Monte Carlo replications.

Density	Kernel	Estimator	$n = 25$	$n = 50$	$n = 100$	$n = 200$
Burr	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.01799	0.00692	0.00155	0.00101
		$\widehat{f}_{TS\text{-}GBS}$	0.01510	0.00491	0.00134	0.00088
		$\widehat{f}_{JLN\text{-}GBS}$	0.01448	0.00410	0.00131	0.00055
Burr	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.01863	0.00600	0.00218	0.00158
		$\widehat{f}_{TS\text{-}GBS}$	0.01647	0.00488	0.00182	0.00149
		$\widehat{f}_{JLN\text{-}GBS}$	0.01554	0.00381	0.00118	0.00049
Pareto	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.00607	0.00228	0.00208	0.00118
		$\widehat{f}_{TS\text{-}GBS}$	0.00733	0.00153	0.00144	0.00092
		$\widehat{f}_{JLN\text{-}GBS}$	0.00371	0.00104	0.00125	0.00047
Pareto	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.01183	0.00384	0.00214	0.00141
		$\widehat{f}_{TS\text{-}GBS}$	0.00877	0.00300	0.00179	0.00095
		$\widehat{f}_{JLN\text{-}GBS}$	0.00411	0.00220	0.00165	0.00074
GBS	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.00817	0.00702	0.00534	0.00322
		$\widehat{f}_{TS\text{-}GBS}$	0.01591	0.01486	0.01173	0.00915
		$\widehat{f}_{JLN\text{-}GBS}$	0.00608	0.00518	0.00320	0.00195
GBS	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.00865	0.00681	0.00476	0.00315
		$\widehat{f}_{TS\text{-}GBS}$	0.00816	0.00629	0.00445	0.00282
		$\widehat{f}_{JLN\text{-}GBS}$	0.00554	0.00474	0.00228	0.00105

that in our study, we have divided these data by 10. Table 8 and Fig. 3 provide respectively the summary statistics and the histogram of these real data. We can observe that the considered data are nonnegative and have heavy tails. Then, the GBS kernels are appropriate for analyzing them.

Now, we apply the TS-GBS and JLN-GBS kernel estimators based on BS-PE($\nu = 2$) and BS-t($\nu = 5$) kernels to estimate the density for O_3 data. For TS-GBS kernel estimator, the value of a is fixed at 0.5. For comparison, the standard GBS kernel estimators are also employed for estimating the density of considered real data. The corresponding bandwidths for the estimators are chosen using the UCV procedure. The obtained values are: $h_{BS\text{-}PE} = 0.3784$, $h_{BS\text{-}t} = 0.1577$, $h_{TS\text{-}BS\text{-}PE} = 0.8327$, $h_{TS\text{-}BS\text{-}t} = 0.3179$, $h_{JLN\text{-}BS\text{-}PE} = 0.4905$ and $h_{JLN\text{-}BS\text{-}t} = 0.2948$. The density estimates are plotted in Fig. 4. We can see that the smoothing of the MBC-GBS kernel estimators is diffident, in comparison to corresponding standard GBS kernel estimators, which tend to under smooth the density of these data.

6. Proofs

We present a sketch of proofs of Theorems 1 and 2.

Table 6
Empirical IV values for the Burr, Pareto and GBS distributions based on 1000 Monte Carlo replications.

Density	Kernel	Estimator	$n = 25$	$n = 50$	$n = 100$	$n = 200$
Burr	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.04319	0.01686	0.01290	0.00661
		$\widehat{f}_{TS\text{-}GBS}$	0.03766	0.01679	0.01262	0.00592
		$\widehat{f}_{JLN\text{-}GBS}$	0.03927	0.01570	0.01427	0.00676
Burr	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.05730	0.03114	0.01656	0.00563
		$\widehat{f}_{TS\text{-}GBS}$	0.04877	0.03271	0.01442	0.00454
		$\widehat{f}_{JLN\text{-}GBS}$	0.05422	0.03428	0.01661	0.00677
Pareto	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.04257	0.01384	0.01325	0.00576
		$\widehat{f}_{TS\text{-}GBS}$	0.04777	0.01752	0.01467	0.00625
		$\widehat{f}_{JLN\text{-}GBS}$	0.04294	0.01568	0.01435	0.00520
Pareto	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.03681	0.01228	0.01319	0.00553
		$\widehat{f}_{TS\text{-}GBS}$	0.03746	0.01617	0.01477	0.00515
		$\widehat{f}_{JLN\text{-}GBS}$	0.04331	0.01670	0.01506	0.00483
GBS	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.02636	0.01447	0.00854	0.00785
		$\widehat{f}_{TS\text{-}GBS}$	0.02354	0.01877	0.01692	0.01773
		$\widehat{f}_{JLN\text{-}GBS}$	0.02938	0.01893	0.00952	0.00661
GBS	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.02473	0.01643	0.01005	0.00850
		$\widehat{f}_{TS\text{-}GBS}$	0.02647	0.01529	0.00982	0.00803
		$\widehat{f}_{JLN\text{-}GBS}$	0.03175	0.01823	0.01149	0.00820

Table 7
Means and standard errors in parentheses of UCV bandwidths given for the Burr, Pareto and GBS distributions over 1000 replications.

Density	Kernel	Estimator	$n = 25$	$n = 50$	$n = 100$	$n = 200$
Burr	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.27578(0.16701)	0.19034(0.09240)	0.15669(0.07617)	0.07491(0.03781)
		$\widehat{f}_{TS\text{-}GBS}$	0.66476(0.45582)	0.40753(0.17100)	0.23097(0.12782)	0.14510(0.05771)
		$\widehat{f}_{JLN\text{-}GBS}$	0.67686(0.54311)	0.34932(0.16269)	0.19328(0.11334)	0.11135(0.05654)
Burr	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.09839(0.07879)	0.05674(0.02768)	0.03269(0.01760)	0.02703(0.00895)
		$\widehat{f}_{TS\text{-}GBS}$	0.18455(0.14423)	0.11800(0.05472)	0.07854(0.04364)	0.05684(0.01966)
		$\widehat{f}_{JLN\text{-}GBS}$	0.15353(0.17251)	0.08235(0.05223)	0.05148(0.04071)	0.04363(0.02442)
Pareto	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	1.26630(0.67478)	1.19034(0.55240)	0.93594(0.45408)	0.70416(0.41845)
		$\widehat{f}_{TS\text{-}GBS}$	1.45951(0.67195)	1.09179(0.62568)	0.91679(0.58587)	0.75002(0.45915)
		$\widehat{f}_{JLN\text{-}GBS}$	1.48357(0.65951)	0.99164(0.70616)	0.84650(0.59540)	0.69467(0.44864)
Pareto	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.64304(0.55585)	0.36113(0.26078)	0.21475(0.15841)	0.18862(0.13022)
		$\widehat{f}_{TS\text{-}GBS}$	1.07330(0.61883)	0.69961(0.49623)	0.42592(0.35834)	0.30844(0.26483)
		$\widehat{f}_{JLN\text{-}GBS}$	1.01468(0.74528)	0.73226(0.63601)	0.58796(0.59298)	0.41652(0.48706)
GBS	BS-PE($\nu = 2$)	$\widehat{f}_{GBS}$	0.55940(0.60370)	0.29015(0.25574)	0.24146(0.19317)	0.23237(0.18403)
		$\widehat{f}_{TS\text{-}GBS}$	1.10607(0.73218)	1.01990(0.71810)	0.81509(0.64686)	0.69953(0.55975)
		$\widehat{f}_{JLN\text{-}GBS}$	0.73790(0.65399)	0.66146(0.57306)	0.34242(0.22470)	0.23790(0.34549)
GBS	BS-t($\nu = 5$)	$\widehat{f}_{GBS}$	0.23645(0.31120)	0.18532(0.38464)	0.12673(0.07452)	0.09115(0.08124)
		$\widehat{f}_{TS\text{-}GBS}$	0.41466(0.49777)	0.27049(0.37872)	0.16982(0.11719)	0.12475(0.12537)
		$\widehat{f}_{JLN\text{-}GBS}$	0.36692(0.52347)	0.25431(0.41100)	0.18820(0.3474)	0.10692(0.24752)

6.1. *Sketch of the proof of Theorem 1*

6.1.1. *Bias*

We start with the bias of the TS-GBS kernel estimator. First, note that $I_h(x) = \mathbb{E}(\widehat{f}_{GBS,h}) = \mathbb{E}(f(Y))$, where the random variable $Y \sim GBS(h^{1/2}, x; g)$. By using the four moments of the random variable Y and Assumption 1, the Taylor expansion of $I_h(x)$ around $h = 0$ is given by

$$I_h(x) = f(x)\left\{1 + \frac{l_1(x)}{f(x)}h + \frac{l_2(x)}{f(x)}h^2 + o(h^2)\right\}, \tag{21}$$

where $l_1(x) = \frac{1}{2}xu_1(g)f' + \frac{1}{2}x^2u_1(g)f''$ and $l_2(x) = \frac{1}{4}x^2u_2(g)f'' + \frac{1}{4}x^3u_2(g)f''' + \frac{1}{24}x^4u_2(g)f''''$. Similarly $I_{h/a}(x) = \mathbb{E}(\widehat{f}_{GBS,h/a})$ can be approximated by

$$I_{h/a}(x) = f(x)\left\{1 + \frac{1}{a}\frac{l_1(x)}{f(x)}h + \frac{1}{a^2}\frac{l_2(x)}{f(x)}h^2 + o(h^2)\right\}. \tag{22}$$

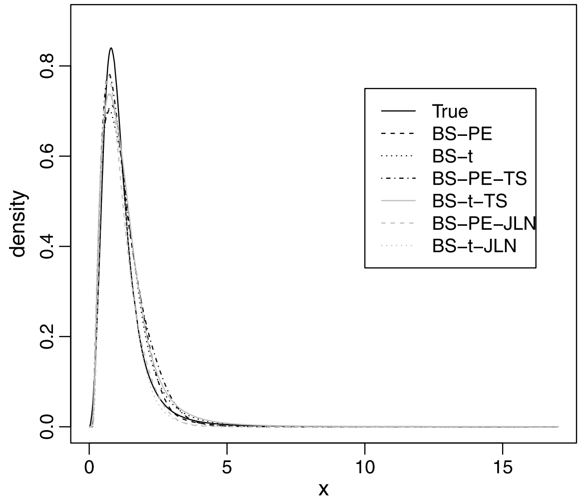

Fig. 2. The estimation of Burr data with $n = 50$ using the standard BS-PE($\nu = 2$), the standard BS-$t(\nu = 5)$, the BS-PE($\nu = 2$)-TS, the BS-$t(\nu = 5)$-TS, the BS-PE($\nu = 2$)-JLN and the BS-$t(\nu = 5)$-JLN kernel density estimators.

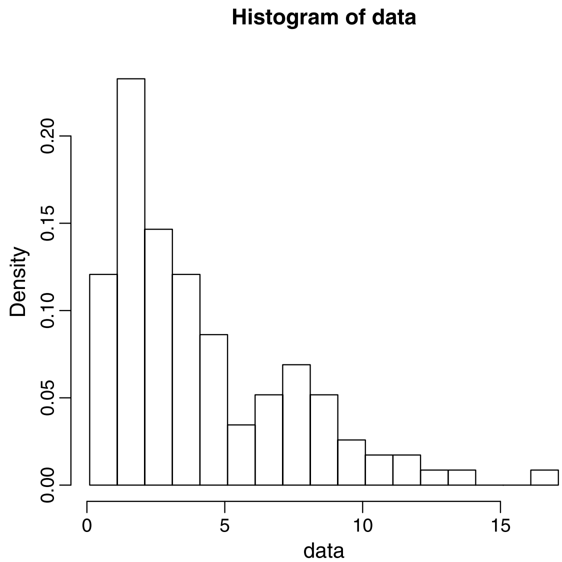

Fig. 3. Histogram of O_3 data.

Now, we define

$$\widehat{f}_{GBS,h}(x) = I_h(x) + Z,$$

and

$$\widehat{f}_{GBS,h/a}(x) = I_h(x) + W.$$

The estimator $\widehat{f}_{TS\text{-}GBS}$ can be written as follows:

$$\widehat{f}_{TS\text{-}GBS} = \{I_h(x)\}^{\frac{1}{1-a}} \left\{1 + \frac{Z}{I_h(x)}\right\}^{\frac{1}{1-a}} \left\{I_{h/a}(x)\right\}^{-\frac{a}{1-a}} \left\{1 + \frac{W}{I_{h/a}(x)}\right\}^{-\frac{a}{1-a}}.$$

Using the expansion $(1+t)^{\alpha} = 1 + \alpha t + o(t^2)$, then we have

$$\widehat{f}_{TS\text{-}GBS}(x) = \{I_h(x)\}^{\frac{1}{1-a}} \left\{I_{h/a}(x)\right\}^{-\frac{a}{1-a}} + \frac{1}{1-a} Z \left\{\frac{I_h(x)}{I_{h/a}(x)}\right\}^{-\frac{a}{1-a}} - \frac{a}{1-a} W \left\{\frac{I_h(x)}{I_{h/a}(x)}\right\}^{\frac{1}{1-a}} + O\left\{(Z+W)^2\right\}. \quad (23)$$

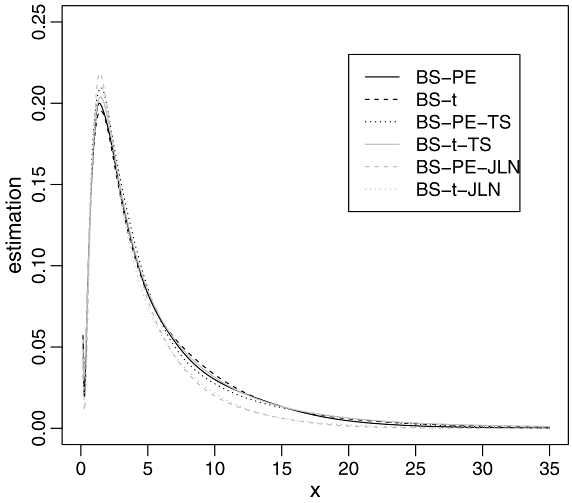

Fig. 4. The estimation of O_3 data with $n = 116$ using the standard BS-PE($\nu = 2$), the standard BS-t($\nu = 5$), the BS-PE($\nu = 2$)-TS, the BS-t($\nu = 5$)-TS, the BS-PE($\nu = 2$)-JLN and the BS-t($\nu = 5$)-JLN kernel density estimators.

Table 8
Descriptive of the analyzed O_3 data.

Data set	n	Max	Min	Median	Mean	SD
O_3	116	16.80	0.10	3.15	4.21	3.29

Based on Assumption 2 and using the same calculations as in Hirukawa (2010) and Terrell and Scott (1980), we can show easily that

$$\mathbb{E}\left(\widehat{f}_{TS\text{-}GBS}(x)\right) = f(x) + \frac{1}{a}\left[\frac{1}{2}\left\{\frac{l_1^2(x)}{f(x)} - l_2(x)\right\}\right]h^2 + o(h^2).$$

6.1.2. Variance

For the variance, from Eq. (23) we have

$$\begin{aligned}\mathrm{Var}\left(\widehat{f}_{TS\text{-}GBS}(x)\right) &= \mathbb{E}\left(\frac{1}{1-a}Z - \frac{a}{1-a}W\right)^2 + O(n^{-1})\\ &= \mathrm{Var}\left(\frac{1}{1-a}\widehat{f}_{GBS,h}(x) - \frac{a}{1-a}\widehat{f}_{GBS,h/a}(x)\right) + O(n^{-1})\\ &= \frac{1}{(1-a)^2}\mathrm{Var}\left(\widehat{f}_{GBS,h}(x)\right) + \frac{a^2}{(1-a)^2}\mathrm{Var}\left(\widehat{f}_{GBS,h/a}(x)\right) - \frac{2a}{(1-a)^2}\mathrm{cov}\left(\widehat{f}_{GBS,h}(x), \widehat{f}_{GBS,h/a}(x)\right).\end{aligned}$$

The terms $\mathrm{Var}\left(\widehat{f}_{GBS,h}(x)\right)$ and $\mathrm{Var}\left(\widehat{f}_{GBS,h/a}(x)\right)$ are the same as in Marchant et al. (2013) and the term $\mathrm{cov}\big(\widehat{f}_{GBS,h}(x), \widehat{f}_{GBS,h/a}(x)\big)$ can be obtained following the same steps as in Hirukawa (2010) (see also Hirukawa & Sakudo, 2014). The proof is completed. □

6.2. Sketch of the proof of Theorem 2

6.2.1. Bias

First the estimator $\widehat{f}_{JLN\text{-}GBS}$ can be written as (see Hirukawa, 2010)

$$\widehat{f}_{JLN\text{-}GBS}(x) = f(x)\left\{1 + \frac{\widehat{f}_{GBS}(x) - f(x)}{f(x)}\right\}\{1 + (\psi(x) - 1)\}$$

where $\psi(x) = n^{-1}\sum_{i=1}^{n} K_{GBS(h^{1/2},x;g)}(X_i)/\widehat{f}_{GBS}(X_i)$. Then, we have

$$\mathbb{E}\left(\widehat{f}_{JLN\text{-}GBS}(x)\right) = f(x) + f(x)\mathbb{E}\left\{\frac{\widehat{f}_{GBS}(x) - f(x)}{f(x)}\right\} + f(x)\mathbb{E}\{\psi(x) - 1\} + f(x)\mathbb{E}\left\{\left(\frac{\widehat{f}_{GBS}(x) - f(x)}{f(x)}\right)(\psi(x) - 1)\right\}.$$

By using Assumption 2 and property of GBS random variable, the terms $\mathbb{E}\left\{\frac{\widehat{f}_{GBS}(x)-f(x)}{f(x)}\right\}$, $\mathbb{E}\{\psi(x)-1\}$ and $\mathbb{E}\left\{\left(\frac{\widehat{f}_{GBS}(x)-f(x)}{f(x)}\right)(\psi(x)-1)\right\}$ can be approximated following the same procedures as in Hirukawa (2010). Thus, $\mathbb{E}(\widehat{f}_{JLN\text{-}GBS})$ is approximated by

$$\mathbb{E}\left(\widehat{f}_{JLN\text{-}GBS}(x)\right) = f(x) - f(x)\left[\frac{1}{2}xu_1(g)q'(x) + \frac{1}{2}x^2u_1(g)q''(x)\right]h^2 + o(h^2),$$

where $q(x) = l_1(x)/f(x)$ with $l_1(x)$ given in the proof of Theorem 1.

6.2.2. Variance

Note that following Hirukawa (2010) and Jones et al. (1995), we can show that $\widehat{f}_{JLN\text{-}GBS}(x)$ is equivalent to

$$\bar{f}_{JLN\text{-}GBS}(x) = f(x)\frac{1}{n}\sum_{i=1}^{n}\frac{K_{GBC(h^{1/2},x;g)}(X_i)}{f(X_i)}. \tag{24}$$

It follows that

$$\mathrm{Var}\left(\bar{f}_{JLN\text{-}GBS}(x)\right) = f^2(x)\frac{1}{n}\mathrm{Var}\left\{\frac{K_{GBC(h^{1/2},x;g)}(X_i)}{f(X_i)}\right\} \tag{25}$$

$$= f^2(x)\frac{1}{n}\mathbb{E}\left\{\frac{K^2_{GBC(h^{1/2},x;g)}(X_i)}{f^2(X_i)} + O(n^{-1})\right\}. \tag{26}$$

Now, observe that

$$\mathbb{E}\left\{\frac{K^2_{GBC(h^{1/2},x;g)}(X_i)}{f^2(X_i)}\right\} = \frac{c^2}{c_{g^2}2\sqrt{hx}}\mathbb{E}\left(\frac{S^{-1/2}}{f(S)}\right) + \frac{c^2}{c_{g^2}\sqrt{hx^{-1}}}\mathbb{E}\left(\frac{S^{-3/2}}{f(S)}\right),$$

where $S \sim GBS(h^{1/2}, x; g^2)$. Finally, following the procedures as in Marchant et al. (2013), we obtain the approximation for variance given in Theorem 2. □

7. Conclusion

This paper has extended the application of the multiplicative bias correction (MBC) approaches for generalized Birnbaum–Saunders (GBS) kernels in the context of kernel density estimation for nonnegative heavy tailed (HT) data. As in Hirukawa (2010) and Hirukawa and Sakudo (2014), we have shown that these two classes of MBC improve the order of magnitude in bias from $O(h)$ to $O(h^2)$. The performances of the MBC-GBS kernel estimators (TS-GBS and JLN-GBS kernel estimators) with unbiased cross-validation (UCV) bandwidth selectors are investigated through simulation study and real application for nonnegative HT data. In general, both MBC-GBS kernel estimators perform better than the standard GBS kernel estimators in the sense of integrated squared error (ISE) and integrated squared bias (ISB). Finally, notice that the MBC approaches can be easily extended to skew-GBS kernels of Saulo et al. (2013).

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