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Random eigenvibrations of elastic structures by the response function method and the generalized stochastic perturbation technique

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This paper addresses the important question in structural analysis how to efficiently model the eigenvibrations of the spatial structures with random physical and/or geometrical parameters. The entire computational methodology is based on the traditional Finite Element Method enriched with the stochastic perturbation technique in its generalized nth order approach, while the computational implementation is performed by the use of the academic FEM software in conjunction with the symbolic algebra computer system MAPLE. Contrary to the previous straightforward solution techniques, now the response function method is applied to compute any order probabilistic moments and coefficients of the structural eigenvalues. The response function is assumed in the polynomial form, the coefficients of which are computed from the several solutions of the deterministic problem around the mean value of the given input random parameter. This method is illustrated with the stochastic eigenvibrations of the simple single degree of freedom system and small steel tower modelled as the 3D truss structure with random mass density and Young modulus. This technique may find its wide application in reliability analysis of the real existing engineering structures using the commercial Finite Element Method packages as well as the other discrete computational techniques like the Finite Difference Method at least.

Keywords: *stochastic dynamics, Stochastic Finite Element Method, response function method, stochastic perturbation*

1. Introduction

The analysis of structures with random parameters plays an important role in structural design, optimization and reliability modelling. It results from various and numerous sources of randomness like manufacturing processes, static fracture and dynamically driven fatigue of the structural elements, stochastic degradation of material and geometrical parameters of those elements. The important role play here also stochastic vibrations resulting from the possible earthquakes and wind induced vibrations, car accidents and at least but not last, some weather influence on the structures like ice and snow coverage in some colder regions of the world. This variety and the nature of some structures exploitation leaded to the formulation and the solution to the eigenvibrations problems for the spatial structures with random parameters. Although the model presented in the paper is illustrated with the example of the three dimen-

sional model of the telecommunication tower, it can be applied after some small modifications to the other shell and spatial structures as well.

There are several well established both mathematical and numerical models enabling for inclusion of randomness in design parameters into the structural dynamics problems solutions, see [1, 5–9]. Starting from the analytical approaches based on the response spectrum analysis, through the crude Monte-Carlo stochastic simulation till the computational spectral methods based on the Karhunen–Loeve or polynomial chaos expansions of the input random fields. On the other hand, there are the lower order stochastic perturbation methods, however they have fundamental bounds on the input random dispersion level, so that their application to the real engineering problems may be limited. Taking into account those limitations, huge time consumptions for the simulation method, large expansions for the chaos expansions as well as the availability for analytical solutions in the specific problems only, the new method is proposed here.

This new method is based on the Taylor expansion of any desired order with random coefficients of all uncertain parameters and state functions around their expected values. The second new idea here is an application of the response function method in conjunction with this generalized stochastic perturbation technique. We suppose that the output state function, namely the particular eigenvalue may be represented by the polynomial form of the input random parameter. The coefficients for this response polynomium are computed from the several solutions of the original problem obtained for this parameter values taken around its mean value. The polynomial form of the response function leads to the easy determination of its partial derivatives with respect to this random input parameter, which can be finally employed for analytical determination of the probabilistic moments; it essentially differs from the previous straightforward solution to the equations of an increasing order (like zeroth, first, etc.). The computational implementation of this method is realized with the classical Finite Element Method program enriched with some stochastic procedures written in the symbolic algebra computer system MAPLE, where the response function formation, its coefficients determination as well as the final derivation of the probabilistic moments is carried out. The nature of this implementation will allow in the nearest future some similar implementations with the use of the commercial packages of the FEM and the other computer methods. The entire procedure is tested on the example of the 3D truss structures with random Young modulus and, separately, mass density of the structural members. It shows that the method converges relatively fast (eight and tenth order approaches returns almost the same results).

2. The generalized stochastic perturbation

Let us introduce the random variable $b \equiv b(\omega)$ and its probability density function as $p(b)$. Then, the first two probabilistic moments of this field are defined as

$$E[b] \equiv b^0 = \int_{-\infty}^{+\infty} b\, p(b)\, db \tag{1}$$

and

$$Cov\left(b(x_r),b(x_s)\right)=\int_{-\infty}^{+\infty}\left[b(x_r)-b^0(x_r)\right]\left[b(x_s)-b^0(x_s)\right]p(b)dba. \tag{2}$$

The basic idea of the stochastic perturbation approach is to expand all the input variables and all the state functions of the given problem via Taylor series about their spatial expectations using some small parameter $\varepsilon > 0$. In case of random quantity $b \equiv e$, the following expression is employed:

$$e = e^0 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\varepsilon^n \frac{\partial^n e}{\partial b^n}(\Delta b)^n, \tag{3}$$

where:

$$\varepsilon\,\Delta b = \varepsilon(b - b^0) \tag{4}$$

is the first variation of b around its expected value and

$$\varepsilon^2(\Delta b)^2 = \varepsilon^2(b - b^0)^2, \tag{5}$$

denotes the second variation of b about b^0. Symbol $(.)^0$ represents the function value (.) taken at the expectation b^0, while $(.)^{,b}$, $(.)^{,bb}$ denote the first and the second partial derivatives with respect to b evaluated at b^0, respectively. Let us analyze further the expected values of any state function $f(b)$ defined analogously to the formula (3) by its expansion via Taylor series with a given small parameter ε as follows:

$$E\left[f(b);b\right]=\int_{-\infty}^{+\infty} f(b)p(b)db=\int_{-\infty}^{+\infty}\left(f^0+\sum_{n=1}^{\infty}\tfrac{1}{n!}\varepsilon^n f^{(n)}\Delta b^n\right)p(b)db. \tag{6}$$

Let us remind that this power expansion is valid only if the state function is analytic in ε and the series converge and, therefore, any criteria of convergence should include the magnitude of the perturbation parameter; perturbation parameter is taken as equal to 1 in many practical computations. Contrary to the previous analyses in this area, now the quantity ε is treated as the expansion parameter in further analysis, so that it is included explicitly in all the further derivations demanding analytical expressions. Numerical studies performed in the next section demonstrate the influence of this parameter on the expected values and standard deviations in various orders of the perturbation methodology. Both moments are obtained in the form of polynomials of the additional order with respect to the parameter ε.

From the numerical point of view, the expansion provided by Equation (6) is carried out for the summation over the finite number of components. Considering various probability distributions, the essential difference is noticed between symmetric distribution functions, where

$$E\left[f(b);b\right]=f^0+\int_{-\infty}^{+\infty}\left(\sum_{n=1}^{2M}\frac{1}{(2n)!}\varepsilon^{2n}\frac{\partial^{2n}f}{\partial b^{2n}}\Delta b^{2n}\right)p(b)db \tag{7}$$

and non-symmetric probability functions

$$E\left[f(b);b\right]=f^0+\int_{-\infty}^{+\infty}\left(\sum_{n=1}^{N}\frac{1}{(n)!}\varepsilon^{n}\frac{\partial^{n}f}{\partial b^{n}}\Delta b^{n}\right)p(b)db. \tag{8}$$

Let us focus now on analytical derivation of the first two probabilistic moments for the structural response function. According to Equation (6) it yields for the input random variable with symmetric probability density function in the second order perturbation approach

$$\begin{aligned}E\left[f(b);b\right]&=\int_{-\infty}^{+\infty}\left(f^0+\varepsilon f^{,b}\Delta b+\tfrac{1}{2}\varepsilon^2 f^{,bb}\Delta b\Delta b\right)p(b)\,db\\&=f^0(b)+0\times\varepsilon f^{,b}(b)+\tfrac{1}{2}\varepsilon^2 f^{,bb}(b)S_{bb}.\end{aligned} \tag{9}$$

This expected value can be calculated or symbolically computed only if it is given as some analytical function of the random input parameter b. Many existing models in various branches of engineering can be adopted to achieve this goal. Computational implementation of the symbolic calculus programs, combined with powerful visualization of probabilistic output moments, assures the fastest solution of such problems. If higher order terms are necessary (because of a great random deviation of the input random variable about its expected value), then the following extension can be proposed:

$$\begin{aligned}E\left[f(b);b\right]&=f^0(b)+\tfrac{1}{2}\varepsilon^2 f^{,bb}(b)\mu_2(b)+\tfrac{1}{4!}\varepsilon^4 f^{,bbbb}(b)\mu_4(b)\\&\quad+\tfrac{1}{6!}\varepsilon^6 f^{,bbbbbb}(b)\mu_6(b)+\ldots\end{aligned} \tag{10}$$

where $\mu^n(b)$ denotes nth order central probabilistic moment of the quantity b and where all terms with the odd orders are equal to 0 for the Gaussian random deviates and where higher than the 6th order terms are neglected. Thanks to such an extension of the random output, any desired efficiency of the expected values as well as higher probabilistic moments can be achieved by an appropriate choice of the parameters m and ε corresponding to the input probability density function (PDF) type, relations between the probabilistic moments, acceptable error of the computations etc. This choice can be made by comparative studies with long enough (almost infi-

nite) series Monte-Carlo simulations or theoretical results obtained from the direct symbolic integration. Similar considerations lead to the 6th order expressions for a variance; there holds

$$\begin{aligned} Var(f) = \int_{-\infty}^{+\infty} \Big(& f^0 + \varepsilon \Delta b f^{,b} + \tfrac{1}{2}\varepsilon^2 (\Delta b)^2 f^{,bb} + \tfrac{1}{3!}\varepsilon^3 (\Delta b)^3 f^{,bbb} \\ & + \tfrac{1}{4!}\varepsilon^4 (\Delta b)^4 f^{,bbbb} + \tfrac{1}{5!}\varepsilon^5 (\Delta b)^5 f^{,bbbbb} - E[f] \Big)^2 p(f(b)) db. \end{aligned} \tag{11}$$

Hence

$$\begin{aligned} Var(f) \cong & \int_{-\infty}^{+\infty} \varepsilon^2 (\Delta b)^2 \left(f^{,b}\right)^2 p(f(b)) db + \int_{-\infty}^{+\infty} \tfrac{1}{4}\varepsilon^4 (\Delta b)^4 \left(f^{,bb}\right)^2 p(f(b)) db \\ & + \int_{-\infty}^{+\infty} \varepsilon^4 \Delta b f^{,b} \tfrac{1}{3!}(\Delta b)^3 f^{,bbb} p(f(b)) db + \int_{-\infty}^{+\infty} \varepsilon^4 \Delta b f^{,b} \tfrac{1}{3!}(\Delta b)^3 f^{,bbb} p(f(b)) db \\ & + \int_{-\infty}^{+\infty} \tfrac{1}{3!}\varepsilon^6 (\Delta b)^3 f^{,bbb} \tfrac{1}{3!}(\Delta b)^3 f^{,bbb} p(f(b)) db \\ & + \int_{-\infty}^{+\infty} \tfrac{1}{4!}\varepsilon^6 (\Delta b)^4 f^{,bbbb} \tfrac{1}{2}(\Delta b)^2 f^{,bb} p(f(b)) db \\ & + \int_{-\infty}^{+\infty} \tfrac{1}{4!}\varepsilon^6 (\Delta b)^4 f^{,bbbb} \tfrac{1}{2}(\Delta b)^2 f^{,bb} p(f(b)) db \\ & + \int_{-\infty}^{+\infty} \tfrac{1}{5!}\varepsilon^6 (\Delta b)^5 f^{,bbbbb} \Delta b f^{,b} p(f(b)) db \\ & + \int_{-\infty}^{+\infty} \tfrac{1}{5!}\varepsilon^6 (\Delta b)^5 f^{,bbbbb} \Delta b f^{,b} p\big(f(b)\big) db \end{aligned} \tag{12}$$

As it can be recognized here, the first integral corresponds to the second order perturbation, the next three complete 4th order approximation and the rest needs to be included to achieve full 6th order expansion. After multiple integration and indices transformations, one can show that

$$\begin{aligned} Var\big(f(b)\big) = & \varepsilon^2 \mu_2(b) f^{,b} f^{,b} \\ & + \varepsilon^4 \mu_4(b) \Big(\tfrac{1}{4} f^{,bb} f^{,bb} + \tfrac{2}{3!} f^{,b} f^{,bbb} \Big) \\ & + \varepsilon^6 \mu_6(b) \Big(\big(\tfrac{1}{3!}\big)^2 f^{,bbb} f^{,bbb} + \tfrac{1}{4!} f^{,bbbb} f^{,bb} + \tfrac{2}{5!} f^{,bbbbb} f^{,b} \Big). \end{aligned} \tag{13}$$

Quite similarly, using the first and the second order terms only, it is possible to derive third order probabilistic moments as

$$\begin{aligned}\mu_3\left(f\left(b\right);b\right) &= \int_{-\infty}^{+\infty}\left(f\left(b\right)-E[f(b)]\right)^3\ p\left(b\right)db \\ &= \int_{-\infty}^{+\infty}\left(f^0+\varepsilon f^{,b}\Delta b+\tfrac{1}{2}\varepsilon^2 f^{,bb}\Delta b\Delta b+\ldots-E[f(b)]\right)^3 p\left(b\right)db \\ &= \int_{-\infty}^{+\infty}\left(\varepsilon f^{,b}\Delta b+\tfrac{1}{2}\varepsilon^2 f^{,bb}\Delta b\Delta b+\ldots\right)^3 p\left(b\right)db \\ &\cong \tfrac{3}{2}\varepsilon^4\mu_4(b)\left(f^{,b}(b)\right)^2 f^{,bb}+\tfrac{1}{8}\varepsilon^6\mu_6(b)\left(f^{,bb}\right)^3\end{aligned} \tag{14}$$

and the fourth order probabilistic moment also

$$\begin{aligned}\mu_4\left(f\left(b\right);b\right) &= \int_{-\infty}^{+\infty}\left(f\left(b\right)-E[f(b)]\right)^4\ p\left(b\right)db \\ &= \int_{-\infty}^{+\infty}\left(f^0+\varepsilon f^{,b}\Delta b+\tfrac{1}{2}\varepsilon^2 f^{,bb}\Delta b\Delta b+\ldots-E[f(b)]\right)^4 p\left(b\right)db \\ &= \int_{-\infty}^{+\infty}\left(\varepsilon f^{,b}\Delta b+\tfrac{1}{2}\varepsilon^2 f^{,bb}\Delta b\Delta b+\ldots\right)^4 p\left(b\right)db \\ &\cong \varepsilon^4\mu_4(b)\left(f^{,b}(b)\right)^4+\tfrac{3}{2}\varepsilon^6\mu_6(b)\left(f^{,b}f^{,bb}\right)^2 \\ &+\tfrac{1}{16}\varepsilon^8\mu_8(b)\left(f^{,b}\right)^3\left(f^{,bb}\right)^4.\end{aligned} \tag{15}$$

Let us mention that it is necessary to multiply each of these equations by the relevant order probabilistic moments of the input random variables to get the algebraic form convenient for any symbolic computations. Because of a great complexity of such a solution, the second order perturbation approach is usually preferred. Recursive derivation of the particular perturbation order equilibrium equations can be powerful in conjunction with symbolic packages with automatic differentiation tools only; it can extend the area of stochastic perturbation technique applications in computational physics and engineering outside the random processes with small dispersion about their expected values. Hence, there is no need to implement directly exact formulas for a particular nth order equations extracted from the perturbation. They can be symbolically generated in the system MAPLE, and next converted to the FORTRAN source codes of the relevant computer software. Finally, it should be emphasized that the random input variables must express here the uncertainty in space or in time, sepa-

rately. Quite analogous expansion may be recalled as a function of the perturbation parameter ε, a perturbation order m as well as the input random variable b

$$E\left[f(b);b,\varepsilon,m\right]=f^{0}(b)+\tfrac{1}{2}\varepsilon^{2}\frac{\partial^{2}f}{\partial b^{2}}\mu_{2}(b) + \tfrac{1}{4!}\varepsilon^{4}\frac{\partial^{4}f}{\partial b^{4}}\mu_{4}(b)+\tfrac{1}{6!}\varepsilon^{6}\frac{\partial^{6}f}{\partial b^{6}}\mu_{6}(b)+\ldots+\tfrac{1}{(2m)!}\varepsilon^{2m}\frac{\partial^{2m}f}{\partial b^{2m}}\mu_{2m}(b), \tag{16}$$

for any natural m with μ_{2m} being the ordinary probabilistic moment of 2 mth order.

3. Stochastic dynamics with random parameters

3.1. The single degree-of-freedom case

Let us consider the single-degree-of-freedom dynamical system consisting of a mass m mounted using the spring with the constant k and the dashpot c to the stiff wall and excited by the function $a(\tau)$. The system parameters, i.e. k, c and m are separately defined here as the truncated Gaussian random variables with the specified first two probabilistic moments. Let us assume furthermore that (1) the spring is linear in this system, (2) the excitation is given as $a(\tau)=\hat{a}f(\tau)$, where $f(\tau)$ is a deterministic function of time, while the magnitude is time – independent, (3) the initial conditions are homogenous – $u(0) = 0$ and $\dot{u}(0)=0$. The general solution to this problem is provided using the Dirac delta distribution, where the forcing function is express in terms of an infinitive sequence of the adjacent impulses with their sampling intervals approach zero; this system response $u(\tau)$ at any time $\tau = t$ is obtained from the superposition of the unit impulse responses

$$u(t)=\int_{0}^{t}a(\tau)w(t-\tau)d\tau=\hat{a}\int_{0}^{t}f(\tau)w(t-\tau)\,d\tau\ ,\ \tau\in[0,T] \tag{17}$$

where $w(\tau)$ denotes the response to the Dirac-type excitation $\delta(\tau)$

$$w(\tau)=w(m,c,k,\tau)=\frac{1}{m\omega_{c}}\exp(-\xi\omega\tau)\sin\omega_{c}\tau,\qquad \tau\in[0,T], \tag{18}$$

where the damped free vibration frequency ω_c, viscous damping factor ξ and natural frequency ω are defined as

$$\omega_{c}=\omega(1-\xi^{2})^{\frac{1}{2}},\qquad \xi=\frac{c}{2m\omega},\qquad \omega=\sqrt{\frac{k}{m}}. \tag{19}$$

Now we consider the uncertainty in (1) spring stiffness k, (2) the dashpot c and (3) the mass m, separately. Each time we start from the same equation of motion

$$m\ddot{u}(\tau)+c\dot{u}(\tau)+ku(\tau)=\hat{a}f(\tau). \tag{20}$$

For the case (1) we obtain the following hierarchical equations:
0th order:

$$m\ddot{u}^0(\tau)+c\dot{u}^0(\tau)+k^0u^0(\tau)=\hat{a}f(\tau), \tag{21}$$

1st order:

$$m\ddot{u}^{,k}(\tau)+c\dot{u}^{,k}(\tau)+k^0u^{,k}(\tau)=-u^0(\tau), \tag{22}$$

2nd order:

$$m\ddot{u}^{,kk}(\tau)+c\dot{u}^{,kk}(\tau)+k^0u^{,kk}(\tau)=-2u^{,k}(\tau), \tag{23}$$

so for n^{th} order equations one proves easily

$$m\frac{\partial\ddot{u}(\tau)}{\partial k^n}+c\frac{\partial^n\dot{u}(\tau)}{\partial k^n}+k^0\frac{\partial^n u(\tau)}{\partial k^n}=-n\frac{\partial^{n-1}u(\tau)}{\partial k^{n-1}}. \tag{24}$$

Therefore, solving successively those equations one by one and inserting a solution of the previous one into the R.H.S. of the partial differential equation it is possible to collect all components for probabilistic moments expressions for $u(\tau)$, $\dot{u}(\tau)$ and $\ddot{u}(\tau)$. The two remaining case studies return very similar results, so that when c is randomized then the first order equations equals to

$$m\ddot{u}^{,c}(\tau)+c^0\dot{u}^{,c}(\tau)+ku^{,c}(\tau)=-\dot{u}^0(\tau), \tag{25}$$

whereas the recursive relation of the n^{th} order is equal to

$$m\frac{\partial^n\ddot{u}(\tau)}{\partial c^n}+c^0\frac{\partial^n\dot{u}(\tau)}{\partial c^n}+k\frac{\partial^n u(\tau)}{\partial c^n}=-n\frac{\partial^{n-1}\dot{u}(\tau)}{\partial c^{n-1}}. \tag{26}$$

3.2. The general elastodynamic problem with random parameters

Let us consider the following set of partial differential equations adequate to the linear elastodynamic problem [2, 5] consisting of

- the equations of motion

$$\mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma} + \hat{\mathbf{f}} = \boldsymbol{\rho}\ddot{\mathbf{u}}, \quad \mathbf{x} \in \Omega, \quad \tau \in [t_0, \infty), \tag{27}$$

- the constitutive equations

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \quad \mathbf{x} \in \Omega, \quad \tau \in [t_0, \infty), \tag{28}$$

- the geometric equations

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u}, \quad \mathbf{x} \in \Omega, \quad \tau \in [t_0, \infty), \tag{29}$$

- the displacement boundary conditions

$$\mathbf{u} = \hat{\mathbf{u}}, \quad \mathbf{x} \in \partial\Omega_u, \quad \tau \in [t_0, \infty), \tag{30}$$

- the stress boundary conditions

$$\mathbf{N}\boldsymbol{\sigma} = \hat{\mathbf{t}}, \quad \mathbf{x} \in \partial\Omega_\sigma, \quad \tau \in [t_0, \infty), \tag{31}$$

- the initial conditions

$$\mathbf{u} = \hat{\mathbf{u}}^{\mathbf{0}}, \quad \dot{\mathbf{u}} = \hat{\dot{\mathbf{u}}}^{\mathbf{0}}, \quad \tau \in [t_0, \infty). \tag{32}$$

It is assumed that all the state functions appearing in this system are sufficiently smooth functions of the independent variables $\mathbf{x}$ and τ. Let us consider the variation $\mathbf{u}(\mathbf{x}, \tau)$ in some time moment $\tau = t$ denoted by $\delta\mathbf{u}(\mathbf{x}, \tau)$. Using the above equations one can show that

$$-\int_{\Omega} (\mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma} + \hat{\mathbf{f}} - \boldsymbol{\rho}\ddot{\mathbf{u}})^{\mathrm{T}} \delta\mathbf{u}\, d\Omega + \int_{\partial\Omega_\sigma} (\mathbf{N}\boldsymbol{\sigma} - \hat{\mathbf{t}})^{\mathrm{T}} \delta\mathbf{u}\, \mathbf{d}(\partial\Omega) = 0. \tag{33}$$

Assuming further that the displacement function $u(x, t)$ has known values at the initial moment $\mathbf{u}(\mathbf{x}, t_1) = 0$ and at the end of the process $\mathbf{u}(\mathbf{x}, t_2) = 0$, so that the variations of this function also equal 0 at those time moments

$$\delta\mathbf{u}(\mathbf{x}, t_1) = 0, \; \delta\mathbf{u}(\mathbf{x}, t_2) = 0. \tag{34}$$

Integrating by parts with respect to the variables $\mathbf{x}$ and τ we can obtain that

$$\int_{t_1}^{t_2} [\delta\mathbf{T} - \int_{\Omega} \boldsymbol{\sigma}^{\mathrm{T}} \delta\boldsymbol{\varepsilon}\, d\Omega + \int_{\Omega} \hat{\mathbf{f}}^{\mathrm{T}} \delta\mathbf{u}\, d\Omega + \int_{\partial\Omega} \hat{\mathbf{t}}^{\mathrm{T}} \delta\mathbf{u}\, d(\partial\Omega)] d\tau = 0, \tag{35}$$

where the kinetic energy of the region Ω is defined as

$$T = \frac{1}{2}\int \rho \dot{\mathbf{u}}^{\mathrm{T}} \dot{\mathbf{u}}\, d\Omega. \tag{36}$$

Equation (35) is considered together with the following strain relation

$$\delta\boldsymbol{\varepsilon} = \mathbf{D}\delta\mathbf{u},\ \mathbf{x} \in \Omega,\ \tau \in [t_0, \infty). \tag{37}$$

Next, we introduce the assumption that the mass forces $\hat{\mathbf{f}}$ and the surface loadings $\hat{\mathbf{t}}$ are independent from the displacement vector **u**, which means that the external loadings do not follow the changes in the domain initial configuration. Therefore, Equation (35) can be modified to the following statement:

$$\delta \int_{\mathbf{t}_1}^{\mathbf{t}_2} \left(T - J_p\right) d\tau = 0, \tag{38}$$

where $\mathbf{J}_p$ means the potential energy cumulated in the domain Ω

$$J_{\mathbf{p}} = U - \int_{\boldsymbol{\Omega}} \hat{\mathbf{f}}^{\mathrm{T}} \mathbf{u} d\Omega - \int_{\partial\boldsymbol{\Omega}_{\sigma}} \hat{\mathbf{t}}^{\mathrm{T}} \mathbf{u} d(\partial\Omega) = 0, \tag{39}$$

whereas the variation is determined with respect to the displacement function and **U** is the elastic strain energy given by the formula

$$U = \tfrac{1}{2}\int_{\Omega} \boldsymbol{\varepsilon}^{\mathrm{T}} \mathbf{C}\boldsymbol{\varepsilon} d\Omega. \tag{40}$$

It is well known that Equation (38) represents the Hamilton principle widely used in structural dynamics in conjunction with the Finite Element Method approach.

3.3. The generalized perturbation-based eigenvalue problem

Let us consider a deterministic eigenproblem in its matrix description for its further stochastic expansion

$$(\mathbf{K} - \varpi^2 \mathbf{M})\boldsymbol{\varphi} = \mathbf{0}. \tag{41}$$

Its zeroth order version looks like

$$(\mathbf{K}^0 - (\varpi^0)^2 \mathbf{M}^0)\boldsymbol{\varphi}^0 = 0, \tag{42}$$

whereas the first order equation has the following form:

$$\left(\mathbf{K}'^{b}-\frac{d}{db}(\varpi^{0})^{2}\mathbf{M}^{0}-(\varpi^{0})^{2}\mathbf{M}'^{b}\right)\boldsymbol{\varphi}^{0}=-(\mathbf{K}^{0}-(\varpi^{0})^{2}\mathbf{M}^{0})\boldsymbol{\varphi}'^{b}. \tag{43}$$

After some algebraic transformation one can get that

$$(\mathbf{K}'^{b}-2\varpi^{0}\varpi'^{b}\mathbf{M}^{0}-(\varpi^{0})^{2}\mathbf{M}'^{b})\boldsymbol{\varphi}^{0}=-(\mathbf{K}^{0}-(\varpi^{0})^{2}\mathbf{M}^{0})\boldsymbol{\varphi}'^{b}, \tag{44}$$

which finally takes the following form:

$$\mathbf{K}'^{b}\boldsymbol{\varphi}^{0}-2\varpi^{0}\varpi'^{b}\mathbf{M}^{0}\boldsymbol{\varphi}^{0}-(\varpi^{0})^{2}\mathbf{M}'^{b}\boldsymbol{\varphi}^{0}+\mathbf{K}^{0}\boldsymbol{\varphi}'^{b}-(\varpi^{0})^{2}\mathbf{M}^{0}\boldsymbol{\varphi}'^{b}=0. \tag{45}$$

As it was expected, the perturbed first order equation is much more complicated than the first order equation in the linear elastostatics. The next differentiation of Equation (44) with respect to the input random variable b gives

$$\begin{aligned}&(\mathbf{K}'^{bb}-\frac{\partial}{\partial b}(2\varpi^{0}\varpi'^{b}\mathbf{M}^{0})-\frac{\partial}{\partial b}((\varpi^{0})^{2}\mathbf{M}'^{b})\boldsymbol{\varphi}^{0}+\\&(\mathbf{K}'^{b}-2\varpi^{0}\varpi'^{b}\mathbf{M}^{0}-(\varpi^{0})^{2}\mathbf{M}'^{b})\boldsymbol{\varphi}'^{b}=\\&=-(\mathbf{K}'^{b}-2\varpi^{0}\varpi'^{b}\mathbf{M}^{0}-(\varpi^{0})^{2}\mathbf{M}'^{b})\boldsymbol{\varphi}'^{b}-(\mathbf{K}^{0}-(\varpi^{0})^{2}\mathbf{M}^{0})\boldsymbol{\varphi}'^{bb}.\end{aligned} \tag{46}$$

After its simplification we obtain

$$\begin{aligned}&\mathbf{K}'^{bb}\,\boldsymbol{\varphi}^{0}+\mathbf{K}'^{b}\,\boldsymbol{\varphi}'^{b}-\frac{\partial}{\partial b}(2\varpi^{0})\varpi'^{b}\,\mathbf{M}^{0}\boldsymbol{\varphi}^{0}-2\varpi^{0}\varpi'^{bb}\,\mathbf{M}^{0}\boldsymbol{\varphi}^{0}\\&-2\varpi^{0}\varpi'^{b}\,\mathbf{M}'^{b}\,\boldsymbol{\varphi}^{0}-2\varpi^{0}\varpi'^{b}\,\mathbf{M}^{0}\boldsymbol{\varphi}'^{b}-\frac{\partial}{\partial b}((\varpi^{0})^{2})\mathbf{M}'^{b}\,\boldsymbol{\varphi}^{0}\\&-(\varpi^{0})^{2}\mathbf{M}'^{bb}\,\boldsymbol{\varphi}^{0}-(\varpi^{0})^{2}\mathbf{M}'^{b}\,\boldsymbol{\varphi}'^{b}+\mathbf{K}'^{b}\,\boldsymbol{\varphi}'^{b}+\mathbf{K}^{0}\boldsymbol{\varphi}'^{bb}\\&-\frac{\partial}{\partial b}((\varpi^{0})^{2})\mathbf{M}^{0}\boldsymbol{\varphi}'^{b}+(\varpi^{0})^{2}\mathbf{M}'^{b}\,\boldsymbol{\varphi}'^{b}-(\varpi^{0})^{2}\mathbf{M}^{0}\boldsymbol{\varphi}'^{bb}=0\end{aligned} \tag{47}$$

Finally, one can show that the second order equation is equal to

$$\begin{aligned}&\mathbf{K}'^{bb}\boldsymbol{\varphi}^{0}+2K'^{b}\boldsymbol{\varphi}'^{b}+\mathbf{K}^{0}\boldsymbol{\varphi}'^{bb}-2\varpi^{0}(\varpi'^{b})^{2}\mathbf{M}^{0}\boldsymbol{\varphi}^{0}-2\varpi^{0}\varpi'^{bb}\mathbf{M}^{0}\boldsymbol{\varphi}^{0}\\&-4\varpi^{0}\varpi'^{b}(\mathbf{M}'^{b}\boldsymbol{\varphi}^{0}+\mathbf{M}^{0}\boldsymbol{\varphi}'^{b})-(\varpi^{0})^{2}(\mathbf{M}'^{bb}\boldsymbol{\varphi}^{0}+2\mathbf{M}'^{b}\boldsymbol{\varphi}'^{b}+\mathbf{M}^{0}\boldsymbol{\varphi}'^{bb}=0.\end{aligned} \tag{48}$$

It is quite clear here that the generalized version of the stochastic perturbation technique based on the nth order Taylor series expansion may lead to the very complex

equation corresponding to the highest order closure of the entire system [3, 4]. The left hand side of this equation denoted by $LHS(n)$ should be equal to

$$LHS(n) = \sum_{k=0}^{n} \binom{n}{k} k^{(n-k)} \boldsymbol{\varphi}^{(k)}, \tag{49}$$

where obviously

$$\boldsymbol{\varphi}^{(k)} = \frac{\partial^k \boldsymbol{\varphi}}{\partial b^k}. \tag{50}$$

An application of the similar nth order differentiation procedure to the right hand side of zeroth order statement results in

$$RHS(n) = \sum_{k=0}^{n} \binom{n}{k} (\varpi^2)^{(k)} \sum_{m=0}^{n-k} \binom{n-k}{m} \mathbf{M}^{(m)} \boldsymbol{\varphi}^{(n-k-m)}. \tag{51}$$

Therefore, the remaining issue is to give a formula for the k-th order partial derivative of the eigenvalues second with respect to the input random variable. There holds

$$(\varpi^2)^{(k+1)} = \frac{\partial^{k+1}}{\partial b^{k+1}}(\varpi^2) = \frac{\partial^k}{\partial b^k}\left(2\varpi \cdot \frac{\partial \varpi}{\partial b}\right) = \sum_{l=0}^{k} \binom{n}{l} (2\varpi)^{(k-l)} \left(\frac{\partial \varpi}{\partial b}\right)^{(k)}. \tag{52}$$

Therefore, one can easily get

$$(\varpi^2)^{(k)} = \sum_{l=0}^{k-1} \binom{k-1}{l} (2\varpi)^{(k-(l-1))} \varpi^{(l+1)}. \tag{53}$$

Hence, the Equation (51) may be rewritten in a form

$$RHS(n) = \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k-1} \binom{k-1}{l} (2\varpi)^{(k-(l+1))} \cdot \varpi^{(l+1)} \sum_{m=0}^{n-k} \binom{n-k}{m} \mathbf{M}^{(m)} \boldsymbol{\varphi}^{(n-k-m)}. \tag{54}$$

A comparison of the Relations (49) and (54) returns

$$\sum_{k=0}^{n} \binom{n}{k} \mathbf{K}^{(n-k)} \boldsymbol{\varphi}^{k} = \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k-1} \binom{k-1}{l} (2\varpi)^{(k-(l+1))} \cdot \varpi^{(l+1)} \sum_{m=0}^{n-k} \binom{n-k}{m} \mathbf{M}^{(m)} \boldsymbol{\varphi}^{(n-k-m)}. \tag{55}$$

A solution to his equation makes it possible to determine up to the n^{th} order eigenvalues together with the corresponding eigenvectors.

4. Computational implementation

Let us consider a discretization of the displacement field $\mathbf{u}(\mathbf{x},\tau)$ using the following forms [2, 5]:

$$\mathbf{u}^{\alpha}_{3x1}(\mathbf{x},\tau) \cong \varphi_{3\,x\,N_{(e)}}(\mathbf{x})\,\mathbf{q}^{\alpha}_{N_{(e)}\,x1}(\tau),\ \mathbf{u}^{\alpha}_{3x1}(\mathbf{x},\tau) \cong \mathbf{\Phi}_{3\,x\,N}(\mathbf{x})\mathbf{r}^{\alpha}_{N\,x1}(\tau), \tag{56}$$

where:

$\mathbf{q}$ is a vector of the generalized coordinates for the considered finite element,

$\mathbf{r}$ is a vector for the generalized coordinates of the entire discretized system,

$N(e)$ is the total number of the finite element degrees of freedom,

N is the total number of degrees of freedom in the structure model,

φ and $\mathbf{\Phi}$ are the corresponding the shape function matrices (local and global).

The generalized coordinates vector for the entire structure model and for the final element are related by the transformation matrix as

$$\mathbf{r}^{\alpha}_{N\,x\,1} = a_{N\,x\,N_{(e)}}\,\mathbf{q}^{\alpha}_{N_{(e)}\,x\,1}. \tag{57}$$

Contrary to the classical formulations of both FEM and the perturbation-based Stochastic Finite Element Method we introduce here the additional index $\alpha = 1, ..., M$ to distinguish between different solutions of the elastodynamic problem obtained to build up the response function around the mean value of the input random parameter. Therefore, the strain tensor can be expressed as

$$\varepsilon^{\alpha}_{6x1}(\mathbf{x},\tau) = B_{6xN(e)}(\mathbf{x})\mathbf{q}^{\alpha}_{N(e)x1}(\tau) = \widetilde{B}_{6\,x\,N}(\mathbf{x})\mathbf{r}^{\alpha}_{N\,x1}(\tau). \tag{58}$$

Application of those relations to the Hamilton principle in Equation (38) leads to the statement

$$\delta\int_{t_1}^{t_2}\left(\frac{1}{2}\sum_{l=1}^{E}\mathbf{q}^{\alpha T}\mathbf{m}^{\alpha}\mathbf{q}^{\alpha} - \tfrac{1}{2}\sum_{e=1}^{E}\mathbf{q}^{\alpha T}\mathbf{k}^{\alpha}\mathbf{q}^{\alpha} + \sum_{e=1}^{E}\mathbf{Q}^{\alpha T}\mathbf{q}^{\alpha}\right)d\tau = 0, \tag{59}$$

so that the global notation gives here

$$\delta\int_{t_1}^{t_2}\left(\frac{1}{2}\dot{\mathbf{r}}^{\alpha T}\mathbf{M}^{\alpha}\dot{\mathbf{r}}^{\alpha} - \tfrac{1}{2}\mathbf{r}^{\alpha T}\mathbf{K}^{\alpha}\mathbf{r}^{\alpha} + \mathbf{R}^{\alpha T}\mathbf{r}^{\alpha}\right)d\,\tau = 0. \tag{60}$$

The element and global mass matrices are defined as

$$\mathbf{m}^{\alpha}_{N_{(e)}\,x\,N_{(e)}} = \int_{\Omega_e}\rho^{\alpha}(\mathbf{x})\mathbf{B}^{T}_{N_{(e)}\,x6}(\mathbf{x})\mathbf{B}_{6x\,N_{(e)}}(\mathbf{x})d\Omega \tag{61}$$

and

$$\mathbf{M}^{\alpha}_{NxN} = \int_{\Omega} \rho^{\alpha}(\mathbf{x}) \widetilde{\mathbf{B}}^{T}_{Nx6}(\mathbf{x}) \widetilde{\mathbf{B}}_{6xN}(\mathbf{x}) \, \mathrm{d}\Omega. \tag{62}$$

The stiffness matrices at the element and at the global scales are defined as follows

$$\mathbf{k}^{\alpha}_{N_{(e)}xN_{(e)}} = \int_{\Omega_{(e)}} \mathbf{B}^{T}_{N_{(e)}x6} \mathbf{C}^{\alpha}_{6x6} \mathbf{B}_{6xN_{(e)}} \, \mathrm{d}\Omega \tag{63}$$

and

$$\mathbf{K}^{\alpha}_{NxN} = \int_{\Omega_{(e)}} \widetilde{\mathbf{B}}^{T}_{Nx6} \mathbf{C}^{\alpha}_{6x6} \widetilde{\mathbf{B}}_{6xN} \, \mathrm{d}\Omega, \tag{64}$$

the vector $\mathbf{R}$ ($\mathbf{x}$, τ) represents the vector of nodal loadings. The time variation over Equation (59) results in a relation

$$\dot{\mathbf{r}}^{\mathbf{T}} \mathbf{M} \, \delta \mathbf{r} - \int_{t_1}^{t_2} (\ddot{\mathbf{r}}^{\mathbf{T}} \mathbf{M} + \mathbf{r}^{\mathbf{T}} \mathbf{K} - \mathbf{R}^{\mathbf{T}}) \, \delta \mathbf{r} \, \mathrm{d}\tau = 0. \tag{65}$$

Considering the assumptions that

$$\delta \mathbf{r}(t_1) = 0, \; \delta \mathbf{r}(t_2) = 0, \tag{66}$$

we finally obtain the dynamic equilibrium system

$$\mathbf{M}^{\alpha} \ddot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha} \mathbf{r}^{\alpha} = \mathbf{R}^{\alpha}, \tag{67}$$

which represents the equations of motion of the discretized system. When we complete this equation with the component $\mathbf{C}^{\alpha}_{NxN} \mathbf{r}^{\alpha}_{Nx1}$ getting

$$\mathbf{M}^{\alpha} \ddot{\mathbf{r}}^{\alpha} + \mathbf{C}^{\varepsilon} \dot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha} \mathbf{r}^{\alpha} = \mathbf{R}^{\alpha}, \tag{68}$$

then we receive the equations system corresponding to the system with nonzero damping; it is frequently defined as

$$\mathbf{C}^{\alpha} = \alpha_0 \mathbf{M}^{\alpha} + \alpha_1 \mathbf{K}^{\alpha}, \tag{69}$$

where the coefficients α_0 and α_1 are determined using the specific eigenfunctions for this problem, so that

$$\mathbf{M}^{\alpha}\ddot{\mathbf{r}}^{\alpha} + \alpha_0\mathbf{M}^{\alpha}\dot{\mathbf{r}}^{\alpha} + \alpha_1\mathbf{K}^{\alpha}\dot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha}\mathbf{r}^{\alpha} = \mathbf{R}^{\alpha}, \tag{70}$$

where no summation over the doubled indices α is applied here. As it is known, the case of undamped and free vibrations leads to the system

$$\mathbf{M}^{\alpha}\ddot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha}\mathbf{r}^{\alpha} = \mathbf{0}, \tag{71}$$

and the solution $\mathbf{r}^{\alpha} = \mathbf{A}^{\alpha}\sin\boldsymbol{\omega}_{\alpha}t$ leads to the relation

$$-\mathbf{M}^{\alpha}\mathbf{A}^{\alpha}\boldsymbol{\omega}_{\alpha}^{2}\sin\boldsymbol{\omega}_{\alpha}t + \mathbf{K}^{\alpha}\mathbf{A}^{\alpha}\sin\boldsymbol{\omega}_{\alpha}t = \mathbf{0}, \tag{72}$$

so that for $\sin\boldsymbol{\omega}_{\alpha}t \neq \mathbf{0}$ and $\mathbf{A}^{\alpha} \neq \mathbf{0}$ there holds

$$-\mathbf{M}^{\alpha}\boldsymbol{\omega}_{\alpha}^{2} + \mathbf{K}^{\alpha} = \mathbf{0}. \tag{73}$$

When the index α is postponed, then the stochastic problem is solved in a straightforward manner analogously to the previous methods and the methodology follows the successive solutions of the increasing order equations proposed in Section 3.3.

As shown during derivation of equations for the generalized perturbation based approach, one of the most complicated issues is numerical determination of up to nth order partial derivatives of the structural response function with respect to the randomized parameter. It is possible to determine this function first by multiple solutions of the boundary value problem around the expectation of the random parameter to complete this task. The response function for each eigenvalue is built up from uniform symmetric discretization in the neighborhood of this expectation, with equidistant intervals. A set of classical deterministic re-computations of the all the components of the eigenvalues vector leads to the final formation of the responses function for all ω_{α}. That is why we consider further a problem of the unknown response function approximation by the following polynomial of $n-1$ order:

$$\omega_{\alpha} = A_1^{(\alpha)}b^{n-1} + A_2^{(\alpha)}b^{n-2} + ... + A_n^{(\alpha)}b^{0}, \tag{74}$$

having the values of this function determined computationally for n different arguments. With this representation, the algebraic system of equations is formed

$$\begin{cases} A_1^{(\alpha)}b_1^{n-1} + A_2^{(\alpha)}b_1^{n-2} + ... + A_n^{(\alpha)}b_1^{0} = \omega_{\alpha(1)} \\ A_1^{(\alpha)}b_2^{n-1} + A_2^{(\alpha)}b_2^{n-2} + ... + A_n^{(\alpha)}b_2^{0} = \omega_{\alpha(2)} \\ \qquad\qquad\qquad ... \\ A_1^{(\alpha)}b_n^{n-1} + A_2^{(\alpha)}b_n^{n-2} + ... + A_n^{(\alpha)}b_n^{0} = \omega_{\alpha(n)} \end{cases}, \tag{75}$$

where the coefficients $\omega_{\alpha(i)}$ for $i = 1, ..., n$ denote the approximated function values in ascending order of the arguments b_i.

Therefore, the following algebraic system of equations is formed to determine the polynomial coefficients $A_i^{\alpha\beta\gamma\delta}$:

$$\begin{bmatrix} b_1^{n-1} & b_1^{n-2} & \dots & b_1^0 \\ b_2^{n-1} & b_2^{n-2} & & b_2^0 \\ \dots & \dots & & \dots \\ b_n^{n-1} & b_n^{n-2} & \dots & b_n^0 \end{bmatrix} \begin{Bmatrix} A_1^{(\alpha)} \\ A_2^{(\alpha)} \\ \dots \\ A_n^{(\alpha)} \end{Bmatrix} = \begin{Bmatrix} \omega_{\alpha(1)} \\ \omega_{\alpha(2)} \\ \dots \\ \omega_{\alpha(n)} \end{Bmatrix}. \tag{76}$$

The crucial point of this method is a proper determination of the set of input parameters $\{b_1^0, \dots, b_n^0\}$ inserted into this equation. This determination is started with a choice of the computational domain $[b-\Delta b, b+\Delta b]$, where $2\Delta b = 0.05b$. Then, this domain is subdivided into the set of equidistant $n-1$ intervals with the length $\Delta b_{(m,m+1)} = \dfrac{2\Delta b}{n-1}$ for any $m = 1, \dots, n-1$.

So, that assuming that $b_0 = b - \Delta b$ it is obtained that $b_m = b - \Delta b + m\dfrac{2\Delta b}{n-1}$. Let us note that since this linear system of equations is non-symmetric, its solution cannot be done by the integration with the FEM solver, and some separate numerical procedure based on the Gauss–Jordan elimination scheme must be employed. The unique solution for this system makes it possible to calculate up to the nth order ordinary derivatives of the homogenized elasticity tensor with respect to the parameter b at the given b_0 as

1st order derivative:

$$\frac{\partial \omega_\alpha}{\partial b} = (n-1)A_1^{(\alpha)}b^{n-2} + (n-2)A_2^{(\alpha)}b^{n-3} + \dots + A_{n-1}^{(\alpha)}, \tag{77}$$

2nd order derivative:

$$\frac{\partial^2 \omega_\alpha}{\partial b^2} = (n-1)(n-2)A_1^{(\alpha)}b^{n-3} + (n-2)(n-3)A_2^{(\alpha)}b^{n-4} + \dots + A_{n-2}^{(\alpha)}, \tag{78}$$

kth order derivative:

$$\frac{\partial^k \omega_\alpha}{\partial b^k} = \prod_{i=1}^{k}(n-i)A_1^{(\alpha)}b^{n-k} + \prod_{i=2}^{k}(n-i)A_2^{(\alpha)}b^{n-(k+1)} + \dots + A_{n-k}^{(\alpha)}. \tag{79}$$

Providing that the response function of the structural eigenvalue has a single independent argument, that is, the input random variable of the problem, it is possible to employ the stochastic perturbation technique based on the Taylor representation to compute up to the m'-th order probabilistic moments $\mu_m(\omega_\alpha)$. It is clear from the derivation above that to complete the m'-th order approximation we need to solve the initial deterministic problem m times, with its number of degrees of freedom and a single system of algebraic equations $m \times m$, to find a single response function. Including the formulas above for the derivatives of the response function in a definition of the probabilistic moments, one can determine the expectations, variances as well as any order random characteristics of the structural response.

5. Numerical experiments

5.1. The single-degree-of-freedom system with random parameters

The first computational example is devoted to the 1 D.O.F. system without the damping. The mass m and the spring stiffness k are separately considered as the Gaussian random variables with the expected values equal to E[k] = 24E6 and E[m] = 1000. Their coefficients of variations belong to the interval [0.0, 0.3] and sometimes are limited for a more transparent results presentation to the narrower interval [0.1, 0.2]. All the equations and computer visualization has been prepared using the symbolic computations system MAPLE, v.11. Using the methodology described above consisting of the straightforward differentiation approach and the well-known classical formula we study here:

1. The expected values for (a) randomized spring stiffness (all the diagrams at the left) and (b) randomized mass (the right diagrams) computed according to the 2nd, 4th, 6th, 8th and 10th order approaches Figure 1;

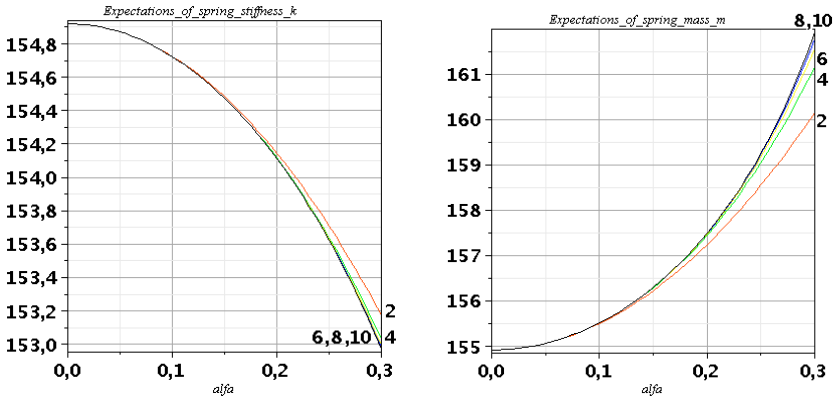

Fig. 1. The expected values for the 1 D.O.F. eigenvalue for the random stiffness (left) and the random mass (right)

2. The variances for (a) the perturbation parameter variability $\varepsilon \in [0.8, 1.2]$ – Figure 2 and (b) for the perturbation parameter arbitrarily defined as $\varepsilon = 1$ – Figure 3 (according to the 2nd, 4th and 6th order methods); the lowest surface on Figure 2 corresponds to the 2nd order results, the intermediate surface results from the 4th order approach and the upper surface is equivalent to the results of the 6th order analysis; Figure 3 contains the solid lines (2nd order analysis), the dot line for the 4th order method and dash-dot line for the 6th order approach;

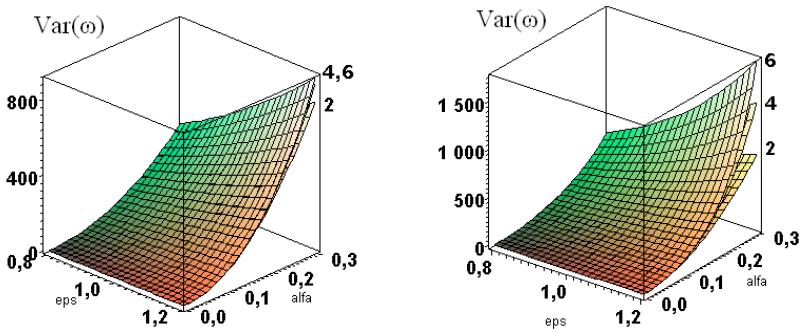

Fig. 2. The variances for the 1 D.O.F. eigenvalue for the random stiffness (left) and the random mass (right)

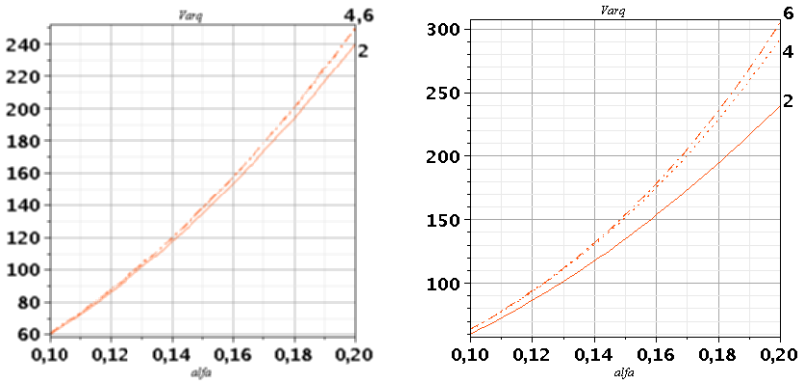

Fig. 3. The variances for the 1 D.O.F. eigenvalue for the random stiffness (left) and the random mass (right) [$\varepsilon = 1$]

3. The standard deviations for (a) the perturbation parameter variability $\varepsilon \in [0.8, 1.2]$ – Figure 4 and (b) for the perturbation parameter arbitrarily defined as $\varepsilon = 1$ – Figure 5 (according to the 2^{nd}, 4^{th} and 6^{th} order methods); the data presentation is exact the same like in Figures 2 and 3;

4. The lowest order approximation of the third central probabilistic moments as the function of both perturbation parameter and the input coefficient of variation varying as before – see Figure 6;

5. The lowest order approximation of the fourth central probabilistic moments as the function of both perturbation parameter and the input coefficient of variation varying as before – in Figure 7.

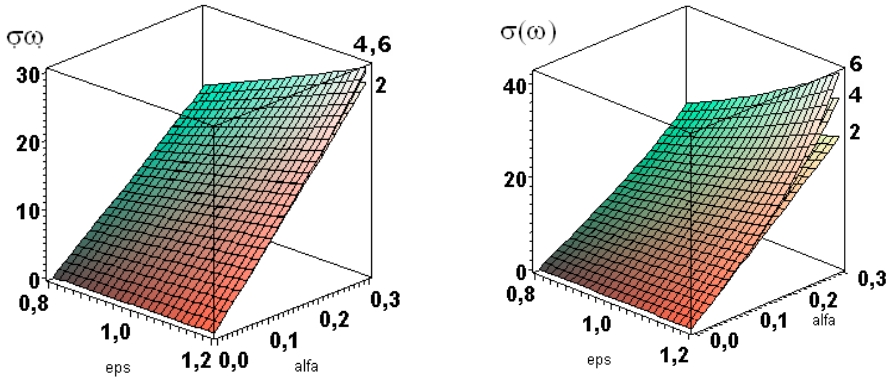

Fig. 4. The standard deviations for the 1 D.O.F. eigenvalue for the random stiffness (left) and the random mass (right)

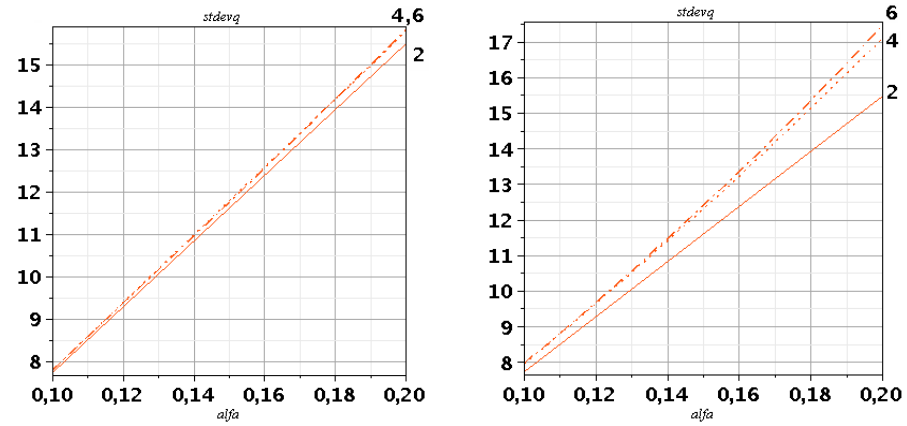

Fig. 5. The standard deviations for the 1 D.O.F. eigenvalue for the random stiffness (left) and the random mass (right)

The expected values shown in Figure 1 nonlinearly decrease together with an increase of the input coefficient of variation of the spring stiffness and systematically and monotonously increase together with an increase of the input coefficient of variation of the random mass. Those changes become more transparent for each next order of the perturbation analysis, however the differences between the neighboring

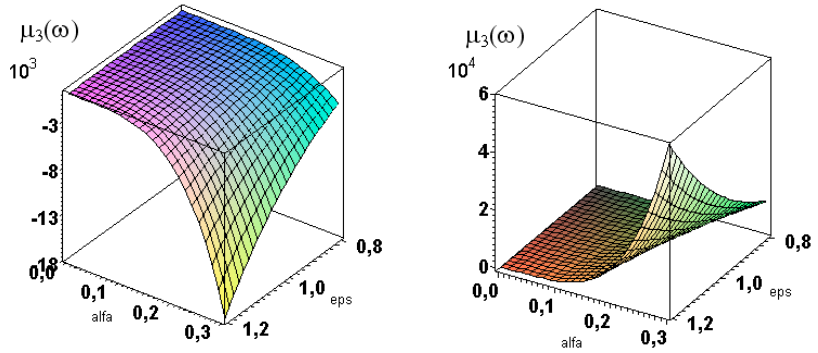

Fig. 6. The third central probabilistic moments for the 1 D.O.F. eigenvalue for the random stiffness (left) and the random mass (right)

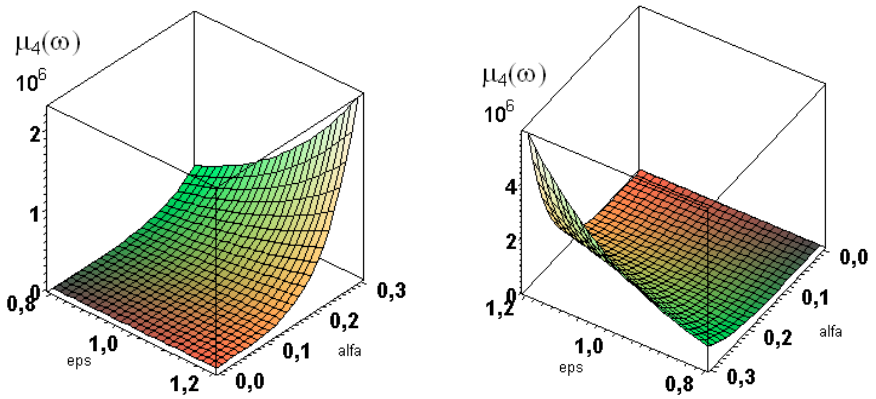

Fig. 7. The fourth central probabilistic moments for the 1 D.O.F. eigenvalue for the random stiffness (left) and the random mass (right)

stochastic expansion orders systematically vanish. It means that the proposed technique based on the stochastic perturbation analysis the Response Function Method converges rather fast even for the very large random dispersion of the input variable as 0.3 in this figure. The variances showed in Figure 3 shows a little different character – as one may expect both variances increase monotonously together with an increase of the input coefficient of variation. Those increases become larger together with the perturbation order taken in the analysis and, as before, the differences between the neighboring orders systematically decrease, however not so fast as in the case of the expectations. Figure 2 brings the information about both variances two-parametric behavior including the perturbation parameter changes. All the three surfaces corresponding to the 2nd, 4th and 6th order computations show that the larger perturbation parameter the larger final value of the variance. The only exception is obtained when the input coefficient of variation equals 0 but this is the limiting deterministic case,

when the perturbation does not matter at all. Let us note that the differences between the neighboring orders are larger for the randomized mass than for the randomized spring stiffness. Analogous observations can be drawn for the standard deviations shown in Figures 4 and 5 quite similarly to the presentation of the variances in Figures 2 and 3, accordingly. The only difference is that the standard deviations do not change so nonlinearly as the variances before. The essential changes are observed for the third central probabilistic moments (Figure 6), where the spring stiffness results in the negative values and the randomized mass of the system gives the positive values. Therefore, the probability density function of the system eigenvalue is non-symmetric in both cases and demonstrates negative skewness for the spring stiffness (more probability mass below the expected value) and the positive skewness for the random mass. The absolute values of the third moments for the randomized mass are about two times larger than for the random spring stiffness. The last figure shows the fourth central probabilistic moments, where their values monotonously and nonlinearly increase together with the increases of the perturbation parameter and the input coefficient of variation. Contrary to the previous case, the fourth central moment must be positive everywhere, but similarly to the third moments, the values obtained for the random mass as twice as much as those computed for the randomized spring stiffness. Finally, let us note that we can easily extract from Figures 6 and 7 the coefficients of asymmetry and kurtosis for probability density functions of the eigenvalue by dividing them by the third and fourth powers of the standard deviations, accordingly. Then, the comparison with the Gaussian probability density function would be more transparent.

5.2. The elastic 3D truss structure

Computational analysis has been tested on the example of the telecommunication tower given schematically in Figure 8 with the height equal to 5.0 meters.

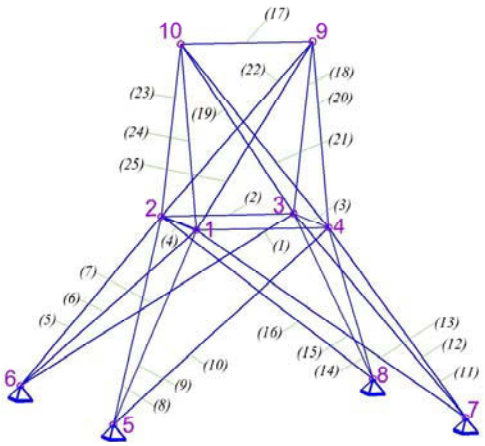

Fig. 8. Structural scheme of the transmission tower

Table 1. Input data for the particular structural members

Element no.	Cross-sectional area [m^2]	Mass density [kg/m]
1	6.450E–06	1.780E–02
2	9.030E–06	2.500E–02
3	6.450E–06	1.780E–02
4	9.030E–06	2.500E–02
5	6.322E–04	1.750E+00
6	1.135E–03	3.143E+00
7	1.574E–03	4.357E+00
8	6.322E–04	1.750E+00
9	1.135E–03	3.143E+00
10	1.574E–03	4.357E+00
11	6.322E–04	1.750E+00
12	1.135E–03	3.143E+00
13	1.574E–03	4.357E+00
14	6.322E–04	1.750E+00
15	1.135E–03	3.143E+00
16	1.574E–03	4.357E+00
17	2.129E–05	5.890E–02
18	1.300E–03	3.598E+00
19	1.821E–03	5.041E+00
20	1.300E–03	3.598E+00
21	1.821E–03	5.041E+00
22	1.300E–03	3.598E+00
23	1.821E–03	5.041E+00
24	1.300E–03	3.598E+00
25	1.821E–03	5.041E+00

The entire structures has been discretized using the two-noded 25 linear space structure finite elements (3D truss elements) joined in 10 nodal points. All the structural members have been manufactured with the stainless steel with Young modulus equal to E = 210 GPa. The cross-sectional areas and mass densities for all those members have been collected in Table 1.

Table 2a. Probabilistic moments of the eigenvalues for the telecommunication tower for the randomized Young modulus

Eigenvalue number	The expected values (2)	3^{rd} order moments	4^{th} order moments
1	22.063E5	–0.076E11	0.521E16
2	34.377E5	–0.180E11	0.308E17
3	70.148E5	–0.196E12	0.532E18
4	75.312E5	–0.264E12	0.710E18
5	95.066E5	–0.805E12	0.180E19
6	114.884E5	–0.990E12	0.388E19
7	136.843E5	–0.626E12	0.773E19
8	152.818E5	–0.290E13	0.120E20
9	157.610E5	–0.439E13	0.143E20
10	226.452E5	–0.851E15	0.243E21

Two groups of numerical tests have been performed (a) with randomized Young modulus of the tower, (b) with randomized mass density of the structural members of it. Each time the parameters collected in Table 1 were treated as the expected values (or their deterministic counterparts), while the coefficient of variation was assumed as the parameter in the perturbation-based expansion and belonged to the interval [0.10, 0.20].

The results of computational analysis are presented in Tables 2a, 2b and 3a, 3b as well as in Figures 9–14.

Table 2b. Probabilistic moments of the eigenvalues for the telecommunication tower for the randomized Young modulus

Eigenvalue number	Expected values (2)	Expected values (4)	Expected values (6)	Expected values (8)	Expected values (10)
1	22.063E5	22.063E5	22.063E5	22.063E5	22.063E5
2	34.377E5	34.377E5	34.376E5	34.377E5	34.377E5
3	70.148E5	70.148E5	70.149E5	70.147E5	70.148E5
4	75.312E5	75.314E5	75.312E5	75.313E5	75.313E5
5	95.066E5	95.069E5	95.065E5	95.067E5	95.067E5
6	114.884E5	114.886E5	114.887E5	114.884E5	114.885E5
7	136.843E5	136.842E5	136.846E5	136.840E5	136.843E5
8	152.818E5	152.823E5	152.817E5	152.820E5	152.819E5
9	157.610E5	157.605E5	157.650E5	157.566E5	157.612E5
10	226.452E5	227.444E5	225.402E5	227.380E5	226.600E5

As we may observe in the attached tables, the expected values of the eigenvalues for the random Young modulus and mass density converge very fast. Of course, this convergence depends on the eigenvalue number being analyzed – the higher eigenvalue the larger differences between its expectations obtained from the neighboring orders of the perturbation. The quality of this method should be further compared against the results of the Monte-Carlo simulation for the same input quantities of random parameters. The comparison of Tables 2 and 3 leads to the second important conclusion that the expected values of all eigenvalues for the randomized Young modulus are somewhat smaller than those obtained for the randomized mass density. This result holds true for the fourth order probabilistic moments of all eigenvalues also. Let us note by the way that all of the probabilistic moments increase together with the eigenvalue number analyzed as it is observed in deterministic analysis. According to the method character, those results are supported by the additional response functions we have collected in Figures 9–11 (1^{st}, 2^{nd} and 4^{th} eigenvalues) for the randomized Young modulus as well as in Figures 13 and 14 (1^{st} and 2^{nd} eigenvalues) for the randomized mass density. Figure 12 shows the expected value of the tenth eigenvalue for the random Young modulus, where the coefficient of variation is taken as the parameter belonging to the interval [0.1, 0.2], out of the second order theory validity.

Table 3a. Probabilistic moments of the eigenvalues for the telecommunication tower for the randomized mass density

Eigenvalue numbers	Expected values (2)	3rd order moments	4th order moments
1	22.134E5	0.615E13	0.304E17
2	34.483E5	0.212E14	0.222E18
3	70.364E5	0.173E15	0.329E19
4	75.532E5	0.188E15	0.511E19
5	95.369E5	0.471E15	0.124E20
6	115.230E5	0.721E15	0.268E20
7	137.257E5	0.124E16	0.498E20
8	153.013E5	0.160E15	0.778E19
9	158.080E5	0.180E16	0.918E20
10	227.243E5	0.510E16	0.422E21

Table 3b. Probabilistic moments of the eigenvalues for the telecommunication tower for the randomized mass density

Eigenvaluenumbers	Expected values (2)	Expected values (4)	Expected values (6)	Expected values (8)	Expected values (10)
1	22.134E5	21.999E5	22.197E5	22.026E5	22.092E5
2	34.483E5	34.297E5	34.555E5	34.344E5	34.420E5
3	70.364E5	69.952E5	70.546E5	70.051E5	70.231E5
4	75.532E5	75.139E5	75.690E5	75.237E5	75.401E5
5	95.369E5	94.814E5	95.600E5	94.950E5	95.186E5
6	115.230E5	114.618E5	115.494E5	114.736E5	115.035E5
7	137.257E5	136.506E5	137.566E5	136.693E5	137.009E5
8	153.013E5	152.861E5	152.954E5	152.920E5	152.926E5
9	158.080E5	157.224E5	158.443E5	157.433E5	157.800E5
10	227.243E5	226.054E5	227.736E5	226.349E5	226.852E5

As it can be recognized from Figures 9 and 10, the polynomium obtained for the first and second eigenvalues is very smooth (almost linear), so that is guarantees a reliable determination of its derivatives (no saddle points and fast oscillations).

Figure 11 shows the same tendency for the fourth eigenvalue polynomial response function around the mean value of the randomized parameter. One of the final results of the stochastic perturbation technique is given in Figure 12, where the expected values computed as the function of an input coefficient of variation converge very well together with the perturbation order. It is seen that the difference between the second order approach results (given by the red curve) and the remaining results cannot be neglected. The green curve reflects the fourth order theory, the yellow one – the sixth order approach, the blue and the black curves are plotted for the eight and tenth order theories. Obviously, the differences between the neighboring orders increase together with an increase of the coefficient of variation input value, however any differences between the last two orders cannot be noticed from this figure. Of course, the larger

coefficient of variation, the higher expected value of the tenth eigenvalue being analyzed. It means that the technique presented here is free from a limitation on the input random dispersion and may be reliably applied for any random variables and converges at the same time. At last we compare the response functions for the first and second eigenvalues, where the mass density of the structural elements is randomized.

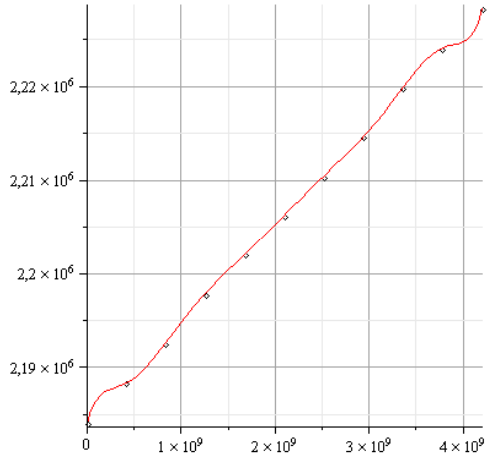

Fig. 9. The approximating polynomium for the first eigenvalues

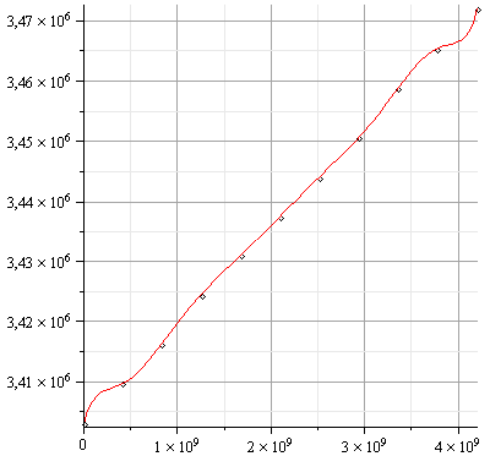

Fig. 10. The approximating polynomium for the second eigenvalues

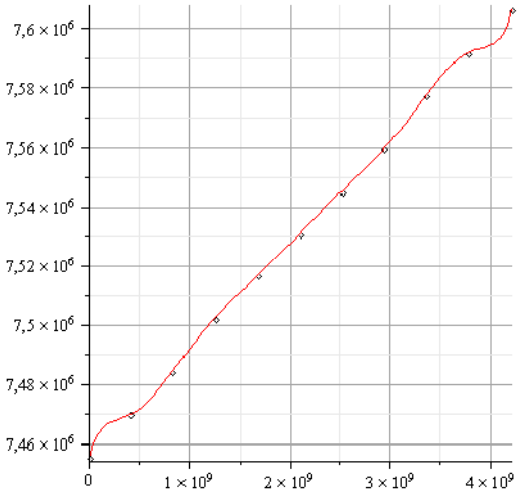

Fig. 11. The approximating polynomium for the fourth eigenvalues

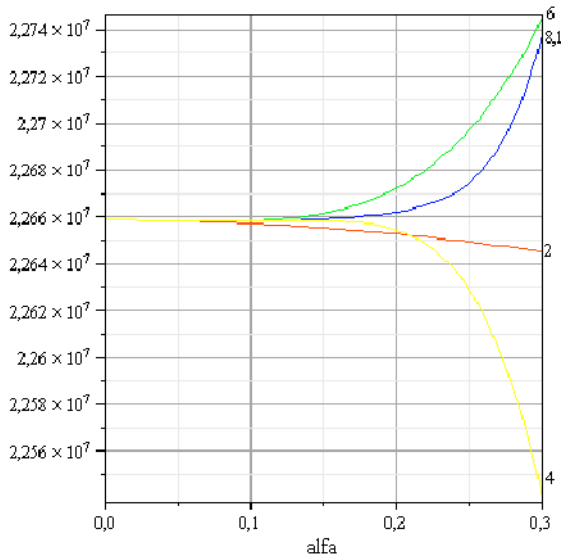

Fig. 12. The expected values for the tenth eigenvalue; random Young modulus

The obtained functions in Figures 13 and 14 are not so smooth like those obtained before, so that the corresponding probabilistic moments may be accompanied by the relatively larger errors.

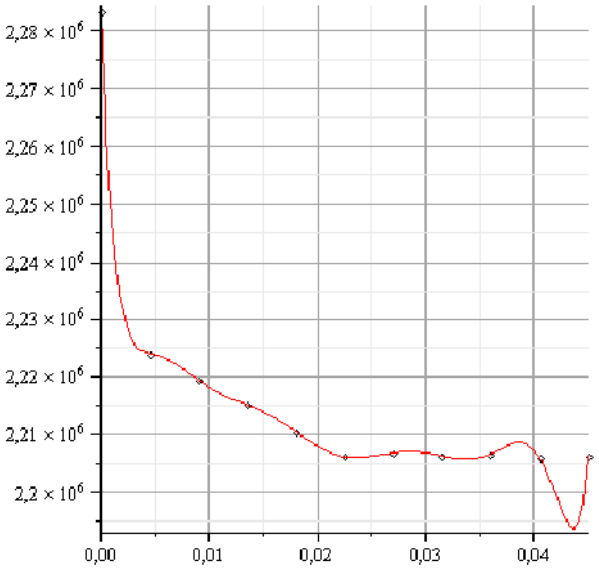

Fig. 13. The response function for the first eigenvalues; random mass density

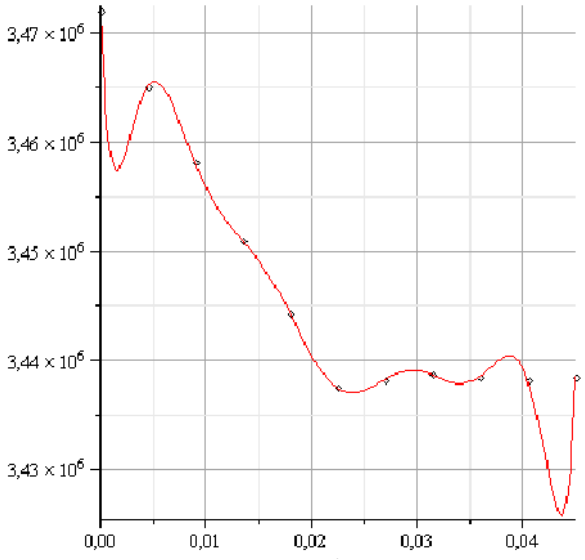

Fig. 14. The response function for the second eigenvalues; random mass density

6. Concluding remarks

1. As it was demonstrated here, the elastodynamic problems with random parameters may be solved efficiently using the generalized Stochastic Finite Element Method. A replacement of the straightforward technique with the response function method enables the relatively easy computations of any probabilistic moments for various eigenfrequencies of the engineering structures. The computational implementation of the traditional Finite Element Method code in conjunction with the symbolic algebra system MAPLE makes it possible to visualize the response functions for various eigenvalues as well as their probabilistic moments as the functions of the initial random input coefficients of variation.

2. This methodology may be further employed for more complex computational problems in stochastic mechanics after a successful comparison against the Monte-Carlo simulations. In particular, an application in the field of reliability engineering analysis seems to be the very promising. The main value of this technique, contrary to the straightforward approach implementation [3–5], is the opportunity to provide its computational realization also for the commercial FEM packages with no access to their source codes.

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Losowe drgania własne konstrukcji wyznaczone metodą funkcji odpowiedzi i uogólnioną metodą perturbacji stochastycznej

Artykuł ukazuje metody analizy konstrukcji pozwalające efektywnie modelować drgania własne konstrukcji przestrzennych z losowym parametrem fizycznym bądź geometrycznym. Całkowita metodologia komputerowa jest oparta na tradycyjnej Metodzie Elementów Skończonych, wzbogaconej metodą perturbacji stochastycznej i jej podejściem *n*-tego rzędu. Komputerowa implementacja została wykonana w programie Metody Elementów Skończonych w powiązaniu z systemem komputerowym algebry symbolicznej MAPLE. W przeciwieństwie do poprzednich rozwiązań bezpośrednich, metoda funkcji odpowiedzi jest zastosowana do obliczeń probabilistycznych momentów dowolnego rzędu i współczynników wartości własnych konstrukcji. Funkcja odpowiedzi jest przyjęta w formie wielomianowej, a współczynniki zostały wyznaczone na podstawie kilku rozwiązań zagadnienia deterministycznego w otoczeniu wartości średniej odpowiedniego parametru losowego. Metoda ta jest zilustrowana na przykładzie stochastycznych drgań własnych prostego układu z jednym stopniem swobody i małej wieży stalowej modelowanej, jako kratowa konstrukcja 3D z losową gęstością masy, a także losowym modułem Younga. Metoda może zostać szeroko zastosowana w analizach niezawodności istniejących konstrukcji inżynierskich przy użyciu komercyjnych programów MES, jak również innych dyskretnych metod obliczeniowych, np. Metody Różnic Skończonych, czy Metody Elementów Brzegowych.