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A semi-analytical solution on static analysis of circular plate exposed to non-uniform axisymmetric transverse loading resting on Winkler elastic foundation

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ABSTRACT

This paper is concerned with static analysis of functionally graded (FG) circular plates resting on Winkler elastic foundation. The material properties vary across the thickness direction so the power-law distribution is used to describe the constituent components. The differential transforms method (DTM) is utilized to solve the governing differential equations of bending of the thin circular plate under various boundary conditions. By employing this solution method, governing differential equations are transformed into recurrence relations and boundary/regularity conditions are changed into algebraic equations. In this study, the plate is subjected to uniform/non-uniform transverse load in two cases of boundary conditions (clamped and simply-supported). Some numerical examples are presented to show the influence of functionally graded variation, different elastic foundation modulus, and variation of the symmetrical transverse loads on the stress and displacement fields. Based on the results, the obtained out-plane displacement coincide with the available solution for a homogenous circular plate. It can be concluded that the applied method provides accurate results and it is easily used for static analysis of circular plates on an elastic foundation.

1. Introduction

Extensive use of circular plates in particular purposes such as bridge decks, turbine disk, thrust bearing plates and clutches, tanks, structural components for diaphragms, and deck plates in launch vehicles, engineering and spatial structures reflects the importance of circular plates. Since scientists focus on functionally graded material (FGM) to such an extent in engineering field recently, in this paper, FG circular plate is considered. FGMs are new materials, microscopically inhomogeneous continua, where continuous variation of the mechanical properties, from metal to ceramic, happens gradually without any sudden changes. For the first time in an industrial application, Japanese scientists proposed FGM for thermal barriers in aerospace structures [1].This kind of new composites can be found in aerospace structures, nuclear reactors, chemical plants, semiconductors and biomedical

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industries. Comprehensive works on static and dynamic responses of FG plates are available in the literature. Next, we briefly concentrate on some recent works related to the static behavior of FG circular plates.

Reddy et al. [2] investigated the axisymmetric bending of functionally graded circular and annular plates. They studied the bending behavior of plate based on the first order shear deformation Mindlin plate theory. In their study, the Mindlin solution of FG circular plate was obtained for the conditions where the Kirchhoff solution for thin plate was formerly known. Li et al. [3] developed the incremental load technique for solving the governing differential equation of thin circular plate bending with large deformations. In this technique, total applied load was divided into different small steps so that linear stress analysis for the plate was reasonable. Civalek [4] employed differential quadrature method (DQM) and harmonic differential quadrature method (HDQM) in analyzing static and vibration of columns as well as circular and rectangular plates. He compared accuracy of the two methods in structural analysis and showed that HDQM needs less grid points than DQM to achieve accurate results. Li and Ding [5], investigated bending of transversely isotropic circular plates, whose elastic compliance coefficients are arbitrary functions of the thickness coordinate, exposed to a transverse load as a power function of radius. Zheng and Zhong [6] investigated axisymmetric bending problem of FG circular plates under two boundary conditions, rigid slipping and elastically supported, subjected to transverse normal and shear loadings. They utilized Fourier–Bessel series as the displacement function. Civalek and Ersoy [7] studied free vibration and bending of Mindlin circular plates based on the discrete singular convolution method (DSCM) with the use of regularized Shannon's delta kernel. They obtained the frequency parameters, deflections, and bending moments and showed that the singular convolution method is an exact method. Sahraee and Saidi [8] investigated axisymmetric bending of functionally graded circular plates under uniform transverse loadings using the fourth-order shear deformation plate theory. They studied the effect of various percentages of ceramic–metal volume fractions on maximum out-plane displacement and shear stress. Their results were compared with those obtained based on the first-order shear deformation plate theory, the third-order shear deformation plate theory of Reddy and the exact three-dimensional elasticity solution and found good agreement between them. Sahraee et al. [9] analyzed bending and buckling of thick circular FG plate based third-order shear deformation plate theories. They applied the shear–free constraint on the top and bottom of the plate and obtained the static response and critical buckling loads in bending and bucking analysis of functionally graded circular plates using unconstrained shear stress theory in terms of the corresponding quantities of the homogeneous plates based on the classical plate theory. Yun et al. [10] carried out bending analysis of transversely isotropic circular plates under arbitrary symmetric transverse loads. They expanded the transverse loading as Fourier–Bessel series. In their work, the material properties varied arbitrarily along the thickness of the plate. They used the direct displacement method for obtaining the analytical solution. Chen [11] suggested an innovative technique for solving nonlinear differential equations for bending problem of a circular plate. He used a type of pseudo-linearization to obtain the final solution for large deformations of the circular plate. Alipour and Shariat [12] proposed stress analysis for axisymmetric bending of circular FG sandwich plates subjected to transversely distributed loads. They derived the governing equations based on elasticity-equilibrium-based zigzag theory. They employed a semi-analytical Maclaurin-type power-series solution.

In numerous engineering applications, the plate is continuously supported within the span. In the case where the support is linear elastic, its reaction is proportional to the local deflection of the structure (so-called Winkler's elastic foundation). Accordingly, if the plate is supported by an elastic foundation, it experiences a local deflection w, and the reaction (counter-pressure) applied by the foundation to the plate is kw where k is a proportionality coefficient called the modulus of the foundation [13]. In other words, Winkler's elastic foundation is assumed to behave linearly. It should be noted that interaction between plate and elastic foundation is a complicated issue which is not easy to be explored. In many practical engineering applications, this kind of model provides satisfied results. It is worth mentioning that the plates resting on an elastic foundation have been greatly used in modern engineering structures such as building footings, reinforced concrete pavements of high runways, foundation of deep wells, storage tanks, base of machines, aerospace, biomechanics, petrochemical, civil, mechanical, electronic, nuclear and foundation engineering. Providing the exact solution for governing equations of static behavior and dynamic response of any kind of plate in shape under various form of loading is not always feasible. So the researchers attempt to employ the semi-analytical and numerical methods when involved the problems in this field of study. For the first time, differential transformation method (DTM) was introduced by Zhou [14] for solving linear and nonlinear initial value problems in electric circuit analysis. This method is a semi-analytical-numerical technique based on Taylor series expansion developed for various types of differential equations. Differential transforms method solves a series extremely shorter and faster than high order Taylor series method. It also significantly reduces the computation cost of linear and nonlinear problems and is easily applicable. By using DTM, governing differential equations are reduced to the recursive relations together with associated boundary conditions which can be transformed to a set of algebraic equations. Furthermore, this method reduces the computational difficulties of the other methods since all the calculations can be made with a simple iterative process [15]. Another advantage of this method is exact results which can be obtained with a rapid convergence.

DTM has recently attracted the attention of scientists in various fields of engineering. Yalcin et al. [16] represented free vibration analysis of circular plates by differential transformation method. Özdemir and Kaya [17] investigated flap wise bending vibration of a rotating tapered cantilever Bernoulli–Euler beam by differential transforms method. Balkaya and Kaya [18] employed differential transforms method to predict the vibrating behavior of Euler–Bernoulli and Timoshenko beams resting on an elastic foundation (elastic soil). They showed that it is a useful tool for analytical and numerical solutions and that the solution procedure can be easily applied to governing equation of beam vibration and. Attarnejad et al. [19] utilized DTM for calculating natural frequency of a Timoshenko beam resting on two-parameter elastic foundation. Soltanizadeh [20] utilized two dimensional DTM for solving the hyperbolic telegraph equation. He carried out some numerical tests to show the advantages and disadvantages of the proposed method.

As seen in the literature above, the differential transforms method (DTM) has been used for solving a vast range of problems in different fields of engineering. To the best knowledge of the authors, no research effort has been devoted so far to find the solution of bending of a functionally graded circular plate resting on an elastic foundation by employing DTM. In our work, bending analysis of functionally graded circular plates resting on Winkler elastic foundation is carried out using differential transforms method (DTM). The plate is subjected to axisymmetric transverse load that is assumed to be represented by a power law distribution along the radial direction of the plate. This study attempts to incorporate both the effect of elastic foundation modulus and FG power index on out-plane displacement of the plate resting on elastic foundation. Also, the distributions of radial and circumferential stresses along the radius and across the thickness are obtained. The results are compared with the published literature and Finite Element Method to demonstrate the applicability and the computational efficiency of the proposed method.

2. Governing equation of bending of FG circular plate subjected to symmetric transverse load

Consider a circular plate subjected to uniform/non-uniform transverse loading while resting on an elastic foundation as shown in Fig. 1. Geometric parameters R and h are radius and thickness of the plate, respectively, and k_w is the foundation elastic modulus.

Differential equation of a circular plate subjected to the symmetric transverse load in the form of $q_0(r/R)^p$, where $p\geq0$ and is a finite even number, rested on Winkler elastic foundation, according to the classical plate theory (CPT) [13], is given as follows:

$$\nabla^4 w=\frac{q_0(r/R)^p-k_w w}{D} \tag{1}$$

where w stands for the out-plane displacement (deflection) of any point of the plate mid-surface, the radial coordinate is denoted by r, k_w is the Winkler foundation modulus, q_0 is a

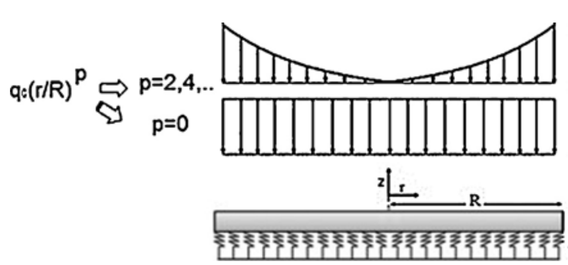

Fig. 1 – Geometric and foundation parameters of FG circular plate.

constant value, D is the flexural rigidity of plate, and ∇^4 is the bi-harmonic operator, which is defined as follows in a polar cylindrical coordinate system:

$$\nabla^4 w=\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right) \tag{2}$$

Upon substituting $\nabla^4 w$ into Eq. (2), the simplified form is obtained as

$$\frac{d^4w}{dr^4}+\frac{2}{r}\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{d^2w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}=\frac{q_0(r/R)^p-k_w w}{D} \tag{3}$$

By definition the following non-dimensionless parameters:

$$\varphi=\frac{r}{R},\quad W=\frac{w}{R},\quad K_w=\frac{k_w R^4}{D},\quad q=\frac{q_0R^3}{D}$$

The governing Eq. (1) can be rearranged in the non-dimensional form below:

$$\varphi^3\frac{d^4W}{d\varphi^4}+2\varphi^2\frac{d^3W}{d\varphi^3}-\varphi\frac{d^2W}{d\varphi^2}+\frac{dW}{d\varphi}+K_w\varphi^3W-q\varphi^{m+3}=0 \tag{4}$$

Young modulus of a functionally graded plate, $E(z)$, smoothly changes based on the power-law distribution across the thickness direction from metal to ceramic, i.e.,

$$E(z)=(E_c-E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^g+E_m \tag{5}$$

where E_m and E_c are the Young modulus of metal and ceramics, respectively, and (g is volume fraction index in which g=0 ($g\to\infty$) represents a fully ceramic (homogeneous metal) plate. Poisson's ratio is considered as a constant ratio throughout the thickness.

The differential Eq. (1) is utilized for an isotropic homogenous plate in which physical neutral surface and geometric middle surface are the same. Based on the asymmetric mechanical properties of FGM plates with respect to the middle plane, the position of the physical neutral plane (where the strain and stress are zero), is not located on the middle plane. Also, there is stretching–bending coupling effect in the governing equations of functionally graded plate for static behavior and dynamic response. By selecting the proper reference plane (which is called physical neutral surface), the governing differential equations of FG thin plates have the simple form as those of classical thin plate theory for homogeneous isotropic materials. The position of this plane (z_0) from the middle surface is introduced as follows [21]:

$$z_0=\frac{\int_{-h/2}^{h/2} zE(z)dz}{\int_{-h/2}^{h/2} E(z)dz} \tag{6}$$

where z is the direction along the thickness.

Consequently, the elastic flexural rigidity is determined as follows:

$$D=\int_{-h/2}^{h/2}\frac{(z-z_0)^2E(z)}{1-\nu^2}dz \tag{7}$$

D can be derived in terms of thickness, Poisson ratio, neutral position from middle plane, ceramic and metal Young modulus, as follows [22]:

Table 1 – The dimensional/ non-dimensional boundary/regularity conditions.

Type of boundary condition	Dimensional boundary condition	Non-dimensional boundary condition	Regularity condition (R.C.)	
			Dimensional R.C.	Non-Dimensional R.C.
Clamped edge	$w\|_{r=R}=0$ $\frac{dw}{dr}\|_{r=R}=0$	$W\|_{\varphi=1}=0$ $\frac{dW}{d\varphi}\|_{\varphi=1}=0$	$\frac{dw}{dr}\|_{r=0}=0$	$\frac{dW}{d\varphi}\|_{\varphi=0}=0$
Simply Supported	$w\|_{r=R}=0$ $M_r\|_{r=R}=-D(\frac{d^2w}{dr^2}+\frac{\upsilon}{r}\frac{dw}{dr})=0$	$W\|_{\varphi=1}=0$ $-D(\varphi\frac{d^2W}{d\varphi^2}+\upsilon\frac{dW}{d\varphi})=0$		

$$D=\frac{h}{(1-\nu^2)}\left\{\frac{E_mh^2}{12}\left(1+\frac{9}{(g+1)(g+2)(g+3)}\right)+\frac{(E_c-E_m)h^2}{4}\left(1-\frac{6g}{(g+1)(g+2)(g+3)}\right)\right\}-z_0h\frac{g(E_c-E_m)}{(g+1)(g+2)}+z_0^2\left(E_m+\frac{(E_c-E_m)}{g+1}\right) \quad (8)$$

3. Boundary and regularity conditions

Out-plane displacement (deflection), w, must satisfy the boundary conditions at the outer edge of the circular plate ($r=R$) for clamped, simply supported plate and regularity condition at the center of the circular plate. The boundary/regularity conditions can be written in terms of dimensionless deflection as shown in Table 1.

4. The definition and operation of differential transforms method

The differential transforms method (DTM) provides an analytical solution procedure in the form of polynomials to solve ordinary and partial differential equations. In this method, differential transformation of mth derivative function $f(r)$ and differential inverse transformation (DT) of $F[m]$ are respectively defined as follows:

$$F[m]=\frac{1}{m!}\left(\frac{d^m f(r)}{dr^m}\right)_{r=r_0} \quad (9)$$

where

$$f(r)=\sum_{m=0}^{\infty}F[m](r-r_0)^m \quad (10)$$

In Eq. (9), $F[m]$ is denoted as the tranformed function (T-function). The lower case and upper case letters represent the original and transformed functions, respectively, and $r=r_0$ represent any point in the domain. The function $f(r)$ is considered analytic in a domain R and is intruduced as a finite power series whose center is located at r_0. Therefore, Eq. (10) can be expressed as

$$f(r)=\sum_{m=0}^{N}F[m](r-r_0)^m \quad (11)$$

Table 2 – Fundamental theorems of one-dimensional DTM.

Original function	Transformed function
$f(r)=y(r)\pm z(r)$	$F[m]=Y[m]\pm Z[m]$
$f(r)=\lambda y(r)$	$F[m]=\lambda Y[m]$
$f(r)=y(r).z(r)$	$F[m]=\sum_{m_1=0}^{m}Y[m_1]Z[m-m_1]$
$f(r)=\frac{d^py(r)}{dr^p}$	$F[m]=\frac{(p+m)!}{m!}Y[m+p]$
$f(r)=r^p$	$F[m]=\delta(m-p)=\begin{cases}0 & \text{if} m\neq p\\ 1 & \text{if} m=p\end{cases}$

in which N determines the convergence of non-dimensional deflection which implies that $f(r)=\sum_{m=N+1}^{\infty}F[m](r-r_0)^m$ is negligible. By combining Eqs. (9) and (10), the following relation is obtained:

$$f(r)=\sum_{m=0}^{\infty}\frac{(r-r_0)^m}{m!}\left(\frac{d^m f(r)}{dr^m}\right)_{r=r_0} \quad (12)$$

As is seen, the concept of differential transformation is based upon the Taylor series expansion. From the definitions of DTM in Eqs. (9) and (10), fundamental theorems of differential transforms method [23–25] can be performed that are listed in Table 2.

5. Application of DTM in the governing equations and boundary/regularity conditions

5.1. Differential transformation of the governing equation

The non-dimensional form of the differential equation of circular plates resting on Winkler elastic foundation (Eq. (5)) can be solved using the above differential transforms theorems at r_0=0. Let $W[m]$ to be the differential transform of $w(r)$; Applying Table 1, the differential transform version of Eq. (3) can be exploited from the solution approach below:

$$\begin{aligned}&\sum_{m_1=0}^{m}\delta(m_1-3)(4+m-m_1)(3+m-m_1)(2+m-m_1)(1+m-m_1)W[4+m-m_1]\\&+2\sum_{m_1=0}^{m}\delta(m_1-2)(3+m-m_1)(2+m-m_1)(1+m-m_1)W[3+m-m_1]\\&-\sum_{m_1=0}^{m}\delta(m_1-1)(2+m-m_1)(1+m-m_1)W[2+m-m_1]+(1+m)W[m+1]\\&+K_w\sum_{m_1=0}^{m}\delta(m_1-3)W[m-m_1]-q\delta(m-(p+3))=0 \text{for} m\geq 0\end{aligned} \quad (13)$$

By utilizing the last theorem of Table 1 and simplified form of the equation above, the following recurrence relation is acquired:

$$W[m+1] = \frac{1}{(m+1)^2(m-1)^2}(q\delta(m-(p+3)) - K_w W[m-3]) \quad \text{for} \quad m \geq 3 \tag{14}$$

Eq. (14) can be classified according to the last theorem in Table 1 as

$$W[m+1] = \frac{1}{(m+1)^2(m-1)^2}[q - K_w W[m-3]] \quad \text{for } m = p+3 \tag{15a}$$

$$W[m+1] = \frac{1}{(m+1)^2(m-1)^2}(-K_w W[m-3]) \quad \text{for } m > p+3 \tag{15b}$$

Obviously, letting m=p+3 in the recurrence relation (15a), it can be more specific for various power values in function of transverse loads:

$$\begin{aligned}
&if\ p = 0 \Rightarrow W[4] = \frac{1}{(4)^2(2)^2}(q - K_w W[0]) \\
&if\ p = 2 \Rightarrow W[6] = \frac{1}{(6)^2(4)^2}(q - K_w W[2]) \\
&if\ p = 4 \Rightarrow W[8] = \frac{1}{(8)^2(6)^2}(q - K_w W[4]) \\
&if\ p = 6 \Rightarrow W[10] = \frac{1}{(10)^2(8)^2}(q - K_w W[6]) \\
&if\ p = 8 \Rightarrow W[12] = \frac{1}{(12)^2(10)^2}(q - K_w W[8])
\end{aligned} \tag{16}$$

It should be noted that p=0 corresponds to uniform transverse loads. It is deduced that, depending to p, only W [p+4] gets influence from the constant q.

5.2. *Differential transformation of boundary conditions*

The differential transformed of boundary and regularity conditions are shown in Table 3 using the DTM rules listed in Table 3.

6. Solution procedure of the problem

To derive deflection equation for the circular FG plate resting on Winkler elastic foundation, the recurrence relation (14) and boundary/regularity conditions in Table 3 are imposed simultaneously. As a result, for m=0,1 DT of deflection function W[0], W[2] is expressed in terms of geometric parameters and mechanical properties of the plate. For m=2, it yields W[3]=0. It is possible to evaluate each T-function W[m] in terms of two terms namely W[0], W[2]. To show this, we calculate W[4],… W [10] from the recurrence relation (14) for m=4,…,10 as follows:

For p=0 (or uniform transverse loading):

$$\begin{aligned}
&W[4] = \frac{q - W[0]K_w}{4^2 \times 2^2}, \quad W[5] = \frac{-W[1]K_w}{5^2 \times 3^2}, \quad W[6] = \frac{-W[2]K_w}{6^2 \times 4^2} \\
&W[7] = \frac{-W[3]K_w}{7^2 \times 5^2}, \quad W[8] = \frac{-W[4]K_w}{8^2 \times 6^2}, \quad W[9] = \frac{-W[5]K_w}{9^2 \times 7^2}, \\
&W[10] = \frac{-W[6]K_w}{10^2 \times 8^2}
\end{aligned} \tag{17}$$

For $p = 2$

$$\begin{aligned}
&W[4] = \frac{-W[0]K_w}{4^2 \times 2^2}, \quad W[5] = \frac{-W[1]K_w}{5^2 \times 3^2}, \quad W[6] = \frac{q - W[2]K_w}{6^2 \times 4^2} \\
&W[7] = \frac{-W[3]K_w}{7^2 \times 5^2}, \quad W[8] = \frac{-W[4]K_w}{8^2 \times 6^2}, \quad W[9] = \frac{-W[5]K_w}{9^2 \times 7^2}, \\
&W[10] = \frac{-W[6]K_w}{10^2 \times 8^2}
\end{aligned} \tag{18}$$

For the sake of brevity, the procedure for finding T-function W[k] corresponding to p=4, 6, 8… are omitted here. From the relations above, it can be concluded that W[5],W[7], W[9] get zero values. In general, for odd values of k, W[k] equals zero. Therefore, by expanding Eq. (11) as non-dimensional out-plane displacement W in terms of φ as non-dimensional parameter, we have respectively for p=0, 2, 4, 6 and 8:

$$\begin{aligned}
W(\varphi) &= a + b\varphi^2 + \frac{q - aK_w}{4^2 \times 2^2}\varphi^4 - \frac{bK_w}{6^2 \times 4^2}\varphi^6 - \frac{(q - aK_w)K_w}{8^2 \times 6^2 \times 4^2 \times 2^2}\varphi^8 \\
&\quad + \frac{bK_w{}^2}{10^2 \times 8^2 \times 6^2 \times 4^2}\varphi^{10} + \ldots \\
W(\varphi) &= a + b\varphi^2 - \frac{aK_w}{4^2 \times 2^2}\varphi^4 + \frac{q - bK_w}{6^2 \times 4^2}\varphi^6 + \frac{aK_w{}^2}{8^2 \times 6^2 \times 4^2 \times 2^2}\varphi^8 \\
&\quad - \frac{(q - bK_w)K_w}{10^2 \times 8^2 \times 6^2 \times 4^2}\varphi^{10} + \ldots \\
W(\varphi) &= a + b\varphi^2 - \frac{aK_w}{4^2 \times 2^2}\varphi^4 - \frac{bK_w}{6^2 \times 4^2}\varphi^6 + \frac{8^2 q + aK_w{}^2}{8^2 \times 6^2 \times 4^2 \times 2^2}\varphi^8 \\
&\quad + \frac{bK_w{}^2}{10^2 \times 8^2 \times 6^2 \times 4^2}\varphi^{10} + \ldots \\
W(\varphi) &= a + b\varphi^2 - \frac{aK_w}{4^2 \times 2^2}\varphi^4 - \frac{bK_w}{6^2 \times 4^2}\varphi^6 + \frac{aK_w{}^2}{8^2 \times 6^2 \times 4^2 \times 2^2}\varphi^8 \\
&\quad + \frac{24^2 q + bK_w{}^2}{10^2 \times 8^2 \times 6^2 \times 4^2}\varphi^{10} + \ldots \\
W(\varphi) &= a + b\varphi^2 - \frac{aK_w}{4^2 \times 2^2}\varphi^4 - \frac{bK_w}{6^2 \times 4^2}\varphi^6 + \frac{aK_w{}^2}{8^2 \times 6^2 \times 4^2 \times 2^2}\varphi^8 \\
&\quad + \frac{bK_w{}^2}{10^2 \times 8^2 \times 6^2 \times 4^2}\varphi^{10} \\
&\quad + \frac{1}{(12)^2(10)^2}\left(q - \frac{aK_w{}^3}{8^2 \times 6^2 \times 4^2 \times 2^2}\right)\varphi^{12} + \ldots
\end{aligned} \tag{19}$$

where W[0] and W[2] are introduced as a and b, respectively.

As mentioned before, DT method implies an iterative procedure to obtain the high-order Taylor series solution of differential equations. The Taylor series method requires a long computational time for large orders, whereas one advantage of employing DTM in solving differential equations is a fast convergence rate and a small calculation error [23]. By neglecting the terms greater than N, we have:

Table 3 – Transformed conditions for clamped and simply supported circular plate.

Boundary/regularity condition	DT form
Clamped edge	$\begin{cases} \sum_{m=0}^{N} W[m] = 0 \\ \sum_{m=0}^{N} mW[m] = 0 \end{cases}$
Simply supported	$\begin{cases} \sum_{m=0}^{N} W[m] = 0 \\ \sum_{m=0}^{N} m(m-1+\nu)W[m] = 0 \end{cases}$
Regularity condition	W[1]=0

$$W(\varphi)=\sum_{m=0}^{N}W[m]\varphi^{m} = W[0]\varphi^{0}+W[1]\varphi^{1}++W[2]\varphi^{2}+\ldots W[N]\varphi^{N} \tag{20}$$

6.1. Convergence and correctness of the solution method

In order to show that differential transform method is an effective and reliable tool for calculating the out-plane displacement (deflection) of circular plate resting on elastic foundation, homogeneous isotropic clamped circular plate subjected to uniform transverse loading is considered.

6.1.1. Clamped circular plate

Using the transformed boundary and regularity conditions in Table 3, W[0] and W[2] are determined as follows:

$$W[0]=\frac{576q+0.25qK_w-(512qK_w^2-qK_w^3)/14745600}{36864+384K_w+0.3K_w^2-(512K_w^3-K_w^4)/14745600} \tag{21a}$$

$$W[2]=\frac{-1152q+qK_w}{36864+384K_w+0.3K_w^2-(512K_w^3-K_w^4)/14745600} \tag{21b}$$

Using Eqs. (21a) and (21b) in (19) for p=0 and assuming (K_w=0), the non-dimensional out-plane displacement function $W(\varphi)$ is achieved as

$$W(\varphi)=\frac{q}{64}-\frac{q\varphi^2}{32}+\frac{q\varphi^4}{64} \tag{22}$$

Substituting the non-dimensional variables $\varphi=r/R$, $W=w/R$, $q=q_0R^3/D$, we get:

$$w=\frac{q_0}{64D}(R^2-r^2)^2 \tag{23}$$

This deflection equation is exactly the same as the one reported by Timoshenko and Woinowsky [26].

6.1.2. Simply supported circular plate

For simply supported circular plate, W[0] and W[2] are determined as follows:

$$W[0]=\frac{576(\nu+5)q+0.25(\nu+9)qK_w-(512(\nu+13)qK_w{}^2-(\nu+17)qK_w{}^3)/14745600}{36864(1+\nu)+384(5+\nu)K_w+0.3(9+\nu)K_w{}^2-(512(\nu+13)K_w{}^3-(\nu+17))K_w{}^4)/14745600} \tag{24a}$$

$$W[2]=\frac{-1152(\nu+3)q+(\nu+7)qK_w}{36864(1+\nu)+384(5+\nu)K_w+0.3(9+\nu)K_w{}^2-(512(\nu+13)K_w{}^3-(\nu+17)K_w{}^4)/14745600} \tag{24b}$$

By substituting relations (24a) and (24b) into (20) and assuming (K_w=0), non-dimensional deflection function is obtained as

$$W(\varphi)=\frac{5q_0R^3+q_0R^3\nu}{64D(1+\nu)}-\frac{\varphi^2(3q_0R^3+q_0R^3\nu)}{32D(1+\nu)}+\frac{q_0R^3\varphi^4}{64D} \tag{25}$$

Finally, the deflection equation can be summarized as follows:

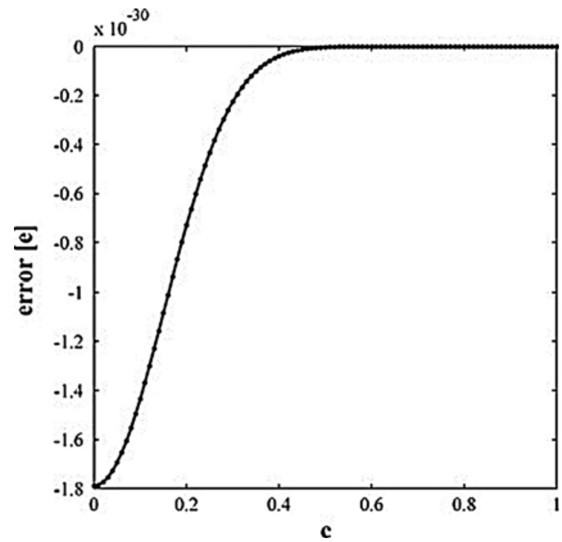

Fig. 2 – Evaluation of error in the interval 0≤c≤1.

$$w=\frac{q_0(R^2-r^2)}{64D}\left(\frac{(5+\nu)}{(1+\nu)}R^2-r^2\right) \tag{26}$$

The equation above for out-plane displacement of circular isotropic homogenous plates is exactly the same as the one reported by Timoshenko and Woinowsky [26].

7. Numerical results and discussion

7.1. Error analysis of the solution method

To indicate the accuracy of the solution approach, error of the solution method (DTM) is evaluated according to Taylor's theorem in the following form [27]:

$$e=\frac{W^{N+1}(c)}{(N+1)!}(\varphi-\varphi_0)^{N+1} \tag{27}$$

where c is an arbitrary number in the interval $\varphi_0\le c\le\varphi$. In this paper, φ_0=0,=1.

Based on the proposed formulation, a MATLAB program is developed to investigate the accuracy of the presented procedure. The calculations presented in the following examples adopt a value of h=0.01 m, R=0.6 m, q_0=0.1 MPa. The FGM circular plate is considered here to consist of Alumina (Al_2O_3) as the ceramic ingredient and Aluminum as the metallic one. Young's moduli of Alumina and Aluminum are E_c=380 GPa, E_m=70 GPa, respectively, whereas Poisson's

Table 4 – Absolute value of non-dimensional central deflection (W_1) of ceramic, FG and homogenous isotropic circular plates for different Winkler foundation modulus [k_w(MPa/m)].

k_w	Ceramic (g=0)		FGM (g=1)		FGM (g=10)		FGM (g=100)		Metal	
	C	S	C	S	C	S	C	S	C	S
0	0.0097	0.0395	0.0195	0.0793	0.0324	0.1320	0.0470	0.1915	0.0526	0.2146
5	0.0082	0.0221	0.0141	0.0306	0.0198	0.0358	0.0243	0.0386	0.0257	0.0393
10	0.0070	0.0153	0.0110	0.0186	0.0141	0.0201	0.0161	0.0206	0.0167	0.0207
20	0.0055	0.0093	0.0076	0.0102	0.0088	0.0104	0.0093	0.0102	0.0095	0.0101
50	0.0033	0.0041	0.0038	0.0041	0.0039	0.0039	0.0038	0.0037	0.0038	0.0036
100	0.0019	0.0020	0.0019	0.0019	0.0019	0.0018	0.0018	0.0017	0.0018	0.0017

Table 5 – Absolute value of non-dimensional central deflection (W_1) of FG circular plates for different values g, p, k_w=0.

p	g=0		g=1		g=5		g=10	
	C	S	C	S	C	S	C	S
0	0.0097	0.0395	0.0195	0.0793	0.0295	0.1202	0.0324	0.1320
2	0.0022	0.0121	0.0043	0.0243	0.0066	0.0368	0.0072	0.0404
4	8.0822e-4	0.0058	0.0016	0.0116	0.0025	0.0176	0.0027	0.0193
6	3.8795e-4	0.0034	7.7832e-4	0.0068	0.0012	0.0103	0.0013	0.0113
8	2.1553e-4	0.0022	4.3240e-4	0.0044	6.5526e-4	0.0067	7.1939e-4	0.0074

ratio remains constant υ=0.3. In this paper, we use convergence test to confirm (N=4) is sufficient to get precise values when determining the deflection of a circular plate exposed to uniform transverse loading, p=0. Also, for other values of p, more terms are sufficient to achieve the converged results. For instance, for p=2, 6 terms are needed to provide the accurate solution. In case of non-zero elastic foundation, $k_w \neq 0$, with N=10 convergence is obtained. For N=10, Therefore, the absolute value of error can be obtained from Eq. (27) as follows:

$$e = \frac{W^{11}(c)}{(11)!} \tag{28}$$

The error is evaluated and graphical results are depicted in Fig. 2. As it is shown, the maximum error value in this interval occurs at c=0. The error is reduced during the interval so that at c=1 then reaches to the infinitesimal value. Accordingly, it can be concluded that this method is very precise.

However, the present results indicate that only the summation of a limited number of terms is required to achieve the converged results. Also, it is worth mentioning that the computational time on a standard PC is less than 3 s for these numerical examples.

7.2. Bending analysis

For different values of Winkler foundation modulus, absolute value of non-dimensional central deflection of ceramic, FGM and metallic isotropic circular plates are presented in Table 4 for clamped and simply-supported as denoted by C and S, respectively. It can be observed that the plate experiences less deflection under clamped boundary condition when compared to simply-supported case. This difference reduces with increasing the elastic foundation modulus k_w.

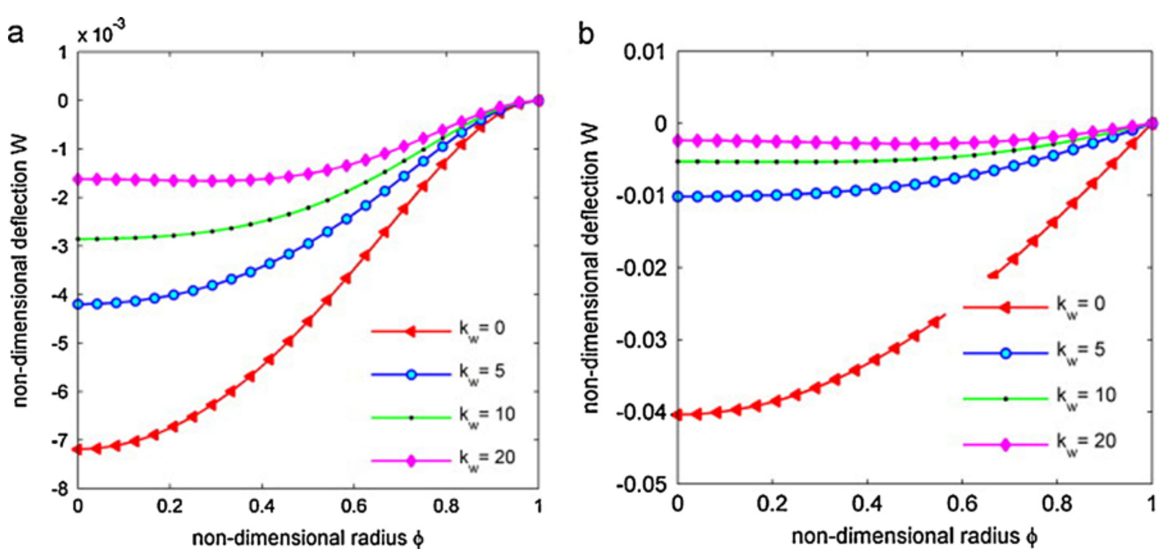

Fig. 3 – Non-dimensional deflection of FG circular plates along non-dimensional radius direction for p=2 and different values of k_w, in the case of g=10, (a) for clamped and (b) for simply supported boundaty condition.

To study the influence of power load index, p, on central deflection of FG plate, Table 5 is presented. Increasing the values of m decreases the non-dimensional central deflection (W_1). This is caused by reduction in the resultant transverse load. To explain more, the resultant transverse load, that is cited here as F_v, is derived as follows:

$$F_v = \int_0^R q_0\left(\frac{r}{R}\right)^p 2\pi r dr = \frac{2}{p+2} q_0 \pi R^2 \tag{29}$$

$$F_v = \begin{cases} p=0 & q_0\pi R^2 \\ p=2 & \frac{1}{2}q_0\pi R^2 \\ p=4 & \frac{1}{3}q_0\pi R^2 \\ & \vdots \end{cases} \tag{30}$$

For more numerical examples, Figs. 3 and 4 are presented to illustrate variations of the non-dimensional deflection versus the non-dimensional radius direction of FGM circular plates under clamped and simply boundary condition for various values of Winkler foundation modulus k_w and corresponding to p=2,6. It is seen that, for a constant power load index p, increase in Winkler foundation modulus in the range from zero up to 20 MPa/m, significantly restricts deflection of the plate. One remarkable result indicates that maximum deflection occurs in a location other than the center of the plate.

7.3. *Stress analysis*

One of the important issues in static analysis of the plates is determination of stress components. Following to obtaining the deflection function in Eq. (19), one can find relations which

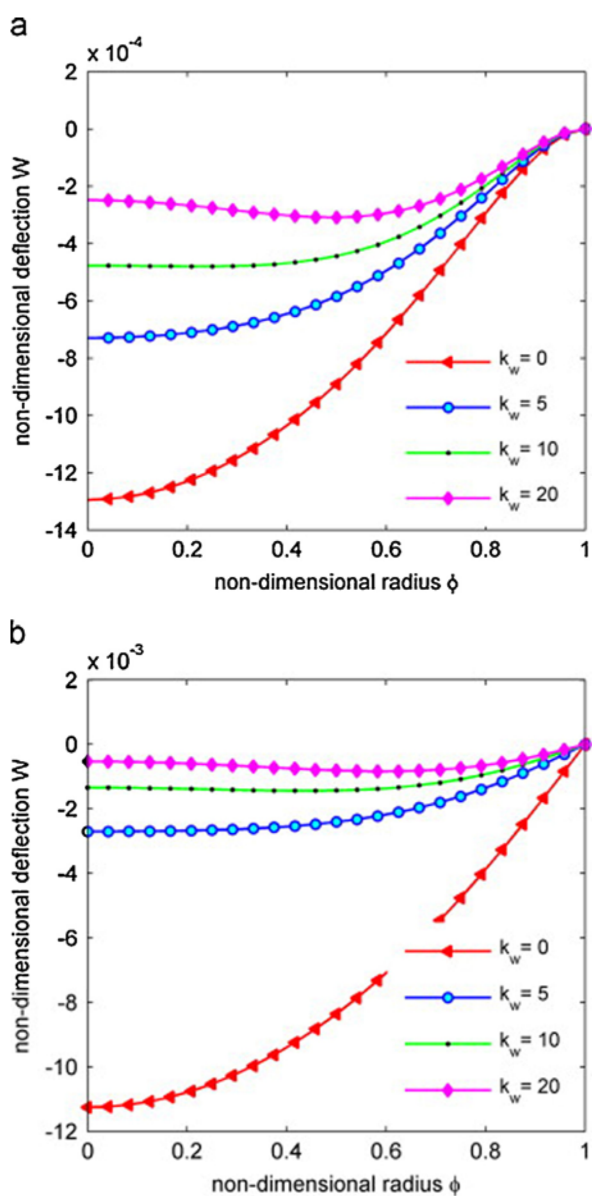

Fig. 4 – Non-dimensional deflection of FG circular plates along non-dimensional radius direction for p=6 and different values of k_w in the case of g=10, (a) for clamped edge and (b) for simply supported.

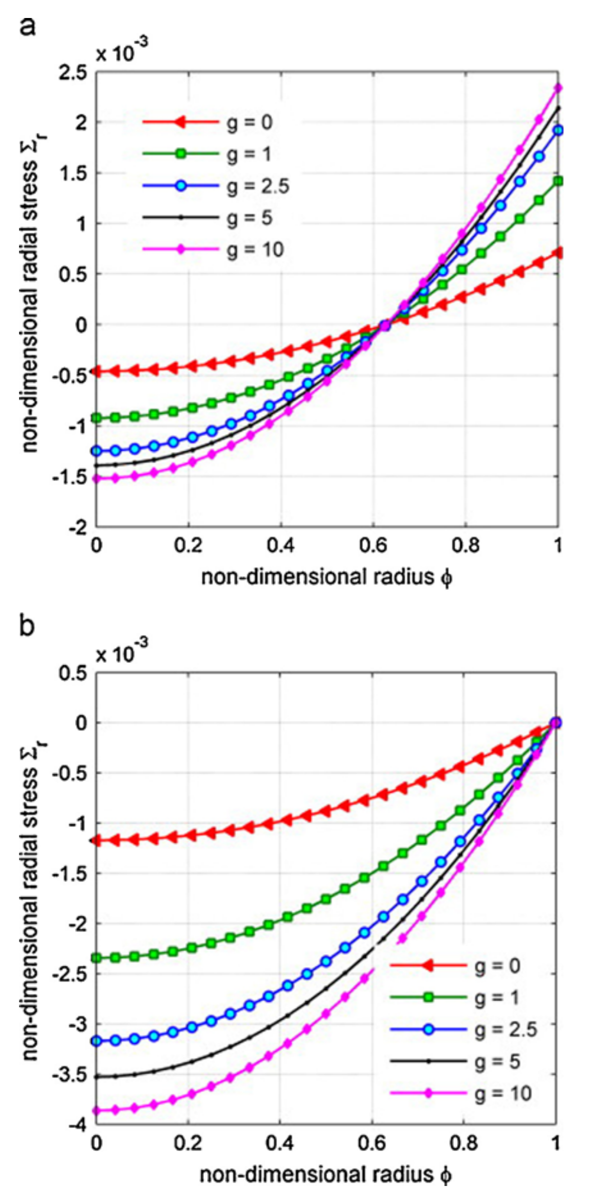

Fig. 5 – Non-dimensional radial stress of FG circular plate along non-dimensional radius with p=0, k_w=0 and ξ=0.5, (a) for clamped plate and (b) for simply supported circular plate.

describe the radial and circumferential stresses in circular plates using appropriate derivatives with respect to the deflection function as follows:

$$\sigma_r(r,z) = -\frac{zE(z)}{1-\nu^2}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) \tag{31a}$$

$$\sigma_\theta(r,z) = -\frac{zE(z)}{1-\nu^2}\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) \tag{31b}$$

in which σ_γ and σ_θ are radial and circumferential stresses, respectively. Upon substituting Eq. (20) and putting zero for the odd values of T-function $W[k]$ as mentioned before, expressions of the radial and circumferential stresses can be obtained as

$$\sigma_r(r,z) = -\frac{zE(z)}{1-\nu^2}\sum_{k=0}^{N} 2k(-1+\upsilon+2k)W[2k]r^{2k-2} \tag{32}$$

$$\sigma_\theta(r,z) = -\frac{zE(z)}{1-\nu^2}\sum_{k=0}^{N} 2k(1-\upsilon+2k\upsilon)W[2k]r^{2k-2} \tag{33}$$

In this stage, to show the capabilities of the presented method, distribution of stress components along the radial direction across the thickness are illustrated in an FG circular plate for various elastic foundation modulus, gradient index materials, power load indices and two boundary conditions. By introducing the non-dimensional parameter $\Sigma_\gamma=\sigma_\gamma/E_c$, $\Sigma_\theta=\sigma_\theta/E_c$, $\xi=z/h$ and assuming that the elastic foundation modulus is zero (k_w=0) and thickness ratio is ξ=0.5, the results are presented in non-dimensional form in Fig. 5 for different values of the volume fraction index g from zero to 10. The plate is subjected to a uniform transverse load under clamped and simply supported boundary conditions. It is observed that

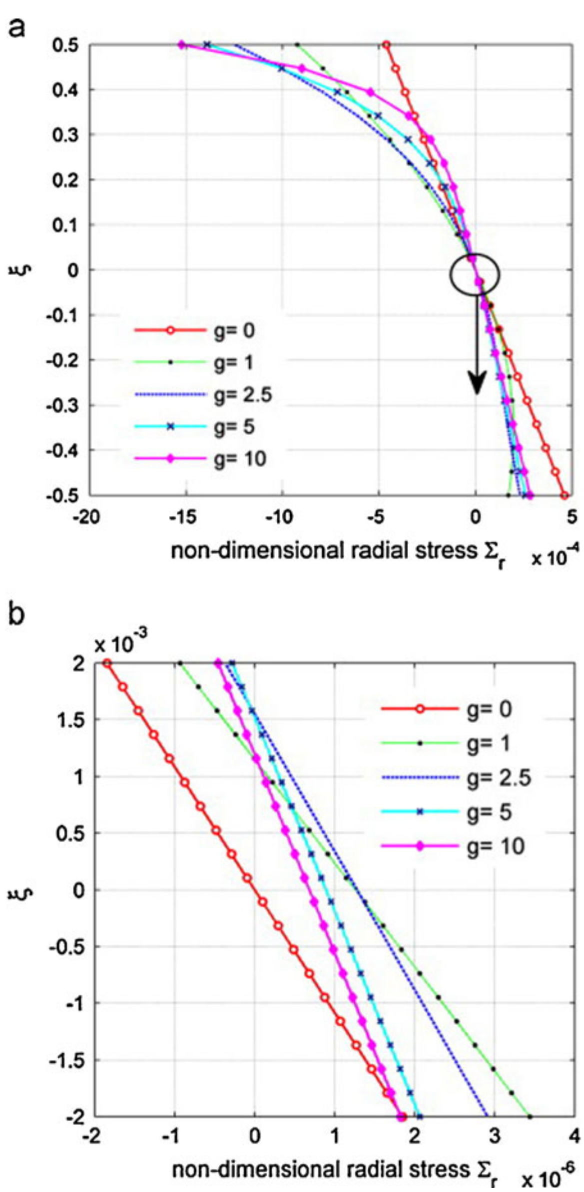

Fig. 6 – Non-dimensional radial stress of FG plate across the non-dimensional thickness under clamped edge, with p=0, k_w=0 and φ=0.

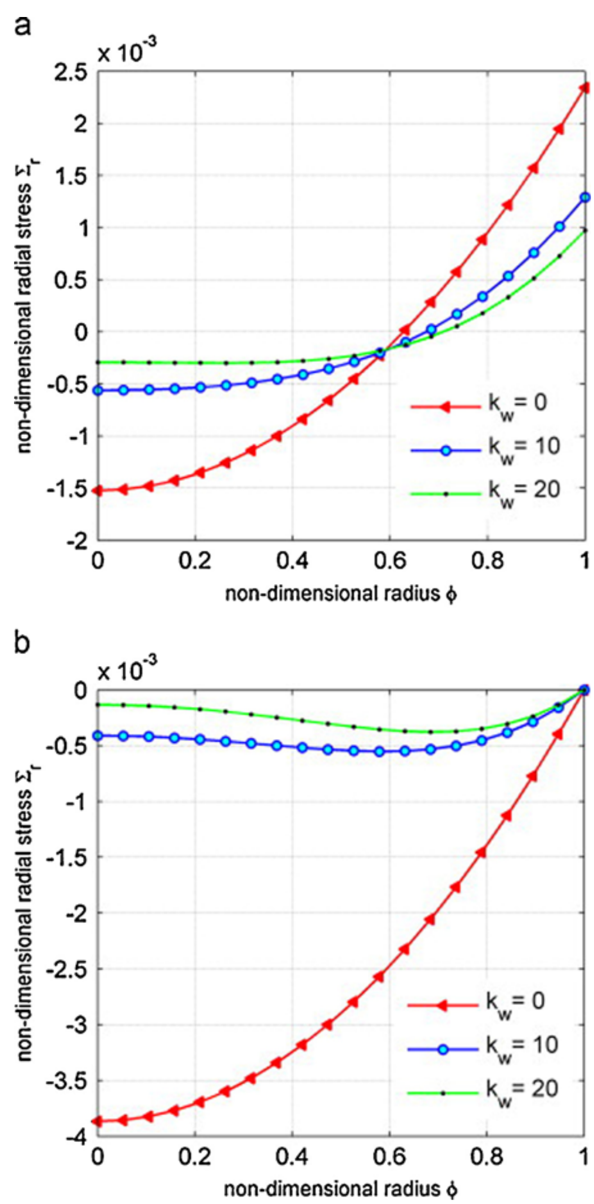

Fig. 7 – Non-dimensional radial stress of FGM circular plates along non-dimensional radius with p=0, g=10 and ξ=0.5, (a) for clamped circular plate and (b) for simply supported circular plate.

radial and circumferential stresses increase with the larger amount of volume fraction index, g, due to the reduction of flexural rigidity that enhances the deflection and curvature.

The asymmetric mechanical properties of FG plates with respect to the middle plane causes the physical neutral plane not to be located on the middle plane. Consequently, stress components of FG plate in the middle surface would not be zero as it can be observed in Fig. 6 unless in the case of g=0 corresponsin to a homogeneous isotropic ceramic plate. As is shown, the non-dimensional radial stress across the FGM thickness is not linear ($g\neq0$) whereas it is linear for the ceramic one (g=0). To show the effect of elastic foundation on the radial stress variation along r and z directions for an FG clamped plate, Figs. 7–9 are depicted for different values of the Winkler foundation modulus k_w[MPa/m]. As it shown in Fig. 7 the values of radial stress are significantly reduced by increasing the foundation modulus. Variation of radial stress across the dimensionless thickness of the clamped FG plate are shown for different values of the Winkler foundation modulus for g=10 in the center of the plate in Fig. 8. It can be seen that stress of FG plate would not be zero in the middle plane for nonzero volume fraction FG index. In addition, dimesionless radial stress deceases effectively with increase in the amount of K_w.

In Fig. 9, variation of non-dimensional radial stress in middle plane of FG circular plate is plotted for three different

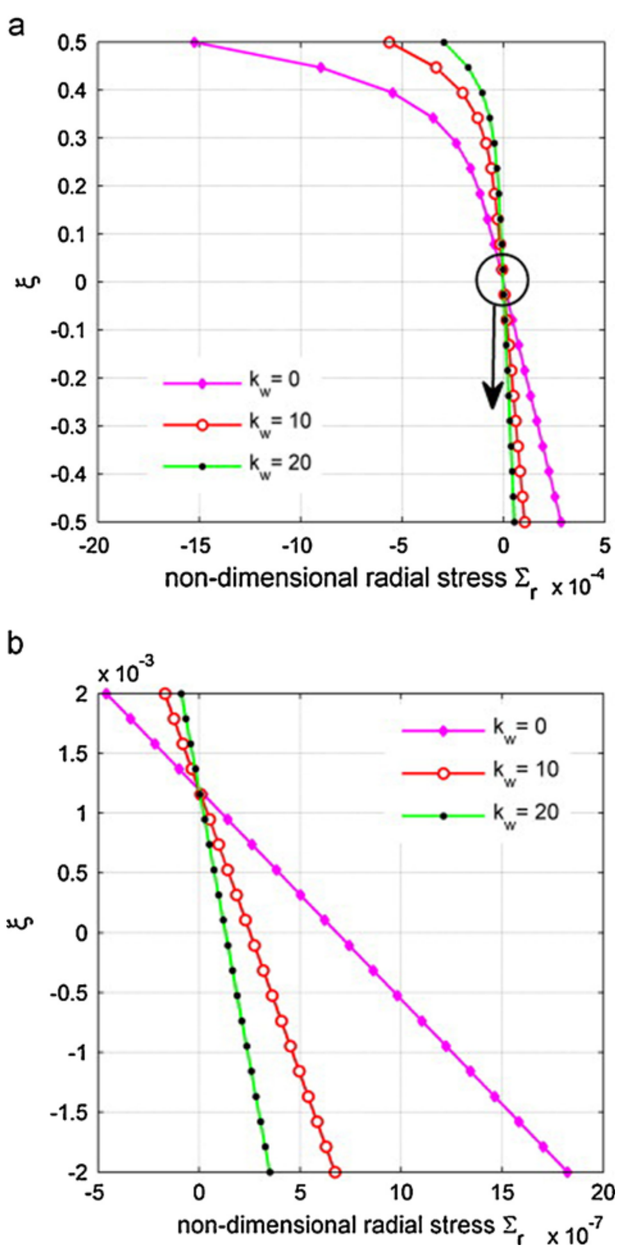

Fig. 8 – Non-dimensional radial stress of FG circular clamped plate across non-dimensional thickness(ξ), with g=10 and φ=0.

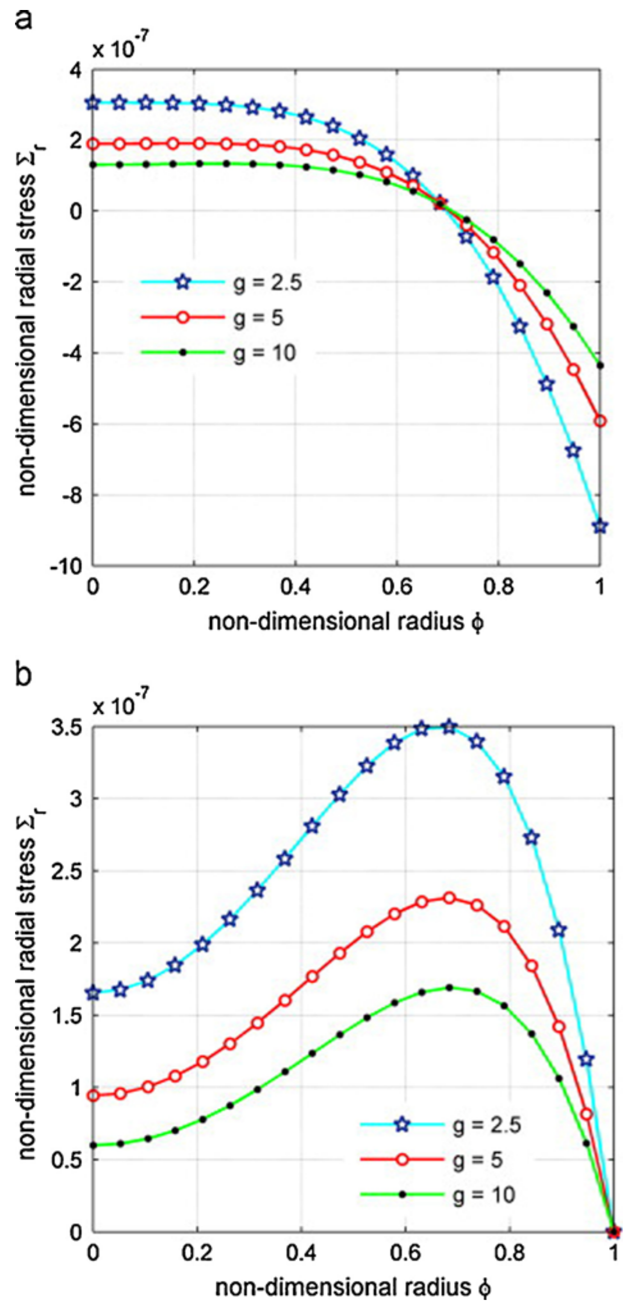

Fig. 9 – Non-dimensional radial stress of FG circular plates along non-dimensional radius in the middle plane (ξ=0) with k_w=20, (a) for clamped circular plate and (b) for simply supported circular plate.

sets of volume fraction index g (g=2.5, 5, 10). As it is seen, due to asymmetrical mechanical properties respect to middle surface, the radial stress is not zero for various values of FG volume index g.

To illustrate the influence of transverse load power index, p, on radial and circumferential stresses along radial direction, Figs. 10 and 11 are depicted. As observed, the point, at which non-dimensional radial and circumferential stresses are minimum, approaches to the outer edge as m increases. Moreover, the stresses reduce due to decrease in the resultant transverse load by increasing p.

In order to validate the present solutions, we compare them with the results taken from reference [2] and reference [5], as depicted Table 6. The following FGM is used in this case:

$$E(z) = E_m\left(\frac{h-2z}{2h}\right)^g + E_c\left(1-\left(\frac{h-2z}{2h}\right)^g\right), \quad \nu = \text{constant} \tag{34}$$

In which the numerical values are [2,5]
$q_0 = 1\text{MPa}, R = 0.1\text{m}, h = 0.03m, W_1 = w/w_1(0),$
$w_1(0) = q_0R^4/64D^{x204e;}, D^{x204e;} = E_ch^3/12(1-\nu^2), \nu = 0.288,$
$E_m = 110.25\text{GPa}, E_c = 278.41\text{GPa},$

where $w_1(0)$ is the central deflection of a homogeneous isotropic plate subjected to uniform transverse load under clamped boundary condition. As is seen, the results are in full agreement with the ones obtained in the reference papers.

To show the correctness and validity of DTM results, central deflection of homogenous isotropic plate is compared

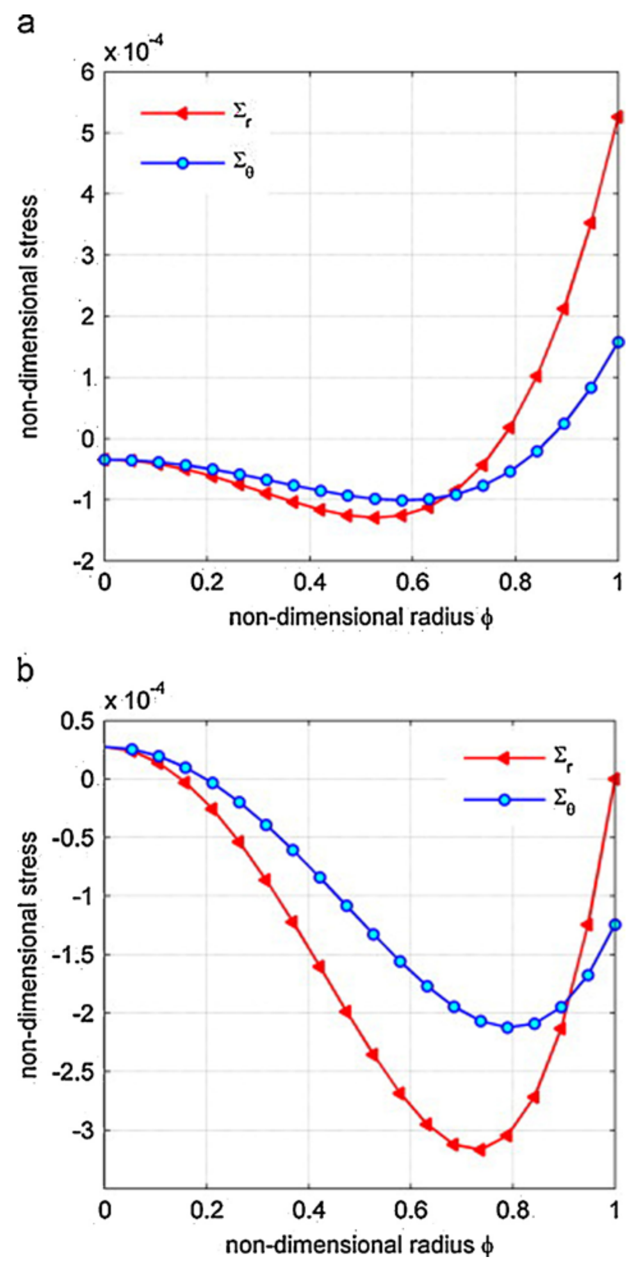

Fig. 10 – Non-dimensional radial and circumferential stresses of FG circular plates along non-dimensional radius with p=2, ξ=0.5, k_w=10 and g=10 (a) for clamped and (b) for simply supported boundary conditions.

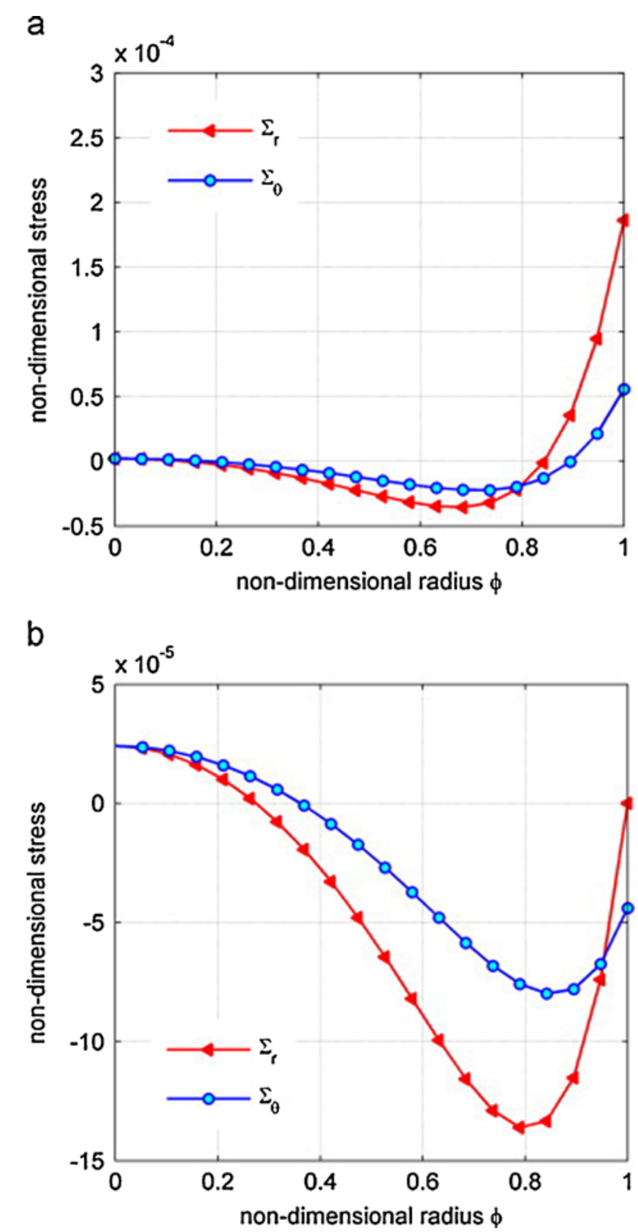

Fig. 11 – Non-dimensional radial and circumferential stresses of FG circular plates along non-dimensional radius with p=6, ξ=0.5, k_w=10 and g=10 (a) for clamped and (b) for simply supported boundary condition.

Table 6 – Absolute non-dimensional central deflection W_1 of FG clamped circular plate.

Source	g									
	0	2	4	8	10	50	10^2	10^3	10^4	10^5
Present	2.5253	1.3882	1.2690	1.1692	1.1427	1.0344	1.0177	1.0018	1.0002	1.0000
Ref. [2]	2.525	1.388	1.269	1.169	1.143	1.034	1.018	1.002	1.000	1.000
Ref. [5]	2.525	1.388	1.269	1.169	1.143	1.034	1.018	1.002	1.001	1.001

Table 7 – Absolute non-dimensional central deflection (W_1) of homogenous isotropic circular plate (aluminum) for different values of k_w (MPa/m), p=0.

Boundary condition	k_w=0		k_w=5		k_w=10		k_w=50		k_w=100	
	DTM	FEM	DTM	FEM	DTM	FEM	DTM	FEM	DTM	FEM
C	0.0526	0.0523	0.0257	0.0257	0.0167	0.0166	0.0038	0.0038	0.0018	0.0018
S	0.2146	0.2145	0.0393	0.0393	0.0207	0.0207	0.0036	0.0037	0.0017	0.0018

Table 8 – Comparison of absolute central deflections (W_1D/q_0R^4) of a circular plate with clamped edge, g=0, k_w=0.

h/a	Source	
	DTM	DSCM [7]
0.01	0.01563	0.01562
0.02	0.01561	0.01563
0.05	0.01564	0.01565

with the results acquired by finite element method (FEM) using ABAQUS 6.10 under two boundary conditions as depicted in Table 7. In the FE results, 4337 Standard-Quadratic 3D stress elements were used. The obtained results were compared with FEM ones that gave good agreement between them.

Table 8 compares the present results for central deflections (W_1D/q_0R^4) with the numerical findings of discrete singular convolution method (DSCM) reported by Civalek and Ersoy [7] for various thickness-to-radius ratios. A good agreement is seen between the results.

8. Conclusions

In this study, the applicability of differential transformation method (DTM) to investigate the static analysis of functionally graded plates resting on Winkler foundation subjected to variable symmetric transverse loading is investigated. This method provides a semi-analytical solution which is able to consider the influence of load power index on the static response of FG plate too. In order to evaluate the validity and high-performance of this method, some numerical examples are presented. The obtained results exactly match with the results of classical plate theory (CPT) for homogeneous isotropic circular plates. Furthermore, the obtained results in FG plates are compared with the ones in the literature. It is shown that the gradient of material properties has a great effect on stress distribution and bending response of uniformly loaded FG circular plates. Moreover, the influence of elastic foundation on restricting the amount of stresses and bending results is studied under simply supported and clamped edges. Consequently, it is observed that stress and out-plane displacement of a plate can be easily controlled by changing the elastic foundation modulus and material gradient index. It is also found that variation of volume fraction index of FG circular plates has a remarkable influence on static behavior of plate. Moreover, elastic foundation effectively reduces bending deflections and stresses of circular plates.

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