Journal of Nonlinear Mathematical Physics (2024) 31:55
https://doi.org/10.1007/s44198-024-00218-9

RESEARCH ARTICLE

Symmetries and Separation of Variables

V. Rosenhaus[1] · **Ravi Shankar**[2] · **Cody Squellati**[1]

Received: 6 June 2023 / Accepted: 15 July 2024

Abstract

In this paper, we look at the method of separation of variables of a PDE from its symmetry transformation point of view. Specifically, we discuss the relation between the existence of additively and multiplicatively separated variables of a PDE, and the form of its symmetry operators. We show that solutions in the form of separated variables are in fact, invariant solutions, i.e. solutions invariant under some subalgebra of the symmetry operators of the equation. For the case of two independent variables, we obtain the form of Lie point symmetry operators corresponding to additively and multiplicatively separated solutions, and generalize our results for the case when separated variables are any functions of independent variables. We also discuss the role of contact symmetry transformations and differential invariants for the existence of separated solutions, and outline the role of variational symmetries, as well as conditional (non-classical) symmetry operators. We demonstrate that the symmetry approach is a valuable tool for obtaining information regarding existence of solutions with separated variables.

Keywords PDE · Separation of variables · Lie symmetry · Exact solutions

Mathematics Subject Classification 35Qxx · 35C05 · 17B81 · 35B06 · 22E70

V. Rosenhaus, Ravi Shankar and Cody Squellati contributed equally to this work.

✉ Ravi Shankar
rs1838@princeton.edu

V. Rosenhaus
vrosenhaus@csuchico.edu

Cody Squellati
csquellati@hotmail.com

1 Department of Mathematics and Statistics, California State University Chico, 400 W. First St., Chico, CA 95929, USA

2 Department of Mathematics, Princeton University, Washington Rd., Princeton, NJ 08544, USA

Published online: 08 August 2024

1 Introduction

The method of separation of variables plays a central role in many topics concerning differential equations, and there is a large number of works related to various aspects of this method. But, in spite of being studied for a very long time, the problem of separation of variables for partial differential equations in general has not been completely solved, even for integrable systems, see, e.g. [26].

Different aspects of the relationship between separation of variables and symmetry properties of a differential system received a lot of attention in the literature; here, we give only some such references: [6, 11, 16, 17, 25, 28, 30] (new coordinate systems, nonlinear separation) [4, 5, 18, 21–23, 29] (functional or generalized separation). The monographs [11, 16, 22], as well as [19], have a lot of relevant information and numerous references.

However, despite many extensive studies, a direct relation between symmetry operators of a general differential system and its solutions in the form of separated variables has not been discussed in the literature so far, and except for some special cases (e.g., Hamilton-Jacobi or Helmholtz equations), such relation did not appear to be well understood [19].

In this paper, we study this relation, and discuss the form of (a subalgebra of) symmetry operators whose invariant solutions are in the form of separated variables. More precisely, given a symmetry of a certain type to be discussed, we show that one can predict the existence of a solution with separated variables. We will mainly be interested in the role of classical point symmetries for generation of invariant solutions in separated variables, but we will also make some observations regarding the role of contact symmetries, differential invariants, as well as variational and conditional symmetries in generation of separated solutions.

In our discussion we show that solutions of many differential equations in the form of separated variables are in fact, invariant solutions, i.e. solutions invariant under some symmetry group operators, and we demonstrate that many important solutions of known equations can be recovered from a symmetry approach.

Let us note the essential differences between our discussion and a well-known Miller–Kalnins approach [11, 16]. Even though both approaches are based on introduction of new coordinates, Miller–Kalnins approach requires the eigenfunctions of the symmetry operators to be in the form of separated variables in new coordinates. Our consideration is free from such a requirement. Miller–Kalnins approach is applicable to linear homogeneous differential equations only. Our approach is more general, is applicable to both linear and nonlinear differential equations, and our main interest is in nonlinear PDE's.

Let us also note that the study of a direct relation between symmetry operators of a differential system and its separated solutions is an important step in understanding of the nature of the method of separation of variables and its solutions.

The paper is organized as follows: In Sect. 2 we review some basic facts regarding classical Lie point symmetries and contact symmetries, as well as separation conditions. In Sect. 3, we derive conditions for separation of variables. We start with the discussion of separation conditions in original variables, (Sect. 3.1), consider separation in an alternate coordinate system (Sect. 3.2), and then derive separation conditions in general "point" coordinates $\varphi(x,t)$, and $\psi(x,t)$, (Sect. 3.3). We give several characteristic examples for each case. In Sects. 3.4, and 3.5 we generalize our results for equations with several dependent variables, and demonstrate applications for Einstein equation, Ricci flow, and Navier-Stokes equations. We discuss the role of contact symmetries, and differential invariants for existence of solutions in separated variables in Sects. 3.6, and 3.7, respectively. In Sect. 4 we consider the situation when an appropriate symmetry of a differential equation together with the existence of a constant solution leads to a separated solution. Section 5 deals with variational problems where its variational symmetries lead to solutions in separated variables. In Sect. 6 we discuss generation of separated solutions using conditional (non-classical) symmetry operators.

2 Overview

2.1 Point Symmetries of Differential Equations

We review some necessary facts about symmetries of a system of differential equations. For a comprehensive review, see e.g., [20] or [19].

Let $\Delta[u]=0$ be a kth order system of differential equations determining m unknown functions $u=\{u_1,\dots,u_m\}$ of n independent variables $x=\{x^1,\dots,x^n\}$. Consider the infinitesimal point transformations given by:

$$\begin{aligned} x^i &\to f^i(x,u;\varepsilon) \simeq x^i + \xi^i(x,u)\varepsilon + O(\varepsilon^2),\\ u^a &\to g^a(x,u;\varepsilon) \simeq u^a + \eta^a(x,u)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{2.1}$$

for $f^i, g^a \in C^\infty$ and sufficiently small ε, that leaves Δ invariant. Each f^i and g^a are the flows of the symmetry

$$\mathbf{X} = \xi^i\,\partial_i + \eta^a\,\partial_a + \eta^a_{j_1\dots j_\ell}(x,u,\,\partial u,\dots,\,\partial_k u)\,\partial_{a,j_1\dots j_\ell}, \tag{2.2}$$

$$\partial_i = \frac{\partial}{\partial x^i},\quad \partial_a = \frac{\partial}{\partial u^a},\quad \partial_{a,j_i\dots j_k} = \frac{\partial}{\partial u^a_{j_1\dots j_k}}, \tag{2.3}$$

$$u^a_{j_1\dots j_\ell} = \frac{\partial^\ell u^a}{\partial x^{j_\ell}\dots\partial x^{j_1}},\qquad \partial_\ell u = \{u^a_{j_1\dots j_\ell}\ :\ \forall a, j_b\}, \tag{2.4}$$

where

$$\eta^a_j = D_j(\eta^a) - u^a_i D_j(\xi^i), \tag{2.5}$$

$$\eta^a_{j_1 j_2} = D_{j_1}(\eta^a_{j_2}) - u^a_{ij_2} D_{j_1}(\xi^i), \tag{2.6}$$

$$\vdots$$
$$\eta^a_{j_1\dots j_k} = D_{j_1}(\eta^a_{j_2\dots j_k}) - u^a_{ij_2\dots j_k} D_{j_1}(\xi^i), \tag{2.7}$$

for any fixed $1 \le \ell \le k$, and,

$$D_i = \partial_i + u^a_i\,\partial_a + u^a_{ji}\,\partial_{a,j} + \cdots + u^a_{j_1\dots j_\ell i}\,\partial_{a,j_1\dots j_\ell} + \cdots . \tag{2.8}$$

Operator $\mathbf{X}$ (2.2) is called a point symmetry of the system of differential equations $\Delta[u] = 0$ if and only if

$$\mathbf{X}(\Delta)|_{\Delta=0} = 0. \tag{2.9}$$

2.2 Contact Symmetries

Denote

$$\begin{aligned} p &= u_x, \\ q &= u_t. \end{aligned} \tag{2.10}$$

A *contact vector field* $\mathbf{X}_\alpha$ with infinitesimal $\alpha = \alpha(x, t, u, p, q)$

$$\mathbf{X}[\alpha] = -\alpha_p\,\partial_x - \alpha_q\,\partial_t + (\alpha - p\alpha_p - q\alpha_q)\,\partial_u + (\alpha_x + p\alpha_u)\,\partial_p + (\alpha_t + q\alpha_u)\,\partial_q \tag{2.11}$$

is called a *contact symmetry* of the system $\Delta[u] = 0$ if and only if its prolongation pr $\mathbf{X}[\alpha]$ (definition omitted, as explained below) satisfies

$$\text{pr}\,\mathbf{X}[\alpha]\Delta\big|_{\Delta=0} = 0. \tag{2.12}$$

Contact vector fields are closed under commutation:

$$\begin{aligned} \big[\mathbf{X}[\alpha]\,,\,\mathbf{X}[\beta]\big] &= \mathbf{X}\big[\Phi(\alpha,\beta) - \Phi(\beta,\alpha)\big], \\ \Phi(\alpha,\beta) &= \alpha_p\beta_x + \alpha_q\beta_t + \beta\alpha_u + (p\alpha_p + q\alpha_q)\beta_u. \end{aligned} \tag{2.13}$$

As a consequence, if $A = <\mathbf{X}[\alpha^i]>_{i=1}^a$ is a contact Lie algebra, there exist structure constants $C^k_{ji} = -C^k_{ij}$ such that the following differential equations hold:

$$\Phi(\alpha^i, \alpha^j) - \Phi(\alpha^j, \alpha^i) = \sum_{k=1}^a C^k_{ij}\alpha^k, \qquad i,j = 1, \dots, a. \tag{2.14}$$

The structure constants must satisfy the Jacobi identity:

$$\sum_{k=1}^a \left[C^k_{ij}C^m_{k\ell} + C^k_{\ell i}C^m_{kj} + C^k_{j\ell}C^m_{ki}\right] = 0, \qquad i,j,\ell,m = 1, \dots, a. \tag{2.15}$$

In the special case that $\mathbf{X}[\alpha]$ is an infinitesimal *point transformation*, or $\alpha = \eta - p\xi - q\tau$, where η, ξ, and τ are functions of (x, t, u), it takes the form

$$\mathbf{X}[\alpha] = \xi\,\partial_x + \tau\,\partial_t + \eta\,\partial_u + (D_x\eta - pD_x\xi - qD_x\tau)\,\partial_p + (D_t\eta - pD_t\xi - qD_t\tau)\,\partial_q. \tag{2.16}$$

In this case, we identify $\mathbf{X}[\alpha]$ with its first three terms:

$$\mathbf{X} = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \eta(x,t,u)\,\partial_u. \tag{2.17}$$

We will also have occasion to use a *vertical* or *evolutionary* formulation of contact vector fields. For each prolonged $\mathbf{X}[\alpha]$, we define

$$\begin{aligned} \mathbf{X}_\alpha &= \mathrm{pr}\,\mathbf{X}[\alpha] + \alpha_{u_x} D_x + \alpha_{u_t} D_t, \\ D_x &= \partial_x + u_x\,\partial_u + u_{xx}\,\partial_{u_x} + u_{xt}\,\partial_{u_t} + \dots, \\ D_t &= \partial_t + u_t\,\partial_u + u_{xt}\,\partial_{u_x} + u_{tt}\,\partial_{u_t} + \dots, \end{aligned} \tag{2.18}$$

such that

$$\mathbf{X}_\alpha = \alpha\,\partial_u + D_x\alpha\,\partial_{u_x} + D_t\alpha\,\partial_{u_t} + \dots \tag{2.19}$$

When prolonging contact vector fields, using $\mathbf{X}_\alpha$ is usually more convenient than using $\mathbf{X}[\alpha]$. Moreover, as shown in [19],

$$\mathrm{pr}\,\mathbf{X}[\alpha]\Delta\big|_{\Delta=0} = 0 \text{ if and only if } \mathbf{X}_\alpha\Delta\big|_{\Delta=0} = 0, \tag{2.20}$$

so the two vector fields satisfy the same symmetry condition. Thus, a vertical field $\mathbf{X}_\alpha$, such that

$$\mathbf{X}_\alpha\Delta\big|_{\Delta=0} = 0 \tag{2.21}$$

is also called *contact symmetry*.

The following obvious lemma is useful.

Lemma 1 *If $F(x,t,T(t),T'(t)) = 0$ for any smooth function $T(t)$, then $F(x,t,u,q) = 0$ for each $(x,t,u,q) \in \mathbb{R}^4$.*

Proof If there exists $(x_0,t_0,u_0,q_0) \in \mathbb{R}^4$ such that $F(x_0,t_0,u_0,q_0) \neq 0$, then choose $T(t) = q_0(t-t_0) + u_0$ to arrive at a contradiction. □

2.3 Separation Conditions

We will primarily consider the case of a scalar differential equation $\Delta[u] = 0$ for function $u = u(x,t)$.

By separation of variables in the original coordinates $(x,\ t,\ u)$, we will mean either additive separation:

$$u(x,t) = \Phi(x) + \Psi(t), \tag{2.22}$$

or multiplicative separation

$$u(x,t) = \Phi(x)\Psi(t). \tag{2.23}$$

We will call solutions derived from these conditions *Separation Solutions*.

If $u(x, t)$ is an additive separated solution (2.22) then

$$v = e^u = e^{\Phi(x)+\Psi(t)} = e^{\Phi(x)}e^{\Psi(t)} = \tilde{\Phi}(x)\tilde{\Psi}(t), \tag{2.24}$$

is a multiplicative separated solution for any smooth functions $\Phi(x)$ and $\Psi(t)$.

Let us consider the *additive separation* (2.22). In this case we have:

$$u_{xt} = 0. \tag{2.25}$$

Applying the transformation

$$u \to \ln(u), \tag{2.26}$$

we can find the *multiplicative separation* condition similar to (2.25):

$$uu_{xt} - u_x u_t = 0. \tag{2.27}$$

Imposing the (additive) separation condition for the scalar equation $\Delta(x, t, u) = 0$ results in the overdetermined system:

$$\begin{aligned} &\Delta(x,t,u) = 0, \\ &u_{xt} = 0. \end{aligned} \tag{2.28}$$

We may consider solutions separated with respect to more general variables $p = p(x, t)$ and $q = q(x, t)$.

3 Invariant Solutions and Separated Variables

Can symmetry tell us if solutions with separated variables exist for a given differential equation? We will start with classical Lie point symmetry groups and see if invariant solutions lead to solutions with separated variables.

We will mainly consider *additive* separation, $u = X(x) + T(t)$ or $u_{xt} = 0$. Note that the case of *multiplicative separation* $u(x, t) = X(x)T(t)$ can be reduced to additive separation by a simple change of variables, $u = \ln|v|$: $\ln|v(x, t)| = \ln|X(x)| + \ln|T(t)|$ (for some nonzero functions $X(x)$ and $T(t)$), such that $(\ln|v|)_{xt} = 0$. Note that the operator $\mathbf{X} = \partial_u$ in the additive framework would correspond to $\mathbf{X} = v\,\partial_v$ in the multiplicative one.

3.1 Separation in (x,t) Coordinates

Consider the vector field

$$\mathbf{X} = \partial_t + \partial_u. \tag{3.1}$$

For each function $g(x)$, the separated function

$$u = g(x) + t, \tag{3.2}$$

is invariant under the action of **X**. Indeed,

$$\mathbf{X}(u - X(x) - t)\big|_{u=X(x)+t} = 1 - 1 = 0. \tag{3.3}$$

Conversely, if $u = f(x, t)$ is an invariant of the operator **X**, then

$$\mathbf{X}(u - f(x, t))\big|_{u=f} = 1 - f_t(x, t) = 0. \tag{3.4}$$

This implies that

$$f(x, t) = g(x) + t, \tag{3.5}$$

with any function $g(x)$. Therefore, *all* invariants of the operator **X** (3.1) are of the separated form (3.2).

Since solutions invariant under classical Lie point symmetries are known to exist in the great majority of regular cases, solutions in separated variables in most situations would also exist.

Let us consider a more general symmetry vector field of the form

$$\mathbf{X} = A(x, t, u)\big[\,\partial_t + T'(t)\,\partial_u\big], \tag{3.6}$$

where A and T are some functions, and $A \neq 0$. The form of solutions invariant under operator **X** can be found by solving an invariant surface condition:

$$\mathbf{X}\big(u - F(x, t)\big)\big|_{u=F} = 0. \tag{3.7}$$

Explicitly,

$$\big[A(x, t, u)\,T'(t) - A(x, t, u)F_t(x, t)\big]\big|_{u=F} = A(x, t, F)\big[T'(t) - F_t(x, t)\big] = 0. \tag{3.8}$$

Since $A \neq 0$, we find that

$$F_t(x, t) = T'(t). \tag{3.9}$$

Therefore, $F(x, t)$ must be of the form

$$F(x, t) = T(t) + X(x), \tag{3.10}$$

where $X(x)$ is an unknown function, assumed arbitrary before substitution into original differential equation. Conversely, operator **X** leaves our solution $u = F(x, t)$ invariant.

Note that in (3.10) and (3.12) function $T(t)$ is determined by the form of the symmetry operator **X** (3.6), and unknown function $X(x)$ can be found by substitution of this solution into original differential equation, and solving corresponding reduced system. Different symmetry operators will lead to different separated solutions.

Note also that the condition for the existence of additively separated invariant solutions (3.6) can be given another form

$$\mathbf{X} = A(x,t,u)\left[\,\partial_x + X'(x)\,\partial_u\right], \tag{3.11}$$

with some nonzero functions $A(x, t, u)$ and $X(x)$.

Thus, the functions invariant under operators $\mathbf{X}$ (3.6) or (3.11) are all *additively separated.*

$$u(x,t) = T(t) + X(x). \tag{3.12}$$

We can show that the reverse statement is also true, namely that the requirement that all invariants of a symmetry operator X be additively separated leads to the operator (3.6) or (3.11).

Let us consider a general vector field

$$\mathbf{X} = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \eta(x,t,u)\,\partial_u, \tag{3.13}$$

with coefficients that do not vanish simultaneously. We assume that operator $\mathbf{X}$ leads to a set of additively separated solutions $u - F(x,t) = 0$, where $F(x,t) = X(x) + T(t)$, and function $X(x)$ is unknown (assumed arbitrary a priori). Then

$$\begin{aligned}\mathbf{X}(u - X - T)\big|_{u=X+T} &= \eta(x,t,X+T) - \xi(x,t,X+T)\,X'(x) - \tau(x,t,X+T)\,T'(t)\\ &= 0.\end{aligned} \tag{3.14}$$

Since function $X(x)$ is arbitrary, then

$$\xi(x,t,X(x)+T(t)) = 0, \tag{3.15}$$

and

$$\eta(x,t,X(x)+T(t)) = T'(t)\,\tau(x,t,X(x)+T(t)), \qquad \tau \neq 0. \tag{3.16}$$

We conclude that (see Lemma 1)

$$\begin{aligned}&\xi(x,t,u) \equiv 0,\\ &\eta(x,t,u) \equiv T'(t)\,\tau(x,t,u).\end{aligned} \tag{3.17}$$

Therefore, our symmetry (3.13) must be of the form (3.6).

$$\mathbf{X} = \tau(x,t,u)\left[\,\partial_t + T'(t)\,\partial_u\right], \tag{3.18}$$

and all its invariants are additively separated.

To generate multiplicatively separated solutions, we consider a symmetry operator similar to (3.6)

$$\mathbf{X} = A(x,t,u)\left[\,\partial_t + \frac{T'(t)}{T(t)}\,u\,\partial_u\right], \tag{3.19}$$

with some nonzero functions $A(x, t, u)$ and $T(t)$. From the invariance condition

$$\mathbf{X}(u - F(x,t))\big|_{u=F} = A(x,t,F)\left[\frac{T'(t)}{T(t)}\,F(x,t) - F_t(x,t)\right] = 0, \tag{3.20}$$

where $A \neq 0$, we have the following equation for F:

$$T'(t)\,F(x,t) - T(t)\,F_t(x,t) = 0. \tag{3.21}$$

Thus,

$$F(x,t) = X(x)T(t). \tag{3.22}$$

We see that invariants of $\mathbf{X}$ are *multiplicatively separated*. As in additive separation case the function $T(t)$ is determined by the symmetry operator $\mathbf{X}$ (3.19), while the function $X(x)$ will be determined by solution of the reduced system after substitution of solution (3.22) into original differential equation.

The condition for the existence of multiplicative separated invariant solutions (3.19) can be given another form

$$\mathbf{X} = A(x,t,u)\left[\partial_x + \frac{X'(x)}{X(x)}\,u\,\partial_u\right], \tag{3.23}$$

with some nonzero functions $A(x, t, u)$ and $X(x)$.

Note that the conditions for existence of additively separated solutions (3.6) or (3.11), and multiplicatively separated solutions (3.19) or (3.23) are sufficient conditions; if a non-degenerate PDE has such symmetry, some solution with separated variables will exist. However, even if the equation does not have such symmetry it might still have separated solutions.

Let us show some examples.

Example 1 Suppose a differential equation has a symmetry subalgebra generated by:

$$X_1 = \partial_t, \quad X_2 = u\,\partial_u. \tag{3.24}$$

Then $X = X_1 + aX_2$ with $a \in \mathbb{C}$ is a symmetry operator in the form of (3.19), where $T'(t)/T(t) = a$, and therefore, $T(t) = e^{at}$. To find solution invariant with respect to X, we write:

$$X\phi(x,t,u) = \phi_t + au\phi_u = 0. \tag{3.25}$$

Solving the characteristic system

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{au}, \tag{3.26}$$

we find the following invariants:

$$I^1 = x, \quad I^2 = ue^{-at}. \tag{3.27}$$

The invariant manifold is described by the relation $\phi(x, ue^{-at}) = 0$, and solving for u, we obtain the form of invariant solution in multiplicative separated form (3.22)

$$u = e^{at}X(x), \tag{3.28}$$

where the function $T(t) = e^{at}$ is determined by the symmetry operator (3.25), and an unknown (and a priori arbitrary) function $X(x)$ will be determined by substituting (3.28) into original differential equation.

There are many second order PDE's $\omega[u] = 0$, that admit symmetries (3.24):

$$\mathbf{X}_1\omega[u] = 0, \tag{3.29}$$

$$\mathbf{X}_2\omega[u] = 0, \tag{3.30}$$

where $\mathbf{X}_i$ is a correspondingly prolonged vector field X_i. The first equation states that ω does not explicitly depend on t. The second, has invariants of the forms u_i/u and u_{ij}/u. Thus, the class of equations yielding these symmetries are:

$$\omega\left(x, \frac{u_x}{u}, \frac{u_t}{u}, \frac{u_{xx}}{u}, \frac{u_{xt}}{u}, \frac{u_{tt}}{u}\right) = 0. \tag{3.31}$$

Example 2 Suppose a differential equation admits a dilatation of the form:

$$X = x\,\partial_x + au\,\partial_u, \tag{3.32}$$

for some non-zero $a \in \mathbb{C}$. Note that this symmetry operator is of the form (3.23)

$$X = x\left[\partial_x + \frac{X'(x)}{X(x)}\,u\,\partial_u\right], \tag{3.33}$$

where $X'/X = a/x$, and therefore, $X(x) = x^a$.

We have

$$X\phi(x,t,u) = x\phi_x + au\phi_u = 0, \tag{3.34}$$

and the invariants of the transformation with operator X are

$$I^1 = t, \quad I^2 = ux^{-a}. \tag{3.35}$$

The invariant manifold is described by $\phi(t, ux^{-a}) = 0$, and the invariant solution will have multiplicatively separated form:

$$u = x^a T(t). \tag{3.36}$$

Example 3 Consider PDE admitting the symmetry of the form

$$X = \partial_t + (2at + b)\,\partial_u. \tag{3.37}$$

This operator is a special case of (3.6) with $T(t) = at^2 + bt$. The class of equations that admit such symmetry:

$$u_{tt} - \Gamma(x, u_x, u_{xx}, u_{xt}) = 0. \tag{3.38}$$

In addition to known linear equations, e.g.,

$$u_x = ku_{tt}, \qquad u_{tt} = k^2 u_{xx} + f(x), \qquad u_{tt} + u_{xx} = 0, \;\; k = const, \tag{3.39}$$

class (3.38) includes fully nonlinear Hessian equations

$$u_{tt} = f(u_{xt}, u_{xx}), \tag{3.40}$$

as well as a nonhomogeneous Monge-Ampère equation

$$u_{tt}u_{xx} - u_{xt}^2 = f(x), \tag{3.41}$$

Correspondingly, all equations of this class will have solutions in the form of additively separated variables $u(x,t) = X(x) + T(t)$.

Example 4 The equation

$$u_{xt} = 0 \tag{3.42}$$

admits the following symmetry operator (e.g. [7])

$$X = \alpha(x)\,\partial_x + \beta(t)\,\partial_t + (a(x) + b(t))\,\partial_u, \tag{3.43}$$

where $\alpha(x)$, $\beta(x)$, $a(x)$, and $b(x)$ are arbitrary functions. Here we have two symmetry operators of the form (3.6):

$$X = \alpha(x)(\,\partial_x + A(x)\,\partial_u) \tag{3.44}$$

and

$$X = \beta(t)(\,\partial_t + B(t)\,\partial_u), \tag{3.45}$$

with arbitrary functions $A(x)$, and $B(t)$. According to (3.10) the invariant solution under each symmetry operator is

$$X = F(x) + G(t), \tag{3.46}$$

with arbitrary functions $F(x)$, and $G(t)$.

Example 5 The homogeneous Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = 0 \tag{3.47}$$

has both additively and multiplicatively separated solutions. Let us show how additive and multiplicative separated solutions are determined by the classical symmetries of the equation.

The Lie symmetry group of the Eq. (3.47) is formed by the following 15 operators:

$$\begin{array}{lllll}\mathbf{X}_1 = \partial_x, & \mathbf{X}_2 = \partial_y, & \mathbf{X}_3 = \partial_u, & \mathbf{X}_4 = x\,\partial_x, & \mathbf{X}_5 = y\,\partial_y, \\ \mathbf{X}_6 = u\,\partial_u, & \mathbf{X}_7 = y\,\partial_x, & \mathbf{X}_8 = x\,\partial_y, & \mathbf{X}_9 = u\,\partial_x, & \mathbf{X}_{10} = u\,\partial_y, \\ \mathbf{X}_{11} = x\,\partial_u, & \mathbf{X}_{12} = y\,\partial_u, & \mathbf{X}_{13} = xD, & \mathbf{X}_{14} = tD, & \mathbf{X}_{15} = uD, \end{array}$$
$$D = \mathbf{X}_4 + \mathbf{X}_5 + \mathbf{X}_6. \tag{3.48}$$

Consider the following operator

$$\mathbf{X}_{add} = \mathbf{X}_1 + a\mathbf{X}_3 = \partial_x + a\,\partial_u, \quad a = const. \tag{3.49}$$

Since this operator has a form of (3.11) then the corresponding invariant solution is in additively separated form with $X'(x) = a$:

$$u(x, y) = X(x) + \psi(y) = ax + \psi(y), \tag{3.50}$$

where the function $\psi(y)$ is a priori arbitrary. Substitution into original Eq. (3.47) shows that (3.50) with arbitrary $\psi(y)$ is indeed, a (additively separated) solution of the Monge-Ampère equation.

We could also consider another operator

$$\mathbf{X}_{ad} = \mathbf{X}_2 + b\mathbf{X}_3 = \partial_y + b\,\partial_u, \quad b = const, \tag{3.51}$$

in the form (3.6). Then the $\mathbf{X}_{ad}$-invariant solution will also be in additively separated form with $Y'(y) = b$, and

$$u(x, y) = by + \phi(x). \tag{3.52}$$

Substituting this form into Eq. (3.47) we see that (3.52) (with arbitrary function $\phi(x)$) is another additively separated solution of the Monge-Ampère equation.

Consider now the operator

$$\mathbf{X}_m = \mathbf{X}_2 + c\mathbf{X}_6 = \partial_y + cu\,\partial_u, \quad c = const. \tag{3.53}$$

Since this operator has a form of (3.19) then the corresponding invariant solution is in multiplicatively separated form with $Y'(y)/Y(y) = c$, or $Y(y) = Ke^{cy}$. Thus, we will get a multiplicatively separated solution in the form

$$u(x, y) = K(x)e^{cy}, \tag{3.54}$$

where the function $K(x)$ is (a priori) arbitrary.

We could get a similar form by considering the solution invariant under the operator

$$\mathbf{X}_l = \mathbf{X}_1 + r\mathbf{X}_6 = \partial_x + ru\,\partial_u, \quad r = const. \tag{3.55}$$

Indeed, $\mathbf{X}_{mul}$-invariant solution gives an alternative form of multiplicatively separated solution:

$$u(x, y) = L(y)e^{rx}. \tag{3.56}$$

Thus, the $(\mathbf{X}_m, \mathbf{X}_l)$-invariant solution has the multiplicatively separated form

$$u(x,y) = ke^{rx+cy}, \tag{3.57}$$

where r, c, k are constants. Substitution of this form into original Eq. (3.47) shows that (3.57) is solution of the Monge–Ampère equation for any constants r, c, k.

Another multiplicatively separated solution of the Monge–Ampère equation can be obtained as the solution invariant under the operator:

$$\mathbf{X}_n = \mathbf{X}_4 + a\mathbf{X}_6 = x\,\partial_x + au\,\partial_u = x[\,\partial_x + \frac{a}{x}u\,\partial_u], \quad a = const. \tag{3.58}$$

This operator has a form of (3.19) with $X'(x)/X(x) = \frac{a}{x}$, or $X(x) = x^a$. Thus, the $\mathbf{X}_n$ -invariant solution leads to the form:

$$u(x,y) = kx^a Y(y). \tag{3.59}$$

Similarly, we can look for the solution invariant under the operator:

$$\mathbf{X}_p = \mathbf{X}_5 + b\mathbf{X}_6 = y\,\partial_y + bu\,\partial_u = y\left[\partial_y + \frac{b}{y}u\,\partial_u\right] \quad a = const. \tag{3.60}$$

This operator has a form of (3.19) with $Y'(y)/Y(y) = \frac{b}{y}$, or $Y(y) = y^b$, or

$$u(x,y) = ly^b X(x). \tag{3.61}$$

The $(\mathbf{X}_n, \mathbf{X}_p)$-invariant solution then will have a multiplicatively separated form

$$u(x,y) = kx^a y^b. \tag{3.62}$$

Substitution into original Eq. (3.47) leads to the following restriction: $b = a - 1$, and the solution

$$u(x,y) = kx^a y^{a-1}, \qquad a = const. \tag{3.63}$$

Let us show an example of Eq. which has additively separated solutions, but conditions (3.6) or (3.11) are not satisfied.

Example 6 The Minimal surface equation:

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0, \tag{3.64}$$

has additively separated solutions. Indeed, the additive separation condition (2.22) $u(x,t) = \Phi(x) + \Psi(t)$ leads to the following constraint:

$$\frac{\Phi''(x)}{1 + \Phi'(x)^2} + \frac{\Psi''(y)}{1 + \Psi'(y)^2} = 0. \tag{3.65}$$

Each term in the L.H.S. must be a constant, and we will get two additive separated solutions:

$$u(x,y) = \frac{1}{c}\left[\ln|\cos(cy-k)| - \ln|\cos(cx-l)|\right] + u_0, \quad c \neq 0, \tag{3.66}$$

and

$$u(x,y) = k_1 x + k_2 y + k_3, \quad c = 0, \tag{3.67}$$

where $c, k, l, u_0, k_1, k_2, k_3$ are constants. However, the classical symmetry group of the Eq. (3.64) with generators:

$$\begin{aligned} &\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_y, \quad \mathbf{X}_3 = \partial_u, \quad \mathbf{X}_4 = x\,\partial_y - y\,\partial_x, \\ &\mathbf{X}_5 = u\,\partial_x - x\,\partial_u, \quad \mathbf{X}_6 = u\,\partial_y - y\,\partial_u, \quad \mathbf{X}_7 = x\,\partial_x + y\,\partial_y + u\,\partial_u, \end{aligned} \tag{3.68}$$

has no combination of its operators that would give rise to the (sufficient) condition of existence of additively separated solutions, either (3.6) or (3.11).

We will show below (see section Contact Symmetries and Separation of Variables) that additively separated solutions of the minimal surface equation are related to its contact symmetries.

We conclude this section with a discussion of the converse: whether every PDE with separated solutions has a symmetry of the form (3.6) or (3.11) (or (3.19) or (3.23)). In general, this is false; see (6.1). However, such operators are symmetries of the stronger system

$$\begin{aligned} &\omega(x,y,u,u_x,u_y,\dots) = 0, \\ &u_{xy} = 0. \end{aligned} \tag{3.69}$$

In other words, they are symmetries on the submanifold of separated solutions. To confirm this statement, consider the symmetry operator (3.11):

$$\mathbf{X}_1 = A(x,t,u)\left[\partial_x + X'(x)\,\partial_u\right], \tag{3.70}$$

where $A \neq 0$. Let us evaluate this operator on the set of additively separated solutions (3.6)

$$u(x,y) = X(x) + Y(y). \tag{3.71}$$

We will have

$$\mathbf{X}_1 = A\left[\partial_x + X'(x)\,\partial_u\right] \doteq A\left[\partial_x + u_x\,\partial_u\right] \doteq A\,D_x, \tag{3.72}$$

where D_x is a total derivative operator. Clearly, the operator AD_x is a symmetry of any differential equation whose terms do not explicitly depend on x. Therefore the operator $\mathbf{X}_1$ restricted to the submanifold (3.71) will be a symmetry of any differential equation $\omega(x,y,u,\dots) = 0$ that allows additive separation of original variables.

3.2 Separation in Alternate Coordinate Systems

Separated solutions can be discovered by using alternate coordinate systems. Consider a differential equation $\Delta[u] = \Delta(x, u, \partial u, \dots) = 0$ (where $x = (x^1, \dots, x^n)$ are independent variables and $u = (u^1, \dots, u^m)$ are functions) whose symmetry algebra includes an operator of the form

$$\mathbf{X} = A^i(x)\,\partial_{x^i}. \tag{3.73}$$

It is known [20, 12, Theorem 17.13] that in this case we can find new local coordinates $(y^1, \dots, y^n)$ such that $\mathbf{X} = \partial_{y^1}$. Then an $\mathbf{X}$ invariant solution is separated in these alternative coordinates, $u = u(y^2, \dots, y^n)$; see [2, 11].

We could also consider operators of the form

$$\mathbf{X}_d = A^i(x)\,\partial_{x^i} + ku^a\,\partial_{u^a} = \partial_{y^1} + ku^a\,\partial_{u^a}. \tag{3.74}$$

The $\mathbf{X}_d$ invariant solution is still separated: $u^a(y) = e^{-ky^1} v^a(y^2, \dots, y^n)$.

Example 7 Consider a differential system $\Delta[u]$ that admits a rotation symmetry

$$\mathbf{X} = -y\,\partial_x + x\,\partial_y. \tag{3.75}$$

The characteristic system

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{du^1}{0} = \frac{du^2}{0} = \cdots = \frac{du^m}{0} \tag{3.76}$$

has the following invariants:

$$x^2 + y^2,\ u^1, \dots, u^m. \tag{3.77}$$

Then in new coordinates

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \theta &= \arctan(y/x), \\ v^a &= u^a, \quad a = 1, \dots, m \end{aligned} \tag{3.78}$$

the symmetry operator (3.75) will transform into translation $\mathbf{X} = \partial_\theta$, and the corresponding $\mathbf{X}$-invariant solution will be separated, and depend only on r, see also [17]:

$$v^a = v^a(r), \quad a = 1, \dots, m. \tag{3.79}$$

Example 8 Consider the stationary nonlinear Schrödinger equation for complex-valued $\psi(x, y)$, a function of two variables:

$$\psi_{xx} + \psi_{yy} + |\psi|^2\psi = 0. \tag{3.80}$$

This equation is scaling invariant under the symmetry generator

$$\mathbf{X} = x\,\partial_x + y\,\partial_y - \psi\,\partial_\psi - \overline{\psi}\,\partial_{\overline{\psi}}, \tag{3.81}$$

where $\overline{\psi}$ is the complex conjugate. The characteristic system

$$\frac{dx}{x} = \frac{dy}{y} = \frac{d\psi}{-\psi} = \frac{d\overline{\psi}}{-\overline{\psi}} \tag{3.82}$$

has the following invariant involving independent variables: y/x. In new (polar) coordinates

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \theta &= \arctan(y/x), \end{aligned} \tag{3.83}$$

the symmetry operator (3.81) will transform into

$$\mathbf{X} = r\,\partial_r - \psi\,\partial_\psi - \overline{\psi}\,\partial_{\overline{\psi}}. \tag{3.84}$$

The characteristic system in new coordinates

$$\frac{dr}{r} = \frac{d\theta}{0} = \frac{d\psi}{-\psi} = \frac{d\overline{\psi}}{-\overline{\psi}}. \tag{3.85}$$

has the following invariants

$$I_1 = r\psi, \qquad I_2 = r\overline{\psi}, \qquad I_3 = \theta. \tag{3.86}$$

The corresponding **X**-invariant solution is thus multiplicatively separated:

$$\psi(r,\theta) = \frac{1}{r}f(\theta), \qquad \overline{\psi}(r,\theta) = \frac{1}{r}\overline{f}(\theta). \tag{3.87}$$

In fact, substitution into (3.80) yields an ordinary differential equation for $f(\theta)$:

$$f''(\theta) + (|f(\theta)|^2 + 1)f(\theta) = 0. \tag{3.88}$$

3.3 Separation in General Variables $\varphi(x,t)$, and $\psi(x,t)$

So far we discussed the case of existence of invariant solutions in the form of additively separated original variables (x, t). Consider now a more general case of solutions in the form of additively separated variables (φ, ψ), where $\varphi = \varphi(x,t)$, $\psi = \psi(x,t)$.

Theorem 1 *Consider a differential equation*

$$\Delta[u] = 0, \tag{3.89}$$

which we assume to be nondegenerate: locally solvable at every point, and of maximal rank [19]. *If the Eq.* (3.89) *admits a symmetry operator in the form*

$$\mathbf{X} = B(x,t,u)\left[\frac{-\varphi_t}{J}\partial_x + \frac{\varphi_x}{J}\partial_t + T'(\psi)\partial_u\right], \tag{3.90}$$

where $B(x, t, u)$ can be any function,

$$J = \varphi_x\psi_t - \varphi_t\psi_x \neq 0, \tag{3.91}$$

and $\varphi(x,t), \psi(x,t), T(\psi)$ are some continuous functions with respect to all their variables, then the equation (3.89) has a solution in the form of additively separated variables $\varphi(x,t)$ and $\psi(x,t)$):

$$u(x,t) = f(\varphi(x,t)) + T(\psi(x,t)). \tag{3.92}$$

Note: if the function $T'(\psi)$ in (3.90) is arbitrary, then the function $T(\psi)$ in (3.92) is also arbitrary. Function f can be determined from the equation obtained by substitution of expression (3.92) into the original differential equation.

Proof:

Let

$$r = \varphi, \;\; s = \psi. \tag{3.93}$$

According to (3.18) if the Eq. (3.89) admits a symmetry operator

$$\mathbf{X} = B(r,s,u)[\,\partial_s + T'(s)\,\partial_u], \tag{3.94}$$

then it will have a solution in additively separated form

$$u = f(r) + T(s), \tag{3.95}$$

with some functions f and T. Therefore, in order to prove the Theorem 1 we need to show that the operator (3.90) can be given the form (3.94). We will prove instead a similar statement: by a change of variables the operator (3.94) can be transformed to (3.90). (Note: $J \neq 0$, (3.91)).

Consider the transformation

$$r \to x = \alpha(r,s), \qquad s \to t = \beta(r,s), \tag{3.96}$$

which is inverse to

$$x \to r = \varphi(x,t), \qquad t \to s = \psi(x,t). \tag{3.97}$$

Using (3.96) we can rewrite the expression for symmetry operator (3.94)

$$\mathbf{X} = B^*(r(x,t), s(x,t), u)[\alpha_s\,\partial_x + \beta_s\,\partial_t + T'(s)\,\partial_u]. \tag{3.98}$$

Let us express derivatives α_s and β_s in terms of x, t. Using (3.96), and finding derivatives of the equations

$$\begin{aligned} x &= \alpha(\varphi(x,t), \psi(x,t)), \\ t &= \beta(\varphi(x,t), \psi(x,t)), \end{aligned} \tag{3.99}$$

with respect to x and t, we will obtain

$$\alpha_s = \frac{-\varphi_t}{J}, \qquad \beta_s = \frac{\varphi_x}{J}, \tag{3.100}$$

where

$$J = \varphi_x \psi_t - \varphi_t \psi_x. \tag{3.101}$$

Substituting these expressions into (3.98) we will get the statement (3.90)

$$\mathbf{X} = B(x,t,u)\left(\frac{-\varphi_t}{J}\partial_x + \frac{\varphi_x}{J}\partial_t + T'(\psi)\partial_u\right). \tag{3.102}$$

Similarly we can prove the following theorem:

Theorem 2 *If a nondegenerate differential equation*

$$\Delta[u] = 0, \tag{3.103}$$

admits a symmetry operator in the form

$$\mathbf{X} = B(x,t,u)\left[\frac{-\varphi_t}{J}\partial_x + \frac{\varphi_x}{J}\partial_t + \frac{T'(\psi)}{T(\psi)}u\partial_u\right], \tag{3.104}$$

where

$$J = \varphi_x \psi_t - \varphi_t \psi_x \neq 0, \tag{3.105}$$

and $\varphi(x,t), \psi(x,t), T(\psi)$ are some continuous functions with respect to all their variables, then the Eq. (3.103) has a solution in the form of multiplicatively separated variables

$$u = f(\varphi(x,t))T(\psi(x,t)). \tag{3.106}$$

Note: if the function $T'(\psi)/T(\psi)$ in (3.104) is arbitrary, then obviously, the function $T(\psi)$ in (3.106) is also arbitrary. As in case of Theorem 1 the form of function f is determined from the equation for f obtained by substitution of expression (3.106) into the original differential equation.

Note also that finding the new variables $\varphi(x,t)$, and $\psi(x,t)$ (or rather function $T(\varphi(x,t))$) is the first step in application of Theorems 1 and 2.

Suppose our equation admits a symmetry operator

$$\mathbf{X} = a\partial_x + b\partial_t + c\partial_u, \tag{3.107}$$

where $a = a(x,t,u),\ b = b(x,t,u),\ c = c(x,t,u)$. From (3.90) we conclude that

$$\frac{b}{a} = -\frac{\varphi_x}{\varphi_t}. \tag{3.108}$$

Therefore, the variable $\varphi(x,t)$ is determined from the condition

$$a\varphi_x + b\varphi_t = 0. \tag{3.109}$$

Using (3.108), we find

$$J = \varphi_x \psi_t - \varphi_t \psi_x = \frac{\varphi_x}{b}\left(a\psi_x + b\psi_t\right). \tag{3.110}$$

We have

$$\begin{aligned} -\frac{\varphi_t}{J} &= \frac{a}{\psi_x + b\psi_t}, \\ \frac{\varphi_x}{J} &= \frac{b}{a\psi_x + b\psi_t}. \end{aligned} \tag{3.111}$$

Therefore,

$$B = a\psi_x + b\psi_t = \frac{c}{T'(\psi)}. \tag{3.112}$$

Thus, we will get the following equation for the variable $\psi(x,t)$ and the function $T(\psi)$

$$a\psi_x + b\psi_t = \frac{c}{T'(\psi)}. \tag{3.113}$$

Practically, the Eq. (3.113) can allow us to find $T(\psi)$ in terms of invariants of the characteristic system

$$\frac{dx}{a} = \frac{dt}{b} = \frac{1}{c}T'(\psi)d\psi. \tag{3.114}$$

In search of multiplicatively separated solutions using Theorem 2 the only difference with additively separated solutions will be in $T(\psi)$ part: $T'(\psi) \to T'(\psi)/T(\psi)$.

We will demonstrate the application of Theorems 1, and 2 on several examples.

Example 9 The wave equation

$$u_{xx} - u_{tt} = 0 \tag{3.115}$$

admits the following symmetry operator (e.g. [9])

$$\mathbf{X} = (\alpha(x+t) + \beta(x-t))\,\partial_x + (\alpha(x+t) - \beta(x-t))\,\partial_t + (\gamma(x+t) + \delta(x-t))\,\partial_u, \tag{3.116}$$

where $\alpha(x+t)$, $\beta(x-t)$, $\gamma(x+t)$, and $\delta(x-t)$ are arbitrary functions of their arguments.

According to Theorem 1, the wave equation has additively separated solution. In order to find $\varphi(x,t)$ we have to solve the Eq. (3.108)

$$a\varphi_x + b\varphi_t = 0. \tag{3.117}$$

For $a = \alpha(x+t)$ we have

$$\alpha(x+t)(\varphi_x + \varphi_t) = 0, \tag{3.118}$$

or

$$\varphi_x + \varphi_t = 0, \tag{3.119}$$

and for φ, we select

$$\varphi = x - t. \tag{3.120}$$

The Eq. (3.113) here is

$$\alpha(x+t)(\psi_x + \psi_t) = \frac{\gamma(x+t)}{T'(\psi)}. \tag{3.121}$$

We can choose

$$\psi = x + t, \tag{3.122}$$

and arbitrary function $T(\psi)$. Note that choosing function $\beta(x-t)$ would result in switching of φ and ψ

$$\varphi = x + t, \quad \psi = x - t. \tag{3.123}$$

Thus we obtain well known result:

$$u(x,t) = F(x-t) + G(x+t), \tag{3.124}$$

with arbitrary functions $F(x-t)$, and $G(x+t)$.

Example 10 Nonlinear wave equation of the type

$$\begin{aligned} &u_{xx} - u_{tt} + F\big(J_1, J_2\big) = 0, \\ &J_1 = (u_x + u_t)(x+t) - (x+t)^4/4, \\ &J_2 = (u_x - u_t)(t-x) + (t-x)^4/4, \end{aligned} \tag{3.125}$$

with some function $F \neq 0$. It is possible to show that equation (3.125) is invariant under the following operator

$$\mathbf{X} = t\,\partial_x + x\,\partial_t + xt(x^2 + t^2)\,\partial_u. \tag{3.126}$$

In order to find $\varphi(x,t)$ we have to solve the Eq. (3.108):

$$a\varphi_x + b\varphi_t = 0, \tag{3.127}$$

or

$$t\varphi_x + x\varphi_t = 0. \tag{3.128}$$

Looking for invariants of

$$\frac{dx}{t} = \frac{dt}{x}, \tag{3.129}$$

we choose

$$\varphi = x^2 - t^2. \tag{3.130}$$

The condition (3.113) for $T(\psi)$ here

$$t\psi_x + x\psi_t = \frac{xt(x^2 + t^2)}{T'(\psi)}. \tag{3.131}$$

A corresponding characteristic equation

$$\frac{dx}{t} = \frac{dt}{x} = \frac{T'(\psi)d\psi}{xt(x^2 + t^2)}. \tag{3.132}$$

We obtain

$$dT = xdx(x^2 + t^2), \tag{3.133}$$

and after integration we will get a special solution

$$T(\psi) = \frac{x^2t^2}{2}. \tag{3.134}$$

Thus, according to Theorem 1 our equation (with symmetry operator (3.126)) should have additively separated solution in the form

$$u(x, t) = f(x^2 - t^2) + T(\psi) = \frac{x^2t^2}{2} + f(x^2 - t^2), \tag{3.135}$$

where function f will be determined from the reduced system obtained by substitution of this solution into the original Eq. (3.125).

Example 11 The potential Burgers equation

$$u_t = u_{xx} + {u_x}^2 \tag{3.136}$$

admits the symmetry operator [10]

$$\mathbf{X} = 4xt\,\partial_x + 4t^2\,\partial_t - (x^2 + 2t)\,\partial_u. \tag{3.137}$$

We will apply Theorem 1, and first find $\varphi(x, t)$:

$$4xt\varphi_x + 4t^2\varphi_t = 0, \tag{3.138}$$

or

$$x\varphi_x + t\varphi_t = 0. \tag{3.139}$$

We select

$$\varphi = x/t. \tag{3.140}$$

The condition for $T(\psi)$ is (3.113)

$$4xt\psi_x + 4t^2\psi_t = -\frac{(x^2+2t)}{T'(\psi)}. \tag{3.141}$$

A corresponding characteristic equation

$$\frac{dx}{4xt} = \frac{dt}{4t^2} = -\frac{T'(\psi)d\psi}{x^2+2t}. \tag{3.142}$$

We obtain a special solution

$$T(\psi) = -\frac{x^2}{4t} - \frac{ln|x|}{2}. \tag{3.143}$$

Therefore, according to Theorem 1 the Eq. (3.136) (with symmetry operator (3.137)) has additively separated solution in the form

$$u = f\left(\frac{x}{t}\right) - \left(\frac{x^2}{4t} + \frac{ln|x|}{2}\right). \tag{3.144}$$

Substituting this form into (3.136) we obtain

$$f'' + f'^2 - \frac{f'}{\xi} + \frac{3}{4\xi^2} = 0, \quad \xi = \frac{x}{t}. \tag{3.145}$$

Notice that the two parts of the solution (3.144) with separated variables are not quite independent, and one of them affects the equation for the other. If $\xi > 0$, the general solution to (3.145) is

$$f(\xi) = C_1 + \log(C_2 + \xi) + \frac{1}{2}\log(\xi), \tag{3.146}$$

where C_1 and $C_2 \geq 0$ are constants. The case $\xi < 0$ is similar.

Example 12 A linear heat equation

$$u_t = ku_{xx} \tag{3.147}$$

admits the symmetry operator [13]

$$\mathbf{X} = 4xt\,\partial_x + 4t^2\,\partial_t - \left(\frac{x^2}{k} + 2t\right)u\,\partial_u. \tag{3.148}$$

We will apply Theorem 2:

$$4xt\varphi_x + 4t^2\varphi_t = 0, \tag{3.149}$$

or

$$x\varphi_x + t\varphi_t = 0, \tag{3.150}$$

and choose

$$\varphi = x/t. \tag{3.151}$$

The condition for $T(\psi)$ is (3.113)

$$4xt\psi_x + 4t^2\psi_t = -\frac{(x^2/k + 2t)}{T'(\psi)/T(\psi)}. \tag{3.152}$$

A corresponding characteristic equation

$$\frac{dx}{4xt} = \frac{dt}{4t^2} = -\frac{T'(\psi)d\psi}{T(\psi)(x^2/k + 2t)}. \tag{3.153}$$

We obtain

$$I = \frac{x}{t}, \quad \frac{dT}{T} = \left(\frac{I^2}{4k} + \frac{1}{2t}\right)dt. \tag{3.154}$$

Thus,

$$lnT = lnR - \left(\frac{x^2}{4kt} - \frac{ln|t|}{2}\right), \tag{3.155}$$

and

$$T(\psi) = R\,\frac{e^{-x^2/4kt}}{t^{1/2}}. \tag{3.156}$$

Therefore, according to Theorem 2 the heat Eq. (3.147) has multiplicatively separated solution in the form

$$u = f\left(\frac{x}{t}\right)\frac{e^{-x^2/4kt}}{t^{1/2}}. \tag{3.157}$$

Note that the case $f(\xi) = 1/(4\pi k)^{1/2}$, $(\xi = \frac{x}{t})$ corresponds to the fundamental solution of heat equation (3.147).

Let us note a short generalization of our approach to additive and multiplicative separation, to certain functional separation. For a diffeomorphism f on some open subsets of $\mathbb{R}$, we say that u is f-separated if $f(u)$ is additively separated, or $f(u) = X(x) + T(t)$. The generalization of Theorem 1 and Theorem 2 follows from the changes $u \to f(u), T \to f(T)$ and $\partial_u \to \partial_u/f'(u)$. Namely, if our PDE has a symmetry operator of the form

$$\mathbf{X} = B(x,t,u)\left[-\frac{\varphi_t}{J}\partial_x + \frac{\varphi_x}{J}\partial_t + \frac{f'(T(\psi))}{f'(u)}T'(\psi)\,\partial_u\right], \tag{3.158}$$

then the PDE has an f-separated solution of the form $u = f^{-1}(X(\varphi) + T(\psi))$. The following example is functionally separated.

Example 13 Given a graph $M = (x, u(x)) \in \mathbb{R}^{n+1}$, its mean curvature H is a function on M. We say that H is rotationally invariant in $\mathbb{R}^{n+1}$ if there is a smooth rotationally invariant function $f = f(\sqrt{|x|^2 + x_{n+1}^2})$ on $\mathbb{R}^{n+1}$ such that $H = f$ on M, or

$$H := -\frac{1}{\sqrt{1+|Du|^2}}\left(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}\right)u_{ij} = f(\sqrt{|x|^2 + u(x)^2}). \tag{3.159}$$

Here, we denote $|x|^2 = x_1^2 + \cdots + x_n^2$, and $|Du|^2 = u_1^2 + \cdots u_n^2$, with $u_{ij} = \partial^2 u/\partial x_i \partial x_j$ the Hessian, and summation over repeated indices is assumed.

The mean curvature, a coordinate free object, is rotationally symmetric: if H is the mean curvature for $(x, u(x))$, it is also the mean curvature for rotated graph $(\bar{x}, \bar{u}(\bar{x}))$, or $H_{(x,u(x))} = H_{(\bar{x},\bar{u}(\bar{x}))}$, where for fixed $1 \le i \le n$ and $\theta \in \mathbb{R}$,

$$\begin{aligned} \bar{x}_i &= \cos\theta\, x_i + \sin\theta\, u(x), \\ \bar{u}(\bar{x}) &= -\sin\theta\, x_i + \cos\theta\, u(x), \\ \bar{x}_j &= x_j, \qquad j \neq i. \end{aligned} \tag{3.160}$$

Because $f(\sqrt{|x|^2 + u^2}) = f(\sqrt{|\bar{x}|^2 + \bar{u}^2})$ is rotationally invariant, it follows that rotation is a point transformation which maps solutions to solutions. Therefore, the infinitesimal generators

$$\mathbf{X}_i = u\partial_i - x_i\partial_u, \tag{3.161}$$

along with the spatial rotations $Y_{ij} = x_i\,\partial_j - x_j\,\partial_i$, generate a subalgebra of the symmetry algebra of (3.159). In terms of $w := u^2$, these are vector fields of the additive form in Theorem 1.

We find additively separated solutions invariant under all $\mathbf{X}_i$. The invariant surface conditions are

$$uu_i + x_i = 0, \qquad i = 1, \dots, n, \tag{3.162}$$

which integrate to $|x|^2 + u(x)^2 = c^2$, or

$$u(x)^2 = c^2 - (x_1^2 + \cdots + x_n^2),$$

which confirms additive separation of u^2. A direct calculation shows that if such u solves (3.159) for $c > 0$, then $f(c) = n/c$. Provided c solves this algebraic equation, we obtain invariant solutions of (3.159). Such solutions correspond to spheres $|(x, x_{n+1})|^2 = c^2$ in $\mathbb{R}^{n+1}$.

3.4 Separated Solutions of Systems

We now extend our approach to systems of differential equations for n functions $u^i(x, t)$. We again seek symmetry algebras whose invariant solutions are in the form of additively separated variables

$$u^i = X^i(x) + T^i(t), \qquad i = 1, \dots, n. \tag{3.163}$$

with some functions $X^i(x)$, and $T^i(t)$. In analogy with (3.18) consider symmetry vector fields of the form

$$\mathbf{X} = A(x, t, u^i)\left[\partial_t + \sum_{i=1}^{n} T^i_t(t)\,\partial_{u^i}\right], \tag{3.164}$$

where $A \neq 0$. Indeed, if $u^i = F^i(x, t)$ is an invariant surface: $\mathbf{X}(F^i - u^i)|_{u=F} = 0$, then

$$0 = \mathbf{X}(F^i(x, t) - u^i)\big|_{u^j \equiv F^j} = A[F^i_t - T^i_t]\big|_{u^j \equiv F^j}. \tag{3.165}$$

Since $A \neq 0$, we conclude that $F^i(x, t) = X^i(x) + T^i(t)$ with arbitrary functions $X^i(x)$, which shows that invariant solutions are additively separated.

In the multiplicative framework, we correspondingly consider symmetry vector fields of the form

$$\mathbf{X} = A(x, t, u^i)\left[\partial_t + \sum_{i=1}^{n} \frac{T^i_t(t)}{T^i(t)} u^i\,\partial_{u^i}\right] \tag{3.166}$$

(with some functions $X^i(x)$, and $T^i(t)$) leading to multiplicatively separated invariant solutions

$$u^i = X^i(x)T^i(t), \qquad \forall i = 1, \dots, n. \tag{3.167}$$

Example 14 Consider the nonlinear-Schrödinger-type system

$$\begin{aligned} &i\Psi_t + \Psi_{xx} - V(x, \Psi\overline{\Psi})\Psi = 0,\\ &-i\overline{\Psi}_t + \overline{\Psi}_{xx} - V(x, \Psi\overline{\Psi})\overline{\Psi} = 0, \end{aligned} \tag{3.168}$$

where $\overline{\Psi} = \Psi^*$ (the complex conjugate), and $V(x, |\Psi|^2)$ is a real-valued potential function. This system admits the two symmetries

$$\mathbf{X}_1 = \partial_t, \qquad \mathbf{X}_2 = i(\Psi\,\partial_\Psi - \overline{\Psi}\,\partial_{\overline{\Psi}}). \tag{3.169}$$

Consider a linear combination of $\mathbf{X}_1$ and $\mathbf{X}_2$:

$$\mathbf{X} = \mathbf{X}_1 - E\mathbf{X}_2 = \partial_t - iE\,\Psi\,\partial_\Psi + iE\,\overline{\Psi}\,\partial_{\overline{\Psi}}. \tag{3.170}$$

Comparing with (3.166), we have

$$\begin{aligned} \frac{T^{1\prime}}{T^1} &= -iE.\\ \frac{T^{2\prime}}{T^2} &= iE. \end{aligned} \tag{3.171}$$

Thus,

$$T^1(t) = Ke^{-iEt}, \quad T^2(t) = Le^{-iEt}, \tag{3.172}$$

and we will obtain the well known form of multiplicatively separated solution for equations of the NLS-type:

$$\begin{aligned} \Psi(x,t) &= e^{-iEt}\phi(x), \\ \overline{\Psi}(x,t) &= e^{iEt}\overline{\phi}(x). \end{aligned} \tag{3.173}$$

After substituting this solution into original equation (3.168), we find that $\phi^* = \overline{\phi}$.

Example 15 We consider a Riemannian or Lorentzian manifold (M, g), where the metric g solves the vacuum Einstein equation

$$\mathrm{Ric}(g) = 0. \tag{3.174}$$

In local coordinates, the point symmetries of (3.174) are given by Stephani [27, equation (10.11)], and found first by Marchildon [15]:

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x^i} - \left(\xi^k_{,j} g_{ik} + \xi^k_{,i} g_{kj} - 2a g_{ij}\right) \frac{\partial}{\partial g_{ij}}, \tag{3.175}$$

where $\xi^i(x)$ are arbitrary functions, a is a constant, and we sum over $i \geq j$ in this example. There, Stephani indicates the symmetry invariant solutions, and presents many examples. In fact, they are of separated form. To recall, we write the symmetry invariance condition

$$\xi^k g_{ij,k} + \xi^k_{,j} g_{ik} + \xi^k_{,i} g_{kj} - 2a g_{ij} = 0 \tag{3.176}$$

in terms of the Lie derivative with respect to vector field $V = \xi^i \partial_i$:

$$\mathcal{L}_V g_{ij} = 2a g_{ij}. \tag{3.177}$$

That is, V is a conformal Killing vector. Let c solve the transport equation $\mathcal{L}_V c = 2ac$. Then we find the multiplicative separation

$$g_{ij}(x) = c(x) h_{ij}(x), \tag{3.178}$$

for arbitrary $h(x)$ solving $\mathcal{L}_V h = 0$. In local coordinates such that $V = \partial_1$, then for $x = (x^1, x')$ and $x' = (x^2, \dots, x^n)$, we have the separated variables

$$g_{ij}(x^1, x') = e^{2ax^1} h_{ij}(x'). \tag{3.179}$$

Many important exact solutions of the Einstein equations take this form, including the Kerr metric, and the Schwarzschild metric

$$g = -\left(1 - \frac{b}{r}\right) c^2 dt^2 + \left(1 - \frac{b}{r}\right)^{-1} dr^2 + r^2 dg_{S^2}(\theta, \phi), \tag{3.180}$$

where $g_{S^2} = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the sphere, and b, c are constants. In this case, $V = \partial_t$ is a Killing vector (isometry), with $a = 0$.

Let us also note that many more separated solutions can be constructed from (3.179) by using the diffeomorphism (gauge) invariance of the PDEs: $\psi^* g(y) = e^{2ax^1(y)}\psi^* h(x'(y))$ is also a solution for any diffeomorphism $x = \psi(y) : M \to M$. The most general such construction is (3.178), which is defined only using geometric objects, hence form invariant under the gauge group.

Example 16 We consider a family of Riemannian manifolds $(M, g(t))_{t\geq 0}$ such that metric $g(t)$ solves the Ricci flow

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g(t)). \tag{3.181}$$

Recently, Lopez et al. [14] found the point symmetry generators of (3.181):

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial t}, \\ \mathbf{X}_2 &= t\frac{\partial}{\partial t} + g_{ij}\frac{\partial}{\partial g_{ij}}, \\ \mathbf{X}_3 &= \xi^k \frac{\partial}{\partial x^k} - (\xi^k_{,j} g_{ij} + \xi^k_{,i} g_{kj})\frac{\partial}{\partial g_{ij}}, \end{aligned} \tag{3.182}$$

where we sum over $i \geq j$ in this example, and $\xi(x)$ is an arbitrary function. We suppose that the solution is invariant under the symmetry $\mathbf{X}_3 + a\mathbf{X}_2$. The invariance condition simplifies to

$$\mathcal{L}_V g_{ij} = a g_{ij} - at\frac{\partial g_{ij}}{\partial t}, \tag{3.183}$$

where $\mathcal{L}_V$ is the Lie derivative with respect to vector field $\xi^i\,\partial_i$. Choosing coordinates for which $V = \partial_1$, we let

$$g_{ij}(t, x) = th_{ij}(t, x). \tag{3.184}$$

Then $h_{ij}(t, x)$ solves the transport equation

$$\frac{\partial h_{ij}}{\partial x^1} = -at\frac{\partial h_{ij}}{\partial t}. \tag{3.185}$$

The solution form is therefore

$$g_{ij}(t, x^1, x') = t\, G_{ij}(x^1 - a\ln t, x'). \tag{3.186}$$

Substituting this into (3.181) and using that Ric is homogeneous degree zero in vertical scalings of g (cf. $\mathbf{X}_2$), we find that G solves the Einstein-type equation

$$(1 - a)\mathcal{L}_V G = -2\mathrm{Ric}(G). \tag{3.187}$$

This is known as the Ricci soliton corresponding to time evolution by a diffeomorphism.

3.5 Separation in General Variables for Systems

Two statements similar to Theorems 1, and 2 can be proven for systems of equations:

Theorem 3 *Consider a differential system for n functions $u^i(x,t)$*

$$\Delta^i[u] = 0, \quad i = 1, \ldots, n, \tag{3.188}$$

which we assume to be nondegenerate: locally solvable at every point, and of maximal rank [19]. If the system (3.188) admits a symmetry operator

$$\mathbf{X} = B(x,t,u)\left[\frac{-\varphi_t}{J}\partial_x + \frac{\varphi_x}{J}\partial_t + \sum_{i=1}^{n} T^{i\prime}(\psi)\partial_{u^i}\right], \tag{3.189}$$

where $B(x,t,u) \equiv B(x,t,u^1,u^2,\ldots,u^n)$, $\varphi(x,t), \psi(x,t), T^i(\psi)$ are continuous functions with respect to all their variables, and

$$J = \varphi_x\psi_t - \varphi_t\psi_x \neq 0, \tag{3.190}$$

then the system (3.188) has a solution in the form of additively separated variables $\varphi(x,t)$ and $\psi(x,t)$):

$$u^i(x,t) = f^i(\varphi(x,t)) + T^i(\psi(x,t)), \quad i = 1, 2, \ldots, n. \tag{3.191}$$

Note: Similar to the case of a single equation if our system admits a symmetry operator

$$\mathbf{X} = a\partial_x + b\partial_t + c^i\partial_{u^i}, \tag{3.192}$$

where coefficients $a, b, c^i, (i = 1, \ldots, n)$ are functions of (x, t, u), $(u \equiv u^1, u^2, \ldots, u^n)$, then the variable $\varphi(x,t)$ is determined from the condition (3.108)

$$a\varphi_x + b\varphi_t = 0, \tag{3.193}$$

and the variable $\psi(x,t)$ and the functions $T^i(\psi)$ can be found from the conditions similar to (3.113)

$$a\psi_x + b\psi_t = \frac{c^1}{T^{1\prime}(\psi)} = \frac{c^2}{T^{2\prime}(\psi)} = \cdots = \frac{c^n}{T^{n\prime}(\psi)}. \tag{3.194}$$

Theorem 4 *If a nondegenerate differential system (3.188) admits a symmetry operator in the form*

$$\mathbf{X} = B(x,t,u)\left[\frac{-\varphi_t}{J}\partial_x + \frac{\varphi_x}{J}\partial_t + \sum_{i=1}^{n}\frac{T^{i\prime}(\psi)}{T^i(\psi)}u^i\partial_{u^i}\right], \tag{3.195}$$

where J is determined by (3.190) *and* $\varphi(x,t), \psi(x,t), T^i(\psi)$ *are continuous functions with respect to all their variables, then the system* (3.188) *has a solution in the form of multiplicatively separated variables*

$$u^i(x,t) = f^i(\varphi(x,t))T^i(\psi(x,t)), \quad i = 1,2,\dots,n. \tag{3.196}$$

Note: Similar to the case of Theorem 3, if the system admits a symmetry operator (3.192) then the variable φ is determined from the condition (3.193) and the variable $\psi(x,t)$ and the functions $T^i(\psi)$ can be found from the conditions

$$a\psi_x + b\psi_t = \frac{c^1T^1(\psi)}{T^{1\prime}(\psi)} = \frac{c^2T^2(\psi)}{T^{2\prime}(\psi)} = \cdots = \frac{c^nT^n(\psi)}{T^{n\prime}(\psi)}. \tag{3.197}$$

Example 17 Consider Navier–Stokes equations for an incompressible fluid of viscosity ν and pressure p

$$\begin{aligned} &u^j_t - \nu u^j_{ii} + u^i u^j_i + p_j = 0,\\ &u^i_i = 0, \qquad i,j = 1,2,3, \end{aligned} \tag{3.198}$$

where $\mathbf{u} = (u^1, u^2, u^3)$ is the velocity vector, and $u^i = u^i(x,y,z,t)$. The symmetry algebra of Navier–Stokes equations includes the following operators [3, 8]:

$$\begin{aligned} \mathbf{X}_t &= \frac{\partial}{\partial t},\\ \mathbf{X}_f &= f\frac{\partial}{\partial x} + f'\frac{\partial}{\partial u^1} - xf''\frac{\partial}{\partial p},\\ \mathbf{X}_g &= g\frac{\partial}{\partial y} + g'\frac{\partial}{\partial u^2} - yg''\frac{\partial}{\partial p},\\ \mathbf{X}_h &= h\frac{\partial}{\partial z} + h'\frac{\partial}{\partial u^3} - zh''\frac{\partial}{\partial p}, \end{aligned} \tag{3.199}$$

where $f(t)$, $g(t)$, $h(t)$ are arbitrary functions.

Consider first the symmetry operator

$$\mathbf{X}_1 = \mathbf{X}_f + k\mathbf{X}_t = f\frac{\partial}{\partial x} + k\frac{\partial}{\partial t} + f'\frac{\partial}{\partial u^1} - xf''\frac{\partial}{\partial p}, \qquad k = const. \tag{3.200}$$

This operator has a form of (3.192) and we can look for additively separated solution to the Navier–Stokes equations (3.198) according to Theorem 3. We first need to find variables φ, ψ. Applying (3.193) we will get

$$f(t)\varphi_x + k\varphi_t = 0. \tag{3.201}$$

We find

$$\varphi = \varphi(I), \qquad I = kx - \int f(t)dt, \tag{3.202}$$

and choose

$$\varphi = I = kx - \int f(t)dt. \tag{3.203}$$

The variable ψ can be determined from the Eq. (3.194):

$$f(t)\psi_x + k\psi_t = \frac{f'(t)}{T^{1\prime}(\psi)}. \tag{3.204}$$

The characteristic system

$$\frac{dx}{f(t)} = \frac{dt}{k} = \frac{T^{1\prime}(\psi)d\psi}{f'(t)} = -\frac{T^{2\prime}(\psi)d\psi}{xf''(t)}. \tag{3.205}$$

We find

$$dT^1 = \frac{f'(t)dt}{k}, \tag{3.206}$$

and therefore,

$$T^1(\psi) = \frac{f(t)}{k}. \tag{3.207}$$

For $T^2(\psi)$ we have

$$dT^2 = \frac{-xf''(t)dt}{k}, \tag{3.208}$$

and

$$T^2(\psi) = -\frac{xf'(t)}{k}. \tag{3.209}$$

Thus, according to Theorem 3 the Navier–Stokes equation (3.198) has additively separated solutions in the form

$$\begin{aligned} u^1(x,t) &= F(\varphi) + T^1(\psi), \\ p(x,t) &= R(\varphi) + T^2(\psi). \end{aligned} \tag{3.210}$$

Introducing function $r(t) : f(t) = r'(t)$ from (3.203) we will get

$$\phi = I = kx - r(t). \tag{3.211}$$

Thus, our $\mathbf{X}_1$-invariant additively separated solution has a form

$$u^1(x,t) = F(kx - r(t)) + \frac{r'(t)}{k},$$
$$p(x,t) = R(kx - r(t)) - \frac{xr'(t)}{k}, \tag{3.212}$$

where functions $F(kx - r(t)), R(kx - r(t)), r(t)$ are arbitrary, and $k = const$.

Some solutions related to the class (3.212), and their properties were studied in [?].

We can also consider another symmetry operator

$$\mathbf{X}_2 = \mathbf{X}_g + l\mathbf{X}_t = g(t)\frac{\partial}{\partial y} + l\frac{\partial}{\partial t} + g'(t)\frac{\partial}{\partial u^2} - yg''(t)\frac{\partial}{\partial p}, \qquad l = const. \tag{3.213}$$

Similarly to the case of $\mathbf{X}_1$ operator we can obtain the $\mathbf{X}_2$-invariant additively separated solution in the form

$$u^2(y,t) = G(ly - s(t)) + \frac{s'(t)}{l},$$
$$p(y,t) = S(ly - s(t)) - \frac{ys'(t)}{l}, \tag{3.214}$$

where functions $G(ly - s(t)), S(ly - s(t)), s(t)$ are arbitrary, and $l = const$.

For the operator $\mathbf{X}_3$ we will get

$$\mathbf{X}_3 = \mathbf{X}_h + m\mathbf{X}_t = h(t)\frac{\partial}{\partial z} + m\frac{\partial}{\partial t} + h'(t)\frac{\partial}{\partial u^3} - zh''(t)\frac{\partial}{\partial p}, \qquad m = const, \tag{3.215}$$

and the $\mathbf{X}_3$-invariant additively separated solution will have a form

$$u^3(z,t) = H(mz - q(t)) + \frac{q'(t)}{m},$$
$$p(x,t) = Q(mz - q(t)) - \frac{zq'(t)}{m}, \tag{3.216}$$

where functions $H(mz - q(t)), Q(mz - q(t)), q(t)$ are arbitrary, and $m = const$.

Combining operators $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ and considering $\mathbf{X}_s = \mathbf{X}_f + \mathbf{X}_g + \mathbf{X}_h + a\mathbf{X}_t$ we can generate the $\mathbf{X}_s$-invariant additively separated solution of the Navier–Stokes equations

$$u^1(x,t) = F(kx - r(t)) + \frac{r'(t)}{k},$$
$$u^2(y,t) = G(ly - s(t)) + \frac{s'(t)}{l},$$
$$u^3(z,t) = H(mz - q(t)) + \frac{q'(t)}{m},$$
$$p(x,t) = R(kx - r(t)) + S(ly - s(t)) + Q(mz - q(t)) - \left(\frac{xr''(t)}{k} + \frac{ys''(t)}{l} + \frac{zq''(t)}{m}\right), \tag{3.217}$$

where functions $F(kx - r(t)), G(ly - s(t)), H(mz - q(t)), r(t), s(t), q(t)$ are arbitrary, $k + l + m = a$, and $k, l, m = const$. Substituting (3.217) into the Eq. (3.198) we will get:

$$\begin{aligned} R(\xi) &= k\nu F'(\xi) - F^2(\xi)/2, \\ S(\eta) &= l\nu G'(\eta) - G^2(\eta)/2, \\ Q(\zeta) &= m\nu H'(\zeta) - H^2(\zeta)/2, \end{aligned} \tag{3.218}$$

where

$$\begin{aligned} \xi &= kx - r(t), \\ \eta &= ky - s(t), \\ \zeta &= kz - q(t). \end{aligned} \tag{3.219}$$

In these variables our additively separated solution of the Navier–Stokes equation (3.198) has the form

$$\begin{aligned} u^1(x,t) &= F(\xi) + \frac{r'(t)}{k}, \\ u^2(y,t) &= G(\eta) + \frac{s'(t)}{l}, \\ u^3(z,t) &= H(\zeta) + \frac{q'(t)}{m}, \\ p(x,t) &= \nu\big(kF'(\xi) + lG'(\eta) + mH'(\zeta)\big) - \frac{1}{2}\big(F^2(\xi) + G^2(\eta) + H^2(\zeta)\big) \end{aligned} \tag{3.220}$$

$$- \left(\frac{xr''(t)}{k} + \frac{ys''(t)}{l} + \frac{zq''(t)}{m}\right), \tag{3.221}$$

with arbitrary functions $F(\xi), G(\eta), H(\zeta), r(t), s(t), q(t)$, and $k, l, m = const$.

3.6 Contact Symmetries and Separation of Variables

We can generalize the previous observations to contact symmetries. We will consider nontrivial contact symmetries which cannot be obtained from point symmetries by simple prolongation. Correspondingly, separated solutions obtained from nontrivial contact symmetries do not arise from point symmetries. In addition, some equations, such as the general class in Example 18, have contact symmetries but no point symmetries.

Consider a contact vector field $\mathbf{X}[\alpha]$ with infinitesimal of the form

$$\alpha(x,t,u,p,q) = \gamma(x,t,u,p,q)\beta(x,t,q), \tag{3.222}$$

where

$$\gamma(x, t, u, p, q) \neq 0, \tag{3.223}$$

and for each function $Q(x, t)$

$$\beta(x, t, Q(x, t)) = 0 \qquad \text{if and only if} \qquad Q(x, t) = T'(t), \tag{3.224}$$

for some fixed function $T(t)$. To find the invariant forms $u = F(x, t)$ of $\mathbf{X}[\alpha]$, we compute

$$\begin{aligned} 0 = \mathbf{X}[\alpha](u - F(x, t))\big|_{u=F} &= \big(\alpha - p\alpha_p - q\alpha_q + pF_x + qF_t\big)\big|_{u=F} \\ &= \alpha(x, t, F(x, t), F_x(x, t), F_t(x, t)) \\ &= \gamma(x, t, F, F_x, F_t)\beta(x, t, F_t). \end{aligned} \tag{3.225}$$

Since $\gamma \neq 0$, we see that

$$\beta(x, t, F_t(x, t)) = 0. \tag{3.226}$$

From condition (3.224), we deduce that $F_t = T'(t)$. Hence, there exists some function $X(x)$ such that

$$F(x, t) = X(x) + T(t). \tag{3.227}$$

In other words, the symmetry (3.222) leads to additively separated invariant solution. Conversely, any $F(x, t)$ of an additively separated form (3.227) is invariant under the contact symmetry $\mathbf{X}[\alpha]$, (3.222) (where X is an arbitrary function (and T is fixed via (3.224)).

Infinitesimal (3.222) corresponds to the following vector field:

$$\mathbf{X}[\alpha] = \gamma\mathbf{X}[\beta] + \beta\mathbf{X}[\gamma] - \gamma\beta\,\partial_u, \tag{3.228}$$

where

$$\mathbf{X}[\beta] = -\beta_q\,\partial_t + (\beta - q\beta_q)\,\partial_u + \beta_x\,\partial_p + \beta_t\,\partial_q. \tag{3.229}$$

In the multiplicative framework, the corresponding infinitesimal in multiplicative variables takes the general form

$$\alpha(x, t, u, p, q) = u\ \gamma(x, t, u, p, q)\ \beta(x, t, q/u), \qquad \gamma \neq 0. \tag{3.230}$$

This recovers the point symmetry case. Indeed, if α is linear in derivatives, or

$$\alpha(x, t, u, p, q) = A(x, t, u)[T'(t) - q], \tag{3.231}$$

then $\mathbf{X}[\alpha]$ takes the form (3.6). In general, the dependence on $T'(t)$ can be more complicated. Admissible examples which satisfy (3.224) include

$$\alpha(x, t, q) = A[T'(t) - q]^n, A[T'(t)^{2n-1} - q^{2n-1}], A(e^{T'(t)-q} - 1),$$

where A is any nonzero function of (x, t, u, p, q), and $n = 1, 2, 3, \ldots$

Example 18 Consider PDEs of the form

$$u_{tt} + \frac{u_t - \lambda}{(u_t - \lambda)\Phi\big(x, u_x, u_t, u - \lambda t(u_t - \lambda)/2\big) - t} = 0, \tag{3.232}$$

where λ is a constant and Φ is some function.

The Eq. (3.232) does not have any point symmetries. However, this equation admits the following contact symmetry:

$$\alpha(u_t) = (u_t - \lambda)^2. \tag{3.233}$$

Corresponding invariant solutions

$$u_t - \lambda = 0$$

are of the separated form

$$u(x, t) = X(x) + \lambda t. \tag{3.234}$$

Example 19 Consider PDEs of the form

$$u_{tt} - u_{xt}\,\beta(u_t)\,\kappa(x, u_x) = 0, \tag{3.235}$$

where $\beta(u_t)$ and $\kappa(x, u_x)$ are some functions. It is possible to show that the Eqs. (3.235) admit contact symmetries with the following infinitesimal:

$$\alpha(t, u_t) = \beta(u_t)[t + \lambda(u_t)], \tag{3.236}$$

where $\lambda(u_t)$ is an arbitrary function. The form of corresponding invariant solution is determined by

$$\lambda(u_t) + t = 0. \tag{3.237}$$

Using the inverse function theorem we will get

$$u_t = T'(t), \tag{3.238}$$

where $T''(t) \neq 0$, and otherwise $T(t)$ is *a priori* arbitrary function. Thus,

$$u(x, t) = X(x) + T(t), \qquad T''(t) \neq 0. \tag{3.239}$$

However, we can see that the Eqs. (3.235) do not have invariant solutions of the form (3.239). Instead, these equations have solutions of the form

$$u(x, t) = X(x) + at, \tag{3.240}$$

where $X(x)$ and $a = const$ are arbitrary. Thus, contact symmetries of the Eqs. (3.235) do not yield their invariant separated solutions. We can see that the invariant separated solutions (3.240) are related to the following point symmetry of the Eqs. (3.235)

$$\mathbf{X} = \partial_x + a\,\partial_u, \qquad a = const. \tag{3.241}$$

Example 20 We will consider the minimal surface Eq. (3.64) and the connection between its contact symmetry operators and additive separation solutions.

In case of additive separation we rewrite (3.64) as

$$\begin{aligned} &(1+q^2)p_x + (1+p^2)q_y = 0, \\ &p_y = q_x = 0, \end{aligned} \tag{3.242}$$

where $u_x = p$ and $u_y = q$, and u is no longer treated as part of the system. We look for contact symmetries of the system (3.242) in the form:

$$\mathbf{X} = f(x)\,\partial_x + g(y)\,\partial_y + \eta^1(x, y, p, q)\,\partial_p + \eta^2(x, y, p, q)\,\partial_q. \tag{3.243}$$

We obtain

$$\begin{gathered} \mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_y, \quad \mathbf{X}_3 = x(1+p^2)\,\partial_p - y(1+q^2)\,\partial_q, \\ \mathbf{X}_4 = (1+p^2)\arctan(p)\,\partial_p + (1+q^2)\arctan(q)\,\partial_q, \\ \mathbf{X}_5 = (1+p^2)\,\partial_p, \quad \mathbf{X}_6 = (1+q^2)\,\partial_q. \end{gathered} \tag{3.244}$$

The pair of symmetry combinations

$$\mathbf{X}_a = \mathbf{X}_1 + c\mathbf{X}_5, \tag{3.245}$$

$$\mathbf{X}_b = \mathbf{X}_2 - c\mathbf{X}_6, \tag{3.246}$$

lead to the invariants

$$I^1 = \arctan(p) - cx, \tag{3.247}$$

$$I^2 = \arctan(q) + cy. \tag{3.248}$$

We can construct the manifold

$$\phi_1(\arctan(p) - cx, \arctan(q) + cy) = 0, \tag{3.249}$$

$$\phi_2(\arctan(p) - cx, \arctan(q) + cy) = 0, \tag{3.250}$$

and find invariants. Integrating each equation on $u_x = p$, $u_y = q$ leads to the following additively separated solutions

$$u = \frac{1}{c}\left[\ln\left(\frac{\cos(c(y-y_0))}{\cos(c(x-x_0))}\right)\right]. \tag{3.251}$$

3.7 Differential Invariant Solutions

As we saw in the examples above, solutions obtained from point and contact symmetries do not recover all possible separated solutions. In this section, we identify conditions when **all** separated solutions are determined by the equation's symmetry properties. For both point and contact symmetries, we relax the symmetry invariance condition to include differential invariants. Examples 21 and 22 illustrate our approach.

The Lie algebra

$$A = < \partial_u, \partial_t, t\,\partial_t > \tag{3.252}$$

has differential invariants x and u_x. Consequently, if A has an invariant surface defined by

$$\mathbf{X}\big(u_x - F(x,t,u,u_t)\big)\big|_{u_x=F} = 0, \qquad \mathbf{X} \in A, \tag{3.253}$$

then $F(x,t,u,u_t) = G(x)$ for some G. Therefore, $u_x = G(x)$, which integrates to

$$u(x,t) = X(x) + T(t), \tag{3.254}$$

for some $T(t)$ and $X'(x) = G(x)$. We see that differential invariant solutions of algebra A lead to additively separated solution forms.

Conversely, if a PDE admits symmetry algebra A and possesses an additively separated solution $u(x,t) = X(x) + T(t)$ for some X and T, then this solution forms a differential invariant under algebra A. Indeed, the surface defined by

$$u_x - X'(x) = 0, \tag{3.255}$$

is clearly invariant under A. In this case, we see that *all* separated solutions of a PDE are invariant solutions, in contrast to the previous approach, in which one of the functions $T(t)$ or $X(x)$ would be fixed by the imposed symmetry $\mathbf{X}$.

We seek conditions on a (finite) contact Lie algebra $A = < \mathbf{X}[\alpha^i] >_{i=1}^{a}$ such that all its differential invariant solutions are additively separated, (3.254) as in the example above. We claim this happens if

$$\alpha^i = \alpha^i(t,q), \qquad i = 1, \dots, a \tag{3.256}$$

and

$$rank \begin{pmatrix} \mathbf{X}[\alpha^1]t & \mathbf{X}[\alpha^1]u & \mathbf{X}[\alpha^1]q \\ \mathbf{X}[\alpha^2]t & \mathbf{X}[\alpha^2]u & \mathbf{X}[\alpha^2]q \\ \vdots & \vdots & \vdots \\ \mathbf{X}[\alpha^a]t & \mathbf{X}[\alpha^a]u & \mathbf{X}[\alpha^a]q \end{pmatrix} = rank \begin{pmatrix} \alpha^1_q & \alpha^1 & \alpha^1_t \\ \alpha^2_q & \alpha^2 & \alpha^2_t \\ \vdots & \vdots & \vdots \\ \alpha^a_q & \alpha^a & \alpha^a_t \end{pmatrix} = 3. \tag{3.257}$$

Of course, this last condition requires $a \geq 3$.

To verify that such algebras leave separated solutions invariant, let $\mathbf{X}[\alpha] \in A$. Then for each $X(x)$,

$$\mathbf{X}[\alpha](X'(x) - u_x)\big|_{u_x=X'} = \big(-\alpha_x - \alpha_u p - \alpha_p u_{xx} - \alpha_q u_{xt}\big)\big|_{u_x=X'} = 0, \tag{3.258}$$

where $p = u_x$, $q = u_t$, by (3.256). Therefore, separated solutions to PDEs admitting such algebras A are differential invariant solutions.

We now prove that all differential invariant solutions of Lie algebra A are separated. Suppose for some F and each i that $\mathbf{X}[\alpha^i](F(x,t,u,q)-p)|_{p=F}=0$, or that

$$\mathbf{X}[\alpha^i](F(x,t,u,q)-p)\big|_{p=F}=\big(F_t\,\mathbf{X}[\alpha^i]t+F_u\,\mathbf{X}[\alpha^i]u+F_q\,\mathbf{X}[\alpha^i]q\big)\big|_{p=F}=0. \tag{3.259}$$

By (3.257), we deduce that there exists G such that

$$F(x,t,u,u_t)=G(x),\qquad \text{if } u \text{ solves } u_x=F(x,t,u,u_t). \tag{3.260}$$

Clearly, if u solves $u_x=F$, it solves $u_x=G(x)$, to which $u(x,t)=X(x)+T(t)$ is a solution for arbitrary T, where $X'(x)=G(x)$. Then according to (3.260)

$$F\big(x,t,X(x)+T(t),T'(t)\big)=G(x),\qquad T(t)\text{ arbitrary.} \tag{3.261}$$

Since function $T(t)$ is arbitrary, we conclude that $F(x,t,u,q)=G(x)$ for each $(x,t,u,q)\in\mathbb{R}^4$, or $F(x,t,u,u_t)\equiv G(x)$; cf. Lemma 1. We see then that algebra A leads only to additively separated differential invariant solutions.

Remark 1 Contact vector fields (2.11) in this algebra take the form

$$\mathbf{X}[\alpha^i]=-\alpha^i_q\,\partial_t+(\alpha^i-q\alpha^i_q)\,\partial_u+\alpha^i_t\,\partial_q,\qquad i=1,\dots,a, \tag{3.262}$$

with commutators (2.13) reducing to

$$\big[\mathbf{X}[\alpha^i],\mathbf{X}[\alpha^j]\big]=\mathbf{X}\big[\alpha^i_q\alpha^j_t-\alpha^i_t\alpha^j_q\big],\qquad i,j=1,\dots,a. \tag{3.263}$$

The Lie algebraic conditions (2.14) reduce to

$$\alpha^i_q\alpha^j_t-\alpha^i_t\alpha^j_q=\sum_{k=1}^{a}C^k_{ij}\alpha^k,\qquad i,j=1,\dots,a. \tag{3.264}$$

If $a=3$ (three-dimensional Lie algebra), rank condition (3.257) becomes equivalent to the nonvanishing of the determinant. The minors of this matrix may be computed using commutator relations (3.264), which ultimately yields:

$$\sum_{k=1}^{3}\big(\alpha^1C^k_{23}\alpha^k+\alpha^2C^k_{31}\alpha^k+\alpha^3C^k_{12}\alpha^k\big)\neq 0. \tag{3.265}$$

In particular, not all C^k_{ij}'s can be zero, so commuting Lie algebras are not admissible.

Remark 2 In the case of point symmetry algebras

$$\alpha^i(t,q)=\eta^i(t)-q\tau^i(t),\qquad i=1,\dots,a,$$

system (3.264) becomes a set of nonlinear ODEs:

$$\begin{aligned} \tau^j \eta^i_t - \tau^i \eta^j_t &= \sum_{k=1}^{a} C^k_{ij} \eta^i, \qquad i,j = 1, \dots, a, \\ \tau^j \tau^i_t - \tau^i \tau^j_t &= \sum_{k=1}^{a} C^k_{ij} \tau^i, \qquad i,j = 1, \dots, a. \end{aligned} \tag{3.266}$$

Note that not all τ's can be zero; ($\tau^i = -\alpha^i_q$), see (3.257).

If $a = 3$, rank condition (3.265) stipulates that *at least one* of the following must hold:

$$\begin{aligned} &\sum_{k=1}^{3} \left(\eta^1 C^k_{23} \eta^k + \eta^2 C^k_{31} \eta^k + \eta^3 C^k_{12} \eta^k \right) \neq 0, \\ &\sum_{k=1}^{3} \left((\eta^1 \tau^k + \eta^k \tau^1) C^k_{23} + (\eta^2 \tau^k + \eta^k \tau^2) C^k_{31} + (\eta^3 \tau^k + \eta^k \tau^3) C^k_{12} \right) \neq 0, \\ &\sum_{k=1}^{3} \left(\tau^1 C^k_{23} \tau^k + \tau^2 C^k_{31} \tau^k + \tau^3 C^k_{12} \tau^k \right) \neq 0, \end{aligned} \tag{3.267}$$

each of which corresponds to a coefficient of q^n in (3.265).

3.7.1 Examples

We first consider three-dimensional point symmetry algebras ($a = 3$) which solve system (3.266) and satisfy one of conditions (3.267).

Example 21 Many equations admit translations in both t and u, or $\mathbf{X} = \partial_t, \partial_u$. Suppose that

$$\eta^2 = 0,\ \tau^2 = -1, \qquad \eta^3 = 1,\ \tau^3 = 0, \qquad C^k_{23},\ C^k_{31} \equiv 0.$$

Then system (3.266) reduces to

$$\begin{aligned} \eta^1_t &= C^1_{12} \eta^1 + C^3_{12}, \\ \tau^1_t &= -C^1_{12} \tau^1 + C^2_{12}. \end{aligned} \tag{3.268}$$

If $C^1_{12} = 0$, then $\eta^1 = C^3_{12} t$, $\tau^1 = C^2_{12} t$ up to linear independence in the algebra. We conclude that the point symmetry algebra

$$A = < \partial_t, \partial_u, at\, \partial_t + bt\, \partial_u >, \tag{3.269}$$

leaves invariant the separation condition (3.255), for any a, b such that $a^2 + b^2 > 0$, i.e. the differential consequence of separation is symmetry invariant. Second order equations which admit this symmetry algebra are of the form

$$\omega\left(x, u_x, u_{xx}, \frac{u_{xt}}{b - au_t}, \frac{u_{tt}}{(b - au_t)^2}\right) = 0.$$

For instance, if $b = 1, a = 0$, we see that the wave, Laplace, and heat equations

$$u_{tt} = c^2(x)u_{xx}, \qquad u_{tt} + u_{xx} = 0, \qquad u_x = ku_{tt},$$

have this symmetry algebra. As another example, the Monge-Ampère equation

$$u_{xx}u_{tt} - u_{xt}^2 = 0,$$

contains A as a subalgebra in its (large) point group, for any a, b. We conclude that all additively separated solutions of Monge-Ampère are also (differential) invariant solutions of this three-dimensional subalgebra of its point group (and, of course, the invariant solutions of this subalgebra are separated solutions). Note that Rosenhaus showed that the Monge–Ampère equation is uniquely determined by its point symmetry group, and all its solutions can be obtained from its symmetry algebra and some of its subalgebras [24].

Obviously, analogous statements hold for multiplicative separation with the algebra

$$A =< \partial_t, u\,\partial_u, at\,\partial_t + btu\,\partial_u > .$$

For example, the Monge–Ampère equation also admits the symmetries with $a = 1, b = 0$, so its multiplicatively separated solutions are invariant under this three-dimensional subalgebra.

We next consider a three-dimensional contact symmetry algebra that solves (3.264) and satisfies (3.265).

Example 22 Suppose that

$$\alpha^1 = 1, \qquad \alpha^2 = -q^2, \qquad C^k_{12}, C^k_{31} \equiv 0.$$

Then contact system (3.264) reduces to

$$2q\alpha^3_t = -C^1_{23} + C^2_{23}q^2 - C^3_{23}\alpha^3.$$

If $C^1_{23} = C^3_{23} = 0$, we find that $\alpha^3 = f(q) + C^2_{23}tq/2$ is a solution, where f is an arbitrary function. If we set $f(q) = 0$ and $C^2_{23} = -2$, we have $\alpha^3 = -tq$. Therefore, the contact Lie algebra

$$A =< \partial_u,\, t\,\partial_t - u_t\,\partial_{u_t},\, 2u_t\,\partial_t + u_t^2\,\partial_u >,$$

leaves invariant the separation condition (3.255). For example, along with the Monge-Ampère equation, the quasilinear equation

$$t\,u_{tt} - u_t - g(x, u_x)u_{xt} = 0,$$

admits this symmetry algebra A. Therefore, all separated solutions of this equation are generated by differential invariants of algebra A.

3.8 Further Remarks

The natural converse problem to that considered in this section is whether all separated solutions of a PDE are invariant under a symmetry. This is true for many equations from mathematical physics. In the case of the heat equation, each additively separated solution $u(x,t) = X(x) + T(t)$ is invariant under a corresponding point symmetry of the equation. In the case of the wave, Laplace, heat, and Monge-Ampère equations, all additively separated solutions are invariant solutions under a three-dimensional point symmetry algebra. However, in a later section, an exotic Eq. (6.1) will be exhibited which has no point symmetries but does possess separated solutions.

For certain linear PDEs, Miller [16] earlier considered the problem of finding separated solutions from symmetries; further studies of linear PDEs are in Kalnins–Kress–Miller [11]. It was shown there that given a symmetry of the form $\mathbf{X} = A^i(x)\,\partial_i$, the eigenfunctions of $\mathbf{X}$ will be separated solutions of the linear PDE. Indeed, in new coordinates, $\mathbf{X} = \partial_{y^1}$, so the eigenfunctions $\mathbf{Xu} = ku$ have the separated form $u = e^{ky^1} f(y^2, \dots, y^n)$.

The results of this section generalize this idea. Indeed, since $u\,\partial_u$ is a scaling symmetry valid for linear PDEs, we see that eigenfunctions of $\mathbf{X}$ are invariant under the combination symmetry $\mathbf{X}_2 = \mathbf{X} - ku\,\partial_u$, which is of the multiplicative form (3.195) and hence within the scope of our Theorem 4.

Our theorem is valid for nonlinear PDEs. The symmetry invariance need not correspond to eigenfunctions of a translation operator, and a priori linear structure on the PDE is not required for the symmetry to yield a separated solution. Our discussion includes additive, multiplicative, and functional separation, such as Example 13.

4 Separated Solutions Generated by Mapping of Constant Solutions

We start with a motivating example. The heat equation

$$u_t = u_{xx}, \tag{4.1}$$

admits the symmetry operator [1]

$$\mathbf{X}_\alpha = (xu + 2tu_x)\,\partial_u, \qquad (\mathbf{X} = -2t\,\partial_x + xu\,\partial_u). \tag{4.2}$$

If $u = f(x,t)$ is a solution to (4.1), then for each ε, so is

$$u^\varepsilon(x,t) = e^{\varepsilon \mathbf{X}_\alpha} f(x,t) = e^{-\varepsilon x + \varepsilon^2 t} f(x - 2\varepsilon t, t). \tag{4.3}$$

In particular, if $f(x,t) = U$ is constant, then

$$u^\varepsilon(x,t) = U\ e^{-\varepsilon x+\varepsilon^2 t}, \tag{4.4}$$

is a solution to (4.1) for each ε. We see that symmetry transformation $e^{\varepsilon \mathbf{X}_\alpha}$ maps a constant solution to a nontrivial multiplicatively separated solution that depends on all independent variables. Notice that we did not need to solve the original Eq. (4.1) to deduce the existence of separated solutions; we needed (i) a constant solution to exist, and (ii) an appropriate symmetry ((4.2)).

Let us consider a more general differential equation

$$\Delta[u] := \Delta(x, u, u_{(1)}, \dots, u_{(\ell)}) = 0, \tag{4.5}$$

for function $u(x_1, \dots, x_p)$. Suppose that it admits constant solutions; i.e. for all constants U (e.g. in $\mathbb{R}$ or $\mathbb{C}$),

$$\Delta[U] = \Delta(x, U, 0, \dots, 0) = 0. \tag{4.6}$$

For example,

$$0 = \Delta[u] = \sum_{|I|\geq 1} A^I u_I = A^i u_i + \sum_{i\leq j} A^{ij} u_{ij} + \dots, \tag{4.7}$$

where the A's are smooth functions of $(x, u, u_{(1)}, \dots, u_{(\ell)})$, admits $u = U$ as a solution for all U, since the derivatives u_I vanish when $u = U$. As a special case, if a Lagrangian

$$L = L(u_{(1)}, \dots, u_{(k)}), \tag{4.8}$$

does not depend on x or u, then its Euler–Lagrange equation

$$(-D)_I \frac{\partial L}{\partial u_I} = -D_i \frac{\partial L}{\partial u_i} + \sum_{i\leq j} D_i D_j \frac{\partial L}{\partial u_{ij}} + \dots = 0, \tag{4.9}$$

admits constant solutions, since each term has a u derivative. The prototypical examples would be $L = u_t^2 \pm |\nabla u|^2$.

For a given vector field $\mathbf{X}_\alpha$ (for now, not necessarily a symmetry), we ask when $U^\varepsilon := e^{\varepsilon \mathbf{X}_\alpha} U$ is a nontrivial separated solution of Δ. In other words, we are looking for conditions when our U^ε:

(a) solves the equation

$$\Delta[U^\varepsilon] = 0, \tag{4.10}$$

(b) is additively separated

$$U^\varepsilon_{ij} = 0, \qquad \forall i \neq j, \tag{4.11}$$

and

(c) depends nontrivially on all independent variables for $\varepsilon \neq 0$:

$$U^\varepsilon_1\, U^\varepsilon_2 \cdots\ U^\varepsilon_p \neq 0. \tag{4.12}$$

This question may be rephrased in terms of infinitesimal α by expanding U^ε in powers of ε. We arrive at the system

$$\mathbf{X}_\alpha^n \Delta[u]\big|_{u=U} = 0 \qquad n = 1, 2, \dots, \tag{4.13}$$

$$\mathbf{X}_\alpha^n (u_{ij})\big|_{u=U} = 0, \qquad n = 1, 2, \dots, \quad \forall i \neq j. \tag{4.14}$$

Our objective is to find such infinitesimals α that this infinite-dimensional system is satisfied for some equation $\Delta = 0$. Consider the following cases:

4.1 $\mathbf{X}_\alpha$ is a Symmetry of Each Equation Separately

We suppose that $\mathbf{X}_\alpha$ is a symmetry of the equation itself

$$\mathbf{X}_\alpha \Delta[u]\big|_{\Delta[u]=0} = 0, \tag{4.15}$$

as well as of the separation conditions

$$\mathbf{X}_\alpha (u_{ij})\big|_{u_{k\ell}=0, \forall k \neq \ell} = 0, \qquad \forall i \neq j. \tag{4.16}$$

Since U is a solution of $\Delta[u] = 0$ and $u_{ij} = 0$, imposing these two symmetry conditions satisfies (4.13) and (4.14). Indeed, assuming Δ is of maximal rank, (4.15) is equivalent to the existence of some smooth B's such that

$$\mathbf{X}_\alpha \Delta[u] = B^I D_I \Delta[u], \tag{4.17}$$

which obviously vanishes on solutions of Δ. If we apply $\mathbf{X}_\alpha$ to both sides of the last equation we get:

$$\begin{aligned}\mathbf{X}_\alpha^2 \Delta[u] &= (\mathbf{X}_\alpha B^I) D_I \Delta[u] + B^I D_I (\mathbf{X}_\alpha \Delta[u]) \\ &= (\mathbf{X}_\alpha B^I) D_I \Delta[u] + B^I D_I (B^J D_J \Delta[u]),\end{aligned} \tag{4.18}$$

The RHS clearly vanishes on solutions (in particular, when $u = U$). In similar ways, we see that both (4.13) and (4.14) are satisfied for all $n \geq 1$.

More generally, suppose that

$$\begin{aligned}&\mathbf{X}_\alpha \Delta[u]\big|_{\Delta[u]=u_{k\ell}=0} = 0, \\ &\mathbf{X}_\alpha (u_{ij})\big|_{\Delta[u]=u_{k\ell}=0} = 0, \qquad \forall i \neq j.\end{aligned} \tag{4.19}$$

Since U is a solution of the joint system $\Delta[u] = u_{ij} = 0$, we see that imposing this joint symmetry condition satisfies (4.13) and (4.14).

For a contact infinitesimal $\alpha = \alpha(x, u, u_{(1)})$ the solution to symmetry conditions (4.16) is

$$\alpha = \lambda u + \sum_{i=1}^{p} f^i(x_i, u_i), \tag{4.20}$$

where λ is a constant, and the f's are some functions. In other words, on additively separated solutions u, the infinitesimal itself should be additively separated.

If $\alpha = \eta(x,u) - u_i\,\xi^i(x,u)$ is a point infinitesimal, then the associated point vector field is of the form

$$\mathbf{X} = \sum_{i=1}^{p} \xi^i(x_i)\,\partial_i + \left[\lambda u + \sum_{i=1}^{p} \eta^i(x_i)\right]\partial_u, \tag{4.21}$$

where the ξ's and η's are functions of independent variables.

In the multiplicative case (i.e. $u \to v = e^u$), the required symmetry is instead of the form

$$\mathbf{X} = \sum_{i=1}^{p} \xi^i(x_i)\,\partial_i + [\lambda u \ln u + u \sum_{i=1}^{p} \eta^i(x_i)]\,\partial_u. \tag{4.22}$$

Example 23 Let $f(x) = \sum_{i=1}^{p} f^i(x_i)$. If an equation Δ has a symmetry of the form

$$\mathbf{X} = u f(x)\,\partial_u \tag{4.23}$$

for some function $f(x)$, then for each solution u, the function

$$u^{\varepsilon}(x) = [e^{\varepsilon \mathbf{X}_\alpha} u](x) = e^{\varepsilon f(x)} u(x) \tag{4.24}$$

is also a solution. If Δ admits constant U as a solution, then it also admits the multiplicatively separated function

$$U^{\varepsilon}(x) = U\,e^{\varepsilon f^1(x_1)} \cdots e^{\varepsilon f^p(x_p)} \tag{4.25}$$

as a solution for each ε. Such equations include those of the form

$$\Delta(x, f_i \ln u - f u_i/u\big|_{i=1}^{p}, f_{ii} u_i/u - f_i(u_{ii}/u - u_i^2/u^2)\big|_{i=1}^{p}, u_{ij}/u - u_i u_j/u^2\big|_{i\neq j}) = 0. \tag{4.26}$$

For example, the degenerate equation for $u = u(x,t)$

$$\Delta[u] = u_{tt} + 2c u_{xt} + c^2 u_{xx} - (u_t + c u_x)\Phi(x, t, (u_t + c u_x)/u) = 0. \tag{4.27}$$

has symmetry (4.23) for $f = x - ct$, and admits $u = U$ as a solution if $U \neq 0$. From (4.23), we conclude that $u(x,t) = U\,e^{x-ct}$ is a solution.

4.2 $\mathbf{X}_\alpha$ is a Symmetry of Only Δ

The previous class does not recover (4.2). In this case, $\mathbf{X}_\alpha$ is a symmetry of only Δ, and not $u_{ij} = 0$:

$$\mathbf{X}_\alpha \Delta[u]\big|_{\Delta[u]=0} = 0. \tag{4.28}$$

Since U is a solution to Δ, this condition implies (4.13). It remains to solve (4.14). Let us rewrite it as follows:

$$D_{ij}(\mathbf{X}_\alpha^n u)\big|_{u=U} = 0, \qquad n \geq 1, \quad \forall i \neq j. \tag{4.29}$$

This is an infinite dimensional overdetermined nonlinear system, but it simplifies considerably in the case of polynomial α.

For example, using (4.2) as a model, let us seek a solution of the form

$$\alpha = a^i x_i + b + (c^{ij} x_j + d_i) u_i, \tag{4.30}$$

i.e. a point symmetry with affine coefficients. It can be verified that this actually solves (4.29) for any such constants, since $D_{ij}|_{u=U} = \partial_{ij}$. The corresponding point symmetry vector field is

$$\mathbf{X} = -(c^{ij} x_j + d_i)\,\partial_i + (a^i x_i + b)\,\partial_u. \tag{4.31}$$

To see when nontriviality condition (4.12) is satisfied, we consider the first two orders of ε:

$$\begin{aligned} 0 &\neq D_1\left(\varepsilon\alpha + \frac{1}{2}\varepsilon^2 \mathbf{X}_\alpha \alpha + O(\varepsilon^3)\right) \cdots D_p\left(\varepsilon\alpha + \frac{1}{2}\varepsilon^2 \mathbf{X}_\alpha \alpha + O(\varepsilon^3)\right)\Big|_{u=U} \\ &= \varepsilon^p \left(a^1 + \frac{1}{2}\varepsilon\, c^{i1} a^i + O(\varepsilon^2)\right) \cdots \left(a^p + \frac{1}{2}\varepsilon\, c^{ip} a^i + O(\varepsilon^2)\right) \\ &= \varepsilon^p \left(a^1 \cdots a^p + \frac{\varepsilon}{2} \sum_{i,j} c^{ij} a^i \Pi_{k\neq j} a^k + O(\varepsilon^2)\right). \end{aligned} \tag{4.32}$$

Therefore, if Δ admits $\mathbf{X}$ as a symmetry, and if $\sum_{i,j} c^{ij} a^i \Pi_{k\neq j} a^k$ (or $a^1 \cdots a^p$) is nonzero, then $e^{\varepsilon \mathbf{X}_\alpha} U$ is a nontrivial separated solution of Δ. [If $c^{ij} = \delta^{ij}\sigma^j$, then $\sum_{i,j} c^{ij} a^i \Pi_{k\neq j} a^k = p\Pi_i a^i$, and the previous class applies.]

Let us consider the special case of $p = 2$ and $u = u(x,t)$ and find equations with the symmetry

$$\mathbf{X} = at\,\partial_x + bx\,\partial_t + cxu\,\partial_u, \tag{4.33}$$

which reduces to (4.2) for $a = -2, b = 0, c = 1$. We have $a^1 = c, a^2 = 0, c^{11} = c^{22} = 0, c^{12} = a, c^{21} = b$. Since $a^1 a^2 = 0$, we require that $0 \neq c^{12}(a^1)^2 + c^{21}(a^2)^2 = ac^2$ in order for our equation to possess nontrivial (multiplicatively) separated solutions. Let us set $a = 1$ and $b = 0$. We have the symmetry

$$\mathbf{X} = t\,\partial_x + cxu\,\partial_u, \tag{4.34}$$

for some $c \neq 0$. The seven differential invariants of the second order extended space $(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$ are

$$
\begin{aligned}
&t, \ln u - cx^2/2t, u_x/u - cx/t, u_t/u + u_x^2/(2cu^2),\\
&u_{xx}/u - u_x^2/u^2, u_{xt}/u - u_xu_t/u^2 + u_xu_{xx}/cu^2 - u_x^3/cu^3,\\
&u_{tt}/u^2 - u_t^2/u^2 + 2u_xu_{xt}/cu^2 - 2u_x^2u_t/cu^3 + u_x^2u_{xx}/c^2u^3 - u_x^4/c^2u^4.
\end{aligned} \tag{4.35}
$$

The last four invariants vanish when $u = U$. Therefore, any differential equation of the form

$$
\Delta[u] = A^4(I)I_4 + A^5(I)I_5 + A^6(I)I_6 + A^7(I)I_7 = 0, \tag{4.36}
$$

where $A^k(I) = A^k(I_1, \dots, I_7)$ are smooth functions, admits $u = U$ as a solution. Since

$$
e^{\varepsilon X_\alpha} U = e^{\varepsilon cx - \frac{1}{2}\varepsilon^2 ct}\, U, \tag{4.37}
$$

we conclude that any equation of the form (4.36) admits separated function (4.37) as a solution. The heat equation (4.1) arises from the sub-case $\Delta[u] = I_5 + 2cI_4$.

5 Lagrangians with Separation

5.1 Variational Symmetries

A generalized vector field given by (2.2) has a corresponding evolutionary vector field:

$$
\mathbf{X}_\alpha = \alpha^a[u]\,\partial_a; \quad \alpha^a = \eta^a[u] - u_i^a \xi^i[u], \tag{5.1}
$$

where α^a is the *characteristic* of the symmetry group. The prolongation of $\mathbf{X}_\alpha$ in (5.1) is

$$
\mathbf{X}_\alpha = D_{j_1} \dots D_{j_\ell} \alpha^a[u]\,\partial_{a,j_1\dots j_\ell}. \tag{5.2}
$$

The operator $\mathbf{X}_\alpha$ is a symmetry if and only if

$$
\mathbf{X} = \mathbf{X}_\alpha + \xi^i D_i \tag{5.3}
$$

is a symmetry.

Consider a variational problem given by the functional $\mathscr{L} : C^k|_\Omega \to \mathbb{R}$ defined by the formula

$$
\mathscr{L}[u] = \int_\Omega L(x, u, \partial u, \dots, \partial_k u) dx, \tag{5.4}
$$

where $L = L(x, u, \partial_u, \dots, \partial_k)$ is the *Lagrangian* of $\mathscr{L}$. The vector field (5.1) is a *variational symmetry* of L if and only if

$$
\mathbf{X}_\alpha(L) = D_i M^i[u], \tag{5.5}
$$

for some functions $M^i[u] \neq 0$, or

$$\mathbf{X}(L) + LD_i(\xi^i) + LD_i(\xi^i) = D_i(M^i + \xi^i L). \tag{5.6}$$

If $\mathbf{X}_\alpha(L) = 0$ the corresponding vector field is a *Noether symmetry*

A variational symmetry is also a symmetry of the corresponding Euler–Lagrange equations $E_\alpha(L) = 0$, where the Euler operator is defined by:

$$E_a = (-1)^\ell D_{j_1} \dots D_{j_\ell} \partial_{a,j_1 \dots j_\ell}; \quad \forall \ell = 1, \dots, k. \tag{5.7}$$

However, symmetries of the Euler–Lagrange equations are not necessarily variational symmetries of the Lagrangian.

5.2 Vector Fields That Scale the Lagrangian

Let $\mathbf{X}$ be a vector field and consider the Lagrangian to some variational problem (5.4). Suppose that under $\mathbf{X}$ we have

$$\mathbf{X}(L) + LD_i\xi^i = \mathbf{X}_\alpha(L) + D_i(L\xi^i) = kL, \tag{5.8}$$

for some real number $k \neq 0$. We refer to all $\mathbf{X}$ that have this effect as *semi-symmetries*. We will use the identity

$$E(\mathbf{X}_\alpha L) = \mathbf{X}_\alpha E(L) + \mathcal{D}^\star_\alpha(E(L)), \tag{5.9}$$

where $\mathcal{D}^\star_\alpha$ is the adjoint of the Frechet-derivative with α

$$\mathcal{D}_\alpha = \frac{\partial \alpha}{\partial u^a_{j_1 \dots j_k}} D_{j_1} \dots D_{j_k}. \tag{5.10}$$

Then

$$\mathbf{X}_\alpha E(L) + \mathcal{D}^\star_\alpha(E(L)) = E(\mathbf{X}_\alpha L + D_i(L\xi^i)) = E(kL) = kE(L). \tag{5.11}$$

Thus,

$$\mathbf{X}_\alpha E(L)\big|_{E(L)=0} = 0, \tag{5.12}$$

and all vector fields that scale Lagrangian L are symmetries of its Euler–Lagrange equations.

Many semi-symmetries are dilations.

Example 24 For the linear wave equation

$$u_{tt} - u_{xx} - u_{yy} = 0, \tag{5.13}$$

there is a well defined variation problem

$$\mathscr{L}[u] = \int_\Omega [u_t^2 - u_x^2 - u_y^2] dxdt. \tag{5.14}$$

The dilation $\mathbf{X}_1 = u\,\partial_u$ is a semi-symmetry vector since

$$\widetilde{\mathbf{X}}_1 L + LD_i(\xi^i) = \widetilde{\mathbf{X}}_1(u_t^2 - u_x^2 - u_y^2) = 2(u_t^2 - u_x^2 - u_y^2) = 2, \tag{5.15}$$

where $\widetilde{\mathbf{X}}_1 = \mathbf{X}_1 + u_x\,\partial_{u_x} + u_t\,\partial_{u_t}$ is the prolonged operator $\mathbf{X}_1$. It is known that $\mathbf{X}_1$ is also a symmetry of the wave equation.

The dilation $\mathbf{X}_2 = x\,\partial_x + y\,\partial_y + t\,\partial_t$ is known to be a symmetry of the wave equation, and it is also a semi-symmetry:

$$\widetilde{\mathbf{X}}_2(L) + LD_i\xi^i = \widetilde{\mathbf{X}}_2(u_t^2 - u_x^2 - u_y^2) + 3(u_t^2 - u_x^2 - u_y^2) = (-2+3)(u_t^2 - u_x^2 - u_y^2) = L. \tag{5.16}$$

Example 25 The minimal surface Eq. (3.64) has Lagrangian function

$$L = \sqrt{1 + u_x^2 + u_y^2}, \tag{5.17}$$

and its symmetry operator

$$\mathbf{X} = x\,\partial_x + y\,\partial_y + u\,\partial_u \tag{5.18}$$

is also a semi-symmetry since

$$\widetilde{\mathbf{X}}(L) + LD_i\xi^i = \mathbf{X}\left(\sqrt{1 + u_x^2 + u_y^2}\right) + 2\sqrt{1 + u_x^2 + u_y^2} = 2\sqrt{1 + u_x^2 + u_y^2} = 2L. \tag{5.19}$$

5.3 Separation Lagrangians

Consider the Lagrangian $L = L(x, t, u, u_x, u_t)$. For any variational symmetry $\mathbf{X}_\alpha$ from (5.5) we have

$$\mathbf{X}_\alpha(L) = D_x(M^1) + D_t(M^2), \tag{5.20}$$

where $M^i = M^i(x, t, u,\ \partial u,\ \partial_2 u,\ \partial_3 u)$.

Now we will use symmetries from the classes (3.6) or (3.11), and (3.19) or (3.23) (3.23) to generate Lagrangians admitting those symmetries that would lead to additively or multiplicatively separated solution, respectively.

Example 26 Consider a special case of (3.23), and assume that the following operators are Noether symmetries of the variational problem with Lagrangian L, (and we will look for such Lagrangians)

$$\begin{aligned} &\mathbf{X}_1 = x\,\partial_x + au\,\partial_u, \\ &\mathbf{X}_2 = \partial_x, \quad \mathbf{X}_3 = \partial_t, \end{aligned} \tag{5.21}$$

for some constant $a \in \mathbb{C}$. Their first prolongations are:

$$\begin{aligned}\tilde{\mathbf{X}}_1 &= \mathbf{X}_1 + (a-1)u_x\,\partial_{u_x} + au_t\,\partial_{u_t},\\ \tilde{\mathbf{X}}_2 &= \mathbf{X}_2, \quad \tilde{\mathbf{X}}_3 = \mathbf{X}_3.\end{aligned} \tag{5.22}$$

Applying (5.22) and (5.6) with $M^i = 0$, we find:

$$\begin{aligned}&(a-1)u_xL_{u_x} + au_tL_{u_t} + auL_u + xL_x + L = 0,\\ &L_x = L_t = 0.\end{aligned} \tag{5.23}$$

Thus, $L = L(u, u_x, u_t)$. From the invariants of the first condition

$$Lu^{1/a}, \frac{u_x}{u^{1-1/a}}, \frac{u_t}{u}, \qquad a \neq 1. \tag{5.24}$$

we find:

$$\begin{aligned}L &= u^{-1/a}\phi\left(\frac{u_x}{u^{1-1/a}}, \frac{u_t}{u}\right) \qquad a \neq 1,\\ L &= u^{-1}\phi\left(u_x, \frac{u_t}{u}\right), \qquad a = 1,\end{aligned} \tag{5.25}$$

where functions ϕ are arbitrary.

Thus, the equations with any Lagrangians (5.25) have multiplicatively separated solutions of the form (see (3.23))

$$u(x,t) = x^aT(t). \tag{5.26}$$

Example 27 Consider variational problems with the following Noether symmetries:

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_t, \quad \mathbf{X}_3 = \partial_u. \tag{5.27}$$

These operators are each their own prolongations. Thus, the Lagrangians:

$$L = L(u_x, u_t). \tag{5.28}$$

Thus, the equations with Lagrangians (5.28) have additively separated solutions, including $u(x,t) = X(x) + at$, and $u(x,t) = bx + T(t)$.

Example 28 The requirement that the operator

$$\mathbf{X} = u\,\partial_u \tag{5.29}$$

is a variational symmetry leads to the following Lagrangians:

$$L = \varphi\left(x, t, \frac{u_x}{u_t}, \frac{u_x}{u}\right), \tag{5.30}$$

with an arbitrary function φ.

However, the requirement that the same operator (5.29) is a semi-symmetry would give rise to the Lagrangians

$$L = u^k \phi\Big(x, t, \frac{u_x}{u}, \frac{u_t}{u}\Big), \tag{5.31}$$

with arbitrary function ϕ, and constant k. All equations with Lagrangians (5.29), or (5.31) will have solutions with multiplicatively separated variables.

6 Separation with Conditional Symmetry Operators

There are equations which have no symmetries but possess separated solutions. Indeed, the minimal surface-type equation

$$(1 + u_t^2)u_{xx} - (x + u^2 t + u_x + u_t)u_{xx}^3 u_{xt} + (1 + u_x^2)u_{tt} = 0,$$

has no classical Lie point or contact symmetries, but has several additively separated solutions of the form $u(x, t) = X(x) + T(t)$. If we rewrite this equation in the form

$$\frac{u_{xx}}{1 + u_x^2} + \frac{u_{tt}}{1 + u_t^2} - u_{xt}\,\frac{(x + u^2 t + u_x + u_t)u_{xx}^3}{(1 + u_x^2)(1 + u_t^2)} = 0, \tag{6.1}$$

it is clear that imposing separation condition $u_{xt} = 0$ removes the main problematic term. On the other hand, the minimal surface equation itself

$$(1 + u_t^2)u_{xx} - 2u_x u_t u_{xt} + (1 + u_x^2)u_{tt} = 0$$

has a large symmetry group, but the additively separated solution

$$u(x, t) = \ln\left|\frac{\cos x}{\cos t}\right|$$

is not invariant with respect to any of its point symmetries. For these types of equations, operators of conditional symmetry (with respect to the side condition $u_{xt} = 0$) play a more important role than symmetry vector fields in determining the existence of separated solutions.

Motivated by form (6.1), we consider equations for $u = u(x^1, \dots, x^p)$ of the form

$$\Delta[u] = \lambda u + \sum_{i=1}^{p} R^i(x^i, u^i_i, (u^i_i)_{(1)}, \dots) + \sum_{\substack{i<j,\\ |\beta| \geq 0}} A^{ij\beta}(x, u, u_{(1)}, \dots)u_{ij\beta}, \tag{6.2}$$

where $u_{ij\beta} = \partial_i\,\partial_j\,\partial_\beta u$, the A's and R's are smooth functions, and λ is a constant. [For multiplicative separation, replace u by $\ln u$ throughout]. Such equations can be separated. Indeed, seeking a separated solution of the form $u = \sum_{i=1}^{p} f^i(x_i)$ yields

$$0 = \Delta\Big[\sum f^i\Big] = \sum_{i=1}^{p}\big[\lambda f^i(x^i) + R^i(x^i, f^i_i(x^i), f^i_{ii}(x^i)\delta_{ij}, \dots)\big]. \tag{6.3}$$

Taking partial derivatives of this expression shows that each summand is constant, or

$$\lambda f^i(x^i) + R^i(x^i, f^i_i(x^i), f^i_{ii}(x^i)\delta_{ij}, \dots) = \gamma^i, \qquad i = 1, \dots, p, \tag{6.4}$$

where constants γ^i satisfy $\sum_{i=1}^p \gamma^i = 0$. This is a set of p decoupled, ordinary differential equations for the $f^i(x^i)$'s, and, provided R^i is nonsingular with respect to the highest $f^i(x^i)$ derivative (e.g. quasilinear), it admits local solutions. Therefore, sufficiently nondegenerate equations of the class (6.2) possess separated solutions.

Many well known equations of mathematical physics are of form (6.2), and this is common for those solved with separation of variables. The nonlinear heat equation and beam equation

$$u_t = \partial_x(k(x, u_x)u_x), \qquad u_{tt} = -\partial_x^2(E(x)u_{xx}) + f(x) \tag{6.5}$$

are of this form for additive separation. The Schrödinger equation and the Klein–Gordon equation

$$iv_t = -v_{xx} + V(x)v, \qquad v_{tt} - c(x)^2 v_{xx} + m(x)^2 v = 0 \tag{6.6}$$

are of this form for multiplicative separation (i.e. $v = e^u$). Geometric equations such as Aronsson's equation and the equation for exponentially harmonic maps

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0, \qquad (1 + u_x^2)u_{xx} + 2u_x u_y u_{xy} + (1 + u_y^2)u_{yy} = 0 \tag{6.7}$$

are additive examples with $A^{ij\beta} \neq 0$. The Monge-Ampére equation

$$v_{xx}v_{yy} - v_{xy}^2 = 0 \tag{6.8}$$

is one such multiplicative (and additive) example. Indeed, letting $v = e^u$ and rearranging gives

$$\frac{u_{xx}}{u_{xx} + u_x^2} + \frac{u_{yy}}{u_y^2} - \frac{u_{xy}(u_{xy} + 2u_x u_y)}{u_y^2(u_{xx} + u_x^2)} = 0, \tag{6.9}$$

provided $u \neq \ln[ax + b(y)]$ or $c(x)$, where a is constant, and b and c are functions.

Let us derive (6.2) by supposing that equation $\Delta[u] = 0$ is conditionally invariant with respect to the operators

$$D_{ij} = D_i D_j, \qquad i \neq j \tag{6.10}$$

and the matching side conditions $u_{ij} = 0, i \neq j$. That is, we suppose that

$$D_{ij}\Delta[u]\big|_{u_{k\ell}=0} = 0, \qquad i \neq j, \qquad k \neq \ell. \tag{6.11}$$

[Note: this is an alternative definition of conditional invariance. The typical approach uses just vector fields, not other differential operators. Also, the other definition allows further constraining u to solve $\Delta[u]=0$, but in that case, $D_{ij}\Delta[u]|_{u_{k\ell}=\Delta=0}=0$ would be trivially satisfied.] Let $\mathcal{R}[u]=\Delta[u]|_{u_{k\ell}=0}$. Since

$$D_{ij}\Delta=\Delta_{ij}+u_{i\alpha}\Delta_{ju_\alpha}+u_{j\alpha}\Delta_{iu_\alpha}+u_{i\alpha}u_{j\beta}\Delta_{u_\alpha u_\beta}+u_{ij\alpha}\Delta_{u_\alpha}, \tag{6.12}$$

where the sum is over $\alpha:|\alpha|\geq 0$, conditional invariance requirement (6.11) becomes

$$0=D_{ij}\Delta[u]\big|_{u_{k\ell}=0}=\mathcal{R}_{ij}+u_{(a+1)\cdot i}\mathcal{R}_{ju_{a\cdot i}}+u_{(a+1)\cdot j}\mathcal{R}_{iu_{a\cdot j}}+u_{(a+1)\cdot i}u_{(b+1)\cdot j}\mathcal{R}_{u_{a\cdot i}u_{b\cdot j}}, \tag{6.13}$$

where $u_{a\cdot i}=\partial_i^a u$, and sums are taken over $a,b\geq 0$. If ω is the highest order of derivative R depends on, then $\mathcal{R}$ does not depend on the $\omega+1$ order derivatives, so coefficients of these terms must vanish, such as $\mathcal{R}_{u_{\omega\cdot i}u_{b\cdot j}}$ for all $b\geq 0$. If we continue this process downward on the derivative order, we obtain that

$$\mathcal{R}[u]=\sum_{i=1}^{p}\mathcal{R}^i(x^i,u_i,u_{ii},\dots)+\mathcal{S}(x,u), \tag{6.14}$$

where $S(x,u)$ solves

$$\mathcal{S}_{ij}+u_i\mathcal{S}_{ju}+u_j\mathcal{S}_{iu}+u_iu_j\mathcal{S}_{uu}=0, \qquad i\neq j. \tag{6.15}$$

So each term is zero, and $\mathcal{S}(x,u)=\lambda u+\sum\mathcal{S}^i(x^i)$, the latter terms of which can be absorbed into $\mathcal{R}^i$.

Now, let $R^i[u]=R^i(x^i,u_i,(u_i)_{(1)},\dots)$ be an extension of $\mathcal{R}^i[u]$ away from $u_{k\ell}=0$. Then

$$\left(\Delta[u]-\lambda u-\sum_{i=1}^{p}R^i[u]\right)\Bigg|_{u_{k\ell}=0}=0. \tag{6.16}$$

The Proposition 2.10 in [19] implies that

$$\Delta[u]=\lambda u+\sum_{i=1}^{p}R^i[u]+\sum_{|\alpha|\geq 0}A^{ij\alpha}[u]u_{ij\alpha}, \tag{6.17}$$

for some smooth coefficients $A^{ij\alpha}$, which shows that (6.2) arises precisely from conditional symmetry operator invariance. [Note that the extensions R^i are well defined up to terms of the form $B^{ij\alpha}(x^i,u_i,(u_i)_{(1)},\dots)u_{ij\alpha}$.]

Using this same classification procedure but with fewer conditional symmetry operators D_{ij}, one can derive equations that have slightly weaker separability properties. For a multiplicative example, the multi-dimensional wave equation and the Schrödinger equation

$$u_{tt}-c(x)^2(u_{11}+\cdots+u_{pp})=0, \qquad iu_t=-\Delta u+V(x)u \tag{6.18}$$

can be partially separated as $u=T(t)X(x^1,\dots,x^p)$, but they do not always admit completely separated solutions of the form $u(x,t)=T(t)X^1(x^1)\cdots X^p(x^p)$.

Further studies are needed to understand the role of conditional (non-classical) symmetry in relation to the existence of solutions in separated variables. It would be important to study the inverse problem: if the existence of solutions in separated variables imply some type of conditional or non-classical symmetry of the original system. Indeed, the Laplace-Beltrami equation $(\Delta_{g(x)} + \Delta_{h(y)})u = 0$ induced by the product metric $g_{ij}(x)dx^i dx^j + h_{ij}(y)dy^i dy^j$ with arbitrary smooth functions $g(x)$ and $h(y)$ is known to have both additive and multiplicative separated solutions but does not have required classical symmetries of the form (3.189) or (3.195).

7 Conclusions

We discussed the role of symmetry operators of a differential system in order to determine the existence of solutions in separated variables. We have shown that, under basic non-degeneracy assumptions, certain types of symmetry operators of a differential system not only provide an indication to whether or not the system has a solution in separated variables, but also partially determine the form of such solution.

For differential systems with two independent variables, we obtained the form of Lie point symmetry operators corresponding to separated solutions for the case when separated variables are any functions of independent variables (Theorems 1–4).

We have also shown that, for many PDE's of mathematical physics, all separated solutions are also invariant solutions of some symmetry subalgebras of the original system.

It would be important to study the inverse problem: if the existence of solutions in separated variables imply some type of symmetry of the original system.

Acknowledgements R.S. acknowledges support from Princeton University.

Author Contributions Not applicable.

Funding R.S. acknowledges support from Princeton University related to this publication.

Availability of data and materials All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest We declare no financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

Ethics approval and consent to participate Not applicable.

Consent for publication Not applicable.

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