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Original Research Article

2D tolerance and asymptotic models in elastodynamics of a thin-walled structure with dense system of ribs

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ABSTRACT

The object of analysis is a plane structure reinforced by a system of thin parallel-distributed ribs. It will be assumed that the number of the ribs is very large. The thickness of neighbouring ribs can smoothly change. The aim of contribution is to derive 2D-macroscopic mathematical models describing elastodynamic behaviour of the plate structure in plane-stress state. The consideration will be based on the tolerance averaging technique [9,10]. The general results of the contribution will be illustrated by the analysis of the free vibrations of a structure under consideration.

1. Introduction

Introduce the orthogonal Cartesian coordinate system $Ox^1x^2x^3$ in the physical space occupied by a plate structure under consideration. Let $\Xi = (0, L_1) \times (0, L_2)$ be the midplane (the symmetry plane) of the structure. It is assumed that thickness of the plate h and thickness of the ribs b are small compared to the minimum length dimension of the midplane of the plate, $h, b \ll \min(L_1, L_2)$. At the same time the thicknesses h and b are supposed to be small compared to the width of the stiffened ribs H, h, $b \ll H$ (Figs 1 and 2).

Subsequently it will be assumed that number n of the ribs is very large, $1/n \ll 1$, and the maximum distance l between ribs is very small when compared to L_1. Hence $l = L_1/n$ will be treated as a microstructure length parameter. At the same time, the thickness h of the plate is supposed to be small compared to the microstructure length parameter l, $h \ll l$.

The aim of this contribution is to formulate 2D macroscopic models of dynamic behaviour of the plate under consideration. These models will be referred to as asymptotic and tolerance, respectively. By the 2-dimensional macroscopic model we shall understand mathematical model governed by averaged equations of motion with smooth coefficients and unknown functions dependent on coordinates x^1 and x^2.

The formulation of averaged mathematical models of the considered plane structure will be based on the tolerance averaging technique. The general modelling procedures of this technique are given by Woźniak et. al. in books [9,10]. Some applications of the tolerance averaging technique for the modelling of various dynamic problems for elastic micro-heterogeneous structures are given in a series of papers by

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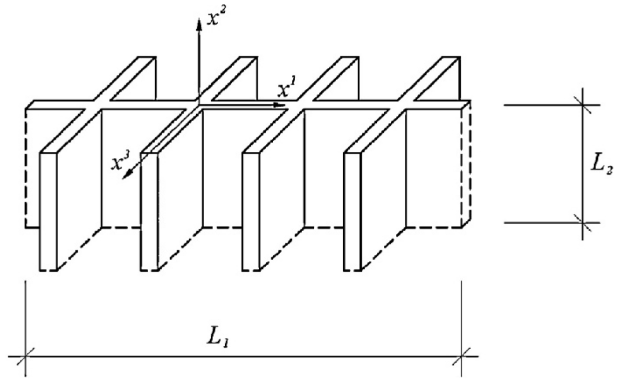

Fig. 1 – Fragment of a plate structure with periodic system of stiffeners.

Baron [1], Jędrysiak [2], Michalak [3], Michalak and Wirowski [4], Nagórko and Woźniak [5], Tomczyk [6], Wągrowska and Woźniak [7], and Wierzbicki and Woźniak [8].

Throughout the paper, indices $i, k, l, \ldots$ run over 1, 2, and 3, indices $\alpha, \beta, \gamma, \ldots$ run over 1, 2 and t stand for the time coordinate. Subsequently we shall use denotations $x = x^1$, $\partial_1 = \partial/\partial x^1$, $\partial_2 = \partial/\partial x^2$. The summation convention holds all aforementioned sub- and superscripts.

2. Formulation of the modelling problem

The considerations will be based on the well-known equations for the plane-stress state in the plate structure. It is assumed that the undeformed midplane of the plate occupies region $\Xi = (0, L_1) \times (0, L_2)$. Denoting by l distance between the ribs of the plate structure, every Δ_i, where $x_i = l/2 + (i-1)l$, $i = 1, 2, \ldots, n$, $(1/n \ll 1)$, will be referred to the cell in Ξ with centre at x_i (Fig. 3). Let $\overline{\Omega} = \cup \Delta_i \times [0, L_2]$ be region in the physical space occupied by plate structure and int$(\cup \Delta_i)$-cross section of Ω by every $x^2 \in (0, L_2)$-plane. Let subcells Δ_i^P, Δ_i^S, and Δ_i^{SP} be parts of every cell $\Delta_i(x)$; belonging to plate, ribs-stiffeners and part belonging both to plate and stiffeners, respectively.

The model equations for the dynamic behaviour of the plate structure under consideration will be obtained for plane-stress state in the plate.

Subcells Δ_i^P. Plane stress in plane Ox^1x^2, $n^{33} = 0$, hence

$$n^{11} = \frac{hE}{1-\nu^2}(e_{11} + \nu e_{22}), \quad n^{22} = \frac{hE}{1-\nu^2}(e_{22} + \nu e_{11}),$$
$$n^{12} = \frac{hE}{1+\nu} e_{12}, \tag{1}$$

where $e_{\alpha\beta}$ is strain tensors.

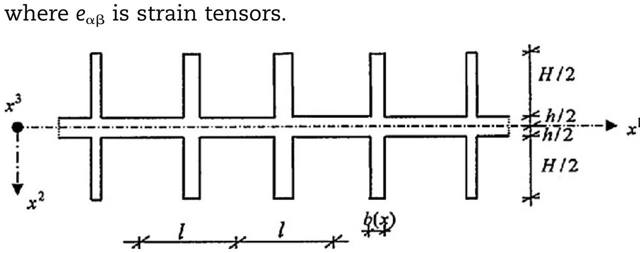

Fig. 2 – Fragment of a cross-section of the stiffened plate structure.

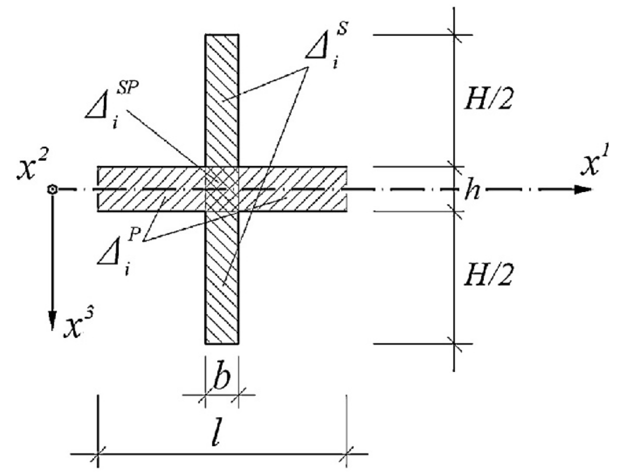

Fig. 3 – The basic cell of the stiffened plate structure.

Subcells Δ_i^{SP}. In this region of the structure we consider 3D-stress state

$$n^{11} = h(\lambda + 2\mu)e_{11} + h\lambda(e_{22} + e_{33}), \quad n^{22} = h(\lambda + 2\mu)e_{22} + h\lambda(e_{11} + e_{33}),$$
$$n^{33} = h(\lambda + 2\mu)e_{33} + h\lambda(e_{11} + e_{22}), \quad n^{12} = \frac{hE}{1+\nu} e_{12}, \tag{2}$$

where λ and μ will be Lame's constants.

Subcells Δ_i^S. Plane stress in plane Ox^2x^3, $n^{11} = 0$, hence

$$n^{22} = \frac{hE}{1-\nu^2}(e_{22} + \nu e_{33}), \quad n^{33} = \frac{hE}{1-\nu^2}(e_{33} + \nu e_{22}), \quad n^{12} = \frac{hE}{1+\nu} e_{12}. \tag{3}$$

Bearing in mind that $h \ll H \ll L_2$ we shall assume approximation $e_{33} \cong -\nu e_{22}$ in subcell Δ_i^S and $e_{33} \cong 0$ in subcell Δ_i^{SP}.

Averaging formulae (2), (3) in Ξ_S (Fig. 4) over $(-(h+H)/2, (h+H)/2)$, with above assumptions, we have

$$N^{11} = h(\lambda + 2\mu)e_{11} + h\lambda e_{22}, \quad N^{22} = [HE + h(\lambda + 2\mu)]e_{22} + h\lambda e_{11}, \quad N^{12} = \frac{hE}{1+\nu} e_{12}, \tag{4}$$

and in Ξ_P averaging formulae (1) over $(-h, h)$

$$n^{11} = \frac{hE}{1-\nu^2}(e_{11} + \nu e_{22}), \quad n^{22} = \frac{hE}{1-\nu^2}(e_{22} + \nu e_{11}),$$
$$n^{12} = \frac{hE}{1+\nu} e_{12}, \tag{5}$$

we derive constitutive equations for 2-dimensional model of the heterogeneous structure under consideration.

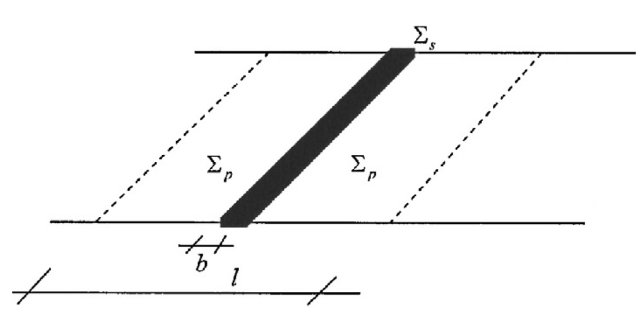

Fig. 4 – Midplane of the plate structure.

2.1. 2-Dimensional model *of the plate structures under consideration*

Let displacement of the midplane of the plate be denoted by $w_\alpha(x^\beta, t)$, external forces by $p_\alpha(x^\beta, t)$ and by $\tilde{\rho}$ the mass density averaged over the plate thickness related to the midplane.

In the framework of the linear theory for plane-stress state we have well-known equations of motion

$$\partial_\alpha \tilde{N}^{\alpha\beta} + p^\beta - \tilde{\rho}\ddot{w}^\beta = 0, \tag{6}$$

where

$$\tilde{\rho} = \begin{cases} \rho h & in\ \Xi_P, \\ \rho(h+H) & in\ \Xi_S, \end{cases} \tag{7}$$

$$\tilde{N}^{\alpha\beta} = \begin{cases} n^{\alpha\beta} & \text{in}\ \Xi_P, \\ N^{\alpha\beta} & \text{in}\ \Xi_S. \end{cases} \tag{8}$$

Constitutive equations we shall write in the form

$$\tilde{N}^{\alpha\beta} = D^{\alpha\beta\gamma\delta} e_{\gamma\delta}, \tag{9}$$

where

$$\tilde{N}^{11} = D^{1111}e_{11} + D^{1122}e_{22}, \quad \tilde{N}^{22} = D^{2211}e_{11} + D^{2222}e_{22},$$
$$\tilde{N}^{12} = D^{1212}e_{12}. \tag{10}$$

It can be seen that the coefficients in the above equations are discontinuous and highly oscillating. These equations are too complicated to be used in the engineering analysis and will be used as a starting point in the tolerance modelling procedure.

3. Modelling technique

In order to derive averaged model equations we applied tolerance-averaging approach. The general modelling procedures of this technique are given in books [9,10]. We mention some basic concepts of this technique [10].

The fundamental concept of the modelling technique in the modelling procedure is the averaging of an arbitrary integrable function $f(\cdot)$ over the cell Δ_i

$$\langle f \rangle = \frac{1}{\Delta_i}\int_{\Delta_i} f(y, x^2)dy, \tag{11}$$

for every $y \in \Delta(x_i)$, $x^2 \in [0, L_2]$.

The important assumption of this technique is that values of functions belonging to region Ω can be determined only within to the assumed accuracy δ. Tolerance relation $\approx$ for an arbitrary positive δ is defined by

$$(\forall (x_1, x_2) \in X^2)[x_1 \approx x_2 \Leftrightarrow \|x_1 - x_2\|_X \leq \delta], \tag{12}$$

where δ will be said to be the tolerance parameter.

Let $\partial^k f$ be the kth gradient of function $f(\mathbf{x}), \mathbf{x} \in \Omega, k = 0, 1, \ldots, \alpha, (\alpha \geq 0), \partial^0 f \equiv f$. Function $f \in H^\alpha(\Omega)$ will be called the tolerance periodic function (with respect to cell $\Delta(x_i)$ and tolerance parameter δ), $f \in TP^\alpha_\delta(\Omega, \Delta_i)$, if the following conditions hold

$$(\forall \mathbf{x} \in \Omega)(\exists\, \tilde{f}^{(k)}(\mathbf{x}, \cdot) \in H^0(\Delta_i))\,[||\partial^k f|_{\Omega_x}(\cdot) - \tilde{f}^{(k)}(\mathbf{x}, \cdot)||_{H^0(\Omega_x)} \leq \delta],$$
$$\int_{\Delta(\cdot)} \tilde{f}^{(k)}(\cdot, y)dy \in C^0(\overline{\Omega}). \tag{13}$$

Function $\tilde{f}^{(k)}(\mathbf{x}, \cdot)$ is referred to as the periodic approximation of $\partial^k f$ in $\Delta(x_i)$.

Function $F \in H^\alpha(\Omega)$ will be called the slowly varying function (with respect to the cell $\Delta(x_i)$ and tolerance parameter δ), $F \in SV^\alpha_\delta(\Omega, \Delta_i)$, if

$$F \in TP^\alpha_\delta(\Omega, \Delta_i),$$
$$(\forall \mathbf{x} \in \Omega)[\tilde{F}^{(k)}(\mathbf{x}, \cdot)|_{\Delta(x_i)} = \partial^k F(\mathbf{x}), \quad k = 0, \ldots, \alpha]. \tag{14}$$

It is possible to notice that periodic approximation $\tilde{F}^{(k)}$ of $\partial^k F(\cdot)$ in $\Delta(x_i)$ is a constant function for every $\mathbf{x} \in \Omega$. If $F \in SV^\alpha_\delta(\Omega, \Delta_i)$ then $(\forall x \in \Omega)(||\partial^k F(\cdot) - \partial^k F(x)||_{H^0(\Delta(x_i))} \leq \delta,\ k = 0, 1, \ldots, \alpha)$.

Function $\varphi \in H^\alpha(\Pi)$ will be called the highly oscillating function (with respect to the cell $\Delta(x_i)$ and tolerance parameter δ), $\varphi \in HO^\alpha_\delta(\Omega, \Delta_i)$, if

$$\varphi \in TP^\alpha_\delta(\Omega, \Delta_i),$$
$$(\forall \mathbf{x} \in \Omega)[\tilde{\varphi}^{(k)}(\mathbf{x}, \cdot)|_{\Delta(x_i)} = \partial^k \tilde{\varphi}(\mathbf{x})]. \tag{15}$$

If $F \in SV^\alpha_\delta(\Omega, \Delta_i)$ then $f \equiv \varphi F \in TP^\alpha_\delta(\Omega, \Delta_i)$ and these functions satisfy condition

$$^{(k)}(\mathbf{x}, \cdot)|_{\Delta(x_i)} = F(\mathbf{x})\partial^k \tilde{\varphi}(\mathbf{x})|_{\Delta(x_i)}. \tag{16}$$

If $\alpha = 0$ then we denote $\tilde{f} \equiv \tilde{f}^{(0)}$.

Let $g(\cdot)$ denote a highly oscillating function, $g \in HO^1_\delta(\Omega, \Delta_i)$, continuous in $\overline{\Omega}$. Its gradient ∂g is a piecewise continuous and bounded. Function $g(\cdot)$ will be called the fluctuation shape function of the first kind, if it depends on l as a parameter and satisfies conditions:

1° $\partial^1 g \in O(l^0)$,

2° $\langle \rho g \rangle(\mathbf{x}) \approx 0$ for every $\mathbf{x} \in \Omega_\Delta$,

where $\rho > 0$ is a certain tolerance periodic function.

4. Macroscopic models

4.1. *Tolerance model*

The first assumption in the tolerance modelling is micro-macro decomposition of the displacement field

$$w_\alpha(x, x^2, t) = u_\alpha(x, x^2, t) + g(x)V_\alpha(x, x^2, t) \tag{17}$$

for $x^\alpha \in \Xi$ and $t \in (t_0, t_1)$.

The modelling assumption states that $u_\alpha(\cdot)$ and $V_\alpha(\cdot)$ are slowly varying functions with respect to the argument $x \in (0, L_1)$. Functions $u_\alpha(\cdot, x^2, t) \in SV^1_\delta(\Xi, \Delta)$ and $V_\alpha(\cdot, x^2, t) \in SV^1_\delta(\Xi, \Delta)$ are the basic unknowns of the tolerance model. Function $g(x)$ is known, dependent on the microstructure length parameter l, fluctuation shape function.

Let $\tilde{g}(\cdot, x)$, $\partial_1\tilde{g}(\cdot, x)$ stand for periodic approximation of $g(\cdot)$, $\partial_1 g(\cdot)$ in Δ_i, respectively. Due to the fact that $w_\alpha(\cdot, x^2, t)$ are tolerance periodic functions, it can be observed that the periodic approximation of $w_{\alpha h}(\cdot, x^2, t)$ and $\partial_\beta w_{\alpha h}(\cdot, x^2, t)$ in $\Delta_i(x)$, $x \in \Xi$ have the form

$$w_{\alpha h}(y, x^2, t) = u_\alpha(x^\beta, t) + \tilde{g}(y, x)V_\alpha(x^\beta, t),$$
$$\partial_\beta w_{\alpha h}(y, x^2, t) = \partial_\beta u_\alpha(x^\gamma, t) + \partial_1\tilde{g}(y, x)V_\alpha(x^\gamma, t) + \tilde{g}(y, x)\partial_2 V_\alpha(x^\gamma, t),$$
$$\dot{w}_{\alpha h}(y, x^2, t) = \dot{u}_\alpha(x^\beta, t) + \tilde{g}(y, x)\,\dot{V}_\alpha(x^\beta, t), \tag{18}$$

for every $x^\beta \in \Xi$, almost every $y \in \Delta_i$ and every $t \in (t_0, t_1)$.

The modelling assumption states that if in every cell $\Delta_i(x)$, $x \in \Xi$ will define residual forces

$$r^\beta = \partial_\alpha \tilde{N}^{\alpha\beta} + p^\beta - \tilde{\rho}\ddot{w}^\beta \tag{19}$$

then the following orthogonality conditions hold

$$\langle r^\beta \rangle_T(x^1) = 0, \quad \langle g r^\beta \rangle_T(x^1) = 0, \tag{20}$$

where operator $\langle \cdot \rangle_T(x)$ stands for tolerance averaging over the cell $\Delta_i(x)$.

Substituting the right-hand side of formula (17) into Eq. (19) and bearing in mind orthogonality conditions (20), we obtain the following system of equations of motion

$$\begin{aligned}
&\partial_\alpha(\langle D^{\alpha\beta\gamma\delta}\rangle \partial_\gamma u_\delta + \langle D^{\alpha\beta1\delta}\partial_1 g\rangle V_\delta + \langle D^{\alpha\beta2\delta} g\rangle \partial_2 V_\delta) + \langle p^\beta \rangle \\
&-\langle \tilde{\rho}\rangle \ddot{u}^\beta - \langle \tilde{\rho} g\rangle \ddot{V}^\beta = 0, \\
&\partial_2(\langle D^{2\beta\gamma\delta} g\rangle \partial_\gamma u_\delta + \langle g D^{2\beta1\delta}\partial_1 g\rangle V_\delta + \langle g D^{2\beta2\delta} g\rangle \partial_2 V_\delta) - \langle D^{1\beta\gamma\delta}\partial_1 g\rangle \partial_\gamma u_\delta \\
&-\langle \partial_1 g D^{1\beta1\delta}\partial_1 g\rangle V_\delta - \langle \partial_1 g D^{1\beta2\delta} g\rangle \partial_2 V_\delta + \langle p^\beta g\rangle - \langle \tilde{\rho} g\rangle \ddot{u}^\beta - \langle g\tilde{\rho} g\rangle \ddot{V}^\beta = 0.
\end{aligned} \tag{21}$$

The above results represent the system equations for averaged displacements $u_\alpha(x^\beta, t)$, and displacements' fluctuation amplitudes $V_\alpha(x^\beta, t)$. These equations, together with micro–macro decomposition of displacement fields (17) and physical condition that solutions have to be slowly varying functions with respect to the argument $x \in (0, L_1)$, constitute the tolerance model of structural plate under consideration.

4.2. *Asymptotic model*

For asymptotic modelling procedure we retain only the concept of highly oscillating function. We shall not deal with the concept of the tolerance periodic function as well as slowly varying function. Using the asymptotic procedure we introduce parameter $\varepsilon = 1/n$, $n = 1, 2, \ldots$. Let εl, εh, εH and εb be the scaled dimensions of the cell $\Delta(x_i)$. A scaled cell will be defined by $\Delta_\varepsilon \equiv (-\varepsilon l/2, \varepsilon l/2)$ and $\Delta_\varepsilon(x_i) = x_i + \Delta_\varepsilon$ is a scaled cell with a centre at $x_i \in \overline{\Xi}$.

The mass density $\tilde{\rho}(\cdot)$ and tensor of elastic moduli $D^{\alpha\beta\gamma\delta}(\cdot)$ are assumed to be highly oscillating discontinuous functions, $\tilde{\rho}(\cdot)$, $D^{\alpha\beta\gamma\delta}(\cdot) \in HO^0_\delta(\Xi, \Delta)$, for almost every $x \in \overline{\Xi}$. If $\tilde{\rho}(\cdot)$, $D^{\alpha\beta\gamma\delta}(\cdot) \in HO^0_\delta(\Xi, \Delta)$ are highly oscillating function then for every $x \in \overline{\Xi}$ there exist functions $\tilde{\rho}(y, x)$ and $\tilde{D}^{\alpha\beta\gamma\delta}(y, x)$ which are periodic approximation of functions $\tilde{\rho}(\cdot)$ and $D^{\alpha\beta\gamma\delta}(\cdot)$, respectively.

The fundamental assumption of the asymptotic modelling is that we introduce decomposition of displacement as family of fields

$$\begin{aligned}
w_{\alpha\varepsilon}(y, x, x^2, t) = u_\alpha(y, x, t) \\
+ \varepsilon\tilde{g}\left(\frac{y}{\varepsilon}, x\right) V_\alpha(y, x, t), \quad y \in \Delta_i(x),\ t \in (t_0, t_1),
\end{aligned} \tag{22}$$

where $\tilde{g}(\cdot, x)$ are periodic approximation of highly oscillating functions $g(\cdot) \in HO^1_\delta(\Xi, \Delta)$. From formula (22) we obtain

$$\begin{aligned}
\partial_\beta w_{\alpha\varepsilon}(y, x, x^2, t) &= \partial_\beta u_\alpha(y, x^2, t) + \partial_1\tilde{g}\left(\frac{y}{\varepsilon}, x\right) V_\alpha(y, x^2, t) \\
&\quad + \varepsilon\tilde{g}\left(\frac{y}{\varepsilon}, x\right) \partial_2 V_\alpha(y, x^2, t), \\
\dot{w}_{\alpha\varepsilon}(y, x, x^2, t) &= \dot{u}_\alpha(y, x^2, t) + \varepsilon\tilde{g}\left(\frac{y}{\varepsilon}, x\right) \dot{V}_\alpha(y, x^2, t).
\end{aligned} \tag{23}$$

Bearing in mind that by means of property of the mean value, Jikov et. al. (1994), function $\tilde{g}(y/\varepsilon, x^2)$, $y \in \Delta_\varepsilon(x)$ is weakly bounded and has under $\varepsilon \to 0$ weak limit. Under limit passage $\varepsilon \to 0$ for $y \in \Delta_\varepsilon(x)$ we obtain

$$\begin{aligned}
&u_\alpha(y, x^2, t) = u_\alpha(x^\beta, t) + O(\varepsilon), \quad \partial_\beta u_\alpha(y, x^2, t) = \partial_\beta u_\alpha(x^\gamma, t) + O(\varepsilon), \\
&V_\alpha(y, x^2, t) = V_\alpha(x^\beta, t) + O(\varepsilon), \quad \partial_\beta V_\alpha(y, x^2, t) = \partial_\beta V_\alpha(x^\gamma, t) + O(\varepsilon), \\
&\dot{u}_\alpha(y, x^2, t) = \dot{u}_\alpha(x^\beta, t) + O(\varepsilon), \quad \dot{V}_\alpha(y, x^2, t) = \dot{V}_\alpha(x^\beta, t) + O(\varepsilon).
\end{aligned} \tag{24}$$

By means of (24) we rewrite formulae (22) and (23) in the form

$$\begin{aligned}
&w_{\alpha\varepsilon}(y, x^2, t) = u_\alpha(x^\beta, t) + O(\varepsilon), \\
&\partial_\beta w_{\alpha\varepsilon}(y, x^2, t) = \partial_\beta u_\alpha(x^\beta, t) + \partial_1\tilde{g}\left(\frac{y}{\varepsilon}\right) V_\alpha(x^\beta, t) + O(\varepsilon), \\
&\dot{w}_{\alpha\varepsilon}(y, x^2, t) = \dot{u}_\alpha(x^\beta, t)
\end{aligned} \tag{25}$$

Using formulae (25) for orthogonality conditions (20) we obtain equations

$$\begin{aligned}
&\partial_\alpha(\langle D^{\alpha\beta\gamma\delta}\rangle \partial_\gamma u_\delta + \langle D^{\alpha\beta1\delta}\partial_1 g\rangle V_\delta) + \langle p^\beta\rangle - \langle\tilde{\rho}\rangle \ddot{u}^\beta = 0, \\
&\langle D^{1\beta\gamma\delta}\partial_1 g\rangle \partial_\gamma u_\delta + \langle \partial_1 g D^{1\beta1\delta}\partial_1 g\rangle V_\delta = 0.
\end{aligned} \tag{26}$$

Eliminating V_δ from Eqs. (26)

$$V_\delta = -\frac{\langle D^{1\beta\gamma\delta}\partial_1 g\rangle}{\langle \partial_1 g D^{1\beta1\delta}\partial_1 g\rangle} \partial_\gamma u_\delta, \tag{27}$$

and denoting effective elastic moduli

$$D^{\alpha\beta\gamma\mu}_{eff} = \langle D^{\alpha\beta\gamma\delta}\rangle - \frac{\langle D^{\alpha\beta1\tau}\partial_1 g\rangle\langle D^{1\mu\gamma\delta}\partial_1 g\rangle}{\langle \partial_1 g D^{1\mu1\tau}\partial_1 g\rangle}, \tag{28}$$

we arrive at the following equation of motion for the averaged displacements of the plate midplane $u_\alpha(x^\beta, t)$

$$\partial_\alpha(D^{\alpha\beta\gamma\delta}_{eff}\partial_\gamma u_\delta) + \langle p^\beta\rangle - \langle\tilde{\rho}\rangle \ddot{u}^\beta = 0. \tag{29}$$

Eqs. (27)–(29) represent the asymptotic model of the structural plate under consideration.

5. Example of application

5.1. *Non-periodic distribution of ribs*

In order to analyse the influence of non-periodic distribution of thickness of the ribs on the free vibration frequencies we consider the simple one-dimensional problem of vibrations. We restricted the analyses to the first vibration frequency for the asymptotic model.

Asymptotic model: After a simple manipulation we obtain from Eq. (29) the following differential equation describing one-dimensional vibrations of the plate structure

$$\partial_1(D^{1111}_{eff}(x)\partial_1 u_1) - \langle\tilde{\rho}\rangle \ddot{u}^1 = 0. \tag{30}$$

We assume that thickness of the ribs is given by function $b(x) = 2(b_e - b_0)x/L + b_0$ where we have denoted: b_0 – thickness of the rib in the middle of plate, b_e – thickness on boundaries, where L – span of the plate band.

The plane-stress state of the plate structure under consideration is described by strain tensor with the first gradient of the displacements. Hence, we assume the saw-like fluctuation shape function $g(\cdot)$ corresponding to the cell $\Delta_i(x)$ (Fig. 5).

Volume fraction $n(x) = b(x)/l$ of material of the ribs is assumed to satisfy condition $l/\partial_1 n(x) \ll 1$.

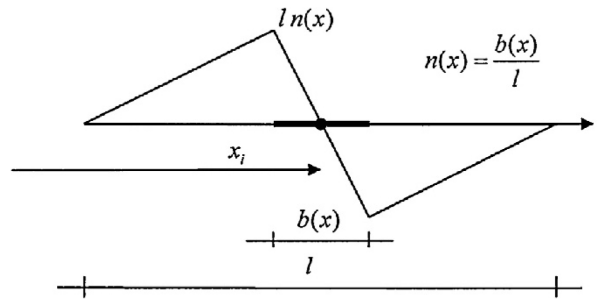

Fig. 5 – Fluctuation shape function.

Restricting consideration to harmonic vibrations we look for a solution to Eq. (30) in the form

$$u_1(x,t) = u_1^0(x)e^{i\omega t}. \tag{31}$$

Substituting (31) into (30) and bearing in mind formulae (28) for effective module, we obtain equation

$$L(u_1^0, x) = 0 \tag{32}$$

with differential operator

$$L(u_1^0, x) = \partial_1\left[\left(\langle D^{1111}\rangle - \frac{\langle D^{1111}\partial_1 g\rangle\langle D^{1111}\partial_1 g\rangle}{\langle \partial_1 g D^{1111}\partial_1 g\rangle}\right)\partial_1 u_1^0\right] - \langle\tilde{\rho}\rangle\omega^2 u_1^0. \tag{33}$$

For non-periodic distribution of the ribs the differential operator (33) has functional coefficients. Hence, we shall look for the approximate solution of Eq. (32) using the Galerkin method

$$\int_{-L/2}^{L/2} L(u_1^0, x)\, f(x)dx = 0. \tag{34}$$

For simply supported plate band with span L we assume the approximate solution in the form

$$u_1^0(x) = \overline{u}\, f(x), \tag{35}$$

where trial function $f(x) = \cos(\pi x/L)$.

This function satisfies the boundary conditions for simply supported plate band.

Substituting the operator (33) with function (35) into Eq. (34) we derive a value of the free vibration frequencies $\omega^2 = (Eh/((1-\nu^2)\rho h L^2))w$ for the plate structure under consideration.

Numerical results: Calculations were conducted for three different distributions of width of the ribs. Let us assume that the volume fraction $n(x) = b(x)/l$ of material of the ribs is given by functions

$$\begin{aligned}
&n1(x) = n_0(n_b/n_0 - 1)(2x/L)^4 + n_0,\\
&n2(x) = n_0(n_b/n_0 - 1)(2x/L)^2 + n_0,\\
&n3(x) = \begin{cases} n_0(n_b/n_0 - 1)(-2x/L) + n_0 & x \in \langle -L/2, 0\rangle,\\ n_0(n_b/n_0 - 1)(2x/L) + n_0 & x \in \langle 0, L/2\rangle,\end{cases}
\end{aligned} \tag{36}$$

where n_0 – part of the ribs in the centre and n_b – part of the ribs on the boundaries of the plate structure.

Fig. 6 shows parameter w of the free vibration frequencies $(\omega^2 = (Eh/(1-\nu^2)\rho h L^2)w)$ versus ratio $m = n_b/n_0$. The height of ribs is 10 times greater than the thickness of the plate, $n_0 = 1/25$ and Poisson's ratio $\nu = 0.2$. The diagrams in Fig. 6 refer appropriately to: w_p – for periodic distribution of thickness of the ribs, $wn1(m)$ – for non-periodic distribution given by function $n1(x)$ from formulae (36), $wn2(m)$ – for non-periodic distribution given by function $n2(x)$ and $wn3(m)$ – by function $n3(x)$. Plots in Fig. 6 show that the biggest differences in relation to the periodic distribution are generated by linear distribution of thickness of the ribs.

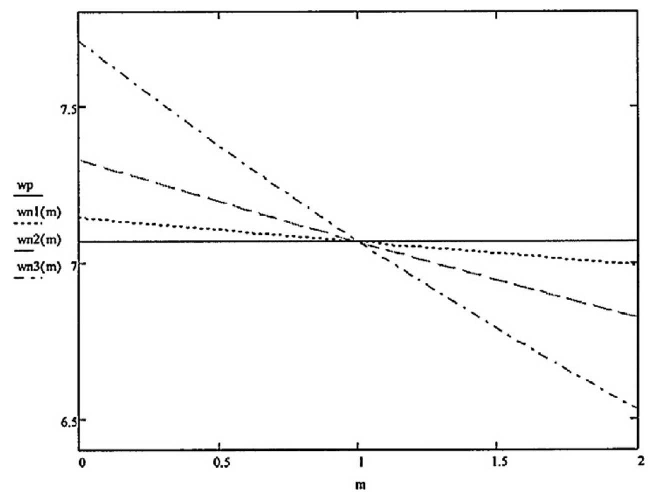

Fig. 6 – Diagrams of free vibration frequency parameters: *wp*, *wn1(m)*, *wn2(m)*, *wn3(m)* for the distribution function of material of the ribs *n(x)* (36) versus ratio $m = n_b/n_0$.

5.2. *Periodic distribution of ribs*

The aim of this subsection is to analyse vibrations and wave propagation in the framework of the tolerance model for the plate with periodic distribution of ribs. It is assumed that the analysis of free vibrations will be restricted to one-dimensional problem. In this case Eq. (21) takes the following form

$$\begin{aligned}
&\langle D^{1111}\rangle\partial_{11}u_1 + \langle D^{1111}\partial_1 g\rangle\partial_1 V_1 - \langle\tilde{\rho}\rangle\ddot{u}^1 = 0,\\
&\langle D^{1111}\partial_1 g\rangle\partial_1 u_1 + \langle \partial_1 g D^{1111}\partial_1 g\rangle V_1 + \langle g\tilde{\rho}g\rangle\ddot{V}^1 = 0.
\end{aligned} \tag{37}$$

We can observe that above equations have solutions

$$u_1(x_1,t) = 0, \quad V_1(x_1,t) = A\cos(\hat{\omega}t) + B\sin(\hat{\omega}t) \tag{38}$$

where A and B are arbitrary constants. The constant $\hat{\omega}$ we can refer as the free micro-vibrations frequency

$$(\hat{\omega})^2 = \frac{\langle \partial_1 g D^{1111}\partial_1 g\rangle}{\langle g\rho g\rangle} \tag{39}$$

We look for a solution to Eqs. (37) in the form

$$u_1(x,t) = u_1^0(x)e^{i\omega t} \quad V_1(x,t) = V_1^0(x)e^{i\omega t} \tag{40}$$

Substituting (40) into (37) we obtain

$$\begin{aligned}
&\langle D^{1111}\rangle\partial_{11}u_1^0 + \langle D^{1111}\partial_1 g\rangle\partial_1 V_1^0 + \langle\tilde{\rho}\rangle\omega^2 u_1^0 = 0,\\
&\langle D^{1111}\partial_1 g\rangle\partial_1 u_1^0 + \langle \partial_1 g D^{1111}\partial_1 g\rangle V_1^0 + \langle g\rho g\rangle\omega^2 V_1^0 = 0.
\end{aligned} \tag{41}$$

Introducing the micro-vibration frequency $\hat{\omega}$ after simple manipulation, Eqs. (41) takes the form

$$\left(\frac{D^{eff}}{\langle D^{1111}\rangle} - \left(\frac{\omega}{\hat{\omega}}\right)^2\right)\partial_{11}u_1^0(x) + \frac{\langle\tilde{\rho}\rangle}{\langle D^{1111}\rangle}\omega^2\left(1 - \left(\frac{\omega}{\hat{\omega}}\right)^2\right)u_1^0(x) = 0, \tag{42}$$

where we have defining

$$D^{eff} = \langle D^{1111}\rangle - \frac{(\langle D^{1111}\partial_1 g\rangle)^2}{\langle \partial_1 g D^{1111}\partial_1 g\rangle}. \tag{43}$$

We can observe that $\hat{\omega} > \omega$ and then from Eq. (42) it follows that

(i) if $(D^{eff}/\langle D^{1111}\rangle) > (\omega/\hat{\omega})^2$ then there exist sinusoidal vibrations $u_1^0 = A\cos(kx)$, $V_1^0 = B\sin(kx)$.
(ii) if $(D^{eff}/\langle D^{1111}\rangle) > (\omega/\hat{\omega})^2$ then there exist exponential vibrations $u_1^0 = A\cosh(kx)$, $V_1^0 = B\sinh(kx)$. This case exists only for micro heterogeneous plate.

For homogeneous plate we have $\langle D^{1111}\partial g\rangle = 0$, then only sinusoidal vibration exists.

Case of sinusoidal vibrations: Substituting $u_1^0(x) = A\cos(kx)$, $V_1^0(x) = B\sin(kx)$ into Eqs. (41) and introducing the micro-vibration frequency $\hat{\omega}$ we obtain

$$\omega^2 = \frac{D^{eff}}{\langle\tilde{\rho}\rangle}k^2 + \left(\omega^2 - \frac{\langle D^{1111}\rangle}{\langle\tilde{\rho}\rangle}k^2\right)\left(\frac{\omega}{\hat{\omega}}\right)^2. \tag{44}$$

The second term in Eq. (44) describes the dispersion effect (the nonlinear relation between ω and k) due to the non-homogeneous structure of the plate under consideration.

Bearing in mind that $u_1^0(\cdot)$ and $V_1^0(\cdot)$ have to be slowly varying functions, the obtained results have a physical sense only if $kl \ll 1$. Treating kl as a small parameter we derive from Eq. (44) the formula for free vibrations frequency

$$\omega^2 = \frac{D^{eff}}{\langle\tilde{\rho}\rangle}k^2\left(1 - (kl)^2\left(\frac{\langle\partial_1 g D^{1111}\rangle}{\langle\partial_1 g D^{1111}\partial_1 g\rangle}\right)^2\right) + O(kl)^4. \tag{45}$$

Case of exponential vibrations: Substituting $u_1^0(x) = A\cosh(kx)$, $V_1^0(x) = B\sinh(kx)$ into Eqs. (41) and introducing the micro-vibration frequency $\hat{\omega}$ we obtain

$$\omega^2 = \left(\frac{\langle D^{1111}\rangle}{\langle\tilde{\rho}\rangle}\left(\frac{\omega}{\hat{\omega}}\right)^2 - \frac{D^{eff}}{\langle\tilde{\rho}\rangle}\right)k^2 + \omega^2\left(\frac{\omega}{\hat{\omega}}\right)^2. \tag{46}$$

Bearing in mind that $u_1^0(\cdot)$ and $V_1^0(\cdot)$ have to be slowly varying functions, the obtained results have a physical sense only if $kl \ll 1$. Treating kl as a small parameter we derive from Eq. (46) free vibrations frequency for case of exponential vibrations.

$$\omega^2 = \frac{D^{eff}}{\langle\tilde{\rho}\rangle}k^2\left(1 + (kl)^2\left(2\frac{\langle D^{1111}\rangle}{\langle\partial_1 g D^{1111}\partial_1 g\rangle} - \frac{\langle\partial_1 g D^{1111}\rangle}{\langle\partial_1 g D^{1111}\partial_1 g\rangle}\right)^2\right) + O(kl)^4. \tag{47}$$

Calculations of the frequency parameters $\beta(kl)$ ($\omega^2 = (E/\rho)k^2\beta(kl)$) are investigated in the framework of these cases. Diagrams of frequency parameters $\beta(kl)$ versus a dimensionless wave number $\gamma = kl$ are presented in Fig. 7. Values of parameter $\beta 1(kl)$ are obtained for the case of sinusoidal vibrations and values of parameter $\beta 2(kl)$ for the case of exponential vibrations. These values are calculated for the Poisson's ratio $\nu = 0.2$ and for volume fraction of the ribs $n = b/l = 0.2$. Diagrams in Fig. 7 show that dispersion effects are much more observable for the case of exponential vibrations.

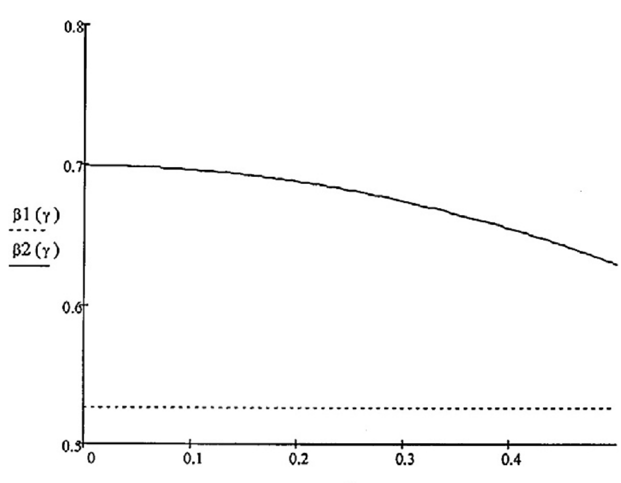

Fig. 7 – Diagrams of free vibration frequency parameters $\beta(kl)$ for a case of sinusoidal $\beta 1(kl)$ and exponential vibrations $\beta 2(kl)$ versus a dimensionless wave number $\gamma = kl$.

6. Conclusions

The obtained results justify formulating the following conclusions:

1. The modelling approach used in this contribution makes it possible to obtain 2D – model equations for the plane structure reinforced by system thin parallel ribs.
2. We can observe that the microheterogeneity of the plane structure under consideration implies the existence of dispersion effect and exponential waves.
3. The tolerance averaging approach makes it possible to replace the governing differential equations with highly oscillating and non-continuous coefficients by equations of motion involving only smooth coefficients.
4. Since the proposed model equations have smooth functional coefficients then solutions to specific problems for the plane structure under consideration can be obtained using well-known numerical methods.
5. The coefficients in the model equations depend on the volume fraction $n(x)$ of material of the ribs. In every specific case this fraction is assumed to be known. However, this fraction can be assumed as unknown if we are going to design the material structure in order to derive the required vibrations frequency.
6. The tolerance model equations describe the dispersion effect due to the microheterogeneous structure of the plate under consideration. These equations lead to the dispersion effect and to exponential vibrations which cannot be analysed in the framework of the asymptotic models.

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