Journal of the Korean Statistical Society 44 (2015) 516–529

Contents lists available at ScienceDirect

Journal of the Korean Statistical Society

journal homepage: www.elsevier.com/locate/jkss

Stationary distribution of the surplus in a risk model with dividends and reinvestments

Sunggon Kim [a], Eui Yong Lee [b,*]

[a] Department of Statistics, University of Seoul, Seoul, 130-743, Republic of Korea

[b] Department of Statistics, Sookmyung Women's University, Seoul, 140-742, Republic of Korea

ARTICLE INFO

Article history:
Received 3 April 2014
Accepted 15 January 2015
Available online 7 February 2015

AMS 2000 subject classifications:
primary 60J25
secondary 60G10

Keywords:
Risk model
Stationary distribution
Surplus process
Level crossing argument
Impulse control

ABSTRACT

A continuous time risk model with dividends and reinvestments is considered. We obtain an explicit formula of the stationary distribution of the surplus and the expected time to ruin after a reinvestment by adopting the level crossing argument. We also propose a scheme to approximate the stationary distribution of the surplus. As an example, we consider the case when the claims are exponentially distributed, Erlang distributed, and generalized hyperexponentially distributed.

1. Introduction

In this paper, we consider a modified Cramér–Lundberg risk model with constant barrier and reinvestments. Whenever the surplus in the risk model reaches barrier $M > 0$, a fraction of the surplus, $M - a$, is immediately paid out as dividend. Meanwhile, if the surplus goes below 0, that is, if there occurs a ruin, an amount of money is immediately reinvested so that the level of the surplus after the ruin becomes b $(0 \leq b < a)$.

The modified risk model has been introduced and studied by Brill and Yu (2011) and Jeong, Lim, and Lee (2009). Brill and Yu (2011) derived a renewal type equation for the stationary distribution of the surplus process and obtain the exact form of the stationary distribution when the claim sizes are exponentially distributed. Under a certain cost structure, Jeong et al. (2009) obtained the long-run average cost as a function of M and a, and illustrated an example how to find the optimal values of M and a which minimize the long-run average cost.

The risk model has been studied by many researchers. For examples, the ruin probability of the classical risk model is well summarized in Klugman, Panjer, and Willmot (2004). Gerber and Shiu (1997) obtained the joint distribution of the time of ruin, the surplus immediately before ruin and the deficit at ruin. Dufresne and Gerber (1991) generalized the classical risk model by assuming that the risk process is perturbed by diffusion and obtained the ruin probabilities. Zhang and Wang (2003) obtained the Gerber–Shiu function in the same model. Li and Lu (2005) studied the Gerber–Shiu function in the risk model with two classes of risk processes. However, most works on the risk model have been focused on the ruin probability

* Corresponding author.
E-mail address: eylee@sookmyung.ac.kr (E.Y. Lee).

http://dx.doi.org/10.1016/j.jkss.2015.01.005

and related characteristics such as the first passage time to the ruin and the levels of the surplus immediately before and/or after the ruin.

In the classical risk model, the surplus process stops if a ruin occurs. However, as pointed by Borch (1969), in practice, though a ruin occurs in an insurance policy, the insurance company keeps the policy operating by a reinvestment or borrowing money from other business. In this case, the surplus process continues even though a ruin occurs. Dickson and Waters (2004) considered the Cramér–Lundberg risk model, in which the amount of deficit is reinvested if the surplus ever becomes negative. Under the barrier policy, i.e. the surplus over a barrier is paid continuously as dividend, they obtained a form of the discounted value function of the coming dividends. For the same model, Kulenko and Schmidli (2008) showed that it is optimal to reinvest the amount of deficit at ruin and to pay the dividends according to the barrier policy.

However, when the payment of each dividend incurs a fixed transaction cost, the payment of the dividend at a constant rate is impossible. In this case, for the Cramér–Lundberg risk model where the reinvestment is not considered, Bai and Guo (2010) studied the impulse control and showed that if the claim sizes are exponentially distributed, it is optimal to pay immediately a constant amount of dividend whenever the surplus surpasses the barrier. Thonhauser and Albrecher (2011) also considered the same model as that of Bai and Guo (2010) and for more general utility function, they obtained the same conclusion as Bai and Gou's, and proposed a numerical scheme to obtain the optimal dividend policy.

To the authors' best knowledge, there is no result on the Cramér–Lundberg risk model where both the payment of the dividends and reinvestment incur fixed transaction costs, while, for the diffusion risk model with the transaction costs, He and Liang (2009) showed that it is optimal to pay immediately a constant amount of dividend whenever the surplus surpasses a barrier and to reinvest up to a prescribed level whenever the surplus becomes negative, which is the same control strategy of surplus as in our model. Hence, for the Cramér–Lundberg risk model with the fixed transaction costs, the control strategy of the surplus considered in our model may be a reasonable choice.

The surplus process goes on after ruins in our model. Hence, it is worth to study the stationary behavior of the surplus process such as the average level of the surplus or the proportion of time where the surplus is in a certain level in the long-run, which is the main result of the paper. The paper is organized as follows. In Section 2, we introduce how to decompose the surplus process into two right continuous Markovian regenerative processes and we give a formula representing the stationary distribution of the surplus process as the weighted sum of those of the two decomposed processes. In Section 3, we introduce how to obtain the stationary distributions of the decomposed processes by the level crossing argument. In Sections 4 and 5, we derive the expected up-crossings of a given level during a cycle in the decomposed processes, and also the expected cycle lengths. By applying the level crossing argument to these values, we obtain the stationary distribution of the decomposed processes. Then, the explicit form of the stationary distribution of the surplus process follows immediately as a result of Section 2. In Section 6, when the claim size distribution is approximated by the generalized hyperexponential (GH) distribution, we show that the stationary distribution of the surplus process can be approximated as a sum of simple functions. In Section 7, we apply the obtained results to the special cases where the claim size is exponentially distributed, Erlang distributed, and GH distributed. Some numerical results are also given.

2. Decomposition of the surplus process

The explicit form of the surplus process analyzed in this paper is as follows. Let $S(t)$, $D(t)$, and $R(t)$ be the accumulated amount of claims, the dividends, and the reinvestments until time t, respectively, and let $\{N_S(t), t \geq 0\}$, $\{N_D(t), t \geq 0\}$, and $\{N_R(t), t \geq 0\}$ be the counting processes of the occurrence of the claims, the payment of the dividends, and the reinvestments, respectively. Then, it follows for $t \geq 0$,

$$S(t) = \sum_{i=1}^{N_S(t)} S_i,$$
$$D(t) = (M - a)\, N_D(t),$$
$$R(t) = \sum_{i=1}^{N_R(t)} R_i,$$

where S_i and R_i are the amount of the ith claim and reinvestment, respectively, for $i = 1, 2, \ldots$. In this paper, we assume that the process $\{N_S(t), t \geq 0\}$ is a Poisson process with rate λ, and that the claim size S_i, $i = 1, 2, \ldots$, is identically distributed with distribution $B(\cdot)$ and independent with any other variables. We also assume that the premium rate is a constant c. Let m be the mean of a claim size. If we define $\{X(t), t \geq 0\}$ as the surplus process, then we have, for $t \geq 0$,

$$X(t) = X(0) + ct - S(t) - D(t) + R(t). \tag{1}$$

The surplus process $\{X(t), t \geq 0\}$ is a Markovian regenerative process. The time epochs of ruins form the regeneration points, i.e. a cycle of $\{X(t), t \geq 0\}$ starts just after a ruin and ends at the next ruin. We assume that the process $\{X(t), t \geq 0\}$ is right continuous except at the time epochs of ruins. If a ruin occurs by a claim of size Y at a time t, then $X(t) = X(t-) - Y$ and the reinvestment makes $X(t+) = b$. Since the process $\{X(t), t \geq 0\}$ is right continuous except the epochs of ruins, $X(t) = a$ if the surplus reaches the barrier M at time t, i.e. $X(t-) = M$. Fig. 1 shows a sample path of the surplus process.

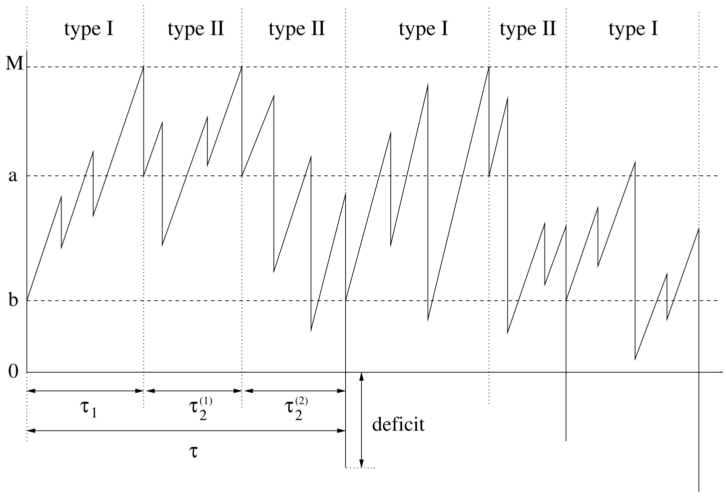
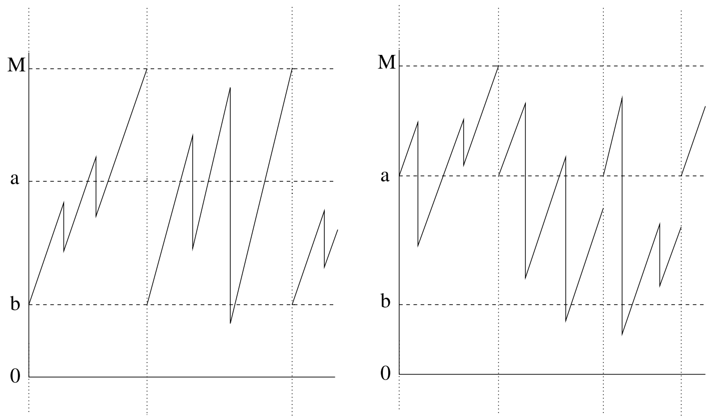


Fig. 1. A sample path of the surplus process $\{X(t), t \geq 0\}$.



(a) A sample path of the connected type I processes.

(b) A sample path of the connected type II processes.

Fig. 2. The decomposition of the process $\{X(t), t \geq 0\}$ into the processes $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$.

We define that the surplus process exits from $[0, M)$ at a time t if and only if $X(t) < 0$ or $X(t-) = M$. In this case, there occurs a ruin or it reaches the barrier M at the time t. Suppose that a cycle of the surplus process starts at time 0 and that the surplus process exits from $[0, M)$ at a time τ_1. If a ruin occurs at the time τ_1, then the cycle ends. Otherwise, the cycle continues with the starting level of a. At the next exit time after the time τ_1, the cycle will end or continue depending on the value of the surplus at the time, and so on.

We denote by the type I process as the part of the surplus process from a ruin to the next exit of the process from $[0, M)$, and denote by the type II process as the part of the surplus process from reaching the barrier M to the next exit of the process from $[0, M)$. Then, a cycle of the surplus process can be decomposed into one type I process and a number of type II processes. Let $\tau_2^{(i)}$, $i = 1, 2, \ldots$, be the length of the ith type II process in a cycle of the surplus process, i.e. for $i = 1, 2, \ldots$,

$$\tau_2^{(i)} = \inf\{t > s_{i-1}; X(t) < 0 \text{ or } X(t-) = M | X(s_{i-1}) = a\} - s_{i-1},$$

where $s_0 = \tau_1$ and $s_i = \tau_1 + \sum_{j=1}^{i} \tau_2^{(j)}$ for $i = 1, 2, \ldots$. Since $\{X(t), t \geq 0\}$ is a Markovian process and the value of $\tau_2^{(i)}$ is determined only by the behavior of $\{X(t), t \geq 0\}$ after time s_{i-1}, we can see that $\tau_2^{(1)}, \tau_2^{(2)}, \ldots$ are independent and identically distributed.

Let $\{X_1(t), t \geq 0\}$ be formed by separating the type I processes from $\{X(t), t \geq 0\}$ and connecting them together. Process $\{X_2(t), t \geq 0\}$ is similarly formed using the type II processes. At the time epochs of ruins, we let the values of $\{X_1(t), t \geq 0\}$ be b so that $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ are right continuous and Markovian regenerative processes. As shown in Fig. 1, a sample path is decomposed into type I and type II processes. Fig. 2 shows the sample paths of $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ formed by decomposing the sample path in Fig. 1. A separated part of $\{X(t), t \geq 0\}$ from a ruin to the next exit from $[0, M)$ forms a cycle of the $\{X_1(t), t \geq 0\}$, and the other separated part of $\{X(t), t \geq 0\}$ from reaching the barrier

M to the next exit from $[0, M)$ forms a cycle of the $\{X_2(t), t \geq 0\}$. The cycle length of the process $\{X_i(t), t \geq 0\}$, $i = 1, 2$, is denoted by τ_i.

If we let τ be the cycle length of the surplus process, then

$$\tau = \tau_1 + \sum_{i=1}^{N_2} \tau_2^{(i)},$$

where N_2 is the number of the type II processes in a cycle of the surplus process. Since N_2 is a stopping time with respect to the sequence of $\tau_2^{(1)}, \tau_2^{(2)}, \ldots$, it follows that

$$E[\tau] = E[\tau_1] + E[N_2]E[\tau_2]. \tag{2}$$

Applying the renewal reward theorem in Ross (1996), the stationary distribution of the surplus process can be represented as the weighted sum of the stationary distributions of the processes $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ as follows.

Theorem 1. *Let $F(x)$ be the stationary distribution of $\{X(t), t \geq 0\}$, and for $i = 1, 2$, let $F_i(x)$ be the stationary distribution of $\{X_i(t), t \geq 0\}$. Then,*

$$F(x) = \frac{E[\tau_1]}{E[\tau]} F_1(x) + \frac{E[N_2]E[\tau_2]}{E[\tau]} F_2(x). \tag{3}$$

In the following sections, we obtain the explicit forms of $E[N_2]$, $E[\tau_i]$ and $f_i(x)$, the p.d.f. of $F_i(x)$ for $i = 1, 2$.

3. Level crossing approach for obtaining the stationary distributions of the decomposed processes

We define that the process $\{X(t), t \geq 0\}$ up-crosses the level x at time t if and only if $X(t) = x$. In the same manner, we also define that the process $\{X_i(t), t \geq 0\}$, $i = 1, 2$, up-crosses the level x at time t if and only if $X_i(t) = x$. Let $U_i(x)$, $i = 1, 2$, be the number of up-crossings of the level x during a cycle of length τ_i in the process $\{X_i(t), t \geq 0\}$, respectively. The level crossing argument in Brill (2008), Cohen (1977) says that the p.d.f. of $F_i(x)$, $i = 1, 2$, is given by

$$f_i(x) = \frac{E[U_i(x)]}{cE[\tau_i]}, \quad 0 \leq x < M. \tag{4}$$

Since the range of $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ is $[0, M)$, integrating $f_i(x)$, $i = 1, 2$, over the interval $[0, M)$ gives the value 1, which implies

$$E[\tau_i] = \frac{1}{c} \int_0^M E[U_i(x)]\, dx, \quad i = 1, 2. \tag{5}$$

Now, we can see that the explicit form of $F_i(\cdot)$, $i = 1, 2$, can be derived directly from the above equations if we obtain the explicit form of $E[U_i(x)]$, $i = 1, 2$, for $x \in [0, M)$.

Let T_y, $0 \leq y < M$, be the first exit time from the interval $[0, M)$ of the surplus process $\{X(t), t \geq 0\}$ when the initial surplus is y, i.e.

$$T_y = \inf\{t > 0; X(t) < 0 \text{ or } X(t-) = M | X(0) = y\}.$$

For $0 \leq x, y < M$, let $U_{y,x}$ be the number of up-crossings of the level x during $(0, T_y)$ in the process $\{X(t), t \geq 0\}$ under the condition of $X(0) = y$.

Suppose that $\{X(t), t \geq 0\}$ and $\{X_1(t), t \geq 0\}$ have the same initial level of b, i.e. $X(0) = X_1(0) = b$. Then, during $(0, T_b)$, the behavior of the process $\{X(t), t \geq 0\}$ is stochastically the same as that of $\{X_1(t), t \geq 0\}$, which implies that the number of up-crossings of the level x $(x \neq b)$ in the process $\{X(t), t \geq 0\}$ has the same distribution as that of the process $\{X_1(t), t \geq 0\}$ during $(0, T_b)$. For $x = b$, there occurs an up-crossing of the level b at time 0, which is not counted in the value of $U_{b,x}$. Since a cycle of the process $\{X_1(t), t \geq 0\}$ is the time period $[0, T_b)$, we have

$$U_1(x) =_D \begin{cases} U_{b,x}, & b \neq x \\ U_{b,x} + 1, & b = x, \end{cases} \tag{6}$$

where $=_D$ means the sameness in distribution. In the same manner, we also have

$$U_2(x) =_D \begin{cases} U_{a,x}, & a \neq x \\ U_{a,x} + 1, & a = x. \end{cases} \tag{7}$$

4. Distribution of the number of up-crossings of the level *x*

In the previous section, we can see that the stationary distributions of the processes $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ can be obtained if for $x \in [0, M)$, the expected values of $U_{b,x}$ and $U_{a,x}$ are derived, respectively. To this end, we obtain the following theorem.

Theorem 2. *Let $P_{y,x}$, $0 \leq x, y < M$, be the probability that there is an up-crossing of the level x in the process $\{X(t), t \geq 0\}$ during $(0, T_y)$ when $\{X(t), t \geq 0\}$ starts at the level y, and let $P_{y,M} = \Pr\{X(T_y-) = M\}$, $0 \leq y < M$, i.e. the probability that the surplus process reaches the barrier M at time T_y. Then, we have*

$$P_{y,x} = \frac{H(y)}{H(x)}, \quad 0 \leq y < x \leq M$$
$$P_{y,x} = \frac{H(y)}{H(x)} - \frac{H(M)\,H(y-x)}{H(x)\,H(M-x)}, \quad 0 \leq x \leq y < M,$$

where

$$H(x) = 1 + \sum_{n=1}^{\infty} \left(\frac{\lambda m}{c}\right)^n B_e^{*(n)}(x), \quad x \geq 0, \tag{8}$$

and $B_e^{*(n)}(\cdot)$ is the n-fold recursive Stieltjes convolution of $B_e(\cdot)$, the equilibrium distribution of $B(\cdot)$.

Proof. Let

$$W(t) = M - X\left(\frac{t}{c}\right). \tag{9}$$

Since there is no dividend and reinvestment during $(0, T_y)$, it follows from Eqs. (1) and (9) that for $0 \leq t < T_y$,

$$W(t) = M - y + \sum_{i=1}^{N_S(t/c)} S_i - t.$$

Then we can see that until the time T_y, $\{W(t), t \geq 0\}$ is the same as the workload process of the $M/G/1$ queue of which arrival rate is λ/c, the service requirement distribution is $B(\cdot)$, and the initial workload is $M - y$. We define that the process $\{W(t), t \geq 0\}$ down-crosses the level x at time t if and only if $W(t) = x$, and denote by $\tilde{P}_{y,x}$, for $0 < x, y \leq M$, the probability that there is a down-crossing of the level x until the workload process $\{W(t), t \geq 0\}$ exits from $(0, M]$ after starting at the level y. Then, Eq. (9) says

$$P_{y,x} = \tilde{P}_{M-y,M-x}. \tag{10}$$

Takács (1967) has shown that

$$\tilde{P}_{y,x} = \frac{H(M-y)}{H(M-x)}, \quad 0 < x < y \leq M.$$

Bae, Kim, and Lee (2002) also have shown that

$$\tilde{P}_{y,x} = \frac{H(M-y)}{H(M-x)} - \frac{H(M)\,H(x-y)}{H(M-x)\,H(x)}, \quad 0 < y \leq x \leq M.$$

Applying the above results to Eq. (10), we have

$$\begin{aligned} P_{y,x} =& \frac{H(y)}{H(x)}, \quad 0 \leq y < x < M, \\ P_{y,x} =& \frac{H(y)}{H(x)} - \frac{H(M)\,H(y-x)}{H(x)\,H(M-x)}, \quad 0 \leq x \leq y < M. \end{aligned} \tag{11}$$

Since the claims occur according to a Poisson process, it can be easily shown that

$$P_{y,M} = \lim_{x \to M} P_{y,x}, \tag{12}$$

which completes the proof. □

Using Theorem 2 and the Markov property of the surplus process, we derive the expected values of $U_{y,x}$ as follows.

Theorem 3. *The expected value of* $U_{y,x}$, $0 \le x, y < M$, *is given by*

$$E[U_{y,x}] = \frac{H(M-x)H(y)}{H(M)}, \quad 0 \le y < x < M,$$
$$E[U_{y,x}] = \frac{H(M-x)H(y)}{H(M)} - H(y-x), \quad 0 \le x \le y < M. \tag{13}$$

Proof. For $0 \le x < M$, $1 - P_{y,x}$ is the probability that there is no up-crossings of the level x during $(0, T_y)$ in the surplus process $\{X(t), t \ge 0\}$ starting at the level of y. Thus,

$$\Pr\{U_{y,x} = 0\} = 1 - P_{y,x}, \quad 0 \le x < M.$$

The surplus process $\{X(t), t \ge 0\}$ is Markovian, which implies that for $0 \le x < M$, $P_{x,x}$ is the probability that after an up-crossing of the level x, there is another up-crossing of the level x during $(0, T_y)$ in the process $\{X(t), t \ge 0\}$. Thus, for $k = 0, 1, 2. \dots,$

$$\Pr\{U_{y,x} = k\} = P_{y,x}P_{x,x}^{k-1}(1 - P_{x,x}), \quad 0 \le x < M.$$

From the above results, the expected value of $U_{y,x}$ is derived as follows, for $0 \le y < M$,

$$E[U_{y,x}] = \frac{P_{y,x}}{1 - P_{x,x}}, \quad 0 \le x < M. \tag{14}$$

Applying Theorem 2 to the above equation, we obtain the desired result. □

5. Stationary distribution of the surplus process

We are ready to obtain the stationary distribution of the processes $\{X_1(t), t \ge 0\}$ and $\{X_2(t), t \ge 0\}$ via the level crossing argument. Eq. (6) and Theorem 3 give the explicit form of $E[U_1(x)]$ as follows,

$$E[U_1(x)] = \frac{H(M-x)H(b)}{H(M)} - H(b-x), \quad 0 \le x < b,$$
$$E[U_1(x)] = \frac{H(M-x)H(b)}{H(M)}, \quad b \le x < M. \tag{15}$$

Eq. (7) and Theorem 3 also give the explicit form of $E[U_2(x)]$ as follows,

$$E[U_2(x)] = \frac{H(M-x)H(a)}{H(M)} - H(a-x), \quad 0 \le x < a,$$
$$E[U_2(x)] = \frac{H(M-x)H(a)}{H(M)}, \quad a \le x < M. \tag{16}$$

Applying the above equations to Eq. (5), we derive the explicit forms of the expected values of the cycle lengths τ_1 and τ_2 given by

$$E[\tau_1] = \frac{1}{c}\left(\frac{H(b)}{H(M)}\int_0^M H(x)\,dx - \int_0^b H(x)\,dx\right), \tag{17}$$

and

$$E[\tau_2] = \frac{1}{c}\left(\frac{H(a)}{H(M)}\int_0^M H(x)\,dx - \int_0^a H(x)\,dx\right). \tag{18}$$

The stationary distribution of the process $\{X_1(t), t \ge 0\}$ is obtained by applying Eqs. (15) and (17) to Eq. (4), and also the stationary distribution of the process $\{X_2(t), t \ge 0\}$ by applying Eqs. (16) and (18) to Eq. (4).

It remains to derive the expected value of N_2 for obtaining the value of $E[\tau]$ and the explicit form of $F(x)$. A cycle of the surplus process $\{X(t), t \ge 0\}$ starts at the level b. If it reaches the barrier M at the exit time T_b, then there is at least one type II process in a cycle, i.e.

$$\Pr\{N_2 \ge 1\} = P_{b,M}.$$

After the surplus process reaches the barrier M, another type II process starts with the level a. From the Markov property of the surplus process, we have

$$\Pr\{N_2 = k\} = P_{b,M}P_{a,M}^{k-1}(1 - P_{a,M}), \quad k = 1, 2, \ldots,$$

which gives

$$E[N_2] = \frac{P_{b,M}}{1 - P_{a,M}}.$$

Applying Theorem 2 to the above equation, we have

$$E[N_2] = \frac{H(b)}{H(M) - H(a)}. \tag{19}$$

From the above results, we have the explicit form of the stationary level of the surplus process $\{X(t), t \geq 0\}$ as follows,

Theorem 4. *Let $f(x)$ be the p.d.f. of the stationary level of the surplus process $\{X(t), t \geq 0\}$. Then,*

$$f(x) = \frac{1}{cE[\tau]}\left\{\frac{H(b)(H(M-x) - H(a-x))}{H(M) - H(a)} - H(b-x)\right\}, \quad 0 \leq x < b,$$
$$f(x) = \frac{1}{cE[\tau]}\frac{H(b)(H(M-x) - H(a-x))}{H(M) - H(a)}, \quad b \leq x < a,$$
$$f(x) = \frac{1}{cE[\tau]}\frac{H(b)H(M-x)}{H(M) - H(a)}, \quad a \leq x < M,$$

where

$$E[\tau] = \frac{1}{c}\left\{\frac{H(b)}{H(M) - H(a)}\int_a^M H(x)\,dx - \int_0^b H(x)\,dx\right\}. \tag{20}$$

Proof. By applying Eqs. (17), (18), and (19) to Eq. (2), we derive the expected cycle length of the surplus process as the form of Eq. (20). It follows from Eq. (3) that the p.d.f. of the stationary level of the surplus process is given by

$$f(x) = \frac{E[\tau_1]}{E[\tau]}f_1(x) + \frac{E[N_2]E[\tau_2]}{E[\tau]}f_2(x), \quad 0 \leq x < M.$$

Since $f_i(x) = E[U_i(x)]/E[\tau_i],\ i = 1, 2$, the above equation is rewritten as

$$f(x) = \frac{E[U_1(x)] + E[N_2]E[U_2(x)]}{E[\tau]}, \quad 0 \leq x < M.$$

By applying Eqs. (17), (18), (19), and (20) into the above equation, we obtain the desired result. □

6. Approximation

As shown in Eq. (8), $H(x)$ is a series of multiple convolutions of $B_e(x)$, which makes it hard to compute the stationary distribution of the surplus process via Theorem 4. Thus, in the practical point of view, we need to find an approximation of $H(x)$ which is easily computable. Generalized hyperexponential (GH) distributions are popularly used as approximating distributions. GH distributions are linear combinations of exponential c.d.f.'s with mixing parameters that sum to unity, i.e. if $G(x)$ is a c.d.f. of a GH distribution, then $G(x)$ has the support of positive values and has the following form,

$$G(x) = 1 - \sum_{i=1}^{k} w_i e^{-\mu_i x}, \quad x \geq 0,$$

where $\mu_i > 0, i = 1, 2, \ldots, n$, $\sum_{i=1}^{k} w_i = 1$ and the values of some w_i's might be negative. Due to Botta and Harris (1986), the family of GH distribution is a dense subset of all distributions with support $[0, \infty)$, i.e. there is a sequence of GH distributions that converges weakly to a given distribution with support $[0, \infty)$. The attractive properties of the GH distributions are discussed by Botta, Harris, and Marchal (1987) and Harris, Marchal, and Botta (1992).

Suppose that $B(x)$, the c.d.f. of a claim size, can be approximated well by

$$G(x) = 1 - \sum_{i=1}^{k} w_i e^{-\mu_i x}, \quad x \geq 0, \tag{21}$$

and let

$$\hat{H}(x) = 1 + \sum_{n=1}^{\infty} \left(\frac{\lambda\, m_g}{c} \right)^n G_e^{*(n)}(x), \tag{22}$$

where $G_e(\cdot)$ is the equilibrium distribution of $G(\cdot)$ and m_g is the mean of $G(\cdot)$, i.e.

$$m_g = \sum_{i=1}^{k} \frac{w_i}{\mu_i}.$$

Then, $\hat{H}(x)$ is an approximation of $H(x)$ in Eq. (8). Laplace transform of $\hat{H}(x)$ is given by

$$\begin{aligned} \psi(s) &= \int_0^\infty e^{-sx} \hat{H}(x)\, dx \\ &= \frac{1}{s} \int_0^\infty e^{-sx}\, d\hat{H}(x) + \frac{1}{s}. \end{aligned}$$

Let $g_e(\cdot)$ be the p.d.f. of $G_e(\cdot)$. Then, it follows from Eq. (22) that

$$\begin{aligned} \psi(s) &= \frac{1}{s} \sum_{n=1}^{\infty} \left(\frac{\lambda\, m_g}{c} \int_0^\infty e^{-sx} g_e(x)\, dx \right)^n + \frac{1}{s} \\ &= \frac{1}{s\left\{1 - (\lambda m_g/c) \int_0^\infty e^{-sx} g_e(x)\, dx\right\}}. \end{aligned} \tag{23}$$

The explicit form of $g_e(x)$ is given by

$$g_e(x) = \frac{1}{m_g} \sum_{i=1}^{k} w_i e^{-\mu_i x}, \quad x \geq 0,$$

and Laplace transform of it is derived to be

$$\int_0^\infty e^{-sx} g_e(x)\, dx = \frac{1}{m_g} \sum_{i=1}^{k} \frac{w_i}{s + \mu_i}.$$

Applying above equation to Eq. (23) gives

$$\psi(s) = \frac{1}{s\{1 - (\lambda/c) \sum_{i=1}^{k} w_i/(s + \mu_i)\}},$$

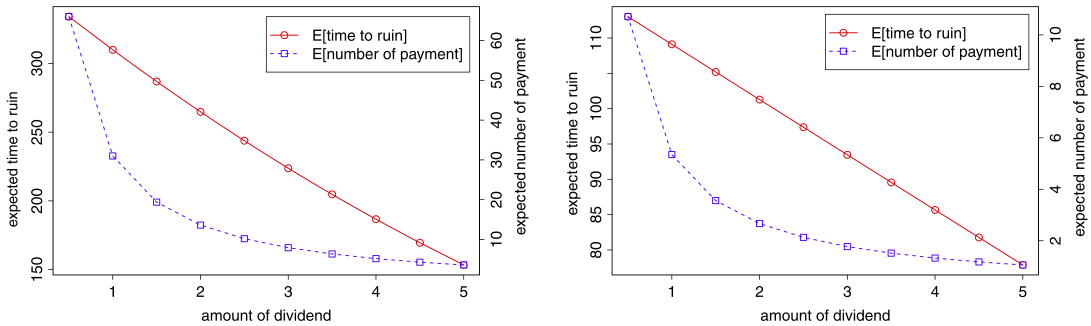
or, we also have

$$\psi(s) = \frac{\prod_{i=1}^{k} (s + \mu_i)}{s\left\{ \prod_{i=1}^{k} (s + \mu_i) - (\lambda/c) \sum_{i=1}^{k} w_i \prod_{j \neq i}^{k} (s + \mu_j) \right\}}. \tag{24}$$

We can see that $\psi(s)$ in Eq. (24) is a rational polynomial and the order of denominator is greater than that of numerator. In this case, the inverse Laplace transform of $\psi(s)$ is obtained by the method of partial fraction expansion or partial fraction decomposition, i.e. we can obtain the exact form of $\hat{H}(x)$ which is a linear combination of simple functions. By applying $\hat{H}(x)$ to Theorem 4, we can approximate the stationary distribution of the surplus process.

7. Case studies

In this section, we consider the cases in which the claim sizes are exponentially distributed, Erlang distributed, and GH distributed. For each case, we derive explicit form of the function $H(x)$, which enables us to obtain the exact forms of $E[\tau]$, $E[N_2]$, and $f(x)$. For some specific values of parameters, we obtain numerically the exact value of $E[\tau]$, i.e. the expected cycle length, or the expected time to ruin after a reinvestment, and also obtain the exact value of $E[N_2]$, which is equal to the expected number of payment of dividends between ruins. $f(x)$ is also computed numerically. In the following, we assume that $c > \lambda m$, i.e. the premium rate is larger than the average claim amount per unit time.



(a) When $\lambda = 11.8$. (b) When $\lambda = 12.29$.

Fig. 3. Expected time to ruin after a reinvestment and expected number of payment of dividends between two ruins with various values of the amount of dividend.

7.1. Exponentially distributed claim sizes

Suppose that the claim sizes are exponentially distributed with rate $1/m$. Then, the equilibrium distribution is also exponential with rate $1/m$, and $B_e^{*(n)}(\cdot)$, $k = 1, 2, \ldots$, is the n-Erlang distribution with rate $1/m$. From the definition of $H(x)$ in Eq. (8), we have for $x \geq 0$,

$$H(x) = 1 + \sum_{n=1}^{\infty} \left(\frac{\lambda m}{c}\right)^n \int_0^x \frac{u^{n-1} e^{-u/m}}{(n-1)! m^n}\, du$$
$$= \frac{1 - \rho e^{-\theta x}}{1 - \rho},$$

where $\rho = \lambda m/c$ and $\theta = 1/m - \lambda/c$. Then, Eqs. (19) and (20) give

$$E[N_2] = \frac{1 - \rho e^{-\theta b}}{\rho(e^{-\theta a} - e^{-\theta M})},$$

and

$$E[\tau] = \frac{1}{c}\left\{\frac{(M-a)(1 - \rho e^{-\theta b})}{\rho(1-\rho)(e^{-\theta a} - e^{-\theta M})} - \frac{1}{\theta} - \frac{b}{1-\rho}\right\}.$$

The p.d.f. of the stationary surplus follows from Theorem 4, i.e.

$$\begin{aligned} f(x) &= \frac{1}{cE[\tau]} \frac{e^{\theta x} - 1}{1 - \rho}, \quad 0 \leq x < b, \\ f(x) &= \frac{1}{cE[\tau]} \frac{(1 - \rho e^{-\theta b}) e^{\theta x}}{1 - \rho}, \quad b \leq x < a, \\ f(x) &= \frac{1}{cE[\tau]} \frac{(1 - \rho e^{-\theta b})(1 - \rho e^{-\theta(M-x)})}{\rho(1-\rho)(e^{-\theta a} - e^{-\theta M})}, \quad a \leq x < M. \end{aligned} \tag{25}$$

This result is exactly the same as that of Brill and Yu (2011).

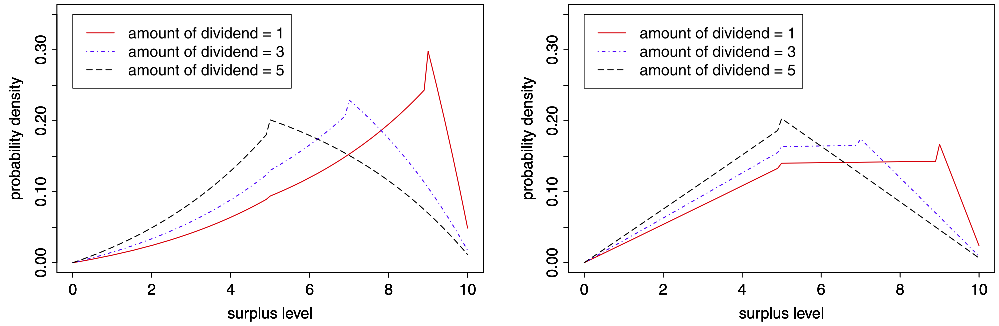
Numerical example: Suppose that the claim size is exponentially distributed with rate $\mu = 6.0$ and $c = 2.05$ for the comparison with the numerical results of Brill and Yu (2011). We assume that the arrival rates are $\lambda = 11.8$ and 12.29. Then, c is larger than λm for each case. $H(x)$ is given by, for $\lambda = 11.8$,

$$H(x) = 24.6 - 23.6e^{-0.2439x}, \quad x \geq 0,$$

and for $\lambda = 12.29$,

$$H(x) = 1230 - 1229e^{-0.004878x}, \quad x \geq 0.$$

We obtain numerically the values of $E[\tau]$ and $E[N_2]$ for various values of $M - a$, the amount of dividend per a time. For $M - a = 0.5, 1.0, \ldots, 5.0$, Fig. 3 shows the results. In this figure, the expected time to ruin decreases almost linearly, while $E[N_2]$ decreases exponentially in both cases of $\lambda = 11.8$ and 12.29. Fig. 4 shows $f(x)$ for $M - a = 1.0, 3.0, 5.0$. As shown in the figure, $f(x)$ has jumps at $x = a$ and $x = b$, and has its peak at $x = a$.



(a) When $\lambda = 11.8$.   (b) When $\lambda = 12.29$.

Fig. 4. The p.d.f. $f(x)$ of the stationary level of the surplus process with various values of the amount of dividend.

7.2. Erlang distributed claim sizes

Suppose that the claim sizes follow the Erlang distribution with shape parameter 2 and rate parameter $2/m$, i.e.

$$B(x) = \int_0^x \left(\frac{2}{m}\right)^2 u e^{-\frac{2}{m}u}\,du, \quad x > 0.$$

Then, the mean claim size is m, and the Laplace transform of a claim size S is given by

$$E[e^{-sS}] = \left(\frac{2}{2+sm}\right)^2. \tag{26}$$

Let $b_e(x)$ be the p.d.f. of the equilibrium distribution of $B(x)$, i.e.

$$b_e(x) = \frac{1-B(x)}{m}, \quad x > 0.$$

Then, the Laplace transform of it is computed to be

$$\begin{aligned}\int_0^\infty e^{-sx} b_e(x)\,dx &= \frac{1}{m}\left(\frac{1-E[e^{-sS}]}{s}\right)\\ &= \frac{4+sm}{(2+sm)^2}.\end{aligned} \tag{27}$$

Let $\phi(s)$ be the Laplace transform of $H(x)$. Then, in the similar manner to derive Eq. (23), we have

$$\phi(s) = \frac{1}{s\left\{1-(\lambda m/c)\int_0^\infty e^{-sx} b_e(x)\,dx\right\}}.$$

Applying Eq. (27) into the above equation, we have

$$\phi(s) = \frac{(ms+2)^2}{s\left\{m^2s^2 + (4-\rho)ms + 4(1-\rho)\right\}}, \tag{28}$$

where $\rho = \lambda m/c$. Then, $\phi(s)$ has the following three poles,

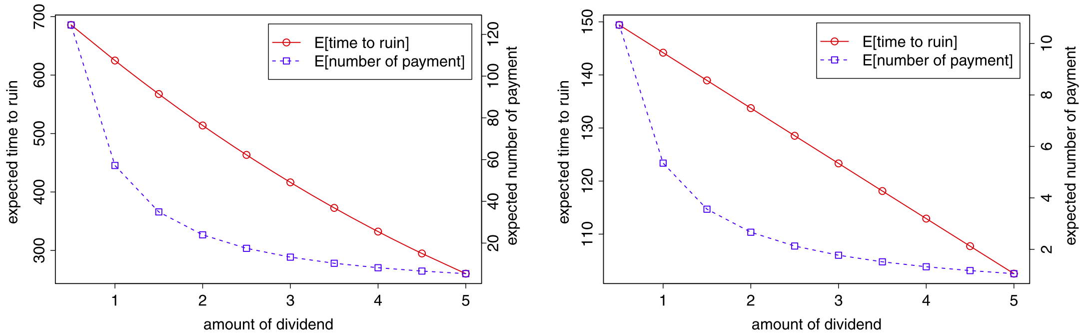
$$\begin{aligned} s_0 &= 0,\\ s_1 &= \frac{-(4-\rho)+\sqrt{\rho^2+8\rho}}{2m},\\ s_2 &= \frac{-(4-\rho)-\sqrt{\rho^2+8\rho}}{2m}.\end{aligned}$$

Due to the fact that $c > \lambda m$, it can be easily checked that s_1 and s_2 are negative real numbers. Then, Eq. (28) is rewritten as

$$\phi(s) = \frac{c_0}{s} + \frac{c_1}{s-s_1} + \frac{c_2}{s-s_2},$$

where

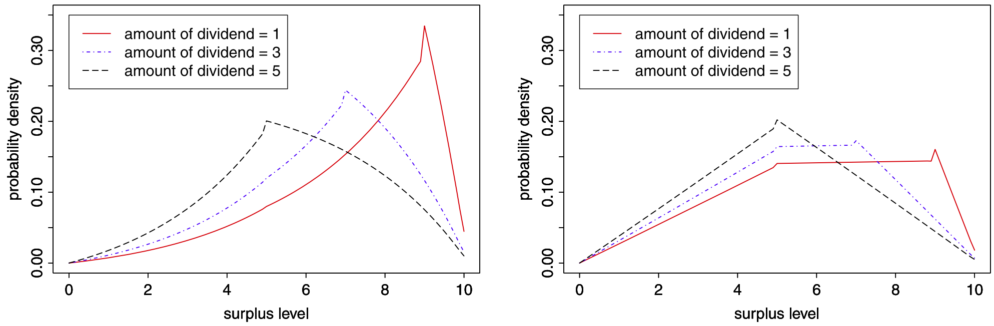
$$c_i = \phi(s_i)(s-s_i), \quad i = 0, 1, 2.$$



(a) When $\lambda = 11.8$.

(b) When $\lambda = 12.29$.

Fig. 5. Expected time to ruin after a reinvestment and expected number of payment of dividends between two ruins with various values of the amount of dividend.



(a) When $\lambda = 11.8$.

(b) When $\lambda = 12.29$.

Fig. 6. The p.d.f. $f(x)$ of the stationary level of the surplus process with various values of the amount of dividend.

Its inverse Laplace transform is given by

$$\begin{aligned} H(x) &= c_0 + c_1 e^{s_1 x} + c_2 e^{s_2 x} \\ &= \frac{1}{1-\lambda m/c} + \frac{(s_1+2/m)^2}{s_1(s_1-s_2)} e^{s_1 x} + \frac{(s_2+2/m)^2}{s_2(s_2-s_1)} e^{s_2 x}. \end{aligned} \tag{29}$$

Then, the stationary distribution of the surplus process is obtained by applying the above equation to Theorem 4.

Numerical example: In this numerical example, $m = 1/6$, $c = 2.05$, and the input rates $\lambda = 11.8$, 12.29 as the same as the exponential distribution case. The claim size is Erlang distributed with rate $\mu = 12.0$ and shape parameter 2 so that the expected claim size $m = 1/6$. If $\lambda = 11.8$, then the other two solutions of Eq. (28) are

$$s_1 = -0.3267, \qquad s_2 = -17.9172,$$

and

$$H(x) = 24.6 - 23.7111 e^{-0.3267x} + 0.1111 e^{-17.9172x}, \quad x \geq 0,$$

and if $\lambda = 12.29$, then the other two solutions of Eq. (28) are

$$s_1 = -0.0065, \qquad s_2 = -17.9984,$$

and

$$H(x) = 1230 - 1229.1111 e^{-0.0065x} + 0.1111 e^{-17.9984x}, \quad x \geq 0.$$

We obtain numerically the values of $E[\tau]$ and $E[N_2]$ for various values of $M-a$. For $M-a = 0.5, 1.0, \ldots, 5.0$, Fig. 5 shows the results. We can find the same feature as the exponential distribution case, i.e. the expected time to ruin decreases almost linearly, while $E[N_2]$ decreases exponentially in both cases of $\lambda = 11.8$ and 12.29. However, due to the lower variance of the claim size compared to the exponential distribution case, both $E[\tau]$ and $E[N_2]$ have the greater value than the exponential distribution case. Fig. 6 gives $f(x)$ for $M - a = 1.0, 3.0, 5.0$. As shown in the figure, $f(x)$ has a negligible size of jump at $x = a$, but a lot larger size of jump at $x = b$ as same as the exponential distribution case.

7.3. GH distributed claim sizes

Suppose that the c.d.f. of the claim sizes can be approximated by a two term GH distribution,

$$G(x) = 1 - w_1 e^{-\mu_1 x} - w_2 e^{-\mu_2 x}.$$

Then, the approximate average claim size is

$$m_g = \frac{w_1}{\mu_1} + \frac{w_2}{\mu_2}.$$

We assume that the premium rate c is greater than λm_g. From Eq. (24), Laplace transform of $\hat{H}(x)$ is computed to be

$$\psi(s) = \frac{(s+\mu_1)(s+\mu_2)}{s\{s^2 + (\mu_1 + \mu_2 - \lambda/c)s + \mu_1\mu_2(1 - \lambda m_g/c)\}}. \quad (30)$$

Since $c > \lambda m_g$, we can see that $s_0 = 0$ is a simple pole of $\psi(s)$. Let s_1 and s_2 be the other two poles. Then,

$$s_1 = \frac{-(\mu_1 + \mu_2 - \lambda/c) + \sqrt{(\mu_1 + \mu_2 - \lambda/c)^2 - 4\mu_1\mu_2(1 - \lambda m_g/c)}}{2},$$

$$s_2 = \frac{-(\mu_1 + \mu_2 - \lambda/c) - \sqrt{(\mu_1 + \mu_2 - \lambda/c)^2 - 4\mu_1\mu_2(1 - \lambda m_g/c)}}{2}.$$

We consider the simplest case that both s_1 and s_2 are real and $s_1 \neq s_2$. Then, we can decompose $\psi(s)$ as follows,

$$\psi(s) = \frac{c_0}{s} + \frac{c_1}{s - s_1} + \frac{c_2}{s - s_2},$$

where

$$c_i = (s - s_i)\psi(s)|_{s=s_i}, \quad i = 0, 1, 2.$$

Its inverse Laplace transform is given by

$$\begin{aligned}\hat{H}(x) &= c_0 + c_1 e^{s_1 x} + c_2 e^{s_2 x} \\ &= \frac{1}{1 - \lambda m_g/c} + \frac{(s_1+\mu_1)(s_1+\mu_2)}{s_1(s_1 - s_2)} e^{s_1 x} + \frac{(s_2+\mu_1)(s_2+\mu_2)}{s_2(s_2 - s_1)} e^{s_2 x}. \end{aligned} \quad (31)$$

By applying the above equation to Theorem 4, we can approximate the stationary distribution of the surplus process.

Numerical example: In this numerical example, $m = 1/6$, $c = 2.05$, and the input rates $\lambda = 11.8$, 12.29 as same as the previous cases. The claim size is two-term GH distributed with c.d.f. $G(x)$,

$$G(x) = 1 - \frac{10}{11} e^{-12x} - \frac{1}{11} e^{-x}, \quad x \geq 0.$$

If $\lambda = 11.8$, then the other two solutions of Eq. (30) are

$$s_1 = -0.0680, \qquad s_2 = -7.1759,$$

and

$$H(x) = 24.6 - 23.0159 e^{-0.0680x} - 0.5841 e^{-7.1759x}, \quad x \geq 0,$$

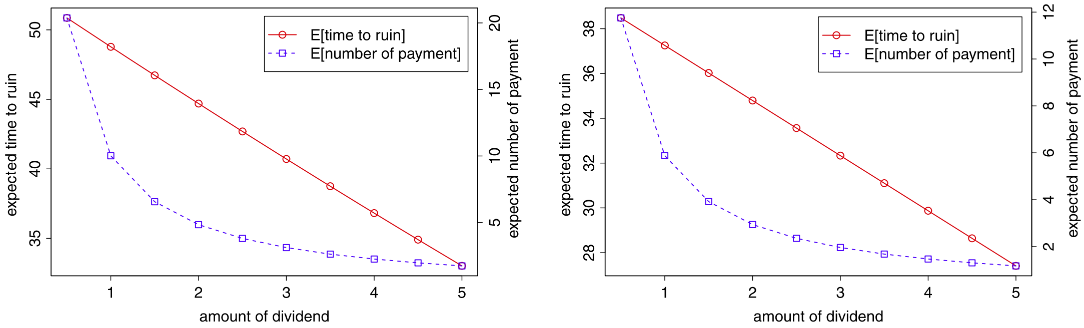
and if $\lambda = 12.29$, then the other two solutions of Eq. (30) are

$$s_1 = -0.0014, \qquad s_2 = -7.0035,$$

and

$$H(x) = 1230 - 1228.3883 e^{-0.0014x} - 0.6117 e^{-7.0035x}, \quad x \geq 0.$$

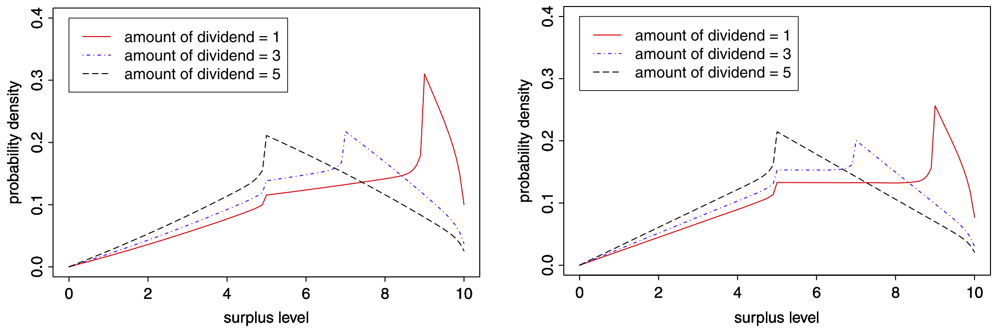
We obtain numerically the values of $E[\tau]$ and $E[N_2]$ for various values of $M - a$. For $M - a = 0.5, 1.0, \ldots, 5.0$, Fig. 7 shows the results. As same as the previous cases, the expected time to ruin decreases almost linearly, while $E[N_2]$ decreases exponentially in both cases of $\lambda = 11.8$ and 12.29. However, both $E[\tau]$ and $E[N_2]$ are lower than those of the previous cases, which is due to the larger variance of the GH distributed claim size. Fig. 8 shows $f(x)$ for $M - a = 1.0, 3.0, 5.0$. Differently from the previous cases, $f(x)$ has no negligible jump at $x = a$, and has larger size of jump at $x = b$ compared to the previous cases.



(a) When $\lambda = 11.8$.

(b) When $\lambda = 12.29$.

Fig. 7. Expected time to ruin after a reinvestment and expected number of payment of dividends between two ruins with various values of the amount of dividend.



(a) When $\lambda = 11.8$.

(b) When $\lambda = 12.29$.

Fig. 8. The p.d.f. $f(x)$ of the stationary level of the surplus process with various values of the amount of dividend.

8. Conclusion

In this paper, a modified Cramér–Lundberg model is analyzed. In the model, a constant amount of dividend is paid immediately whenever the surplus level reaches a barrier, and at ruin a reinvestment is done so that the surplus level is to be a specified level. Such a model is justified in case the costs of payment of dividends and reinvesting are not negligible. We obtain the distribution of the stationary surplus level, the expected time to ruin after a reinvestment, and the expected number of payments of dividends between ruins.

In obtaining the results, we adopt the level-crossing argument and the technique of the decomposition of the surplus process. However, differently from Brill and Yu (2011), we obtain the expected number of the up-crossing of a level, and using this, we drive the distribution of the stationary surplus level. Brill and Yu (2011) also adopted the level-crossing argument to obtain the distribution of the stationary surplus level, while they obtained the renewal type equation (or Volterra integral equation) for the p.d.f. of the stationary surplus level.

In practice, it may be difficult to obtain the value of the function $H(x)$. For such case, the Laplace transform method is helpful for evaluating $H(x)$ if the Laplace transform of $H(x)$ is easily invertible. When a GH distribution gives a good approximation of the distribution of the actual claim size, we show that $H(x)$ can be evaluated using the partial fraction expansion of the Laplace transform of $H(x)$. Through the numerical examples, the cases with various distributions of the claim size are treated, and we can see that the obtained results of this paper can be applied without huge difficulty.

Acknowledgments

This work was supported by the 2013 sabbatical year research grant of the University of Seoul for Sunggon Kim, and also supported by the National Research Foundation of Korea (NRF) Grant funded by the Korean Government (MOE) (NRF-2013R1A1A2006750) for Eui Yong Lee.

References

Bae, J., Kim, S., & Lee, E. Y. (2002). A P_{λ}^{M}-policy of an $M/G/1$ queueing system. *Applied Mathematical Modelling*, *26*, 929–939.

Bai, L., & Guo, J. (2010). Optimal dividend payment in the classical risk model when payment are subject to both transaction costs and taxes. *Scandinavian Acturial Journal*, 36–55.

Borch, K. (1969). The resque of an insurance company after ruin. *ASTIN Bulletin*, *1*, 280–292.

Botta, R. F., & Harris, C. M. (1986). Approximation with generalized hyperexponential distributions: weak convergence results. *Queueing Systems*, *2*, 169–190.
Botta, R. F., Harris, C. M., & Marchal, W. G. (1987). Characterization of generalized hyperexponential distribution functions. *Stochastic Models*, *3*, 115–148.
Brill, P. H. (2008). *Level crossing methods in stochastic models*. New York: Springer.
Brill, P. H., & Yu, K. (2011). Analysis of risk models using a level crossing technique. *Insurance: Mathematics & Economics*, *49*, 298–309.
Cohen, J. W. (1977). On up and down crossings. *Journal of Applied Probability*, *12*, 405–410.
Dickson, D. C. M., & Waters, H. R. (2004). Some optimal dividens problems. *ASTIN Bulletin*, *34*, 49–74.
Dufresne, F., & Gerber, H. U. (1991). Risk theory for the compound Poisson process that is perturbed by diffusion. *Insurance: Mathematics & Economics*, *10*, 51–59.
Gerber, H. U., & Shiu, E. S. W. (1997). The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. *Insurance: Mathematics & Economics*, *21*, 129–137.
Harris, C. M., Marchal, W. G., & Botta, R. F. (1992). A note on generalized hyperexponential distributions. *Stochastic Models*, *8*, 179–191.
He, L., & Liang, Z. (2009). Optimal financing and dividend control of the insurance company with fixed and proportional transaction costs. *Insurance: Mathematics & Economics*, *44*, 88–94.
Jeong, M. O., Lim, K. E., & Lee, E. Y. (2009). An optimization of a continuous time risk process. *Applied Mathematical Modelling*, *33*, 4062–4068.
Klugman, S. A., Panjer, H. H., & Willmot, G. E. (2004). *Loss model: From data to decisions* (2nd ed.). Hoboken: John Wiley & Sons.
Kulenko, N., & Schmidli, H. (2008). Optimal dividend strategies in a Cramér–Lundberg model with capital injections. *Insurance: Mathematics & Economics*, *43*, 270–278.
Li, S., & Lu, Y. (2005). On the expected discounted penalty functions for two classes of risk processes. *Insurance: Mathematics & Economics*, *36*, 170–193.
Ross, S. M. (1996). *Stochastic processes* (2nd ed.). Wiley.
Takács, L. (1967). The distribution of the content of finite dams. *Journal of Applied Probability*, *4*, 151–161.
Thonhauser, S., & Albrecher, H. (2011). Optimal dividend strategies for a compound Poisson process under transaction costs and power utility. *Stochastic Models*, *27*, 120–140.
Zhang, C., & Wang, G. (2003). The joint density function of three characteristics on jump-diffusion risk process. *Insurance: Mathematics & Economics*, *32*, 445–455.