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GUIDED WAVES IN MULTILAYERED PLATES: AN IMPROVED ORTHOGONAL POLYNOMIAL APPROACH**

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ABSTRACT Conventional orthogonal polynomial approach can solve the multilayered plate only when the material properties of two adjacent layers do not change significantly. This paper developed an improved orthogonal polynomial approach to solve wave propagation in multilayered plates with very dissimilar material properties. Through numerical comparisons among the exact solution, the results from the conventional polynomial approach and from the improved polynomial approach, the validity of the improved polynomial approach is illustrated. Finally, it is shown that the conventional polynomial approach can not yield correct continuous normal stress profiles. The improved orthogonal polynomial approach has overcome this drawback.

KEY WORDS multilayered plate, orthogonal polynomial, dispersion curves, stress profiles

I. INTRODUCTION

As early as 1972, orthogonal polynomial approach was developed to solve line acoustic waves in homogeneous semi-infinite wedges[1]. After that, this approach has been used to solve various wave and vibrational problems, from acoustic waves in wedges and ridges[1–3] to surface acoustic waves in layered[4,5] and inhomogeneous[6] semi-infinite structures. Later on, it was extended to investigate Lamb-like guided acoustic waves in multilayered[7] and functionally graded[8] finite-thickness plates.

The polynomial approach has one specificity. It directly incorporates the boundary conditions into the equations of motion by assuming position-dependent material physical constants. The motion equations are then converted into a matrix eigenvalue problem thanks to an expansion of the independent mechanical variables in an appropriate series of orthonormal functions; leading to semi-variational determination of the frequencies of modes and associated profiles. This orthogonal polynomial approach with automatically satisfied boundary conditions is not confined to only flat surfaces but is capable of calculating the vibration modes of curved waveguides. It has been used to calculate axial waves[9,10] and

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circumferential waves[11] in anisotropic functionally graded cylinders. It has also been applied to calculate toroidal waves on the surface of homogeneous[12] and functionally graded[13] spherical curved plates. This polynomial approach is not limited to only either anisotropic elastic media or piezoelectric elastic media. It has also been applied to piezoelectric-piezomagnetic composites to study the magneto-electric coupling effect both in plates[14] and cylinders[15]. Very recently, it has been extended to investigate the generalized thermoelastic waves[16–18] and viscoelastic waves[19,20] in multilayered and graded plates.

From the above simple review, we can see that the orthogonal polynomial approach is highly effective in calculating free guided waves in multilayered plates and functionally graded structures. However, when it is used to solve multilayered plates, there is no significant change in the material properties of two adjacent layers, otherwise the approach would not work satisfactorily. Moreover, to be complete and reliable, the approach must retrieve not only the dispersion curves but also the field profiles. More than anything else, it must, whatever the layer material properties, very similar or not, reliably restitute continuity or discontinuity of any field profile in accordance with the requirements of physical boundary and continuity conditions. The conventional orthogonal polynomial method uses a single polynomial expansion which is continuous in level and in slope over the entire structure even at the frontier between two adjacent layers. This results in level and sloping continuous mechanical displacement distributions and therefore discontinuous stress distributions because of different elastic constants of two adjacent layers. But for such a real structure, the true or physico-mechanical displacement is continuous at the interface between two adjacent layers, but its derivatives are not. These discontinuous derivatives with different elastic constants allow the normal stress components to be continuous.

Considering these points, this paper proposes an improved orthogonal polynomial approach to make it suitable to accurately solve motion equations in multilayered plates whatever the layer material properties, very similar or not. Through numerical comparisons between the exact solution obtained from the transfer matrix method, and the results obtained from the conventional polynomial approach and the improved polynomial approach, the validity of the improved polynomial approach is illustrated. It is also shown that the conventional orthogonal polynomial approach cannot calculate accurately the continuous distribution of the normal stress field, even in multilayered plates with similar layer material properties while the proposed improved polynomial approach has overcome these major drawbacks. In this paper, traction-free boundary conditions are assumed.

II. MATHEMATICS AND FORMULATION OF THE PROBLEM

Consider an orthotropic N-layered plate which is infinitely horizontal with a total thickness h_N. We place the horizontal (x, y)-plane of a cartesian coordinate system on the bottom surface and let the plate be in the positive z-region, as shown in Fig.1, where the medium occupies the region $0 \le z \le h_N$.

For the wave propagation considered in this paper, the body forces are assumed to be zero. Thus, the dynamic equation for the plate is governed by

$$\begin{aligned} \frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z}&=\rho\frac{\partial^2 u_x}{\partial t^2}\\ \frac{\partial T_{xy}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{yz}}{\partial z}&=\rho\frac{\partial^2 u_y}{\partial t^2}\\ \frac{\partial T_{xz}}{\partial x}+\frac{\partial T_{yz}}{\partial y}+\frac{\partial T_{zz}}{\partial z}&=\rho\frac{\partial^2 u_z}{\partial t^2} \end{aligned} \tag{1}$$

where T_{ij}, u_i are the stress and elastic displacements, respectively; ρ is the density of the material.

The relationship between the general strain and general displacement can be expressed as

$$\begin{aligned} &\varepsilon_{xx}=\frac{\partial u_x}{\partial x},\quad \varepsilon_{yy}=\frac{\partial u_y}{\partial y},\quad \varepsilon_{zz}=\frac{\partial u_z}{\partial z},\quad \varepsilon_{yz}=\frac{1}{2}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)\\ &\varepsilon_{xz}=\frac{1}{2}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right),\quad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) \end{aligned} \tag{2}$$

where ε_{ij} is the strain.

The traction-free boundary conditions for a multilayered structure require that: (1) the mechanical displacement and the normal component of stress should be continuous at the interfaces; (2) the normal component of the stress should be zero at the upper and bottom surfaces.

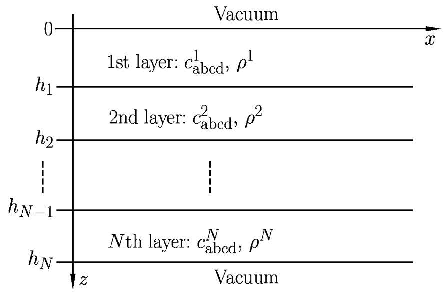


Fig. 1. Schematic diagram of a multilayered plate showing the coordinate system.

By introducing the rectangular window function $\pi_{0,h_N}(z)$

$$\pi_{0,h_N}(z) = \begin{cases} 1, & 0 \le z \le h_N \\ 0, & \text{elsewhere} \end{cases} \tag{3}$$

the stress-free boundary ($T_{zz} = T_{xz} = T_{yz} = 0$ at $z = 0$, $z = h_N$) are automatically incorporated in the constitutive relations of the plate[4]

$$\begin{aligned} T_{xx} &= C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} \\ T_{yy} &= C_{12}\varepsilon_{xx} + C_{22}\varepsilon_{yy} + C_{23}\varepsilon_{zz} \\ T_{zz} &= (C_{13}\varepsilon_{xx} + C_{23}\varepsilon_{yy} + C_{33}\varepsilon_{zz})\pi_{0,h_N}(z) \\ T_{yz} &= 2C_{44}\varepsilon_{yz}\pi_{0,h_N}(z) \\ T_{xz} &= 2C_{55}\varepsilon_{xz}\pi_{0,h_N}(z) \\ T_{xy} &= 2C_{66}\varepsilon_{xy} \end{aligned} \tag{4}$$

where C_{ij} are the elastic coefficients.

For the layered plate they are expressed as

$$C_{ij} = \sum_{n=1}^{N} C_{ij}^n \pi_{h_{n-1},h_n}(z) \tag{5a}$$

where N is the number of the layers and C_{ij}^n is the elastic constant of the Nth material. Similarly, the mass density can be expressed as

$$\rho = \sum_{n=1}^{N} \rho^n \pi_{h_{n-1},h_n}(z) \tag{5b}$$

For a free harmonic plane wave propagating in the x direction in a plate, we assume the displacement components, to be of the form

$$u_x(x,y,z,t) = \exp(\mathrm{i}kx - \mathrm{i}\omega t)U(z) \tag{6a}$$

$$u_y(x,y,z,t) = \exp(\mathrm{i}kx - \mathrm{i}\omega t)V(z) \tag{6b}$$

$$u_z(x,y,z,t) = \exp(\mathrm{i}kx - \mathrm{i}\omega t)W(z) \tag{6c}$$

$U(z)$, $V(z)$, $W(z)$ represent the amplitude of vibration in the x, y, z directions, respectively. k is the magnitude of the wave vector in the propagation direction, and ω is the angular frequency.

By substituting Eqs.(2), (4), (5), (6) into Eq.(1), the governing differential equations in terms of displacement components can be obtained

$$\begin{aligned} &U'' \sum_{n=1}^{N} C_{55}^n \pi_{h_{n-1},h_n}(z) + U' \left(\sum_{n=1}^{N} C_{55}^n \pi_{h_{n-1},h_n}(z) \right)' + \mathrm{i}kW' \left(\sum_{n=1}^{N} C_{13}^n \pi_{h_{n-1},h_n}(z) + \sum_{n=1}^{N} C_{55}^n \pi_{h_{n-1},h_n}(z) \right) \\ &\quad - k^2 U \sum_{n=1}^{N} C_{11}^n \pi_{h_{n-1},h_n}(z) + \mathrm{i}kW \left(\sum_{n=1}^{N} C_{55}^n \pi_{h_{n-1},h_n}(z) \right)' = -\omega^2 U \sum_{n=1}^{N} \rho^n \pi_{h_{n-1},h_n}(z) \end{aligned} \tag{7a}$$

$$V''\sum_{n=1}^{N}C_{44}^{n}\pi_{h_{n-1},h_n}(z)+V'\left(\sum_{n=1}^{N}C_{44}^{n}\pi_{h_{n-1},h_n}(z)\right)'-k^2V\sum_{n=1}^{N}C_{66}^{n}\pi_{h_{n-1},h_n}(z)=-\omega^2V\sum_{n=1}^{N}\rho^n\pi_{h_{n-1},h_n}(z) \tag{7b}$$

$$W''\sum_{n=1}^{N}C_{33}^{n}\pi_{h_{n-1},h_n}(z)+\mathrm{i}kU'\left(\sum_{n=1}^{N}C_{13}^{n}\pi_{h_{n-1},h_n}(z)+\sum_{n=1}^{N}C_{55}^{n}\pi_{h_{n-1},h_n}(z)\right)+W'\left(\sum_{n=1}^{N}C_{33}^{n}\pi_{h_{n-1},h_n}(z)\right)'$$
$$+\mathrm{i}kU\left(\sum_{n=1}^{N}C_{13}^{n}\pi_{h_{n-1},h_n}(z)\right)'-k^2W\sum_{n=1}^{N}C_{55}^{n}\pi_{h_{n-1},h_n}(z)=-\omega^2W\sum_{n=1}^{N}\rho^n\pi_{h_{n-1},h_n}(z) \tag{7c}$$

where the superscript $(\quad)'$ is the partial derivative for z. Obviously, Eq.(7b) is independent of the other two equations. It represents the propagating SH wave. The other two Eqs.(7a) and (7c) control the propagating Lamb-like wave.

To solve the coupled multilayered plate wave equations (7), the conventional orthogonal polynomial approach expands the $U(z)$, $V(z)$, $W(z)$ to three Legendre orthogonal polynomial series[7]

$$U(z)=\sum_{m=0}^{\infty}p_m^1Q_m(z),\quad V(z)=\sum_{m=0}^{\infty}p_m^2Q_m(z),\quad W(z)=\sum_{m=0}^{\infty}p_m^3Q_m(z) \tag{8}$$

where $p_m^i(i=1,2,3)$ are the expansion coefficients and

$$Q_m(z)=\sqrt{\frac{2m+1}{h_N}}P_m\left(\frac{2z-h_N}{h_N}\right) \tag{9}$$

with P_m being the mth Legendre polynomial. Theoretically, m runs from 0 to ∞. In practice, the summation over the polynomials in Eq.(8) can be halted at some finite value $m=M$, when higher order terms become essentially negligible.

As is mentioned above, because of truncation, the conventional orthogonal polynomial approach can only solve the multilayered plate when material properties of two adjacent layers do not change significantly. Here, we improved the orthogonal polynomial approach so as to make it suitable for the multilayered plate with very dissimilar materials. We expand field quantities of each layer to one specific Legendre polynomial

$$\text{for the first layer:}\quad Q_m^1(z)=\sqrt{\frac{2m+1}{h_1}}\,P_m\left(\frac{2}{h_1}z-1\right) \tag{10a}$$

$$\text{for the second layer:}\quad Q_m^2(z)=\sqrt{\frac{2m+1}{h_2-h_1}}\,P_m\left(\frac{2}{h_2-h_1}z-\frac{h_2+h_1}{h_2-h_1}\right) \tag{10b}$$

...

$$\text{for the }N\text{th layer:}\quad Q_m^N(z)=\sqrt{\frac{2m+1}{h_N-h_{N-1}}}\,P_m\left(\frac{2}{h_N-h_{N-1}}z-\frac{h_N+h_{N-1}}{h_N-h_{N-1}}\right) \tag{10c}$$

Therefore, $u_a(a=1,2,3)(u_x,u_y,u_z)$ are expanded as follows (the form is chosen in order to automatically incorporate in the calculation the continuity conditions at the interfaces relative to the components of the mechanical displacement):

$$\begin{aligned}&\text{In the first layer}: \quad u_a^1 = \sum_{m=0}^{\infty} p_{m,1}^a Q_m^1(z)\exp(\mathrm{i}kx)\\&\text{with}: \quad u_a^1(z=h_1) = u_a^{1,h_1} = \sum_{m=0}^{\infty} p_{m,1}^a Q_m^1(z=h_1)\exp(\mathrm{i}kx)\end{aligned} \tag{11a}$$

$$\begin{aligned}&\text{In the 2nd layer}: \quad u_a^2 = u_a^{1,h_1} + (z-h_1)\sum_{m=0}^{\infty} p_{m,2}^a Q_m^2(z)\exp(\mathrm{i}kx)\\&\text{with}: \quad u_a^2(z=h_2) = u_a^{2,h_2} = u_a^{1,h_1} + (h_2-h_1)\sum_{m=0}^{\infty} p_{m,2}^a Q_m^2(z=h_2)\exp(\mathrm{i}kx)\end{aligned} \tag{11b}$$

$$\begin{aligned}&\text{In the third layer}: \quad u_a^3 = u_a^{2,h_2} + (z-h_2)\sum_{m=0}^{\infty} p_{m,3}^a Q_m^3(z)\exp(\mathrm{i}kx)\\&\text{with}: \quad u_a^3(z=h_3) = u_a^{3,h_3} = u_a^{2,h_2} + (h_3-h_2)\sum_{m=0}^{\infty} p_{m,3}^a Q_m^3(z=h_3)\exp(\mathrm{i}kx)\end{aligned} \tag{11c}$$

and so on . . . .

Substituting Eqs.(10) and (11) into Eqs.(7), then multiplied by $Q_j^{1*}(z)$, $Q_j^{2*}(z)\cdots\cdots Q_j^{N*}(z)$, with j running from 0 to M, respectively, integrating over z from 0 to h_N, and taking advantage of the orthonormality of the Legendre polynomial gives the following systems:

$$ {}^nA_{11}^{j,m}p_{m,n}^1 + {}^nA_{12}^{j,m}p_{m,n}^2 + {}^nA_{13}^{j,m}p_{m,n}^3 = -\omega^2\cdot {}^nM_m^j p_{m,n}^1 \tag{12a}$$

$$ {}^nA_{21}^{j,m}p_{m,n}^1 + {}^nA_{22}^{j,m}p_{m,n}^2 + {}^nA_{23}^{j,m}p_{m,n}^3 = -\omega^2\cdot {}^nM_m^j p_{m,n}^1 \tag{12b}$$

$$ {}^nA_{31}^{j,m}p_{m,n}^1 + {}^nA_{32}^{j,m}p_{m,n}^2 + {}^nA_{33}^{j,m}p_{m,n}^3 = -\omega^2\cdot {}^nM_m^j p_{m,n}^1 \tag{12c}$$

where ${}^nA_{\alpha\beta}^{j,m}(\alpha,\beta=1,2,3)$ and ${}^nM_m^j$ are the elements of a non-symmetric matrix. They can be obtained according to Eqs.(7).

Equations (12) can be written as

$$\begin{bmatrix} {}^nA_{11}^{j,m} & {}^nA_{12}^{j,m} & {}^nA_{13}^{j,m}\\ {}^nA_{21}^{j,m} & {}^nA_{22}^{j,m} & {}^nA_{23}^{j,m}\\ {}^nA_{31}^{j,m} & {}^nA_{32}^{j,m} & {}^nA_{33}^{j,m}\end{bmatrix}\begin{Bmatrix} p_{m,n}^1\\ p_{m,n}^2\\ p_{m,n}^3\end{Bmatrix} = -\omega^2\begin{bmatrix} {}^nM_m^j & 0 & 0\\ 0 & {}^nM_m^j & 0\\ 0 & 0 & {}^nM_m^j\end{bmatrix}\begin{Bmatrix} p_{m,n}^1\\ p_{m,n}^2\\ p_{m,n}^3\end{Bmatrix} \tag{13}$$

So, Eqs.(13) yields a form of the eigenvalue problem. The eigenvalue ω^2 gives the angular frequency of the guided wave; eigenvectors $p_{m,n}^i(i = 1, 2, 3)$ allow the components of the particle displacement to be calculated. According to $V_{\rm ph} = \omega/k$ and $V_{\rm g} = {\rm d}\omega/{\rm d}k$, the phase velocity and group velocity can be obtained. The complex matrix Eq.(13) can be solved numerically making use of standard computer programs for the diagonalization of non-symmetric square matrices. $3N(M+1)$ eigenmodes are generated from the order M of the expansion. Acceptable solutions are those eigenmodes for which convergence is obtained as M is increased. It is asserted that the eigenvalues obtained are converged solutions when further increase in the matrix dimension does not result in a significant change in the eigenvalue.

III. NUMERICAL RESULTS

Based on the foregoing formulations, computer programs in terms of both the conventional polynomial approach and the improved polynomial approach have been written using Mathematica to calculate the dispersion curves for the layered plates.

Table 1. **The material properties of the bilayer plate**

Property	C_{11}	C_{13}	C_{33}	C_{55}	ρ
steel	282	113	282	84	7.932
brass	162.6	81.3	162.6	40.7	8.4

Units: $C_{ij}(10^9\ \mathrm{N/m^2})$, ρ ($10^3\ \mathrm{kg/m^3}$).

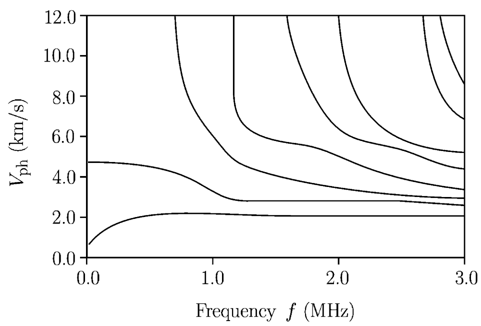
Table 2. **The material properties of the sandwich plate**

Property	C_{11}	C_{22}	C_{33}	C_{12}	C_{13}	C_{23}	C_{44}	C_{55}	C_{66}	ρ
Middle layer	281	349	294	126	84	88	108	132	131	3.59
Top/bottom layer	28.1	34.9	29.4	18.9	12.6	13.2	12.96	15.84	15.72	1.795

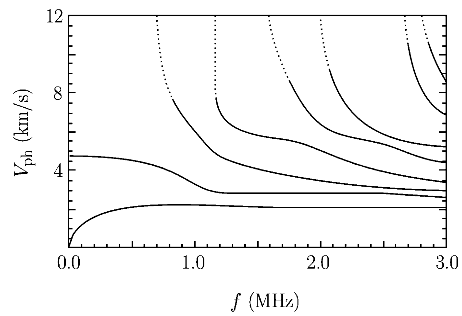
Units: $C_{ij}(10^9\ \mathrm{N/m^2})$, ρ ($10^3\ \mathrm{kg/m^3}$).

3.1. Comparison with the Exact Solution from the Transfer Matrix Method

Firstly, for validation, we calculated a two-layered metal plate to make a comparison between the exact solution obtained from the transfer matrix method and both the conventional and improved polynomial approaches' results. The plate is composed of stainless steel (1 mm thick) and brass (1 mm thick). Their material constants are shown in Table 1. Figure 2(a) shows the exact solution Lamb-like wave dispersion curves from the transfer matrix method. Figure 2(b) is obtained from the conventional polynomial approach. The solution of the improved polynomial approach is the same as in Fig.2(b). In order to save space, it is not shown here. As can be seen, for the two-layered metal plate, the conventional and improved polynomial approaches can yield correct dispersion curves.



(a) Results from the transfer matrix method



(b) Results from the conventional polynomial method

Fig. 2. Phase velocity dispersion curves for the stainless steel-brass two-layer plate.

Next, we show an example of an equal thickness (1 m) three-layer sandwich plate with very dissimilar materials. The material constants of the three-layer materials are shown in Table 2. Figure 3 shows the dispersion curves of the exact solutions from the transfer matrix method using solid lines, the solutions from the improved polynomial approach using dotted lines, and the solutions from the conventional polynomial approach using dashed lines. It can be seen that solid lines and dotted lines agree very well. Dashed lines exhibit differences from solid lines and dotted lines. This illuminates the validity of the

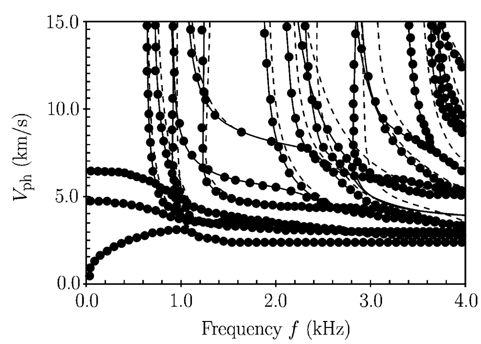


Fig. 3. Phase velocity dispersion curves for the three layer sandwich plate with very dissimilar material: solid lines, from the transfer matrix method; dotted lines, from the improved polynomial method; dashed lines, from the conventional polynomial method.

improved polynomial approach.

3.2. Stress Profiles

This section shows stress profiles for the above two-layered structures. Figures 4 and 5 give the stress profiles of the first two Lamb-like wave modes for the two layer metal plate and the sandwich plate, respectively. The first and second modes are given respectively on the left- and right-hand sides of the figures. From these figures we can see that even for the two-layer metal plate with similar materials, the conventional polynomial approach can not give correct results. The normal stress components T_{xz} and T_{zz} obtained are discontinuous at the interfaces and not zero at the bottom and top surfaces. The computational discontinuity obtained, theoretically non expected from the physical point of view, is the result, as mentioned at the beginning of the paper, of a combination of the following two factors (i) the use, for each mechanical displacement component, of a single truncated polynomial expansion which applies all over the multilayered structure and is unconditionally continuous both in level and in slope and (ii) dissimilar layer material properties with, for any property, a jump at every interface which ends up, in the transition from one layer to the next, in a jump in normal stress components. It can

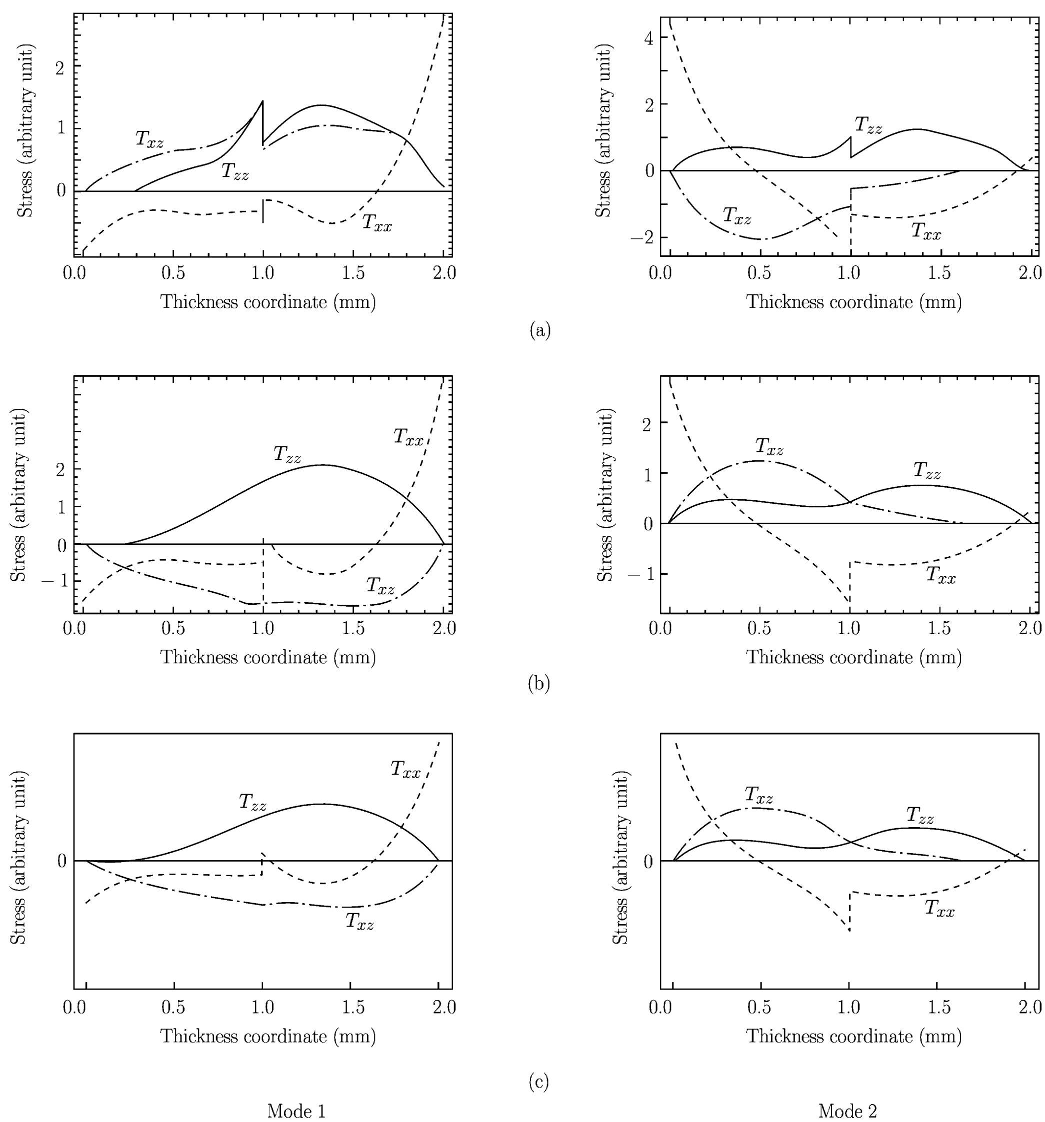


Fig. 4. Stress profiles for the steel-brass two-layer plate at $kh = 6.6$; (a) from the conventional polynomial method, (b) from the improved polynomial method, (c) from the transfer matrix method.

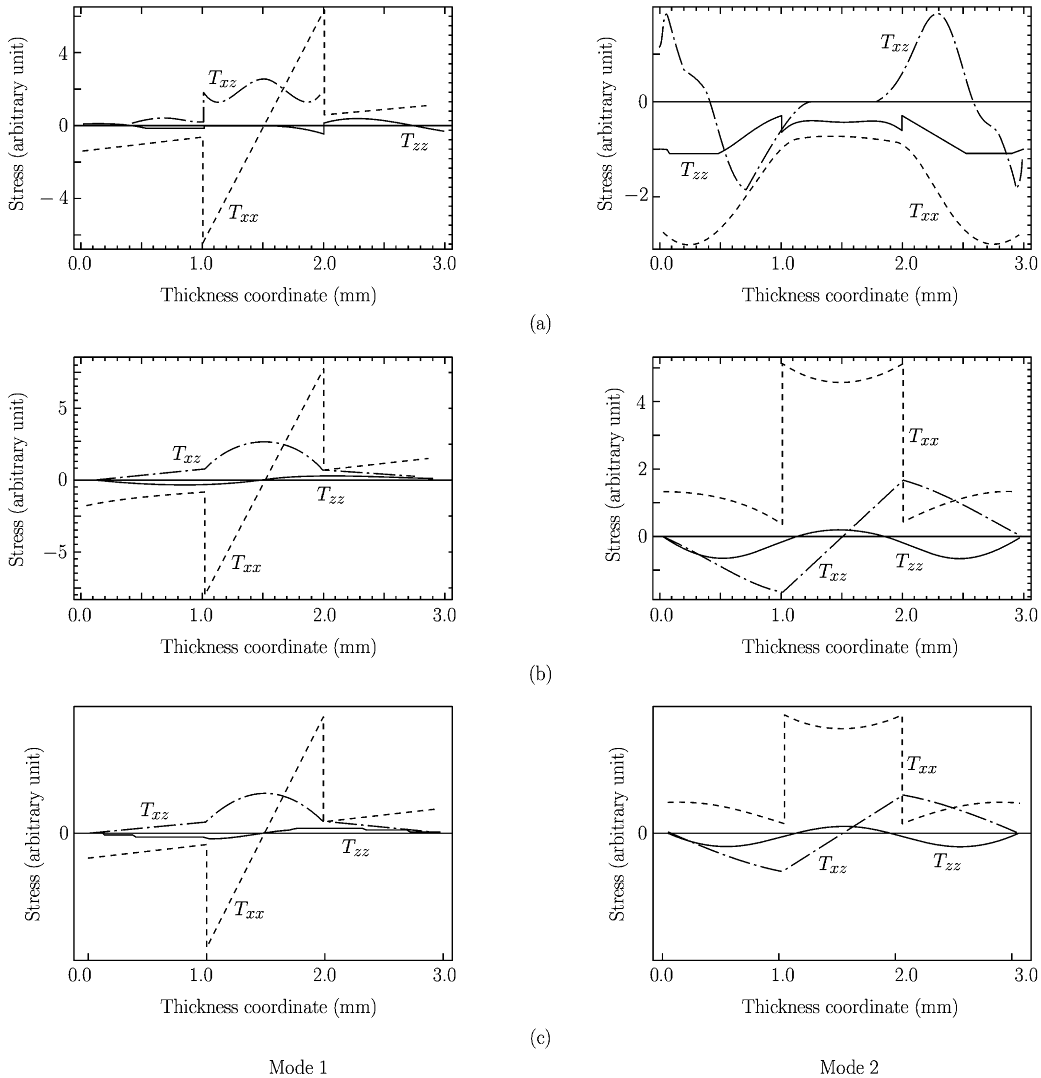


Fig. 5. Stress profiles for the sandwich plate at $kh = 3.3$. (a) from the conventional polynomial method, (b) from the improved polynomial method, (c) from the transfer matrix method.

be seen from Figs.4(b) and 5(b) that the improved polynomial approach has completely overcome this drawback of the conventional polynomial approach. Figures 4(c) and 5(c) also give the exact solutions from the transfer matrix method, which are consistent with the results of the improved method.

IV. CONCLUSIONS

Considering the drawbacks of the conventional orthogonal polynomial approach to solving multilayer plates, this paper improved the orthogonal polynomial approach to make it suitable to solve the multilayered plate whatever the dissimilarities of the layer material properties. Dispersion curves and field profiles, continuous or not, are accurately restituted.

To sum up, we can anticipate three prospects for the proposed approach:

(a) The improved approach can be extended to solve various multi-field coupled multilayered structures, such as piezoelectric multilayered structures, magneto-electro-elastic ones, and so on.

(b) The improvement of the Legendre orthogonal polynomial approach can be transposed to the Laguerre orthogonal polynomial for solving semi-infinite structures.

(c) The improved approach can be extended to deal with curved multilayered structures, such as hollow cylinders, spherical curved plates, and so on.

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