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RESEARCH ARTICLE

Probabilistic Degenerate Fubini Polynomials Associated with Random Variables

Rongrong Xu[1] · Taekyun Kim[1,2] · Dae San Kim[3] · Yuankui Ma[1]

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Abstract

Let Y be a random variable such that the moment generating function of Y exists in a neighborhood of the origin. The aim of this paper is to study probabilistic versions of the degenerate Fubini polynomials and the degenerate Fubini polynomials of order r, namely the probabilisitc degenerate Fubini polynomials associated with Y and the probabilistic degenerate Fubini polynomials of order r associated with Y. We derive some properties, explicit expressions, certain identities and recurrence relations for those polynomials. As special cases of Y, we treat the gamma random variable with parameters $\alpha, \beta > 0$, the Poisson random variable with parameter $\alpha > 0$, and the Bernoulli random variable with probability of success p.

Keywords Probabilistic degenerate Fubini polynomials · Probabilistic degenerate Fubini polynomials of order r · Probabilistic degenerate Stirling numbers of the second kind

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✉ Taekyun Kim
tkkim@kw.ac.kr

✉ Dae San Kim
dskim@sogang.ac.kr

✉ Yuankui Ma
mayuankui@xatu.edu.cn

Rongrong Xu
xurongrong0716@163.com

1 School of Science, Xi'an Technological University, Xi'an 710021, Shaanxi, China

2 Department of Mathematics, Kwangwoon University, Seoul 01897, Republic of Korea

3 Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea

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1 Introduction

In recent years, degenerate versions, λ-analogues and probabilistic versions of many special polynomials and numbers have been investigated by employing various methods such as generating functions, combinatorial methods, umbral calculus, p-adic analysis, differential equations, probability, special functions, analytic number theory and operator theory (see [11–16, 18–21] and the references therein).

Let Y be a random variable satisfying the moment condition (see 20). The aim of this paper is to study probabilistic versions of the degenerate Fubini polynomials and the degenerate Fubini polynomials of order r, namely the probabilisitc degenerate Fubini polynomials associated with Y and the probabilistic degenerate Fubini polynomials of order r associated with Y. We derive some properties, explicit expressions, certain identities and recurrence relations for those polynomials and numbers. In addition, we consider the special cases that Y is the gamma random variable with parameters $\alpha, \beta > 0$, the Poisson random variable with parameter $\alpha(> 0)$, and the Bernoulli random variable with probability of success p.

The outline of this paper is as follows. In Sect. 1, we recall the degenerate exponentials, the degenerate Stirling numbers of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda}$, the degenerate Bell polynomials, the degenerate Fubini polynomials and the degenerate Fubini polynomials of order r. We remind the reader of Lah numbers and the partial Bell polynomials. Assume that Y is a random variable such that the moment generating function of Y, $E[e^{tY}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[Y^n], \quad (|t| < r)$, exists for some $r > 0$. Let $(Y_j)_{j \geq 1}$ be a sequence of mutually independent copies of the random variable Y, and let $S_k = Y_1 + Y_2 + \cdots + Y_k, \ (k \geq 1), \quad \text{with} \quad S_0 = 0$. Then we recall the probabilistic degenerate Stirling numbers of the second kind associated with Y and the probabilistic degenerate Bell polynomials associated with Y, $\phi_{n,\lambda}^{Y}(x)$. Also, we remind the reader of the gamma random variable with parameters $\alpha, \beta > 0$. Section 2 is the main result of this paper. Let $(Y_j)_{j \geq 1}, \ S_k, \ (k = 0, 1, \dots)$ be as in the above. Then we first define the probabilistic degenerate Fubini polynomials associated with the random variable Y, $F_{n,\lambda}^{Y}(x)$. We derive for $F_{n,\lambda}^{Y}(x)$ an explicit expression in Theorem 1 and an expression as an infinite sum involving $E[(S_k)_{n,\lambda}]$ in Theorem 2. In Theorem 3, when $Y \sim \Gamma(1, 1)$, we find an expression for $F_{n,\lambda}^{Y}(x)$ in terms of Lah numbers and Stirling numbers of the first kind. We obtain a representation of $F_{n,\lambda}^{Y}(x)$ as an integral over $(0, \infty)$ of the integrand involving $\phi_{n,\lambda}^{Y}(x)$ in Theorem 4 and its generalization in Theorem 14. In Theorem 5, we express the probabilistic degenerate Fubini numbers associated with Y, $F_{n,\lambda}^{Y} = F_{n,\lambda}^{Y}(1)$, as a finite sum involving the partial Bell polynomials. Then we introduce the probabilistic degenerate Fubini polynomials of order r associated with Y and deduce an explicit expression for them in Theorem 6. We obtain a recurrence relation for $F_{n,\lambda}^{Y}(x)$ in Theorem 7, and another one in Theorem 8 together with its generalization in Theorem 15. In Theorem 9, the rth derivative of $F_{n,\lambda}^{Y}(x)$ is expressed in terms of $F_{i,\lambda}^{(r+1,Y)}(x)$. We get the identity $\frac{1}{1-x}F_{n,\lambda}^{Y}\big(\frac{x}{1-x}\big)=\sum_{i=0}^{\infty}E[(S_i)_{n,\lambda}]x^i$ in Theorem 10 and its generalization in Theorem 13. In Theorem 11, when Y is the Poisson random variable with parameter α, we express $F_{n,\lambda}^{Y}(x)$ in terms of the Fubini polynomials $F_i(x)$ and $\left\{ \begin{matrix} n \\ i \end{matrix} \right\}_{\lambda}$. In Theorem 12, when Y is the Poisson random variable with parameter α, we show $\frac{1}{1-x}F_{n,\lambda}^{Y}\big(\frac{x}{1-x}\big)=\sum_{k=0}^{\infty}\phi_{n,\lambda}(k\alpha)x^k$. Finally, we show in Theorem 16 that $F_{n,\lambda}^{Y}(x)=F_{n,\lambda}(xp)$ if Y is the Bernoulli random variable with probability of success p. For the rest of this section, we recall the facts that are needed throughout this paper.

For any $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by (see [6–21])

$$e_{\lambda}^{x}(t)=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^k}{k!},\quad e_{\lambda}(t)=e_{\lambda}^{1}(t), \tag{1}$$

where

$$(x)_{0,\lambda}=1,\quad (x)_{n,.\lambda}=x(x-\lambda)\cdots\big(x-(n-1)\lambda\big),\quad (n\geq 1). \tag{2}$$

Note that

$$\lim_{\lambda\to 0}e_{\lambda}^{x}(t)=e^{xt}.$$

The Stirling numbers of the first kind are defined by (see [1–3, 5, 24])

$$(x)_n=\sum_{k=0}^{n}S_1(n,k)x^k,\quad (n\geq 0), \tag{3}$$

where

$$(x)_0=1,\quad (x)_n=x(x-1)\cdots(x-n+1),\quad (n\geq 1).$$

Alternatively, they are given by (see [5–24])

$$\frac{1}{k!}\big(\log(1+t)\big)^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}. \tag{4}$$

The Lah numbers are defined by

$$\langle x\rangle_n=\sum_{k=0}^{n}L(n,k)(x)_k,\quad (n\geq 0), \tag{5}$$

where

$$\langle x\rangle_0=1,\quad \langle x\rangle_n=x(x+1)\cdots(x+n-1),\quad (n\geq 1).$$

By (5), we easily get (see [5–24])

$$L(n,k)=\frac{n!}{k!}\binom{n-1}{k-1},\quad (n\geq k\geq 0). \tag{6}$$

From (6), the generating function of the Lah numbers is given by

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!}. \tag{7}$$

In [13], the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda}=\sum_{k=0}^{n}\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda}(x)_k,\quad (n\geq 0). \tag{8}$$

Alternatively, they are given by

$$\frac{1}{k!}\left(e_\lambda(t)-1\right)^k=\sum_{n=k}^{\infty}\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda}\frac{t^n}{n!}. \tag{9}$$

It is well known that the degenerate Bell polynomials are defined by (see [12–14])

$$e^{x(e_\lambda(t)-1)}=\sum_{n=0}^{\infty}\phi_{n,\lambda}(x)\frac{t^n}{n!}. \tag{10}$$

Thus, by (8) and (10), we get (see [12, 17, 21])

$$\phi_{n,\lambda}(x)=\sum_{k=0}^{n}\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda}x^k,\quad (n\geq 0). \tag{11}$$

The degenerate Fubini polynomials are defined by (see [17–19, 27])

$$F_{n,\lambda}(x)=\sum_{k=0}^{n}\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda}k!x^k,\quad (n\geq 0). \tag{12}$$

Thus, by (12), we get (see [4, 19, 21, 27])

$$\frac{1}{1-x(e_\lambda(t)-1)}=\sum_{n=0}^{\infty}F_{n,\lambda}(x)\frac{t^n}{n!}. \tag{13}$$

From (13), we note that (see [18])

$$\frac{1}{1-x}F_{n,\lambda}\left(\frac{x}{1-x}\right)=\left(x\frac{d}{dx}\right)_{n,\lambda}\frac{1}{1-x}=\sum_{k=0}^{\infty}(k)_{n,\lambda}x^k. \tag{14}$$

For $r \in \mathbb{N}$, the degenerate Fubini polynomials of order (see [8, 9, 16]) r are defined by

$$\left(\frac{1}{1-y(e_\lambda(t)-1)}\right)^r = \sum_{n=0}^{\infty} F_{n,\lambda}^{(r)}(y)\frac{t^n}{n!}. \tag{15}$$

Thus, by (15), we get (see [8, 18, 19, 22, 23])

$$F_{n,\lambda}^{(r)}(y) = \sum_{k=0}^{n} \binom{k+r-1}{k} y^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_\lambda k!. \tag{16}$$

From (15), we have

$$\left(\frac{1}{1-x}\right)^{r+1} F_{n,\lambda}^{(r+1)}\left(\frac{x}{1-x}\right) = \left(x\frac{d}{dx}\right)_{n,\lambda}\left(\frac{1}{1-x}\right)^{r+1} = \sum_{k=0}^{\infty}\binom{k+r}{r}(k)_{n,\lambda}x^k, \tag{17}$$

where n, r are nonnegative integers.

For any integer $k \geq 0$, the partial Bell polynomials are given by (see [5])

$$\frac{1}{k!}\left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1})\frac{t^n}{n!}, \tag{18}$$

where

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{l_1+l_2+\cdots+l_{n-k+1}=k \\ l_1+2l_2+\cdots+(n-k+1)l_{n-k+1}=n}} \frac{n!}{l_1 l_2! \cdots l_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{l_1}\left(\frac{x_2}{2!}\right)^{l_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{l_{n-k+1}}. \tag{19}$$

Let Y be a random variable such that the moment generating function of Y

$$E[e^{tY}] = \sum_{n=0}^{\infty} E[Y^n]\frac{t^n}{n!}, \quad (|t| < r) \quad \text{exists for some r>0.} \tag{20}$$

Assume that $(Y_j)_{j\geq 1}$ is a sequence of mutually independent copies of Y and $S_k = Y_1 + Y_2 + \cdots + Y_k$, $(k \geq 1)$ with $S_0 = 0$.

The probabilistic degenerate Stirling numbers of the second kind associated with random variable Y are defined by (see [15, 22])

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[(S_j)_{n,\lambda}], \quad (n \geq k \geq 0). \tag{21}$$

By binomial inversion, the Eq. (21) is equivalent to (see [15])

$$E[(S_k)_{n,\lambda}] = \sum_{j=0}^{k} \binom{k}{j} j! \left\{ \begin{matrix} n \\ jk \end{matrix} \right\}_{Y,\lambda}. \tag{22}$$

From (21), we note that (see [15])

$$\frac{1}{k!}(E[e_{\lambda}^{Y}(t)] - 1)^k = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} \frac{t^n}{n!}, \quad (k \geq 0). \tag{23}$$

In view of (11), the probabilistic degenerate Bell polynomials associated with Y are defined by (see [15, 20])

$$\phi_{n,\lambda}^{Y}(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} x^k, \quad (n \geq 0). \tag{24}$$

When $Y = 1$, we have $\phi_{n,\lambda}^{Y}(x) = \phi_{n,\lambda}(x)$.

By (24), we get (see [15])

$$e^{x(E[e_{\lambda}^{Y}(t)]-1} = \sum_{n=0}^{\infty} \phi_{n,\lambda}^{Y}(x) \frac{t^n}{n!}. \tag{25}$$

We recall that Y is the gamma random variable with parameter $\alpha, \beta > 0$ if probability density function of Y is given by (see [3, 25–28])

$$f(x) = \begin{cases} \frac{\beta}{\Gamma(\alpha)} e^{-\beta x} (\beta x)^{\alpha-1}, & \text{if } x \geq 0, \\ 0, & \text{if} x < 0, \end{cases}$$

which is denoted by $Y \sim \Gamma(\alpha, \beta)$.

Finally, if Y is the Poisson random variable with parameter $\alpha (> 0)$, then the moment generating function is given by:

$$E[e^{tY}] = \sum_{n=0}^{\infty} e^{tn} \frac{\alpha^n e^{-\alpha}}{n!} = e^{\alpha(e^t-1)}.$$

2 Probabilistic Degenerate Fubini Polynomials Associated with Random Variables

Let $(Y_k)_{k \geq 1}$ be a sequence of mutually independent copies of random variable Y, and let

$$S_0 = 0, \quad S_k = Y_1 + Y_2 + \cdots + Y_k, \quad (k \in \mathbb{N}).$$

Now, we consider the *probabilistic degenerate Fubini polynomials associated with random variable Y* which are given by

$$\frac{1}{1-x\big(E[e_\lambda^Y(t)]-1\big)}=\sum_{n=0}^{\infty}F_{n,\lambda}^Y(x)\frac{t^n}{n!}. \tag{26}$$

For $Y=1$, $E[Y]=1$ and we have $F_{n,\lambda}^Y(x)=F_{n,\lambda}(x)$, $(n\geq 0)$. When $x=1$, $F_{n,\lambda}^Y=F_{n,\lambda}^Y(1)$ are called the *probabilistic degenerate Fubini numbers associated with random variable Y*.

From (26) and (23), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}F_{n,\lambda}^Y(x)\frac{t^n}{n!}&=\sum_{k=0}^{\infty}x^k(E[e_\lambda^Y(t)]-1)^k=\sum_{k=0}^{\infty}x^k k!\frac{1}{k!}(E[e_\lambda^Y(t)]-1)^k\\&=\sum_{k=0}^{\infty}x^k k!\sum_{n=k}^{\infty}\left\{\begin{matrix}n\\k\end{matrix}\right\}_{Y,\lambda}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}x^k k!\left\{\begin{matrix}n\\k\end{matrix}\right\}_{Y,\lambda}\frac{t^n}{n!}.\end{aligned} \tag{27}$$

Therefore, by comparing the coefficients on both sides of (27), we obtain the following theorem.

Theorem 1 *For* $n\geq 0$, *we have*

$$F_{n,\lambda}^Y(x)=\sum_{k=0}^{n}\left\{\begin{matrix}n\\k\end{matrix}\right\}_{Y,\lambda}k!x^k.$$

By (26), we get

$$\begin{aligned}\sum_{n=0}^{\infty}F_{n,\lambda}^Y(x)\frac{t^n}{n!}&=\frac{1}{1-x(E[e_\lambda^Y(t)]-1)}=\frac{1}{1+x-xE[e_\lambda^Y(t)]}\\&=\frac{1}{1+x}\frac{1}{1-\frac{x}{1+x}E[e_\lambda^Y(t)]}=\frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k\Big(E[e_\lambda^Y(t)]\Big)^k\\&=\sum_{n=0}^{\infty}\frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k E\big[(Y_1+Y_2+\cdots+Y_k)_{n,\lambda}\big]\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k E[(S_k)_{n,\lambda}]\frac{t^n}{n!}.\end{aligned} \tag{28}$$

Therefore, by (28), we obtain the following theorem.

Theorem 2 *For* $n\geq 0$, *we have*

$$F^Y_{n,\lambda}(x) = \frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k E[(S_k)_{n,\lambda}].$$

In particular, for $Y = 1$, *we have*

$$F^Y_{n,\lambda}(x) = \frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k (k)_{n,\lambda}.$$

Let $Y \sim \Gamma(1,1)$. Then, by using (1), (4) and (7), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F^Y_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{1}{1-x(E[e^Y_\lambda(t)]-1)} = \sum_{k=0}^{\infty} x^k\Big(E[e^Y_\lambda(t)]-1\Big)^k \\
&= \sum_{k=0}^{\infty} x^k\left(\int_0^\infty e^y_\lambda(t)e^{-y}dy - 1\right)^k = \sum_{k=0}^{\infty} x^k\left(\int_0^\infty e^{y(\frac{1}{\lambda}\log(1+\lambda t)-1)}dy - 1\right)^k \\
&= \sum_{k=0}^{\infty} k!x^k\frac{1}{k!}\left(\frac{\frac{1}{\lambda}\log(1+\lambda t)}{1-\frac{1}{\lambda}\log(1+\lambda t)}\right)^k \\
&= \sum_{k=0}^{\infty} k!x^k\sum_{l=k}^{\infty} L(l,k)\frac{1}{l!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^l \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{l} k!x^k L(l,k)\sum_{n=l}^{\infty}\lambda^{n-l}S_1(n,l)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l} k!x^k L(l,k)\lambda^{n-l}S_1(n,l)\frac{t^n}{n!},
\end{aligned} \tag{29}$$

where $S_1(n,l)$ are the Stirling numbers of the first kind. Here we should observe that, for all t with $|t|$ small, we have

$$\left|\frac{1}{\lambda}\log(1+\lambda t)\right| < 1,$$

since $|\frac{\log(1+x)}{x}|$ is bounded on $(0,\infty)$. Therefore, by comparing the coefficients on both sides of (29), we obtain the following theorem.

Theorem 3 *Let* $Y \sim \Gamma(1,1)$. *Then we have*

$$F^Y_{n,\lambda}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l} k!\lambda^{n-l}L(l,k)S_1(n,l)x^k, \quad (n \geq 0).$$

Now, we observe from (24) and Theorem 1 that

$$\int_0^\infty \phi_{n,\lambda}^Y(xy)e^{-y}dy = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} x^k \int_0^\infty y^k e^{-y}dy = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} x^k \Gamma(k+1)$$
$$= \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} x^k k! = F_{n,\lambda}^Y(x), \quad (n \geq 0). \tag{30}$$

Thus, from (30), we obtain the following theorem.

Theorem 4 *For* $n \geq 0$, *we have*

$$\int_0^\infty \phi_{n,\lambda}^Y(xy)e^{-y}dy = F_{n,\lambda}^Y(x).$$

From (23) and (18), we note that

$$\sum_{n=k}^\infty \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} \frac{t^n}{n!} = \frac{1}{k!}(E[e_\lambda^Y(t)] - 1)^k = \frac{1}{k!}(\sum_{i=1}^\infty E[(Y)_{i,\lambda}]\frac{t^i}{i!})^k$$
$$= \sum_{n=k}^\infty B_{n,k}(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], \cdots, E[(Y)_{n-k+1,\lambda}])\frac{t^n}{n!}. \tag{31}$$

Thus, by (31), we get

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} = B_{n,k}(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], \cdots, E[(Y)_{n-k+1,\lambda}]), \quad (n \geq k \geq 0). \tag{32}$$

Hence

$$\phi_{n,\lambda}^Y(y) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} y^k = \sum_{k=0}^n B_{n,k}(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], \cdots, E[(Y)_{n-k+1,\lambda}])y^k. \tag{33}$$

By (30) and (33), we get

$$F_{n,\lambda}^Y = \sum_{k=0}^n B_{n,k}\Big(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], \cdots, E[(Y)_{n-k+1,\lambda}]\Big) \int_0^\infty y^k e^{-y}dy$$
$$= \sum_{k=0}^n k! B_{n,k}\Big(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], \cdots, E[(Y)_{n-k+1,\lambda}]\Big), \quad (n \geq 0). \tag{34}$$

Therefore, by (34), we obtain the following theorem.

Theorem 5 *For* $n \geq 0$, *we have*

$$F^Y_{n,\lambda} = \sum_{k=0}^{n} k! B_{n,k}\Big(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], \cdots, E[(Y)_{n-k+1,\lambda}]\Big).$$

For $r \in \mathbb{N}$, the *probabilistic degenerate Fubini polynomials of order r associated with random variable Y* are defined by

$$\left(\frac{1}{1 - x(E[e^Y_\lambda(t)] - 1)}\right)^r = \sum_{n=0}^{\infty} F^{(r,Y)}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{35}$$

When $Y = 1$, we have $F^{(r,Y)}_{n,\lambda}(x) = F^{(r)}_{n,\lambda}(x)$, $(n \geq 0)$, (see (13)).

From (23) and (35), we note that

$$\begin{aligned}\left(\frac{1}{1 - x(E[e^Y_\lambda(t)] - 1)}\right)^r &= \sum_{i=0}^{\infty}\binom{-r}{i}(-1)^i x^i (E[e^Y_\lambda(t)] - 1)^i = \sum_{i=0}^{\infty}\binom{r+i-1}{i} i! x^i \frac{1}{i!}(E[e^Y_\lambda(t)] - 1)^i \\ &= \sum_{i=0}^{\infty}\binom{r+i-1}{i} i! x^i \sum_{n=i}^{\infty}\left\{ \begin{matrix} n \\ i \end{matrix} \right\}_{Y,\lambda}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{r+i-1}{i} i! x^i \left\{ \begin{matrix} n \\ i \end{matrix} \right\}_{Y,\lambda}\frac{t^n}{n!}.\end{aligned} \tag{36}$$

Therefore, by (35) and (36), we obtain the following theorem.

Theorem 6 *For* $n \geq 0$, *we have*

$$F^{(r,Y)}_{n,\lambda}(x) = \sum_{i=0}^{n}\binom{r+i-1}{i} i! \left\{ \begin{matrix} n \\ i \end{matrix} \right\}_{Y,\lambda} x^i.$$

By (26) and using the Cauchy product of two power series, we get

$$\begin{aligned}\sum_{n=1}^{\infty} F^Y_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{1}{1 - x(E[e^Y_\lambda(t)] - 1)} - 1 = \frac{x(E[e^Y_\lambda(t)] - 1)}{1 - x(E[e^Y_\lambda(t)] - 1)} \\ &= \frac{xE[e^Y_\lambda(t)]}{1 - x(E[e^Y_\lambda(t)] - 1)} - \frac{x}{1 - x(E[e^Y_\lambda(t)] - 1)} \\ &= x\sum_{k=0}^{\infty} E[(Y)_{k,\lambda}]\frac{t^k}{k!}\sum_{l=0}^{\infty} F^Y_{l,\lambda}(x)\frac{t^l}{l!} - x\sum_{n=0}^{\infty} F^Y_{n,\lambda}(x)\frac{t^n}{n!} \\ &= x\sum_{k=1}^{\infty} E[(Y)_{k,\lambda}]\frac{t^k}{k!}\sum_{l=0}^{\infty} F^Y_{l,\lambda}(x)\frac{t^l}{l!} \\ &= \sum_{n=1}^{\infty} x\sum_{k=1}^{n}\binom{n}{k} E\big[(Y)_{k,\lambda}\big] F^Y_{n-k,\lambda}(x)\frac{t^n}{n!}.\end{aligned} \tag{37}$$

Therefore, by comparing the coefficients on both sides of (37), we obtain the following theorem.

Theorem 7 *For* $n \geq 1$, *we have*

$$F_{n,\lambda}^{Y}(x)=x\sum_{k=1}^{n}\binom{n}{k}E[(Y)_{k,\lambda}]F_{n-k,\lambda}^{Y}(x).$$

Here and elsewhere, all differentiations of power series are done term by term. From (26), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty}F_{n+1,\lambda}^{Y}(x)\frac{t^{n}}{n!}&=\frac{d}{dt}\sum_{n=0}^{\infty}F_{n,,\lambda}^{Y}(x)\frac{t^{n}}{n!}=\frac{d}{dt}\left(\frac{1}{1-x(E[e_{\lambda}^{Y}(t)]-1)}\right)\\
&=\frac{xE[Ye_{\lambda}^{Y-\lambda}(t)]}{(1-x(E[e_{\lambda}^{Y}(t)]-1))^{2}}=\frac{x}{1-x(E[e_{\lambda}^{Y}(t)]-1)}\frac{E[Ye_{\lambda}^{Y-\lambda}(t)]}{1-x(E[e_{\lambda}^{Y}(t)]-1)}\\
&=x\sum_{i=0}^{\infty}F_{i,\lambda}^{Y}(x)\frac{t^{i}}{i!}\sum_{j=0}^{\infty}F_{j,\lambda}^{Y}(x)\frac{t^{j}}{j!}\sum_{m=0}^{\infty}E[Y(Y-\lambda)_{m,\lambda}]\frac{t^{m}}{m!}\\
&=x\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}F_{i,\lambda}^{Y}(x)F_{k-i,\lambda}^{Y}(x)\right)\frac{t^{k}}{k!}\sum_{m=0}^{\infty}E[(Y)_{m+1,\lambda}]\frac{t^{m}}{m!}\\
&=\sum_{n=0}^{\infty}x\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{k}{i}\binom{n}{k}F_{i,\lambda}^{Y}(x)F_{k-i,\lambda}^{Y}(x)E[(Y)_{n-k+1,\lambda}]\frac{t^{n}}{n!}.
\end{aligned}\tag{38}$$

Therefore, by comparing the coefficients on both sides of (29), we obtain the following theorem.

Theorem 8 *For* $n\geq 0$, *we have*

$$F_{n+1,\lambda}^{Y}(x)=x\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{k}{i}\binom{n}{k}F_{i,\lambda}^{Y}(x)F_{k-i,\lambda}^{Y}(x)E[(Y)_{n-k+1,\lambda}].$$

Now, we observe from (35) that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{d^{r}}{dx^{r}}F_{n,\lambda}^{Y}(x)\frac{t^{n}}{n!}&=\frac{d^{r}}{dx^{r}}\left(\frac{1}{1-x(E[e_{\lambda}^{Y}(t)]-1)}\right)=r!\frac{\big(E[e_{\lambda}^{Y}(t)]\big)^{r}}{\big(1-x(E[e_{\lambda}^{Y}(t)]-1)\big)^{r+1}}\\
&=r!\sum_{i=0}^{\infty}F_{i,\lambda}^{(r+1,Y)}(x)\frac{t^{i}}{i!}\sum_{k=0}^{\infty}E\big[(Y_{1}+Y_{2}+\cdots+Y_{r})_{k,\lambda}\big]\frac{t^{k}}{k!}\\
&=\sum_{n=0}^{\infty}\left(r!\sum_{i=0}^{n}F_{i,\lambda}^{(r+1,Y)}(x)E[(S_{r})_{n-i,\lambda}]\binom{n}{i}\right)\frac{t^{n}}{n!}.
\end{aligned}\tag{39}$$

Therefore, by (39), we obtain the following theorem.

Theorem 9 *For* $r,n\geq 0$, *we have*

$$\frac{d^r}{dx^r}F^Y_{n,\lambda}(x)=r!\sum_{i=0}^{n}F^{(r+1,Y)}_{i,\lambda}(x)E[(S_r)_{n-i,\lambda}]\binom{n}{i}.$$

From (22) and (26), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{\infty}E[(S_i)_{n,\lambda}]x^i\right)\frac{t^n}{n!}&=\sum_{i=0}^{\infty}x^i\sum_{n=0}^{\infty}E[(S_i)_{n,\lambda}]\frac{t^n}{n!}\\&=\sum_{i=0}^{\infty}x^iE[e^{S_i}_{\lambda}(t)]=\sum_{i=0}^{\infty}x^i\Big(E[e^Y_\lambda(t)]\Big)^i\\&=\frac{1}{1-xE[e^Y_\lambda(t)]}=\frac{1}{1-x}\frac{1}{1-\frac{x}{1-x}\big(E[e^Y_\lambda(t)]-1\big)}\\&=\frac{1}{1-x}\sum_{n=0}^{\infty}F^Y_{n,\lambda}\left(\frac{x}{1-x}\right)\frac{t^n}{n!}.\end{aligned}\tag{40}$$

Therefore, by comparing the coefficients on both sides of (40), we obtain the following theorem.

Theorem 10 *For $n\geq 0$, we have*

$$\frac{1}{1-x}F^Y_{n,\lambda}\left(\frac{x}{1-x}\right)=\sum_{i=0}^{\infty}E[(S_i)_{n,\lambda}]x^i.$$

Taking $x=\frac{1}{2}$, we get

$$\sum_{i=0}^{\infty}E[(S_i)_{n,\lambda}]\left(\frac{1}{2}\right)^i=2F^Y_{n,\lambda},\quad (n\geq 0).$$

Let Y be the Poisson random variable with parameter $\alpha(>0)$. Then we have

$$E[e^Y_\lambda(t)]=\sum_{n=0}^{\infty}e^n_\lambda(t)\frac{\alpha^n}{n!}e^{-\alpha}=e^{\alpha(e_\lambda(t)-1)}.\tag{41}$$

From (41), (26) and (9), we have

$$\begin{aligned}\sum_{n=0}^{\infty}F^Y_{n,\lambda}(x)\frac{t^n}{n!}&=\frac{1}{1-x(E[e^Y_\lambda(t)]-1)}=\frac{1}{1-x(e^{\alpha(e_\lambda(t)-1)}-1)}\\&=\sum_{i=0}^{\infty}F_i(x)\alpha^i\frac{1}{i!}(e_\lambda(t)-1)^i=\sum_{i=0}^{\infty}F_i(x)\alpha^i\sum_{n=i}^{\infty}\left\{\begin{matrix}n\\i\end{matrix}\right\}_\lambda\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}F_i(x)\left\{\begin{matrix}n\\i\end{matrix}\right\}_\lambda\alpha^i\right)\frac{t^n}{n!},\end{aligned}\tag{42}$$

where $F_i(x)$ are the Fubini polynomials given by $\frac{1}{1-x(e^t-1)} = \sum_{i=0}^{\infty} F_i(x)\frac{t^i}{i!}$. Therefore, by (42), we obtain the following theorem.

Theorem 11 *Let Y be the Poisson random variable with parameter* $\alpha(>0)$. *Then we have*

$$F_{n,\lambda}^{Y}(x) = \sum_{i=0}^{n} F_i(x) \left\{ \begin{matrix} n \\ i \end{matrix} \right\}_{\lambda} \alpha^i, \quad (n \geq 0).$$

Let Y be the Poisson random variable with parameter $\alpha > 0$. Then, by (41) and (10), we have

$$\left(E[e_{\lambda}^{Y}(t)]\right)^{k} = e^{k\alpha(e_{\lambda}(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(k\alpha)\frac{t^n}{n!}. \tag{43}$$

and

$$\left(E[e_{\lambda}^{Y}(t)]\right)^{k} = \sum_{n=0}^{\infty} E[(S_k)_{n,\lambda}]\frac{t^n}{n!}. \tag{44}$$

Thus, by (43) and (44), we get

$$E\left[(S_k)_{n,\lambda}\right] = \phi_{n,\lambda}(k\alpha), \quad (n \geq 0). \tag{45}$$

From Theorem 10 and (45), we have

$$\sum_{k=0}^{\infty} \phi_{n,\lambda}(k\alpha)x^k = \sum_{k=0}^{\infty} E\left[(S_k)_{n,\lambda}\right]x^k = \frac{1}{1-x}F_{n,\lambda}^{Y}\left(\frac{x}{1-x}\right). \tag{46}$$

Therefore, by (46), we obtain the following theorem.

Theorem 12 *Let Y be the Poisson random variable with parameter* $\alpha(>0)$. *For* $n \geq 0$, *we have*

$$\sum_{k=0}^{\infty} \phi_{n,\lambda}(k\alpha)x^k = \sum_{k=0}^{\infty} E\left[(S_k)_{n,\lambda}\right]x^k = \frac{1}{1-x}F_{n,\lambda}^{Y}\left(\frac{x}{1-x}\right).$$

By using Theorem 6 and (22), we note that

$$\begin{aligned}\Big(\frac{1}{1-x}\Big)^{r+1}F_{n,\lambda}^{(r+1,Y)}(\frac{x}{1-x}) &=(\frac{1}{1-x})^{r+1}\sum_{l=0}^{n}\begin{Bmatrix} n \\ l \end{Bmatrix}_{Y,\lambda}\binom{l+r}{l}l!(\frac{x}{1-x})^{l}\\ &=\sum_{l=0}^{n}\begin{Bmatrix} n \\ l \end{Bmatrix}_{Y,\lambda}l!\binom{l+r}{l}x^{l}(\frac{1}{1-x})^{l+r+1}=\sum_{l=0}^{n}\sum_{k=0}^{\infty}\begin{Bmatrix} n \\ l \end{Bmatrix}_{Y,\lambda}l!\binom{l+r}{l}\binom{k+l+r}{k}x^{k+l}\\ &=\sum_{l=0}^{n}(\sum_{k=l}^{\infty}\begin{Bmatrix} n \\ l \end{Bmatrix}_{Y,\lambda}l!\binom{l+r}{l}\binom{k+r}{k-l})x^{k}=\sum_{k=0}^{n}\binom{k+r}{k}x^{k}\sum_{l=0}^{k}\begin{Bmatrix} n \\ l \end{Bmatrix}_{Y,\lambda}(k)_{l}\\ &+\sum_{k=n+1}^{\infty}\binom{k+r}{k}x^{k}\sum_{l=0}^{n}\begin{Bmatrix} n \\ l \end{Bmatrix}_{Y,\lambda}(k)_{l}=\sum_{k=0}^{n}\binom{k+r}{k}x^{k}\sum_{l=0}^{k}\begin{Bmatrix} n \\ l \end{Bmatrix}_{Y,\lambda}(k)_{l}\\ &+\sum_{k=n+1}^{\infty}\binom{k+r}{k}x^{k}\sum_{l=0}^{k}\begin{Bmatrix} n \\ l \end{Bmatrix}_{Y,\lambda}(k)_{l}=\sum_{k=0}^{\infty}\binom{k+r}{k}x^{k}\sum_{l=0}^{k}\begin{Bmatrix} n \\ l \end{Bmatrix}_{Y,\lambda}(k)_{l}\\ &=\sum_{k=0}^{\infty}\binom{k+r}{k}x^{k}E[(S_{k})_{n,\lambda}].\end{aligned} \tag{47}$$

Therefore, by (47), we obtain the following theorem.

Theorem 13 *For $n \geq 0$, we have*

$$\left(\frac{1}{1-x}\right)^{r+1}F_{n,\lambda}^{(r+1,Y)}\left(\frac{x}{1-x}\right)=\sum_{k=0}^{\infty}\binom{k+r}{k}x^{k}E\Big[(S_{k})_{n,\lambda}\Big].$$

When $Y=1$, we have

$$\left(\frac{1}{1-x}\right)^{r+1}F_{n,\lambda}^{(r+1)}\left(\frac{x}{1-x}\right)=\sum_{k=0}^{\infty}\binom{k+r}{k}x^{k}(k)_{n,\lambda}.$$

Now, we observe from (24), (16) and Theorem 6 that

$$\begin{aligned}\int_{0}^{\infty}y^{r-1}\phi_{n,\lambda}^{Y}(xy)e^{-y}dy &=\sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}_{Y,\lambda}x^{k}\int_{0}^{\infty}y^{r+k-1}e^{-y}dy\\ &=\sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}_{Y,\lambda}x^{k}\Gamma(r+k)=\Gamma(r)\sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}_{Y,\lambda}x^{k}\binom{r+k-1}{k}k!\\ &=\Gamma(r)F_{n,\lambda}^{(r,Y)}(x),\quad (r\in\mathbb{N}).\end{aligned} \tag{48}$$

Therefore, by (48), we obtain the following theorem.

Theorem 14 *For $n \geq 0$ and $r \geq 1$, we have*

$$F_{n,\lambda}^{(r,Y)}(x) = \frac{1}{\Gamma(r)} \int_0^\infty y^{r-1} \phi_{n,\lambda}^Y(xy) e^{-y} dy.$$

From (35), (25) and Theorem 14, we have

$$\begin{aligned} \left(\frac{1}{1 - x(E[e_\lambda^Y(t)] - 1)} \right)^r &= \frac{1}{\Gamma(r)} \int_0^\infty y^{r-1} \sum_{n=0}^\infty \phi_{n,\lambda}^Y(xy) \frac{t^n}{n!} e^{-y} dy. \\ &= \frac{1}{\Gamma(r)} \int_0^\infty y^{r-1} e^{xy(E[e_\lambda^Y(t)]-1)} e^{-y} dy \\ &= \frac{1}{\Gamma(r)} \int_0^\infty y^{r-1} e^{y(xE[e_\lambda^Y(t)]-1-x)} dy \end{aligned} \tag{49}$$

By (49), we get

$$\frac{1}{\Gamma(r)} \int_0^\infty y^{r-1} e^{y(xE[e_\lambda^Y(t)]-1-x)} dy = \left(\frac{1}{1 - x(E[e_\lambda^Y(t)] - 1)} \right)^r,$$

where r is a positive integer.

The proof of Theorem 15 is similar to that of Theorem 8. So we omit its proof.

Theorem 15 *For $n \geq 0$, we have*

$$F_{n+1,\lambda}^{(r,Y)}(x) = rx \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{k}{j} F_{n-k,\lambda}^{(r,Y)}(x) F_{k-j,\lambda}^Y(x) E[(Y)_{j+1,\lambda}].$$

Let Y be the Bernoulli random variable with probability of success p. Then we have

$$E[e_\lambda^Y(t)] = \sum_{k=0}^1 e_\lambda^k(t) p(k) = 1 - p + p e_\lambda(t) = 1 + p(e_\lambda(t) - 1). \tag{50}$$

By (26), (50) and (13), we get

$$\begin{aligned} \sum_{n=0}^\infty F_{n,\lambda}^Y(x) \frac{t^n}{n!} &= \frac{1}{1 - x(E[e_\lambda^Y(t)] - 1)} = \frac{1}{1 - xp(e_\lambda(t) - 1)} \\ &= \sum_{n=0}^\infty F_{n,\lambda}(xp) \frac{t^n}{n!}. \end{aligned} \tag{51}$$

Therefore, by comparing the coefficients on both sides of (51), we obtain the following theorem.

Theorem 16 *Let Y be the Bernoulli random variable with probability of success p. For $n \geq 0$, we have*

$$F_{n,\lambda}^{Y}(x) = F_{n,\lambda}(xp).$$

3 Conclusion

In this paper, we studied by using generating functions the probabilistic degenerate Fubini polynomials associated with Y and the probabilistic degenerate Fubini polynomials of order r associated with Y, as probabilistic versions of the degenerate Fubini polynomials and the degenerate Fubini polynomials of order r, respectively. Here Y is a random variable such that the moment generating function of Y exists in a neighborhood of the origin. In more detail, we derived several explicit expressions of $F_{n,\lambda}^{Y}(x)$ (see Theorems 1, 2, 4) and those of $F_{n,\lambda}^{r,Y}(x)$ (see Theorems 6, 14). We obtained a recurrence relations for $F_{n,\lambda}^{Y}(x)$ (see Theorem 7), and another one (see Theorem 8) together with its generalization (see Theorem 15). We expressed the rth derivative of $F_{n,\lambda}^{Y}(x)$ in terms of $F_{i,\lambda}^{(r+1,Y)}(x)$ (see Theorem 9). We showed the identity $\frac{1}{1-x}F_{n,\lambda}^{Y}\left(\frac{x}{1-x}\right) = \sum_{i=0}^{\infty} E[(S_i)_{n,\lambda}]x^i$ (see Theorem 10) and its generalization (see Theorem 13). We deduced an explicit expression for $F_{n,\lambda}^{Y}(x)$ when $Y \sim \Gamma(1,1)$ (see Theorem 3) and also that when Y is the Poisson random variable with parameter α (see Theorem 11). We proved that $\frac{1}{1-x}F_{n,\lambda}^{Y}\left(\frac{x}{1-x}\right) = \sum_{k=0}^{\infty} \phi_{n,\lambda}(k\alpha)x^k$ when Y is the Poisson random variable with parameter α (see Theorem 12). We showed $F_{n,\lambda}^{Y}(x) = F_{n,\lambda}(xp)$ when Y be the Bernoulli random variable with probability of success p (see Theorem 16).

As one of our future projects, we would like to continue to study degenerate versions, λ-analogues and probabilistic versions of many special polynomials and numbers and to find their applications to physics, science and engineering as well as to mathematics.

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