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RESEARCH ARTICLE

Large Time Behavior and Stability for Two-Dimensional Magneto-Micropolar Equations with Partial Dissipation

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Abstract

This paper is devoted to the stability and decay estimates of solutions to the two-dimensional magneto-micropolar fluid equations with partial dissipation. Firstly, focus on the 2D magneto-micropolar equation with only velocity dissipation and partial magnetic diffusion, we obtain the global existence of solutions with small initial in $H^s(\mathbb{R}^2)$ $(s > 1)$, and by fully exploiting the special structure of the system and using the Fourier splitting methods, we establish the large time decay rates of solutions. Secondly, when the magnetic field has partial dissipation, we show the global existence of solutions with small initial data in $\dot{B}^0_{2,1}(\mathbb{R}^2)$. In addition, we explore the decay rates of these global solutions are correspondingly established in $\dot{B}^m_{2,1}(\mathbb{R}^2)$ with $0 \le m \le s$, when the initial data belongs to the negative Sobolev space $\dot{H}^{-l}(\mathbb{R}^2)$ (for each $0 \le l < 1$).

Keywords 2D magneto-micropolar equations · Partial dissipation · Large time behavior

Mathematics Subject Classification 35Q35 · 35B40 · 76D03

1 Introduction

The magneto-micropolar equations were introduced in [1] to describe the motion of an incompressible, electrically conducting micropolar fluids in the presence of an arbitrary magnetic field. It belongs to a class of fluids with nonsymmetric stress tensor and includes, as special cases, the classical fluids modeled by the Navier-Stokes equation (see, e.g., [5, 31, 39]), magnetohydrodynamic (MHD)

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equations (see, e.g., [26]) and micropolar equations (see., e.g., [15, 16]). The 3D incompressible magneto-micropolar fluid equations can be written as:

$$\begin{cases} \partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + b \cdot \nabla b + 2\chi \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega - \gamma \nabla \nabla \cdot \omega + 4\chi\omega = \kappa \Delta \omega + 2\chi \nabla \times u, \\ \partial_t b + u \cdot \nabla b = \nu \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0,\ \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \omega(x,0) = \omega_0(x), b(x,0) = b_0(x), \end{cases} \tag{1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $t \geq 0$, $u(x,t), \omega(x,t), b(x,t)$ and $\pi(x,t)$ denote the velocity of the fluid, microrotational velocity, the magnetic field and the hydrostatic pressure, respectively, μ, χ and $\frac{1}{\nu}$ are, respectively, kinematic viscosity, vortex viscosity and magnetic Reynolds number. γ and κ are angular viscosities, and this is an isotropic system. The 3D magneto-micropolar equations reduce to the 2D micropolar equations when

$$\begin{aligned} u &= (u_1(x_1,x_2,t), u_2(x_1,x_2,t), 0), \qquad & \pi &= \pi(x_1,x_2,t), \\ b &= (b_1(x_1,x_2,t), b_2(x_1,x_2,t), 0), \qquad & \omega &= (0,0,\omega_3(x_1,x_2,t)). \end{aligned}$$

More explicitly, the 2D incompressible magneto-micropolar fluid equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + b \cdot \nabla b + 2\chi \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega + 4\chi\omega = \kappa \Delta \omega + 2\chi \nabla \times u, \\ \partial_t b + u \cdot \nabla b = \nu \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \omega(x,0) = \omega_0(x), b(x,0) = b_0(x), \end{cases} \tag{2}$$

where we have written $u = (u_1, u_2)$, $b = (b_1, b_2)$ and ω for ω_3 for notational brevity. It is worth noting that, in the 2D case,

$$\Omega \equiv \nabla \times u = \partial_1 u_2 - \partial_2 u_1$$

is a scalar function representing the vorticity, and $\nabla \times \omega = (\partial_2 \omega, -\partial_1 \omega)$.

The magneto-micropolar equations play an important role in engineering and physics and have attracted considerable attention from the community of mathematical fluids (see, e.g., [20, 25, 28, 29]). When (2) has full dissipation (namely, $\mu, \chi, \kappa, \nu > 0$), the global existence and uniqueness of solutions could be obtained easily (see, e.g., [20, 28]). However, for the inviscid case (namely, (2) with $\mu > 0$, $\chi > 0$, $\kappa = \nu = 0$ and Δu replaced by u), the global regularity problem is still a challenging open problem. Therefore, it is natural to study the intermediate cases, namely (2) with partial dissipation.

This paper aims at a system of the 2D magneto-micropolar equations that is closely related to (2),

$$\begin{cases} \partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + b \cdot \nabla b + 2\chi \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega + 4\chi \omega = 2\chi \nabla \times u, \\ \partial_t b_1 + u \cdot \nabla b_1 = \nu \partial_{22} b_1 + b \cdot \nabla u_1, \\ \partial_t b_2 + u \cdot \nabla b_2 = \nu \partial_{11} b_2 + b \cdot \nabla u_2, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \omega(x,0) = \omega_0(x), b(x,0) = b_0(x). \end{cases} \tag{3}$$

Physically, the partial dissipation assumption is natural in the study of geophysical fluids. It turns out that, in certain regimes and under suitable scaling, certain dissipation can become small and be ignored. Anisotropic magnetic diffusion also arises in the modeling of reconnecting plasmas. When the resistivity of electrically conducting fluids such as certain plasmas and liquid metal is anisotropic and only in the mixed directions, the mixed magnetic diffusion may be relevant. In addition, mathematically, (3) allows us to explore the smoothing effect and the effect on large time behavior of the anisotropic magnetic diffusion. When the b with partial dissipation and zero angular viscosity, the global regularity problem for (3) can be quite difficult. However, many important progresses have recently been made on this direction (see, e.g., [6–14, 17, 23, 30, 36, 38, 42]). In [17, 23, 30], the global regularity of the 2D magneto-micropolar equations with various partial dissipation cases was obtained. Wang, Xu and Liu in [41] proved the uniqueness of global strong solution for the magneto-micropolar equations with zero angular viscosity in a smooth bounded domain. Yamazaki [43] obtained the global regularity of the Cauchy problem for the magneto-micropolar equations with zero angular viscosity

The magneto-micropolar equations share similarities with the Navier-Stokes equations, but they contain much richer structures than Navier–Stokes. It is well-known that the L^2 decay problem of weak solutions to the 3D Navier–Stokes equations, i.e., (1) with $\omega = 0$, $b = 0$ and $\chi = 0$, was proposed by the celebrated work of Leray [19]. By introducing the elegant method of Fourier splitting, the algebraic decay rate for weak solutions was first obtained by Schonbek [33]. Later, the result in [33] is sharpened and extended in [34], see also [35]. Recently, Niu and Shang [24] proved the L^2 -decay estimates of weak solutions, and also proved the optimal decay rates of global solutions in $\dot{H}^s(\mathbb{R}^3)(s > \frac{3}{2})$ and in $\dot{B}^m_{2,1}(\mathbb{R}^3)$ with $0 \le m \le \frac{1}{2}$. Shang and Gu [37] also proved the global existence of classical solutions for (3). Li [21] proved the L^2-decay estimates for global solutions of (8) and their derivate with initial data in $L^1(\mathbb{R}^2)$. In addition, Li [21] also shown the global stability of these solutions in $H^s(\mathbb{R}^2)(s > 1)$ and the decay rates of global solutions and their higher derivates.

Motivated by the results of the magneto-micropolar equations [43] and the related fluid models [11, 18]. In this paper, the first theorem states that system (3) has a unique golbal solution when the initial data (u_0, ω_0, b_0) is sufficiently small in $H^s(\mathbb{R}^2)$, and obtain the upper bounds of time decay rates of the global solution to (3) in $L^2(\mathbb{R}^2)$, as stated in the following theorem.

Theorem 1 *Let* $\mu > 0$, $\chi > 0$, $\nu > 0$ *and* $\kappa > 0$. *Assume that* $(u_0, \omega_0, b_0) \in H^s(\mathbb{R}^2)$ *with* $s > 0$ *and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Then the following two statements hold*:

(*I*) *Let* $s > 1$, *then there exists a positive constant* ϵ_0, *such that for all* $0 < \epsilon < \epsilon_0$, *if*

$$\|u_0\|^2_{H^s(\mathbb{R}^2)} + \|\omega_0\|^2_{H^s(\mathbb{R}^2)} + \|b_0\|^2_{H^s(\mathbb{R}^2)} < \epsilon, \tag{4}$$

then system (3) *has a unique global solution* (u, ω, b) *satisfying*, *for any* $t > 0$,

$$\begin{aligned} &\|u(t)\|^2_{H^s(\mathbb{R}^2)} + \|\omega(t)\|^2_{H^s(\mathbb{R}^2)} + \|b(t)\|^2_{H^s(\mathbb{R}^2)} \\ &\quad + \int_0^t (\|\nabla u(\tau)\|^2_{H^s(\mathbb{R}^2)} + \|\omega(\tau)\|^2_{H^s(\mathbb{R}^2)} + \|\nabla b(\tau)\|^2_{H^s(\mathbb{R}^2)}) d\tau \le C\epsilon, \end{aligned} \tag{5}$$

where $C > 0$ *is a constant independent of* t.

(*II*) *suppose that* $(u_0, \omega_0, b_0) \in L^1(\mathbb{R}^2)$, *then the global solution* (u, ω, b) *has the following upper decay rates*:

$$\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} + \|b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}. \tag{6}$$

Moreover, *when* $\mu < \sqrt{3}\chi$, *then the global solution* (u, ω, b) *of the system* (3) *has the following upper decay rates*

$$\|\nabla u(t)\|_{L^2} + \|\omega(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}. \tag{7}$$

Finally, we consider the 2D magneto-micropolar equations with partial dissipation for the magnetic field, which can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + b \cdot \nabla b + 2\chi \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega + 4\chi\omega = \kappa \Delta \omega + 2\chi \nabla \times u, \\ \partial_t b_1 + u \cdot \nabla b_1 = \nu \partial_{22} b_1 + b \cdot \nabla u_1, \\ \partial_t b_2 + u \cdot \nabla b_2 = \nu \partial_{11} b_2 + b \cdot \nabla u_2, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \omega(x,0) = \omega_0(x), b(x,0) = b_0(x). \end{cases} \tag{8}$$

Motivated by the [21, 24], we establish the global existence results to system (8) in Besov spaces $B^s_{2,1}(\mathbb{R}^2)$. Furthermore, we study the large time decay rates of these global solutions in the Besove spaces $B^s_{2,1}(\mathbb{R}^2)$, as stated in the following theorem.

Theorem 2 *Let* $\mu > 0$, $\chi > 0$, $\nu > 0$ *and* $\kappa > 0$. *Assume that* $(u_0, \omega_0, b_0) \in B^s_{2,1}(\mathbb{R}^2)$ *with* $s > 0$ *and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Then the following two statements hold*:

(*I*) *Let* $s \ge 1$, *then there exists a positive constant* ϵ_0, *such that for all* $0 < \epsilon < \epsilon_0$, *if*

$$\|u_0\|^2_{\dot{B}^0_{2,1}(\mathbb{R}^2)} + \|\omega_0\|^2_{\dot{B}^0_{2,1}(\mathbb{R}^2)} + \|b_0\|^2_{\dot{B}^0_{2,1}(\mathbb{R}^2)} < \epsilon, \tag{9}$$

then system (8) *has a unique global solution* (u, ω, b) *satisfying*, *for any* $t > 0$,

$$\begin{aligned} &\|u(t)\|_{B^s_{2,1}(\mathbb{R}^2)} + \|\omega(t)\|_{B^s_{2,1}(\mathbb{R}^2)} + \|b(t)\|_{B^s_{2,1}(\mathbb{R}^2)} + \|\nabla^2 u(\tau)\|_{L^1_t(B^s_{2,1}(\mathbb{R}^2))} \\ &\quad + \|\nabla^2 \omega(\tau)\|_{L^1_t(B^1_{2,1}(\mathbb{R}^2))} + \|\nabla^2 b(\tau)\|_{L^1_t(B^s_{2,1}(\mathbb{R}^2))} \le C, \end{aligned} \tag{10}$$

where $C > 0$ is a constant independent of t.

(*II*) *Let $s \ge 1$, suppose that $(u_0, \omega_0, b_0) \in \dot{H}^{-l}(\mathbb{R}^2)$ with $0 \le l < 1$. Then for all real numbers m with $0 \le m \le s$, the global solution (u, ω, b) established in (I) satisfies the following decay estimates*:

$$\|u(t)\|_{\dot{B}^m_{2,1}(\mathbb{R}^2)} + \|\omega(t)\|_{\dot{B}^m_{2,1}(\mathbb{R}^2)} + \|b(t)\|_{\dot{B}^m_{2,1}(\mathbb{R}^2)} \le C(1+t)^{-\frac{m}{2}-\frac{l}{2}}. \tag{11}$$

Remark 1

(i) Since $L^p(\mathbb{R}^2) \hookrightarrow \dot{H}^{-l}(\mathbb{R}^2)$ when $l \in [0,1)$ and $p \in (1,2]$, and $L^p(\mathbb{R}^2) \hookrightarrow \dot{B}^{-l}_{2,\infty}(\mathbb{R}^2)$ when $l \in (0,1]$ and $p \in [1,2)$, thus Theorem 2 also hold for $(u_0, \omega_0, b_0) \in L^p(\mathbb{R}^2)$ with $p \in [1,2]$.

(ii) Because of the divergence free condition $\nabla \cdot b = 0$, then $\|\nabla b\|_{L^2(\mathbb{R}^2)} = \|\nabla \times b\|_{L^2(\mathbb{R}^2)}$, thus for the full dissipation 2D magneto-micropolar equations, we also have the same results as Theorem 1 and Theorem 2.

Remark 2

(i) Compared to the classical magneto-micropolar equations (1), the full Laplacian operator is replaced by partial magnetic diffusion in systems (3) and (8). Theorem 1 and Theorem 2 indicate that the mixed partial magnetic diffusion has the same effect as the full Laplacian in deriving the large time behavior, in the sense that the decay rates in Theorem 1 and Theorem 2 coincide with the solutions of system (1).

(ii) In Theorem 2, by assuming the initial data small in the critical Besov space $\dot{B}^0_{2,1}$, we can establish the global well-posedness to (8). However, due to the lack of micro-rotational velocity dissipation and the complex structure of the magneto-micropolar equations, it appears difficult to show the global well-posedness in critical Besov space to the solutions of (3).

To prove Theorem 2, we focus on the uniform bounds of $\|(u,\omega,b)\|_{B^s_{2,1}}$. As preparation, we firstly show the global existence of solutions with small data in $\dot{B}^0_{2,1}(\mathbb{R}^2)$, then used the $\|(u(t),\omega(t),b(t))\|_{\dot{B}^0_{2,1}} \le C\epsilon$, to obtain (10). The rest of this paper is divided into four sections. Sections 2 and 3 state the proofs of Theorem 1 and Theorem 2, respectively. An appendix containing the Littlewood-Paley decomposition, the definition of Besov spaces, and several useful calculus inequalities are also given for the convenience of the readers. To simplify the notation, we will write ∂_1 for ∂_{x_1}, ∂_2 for ∂_{x_2}, $\int f$ for $\int_{\mathbb{R}^2} f dx$, $\|f\|_{L^p}$ for $\|f\|_{L^p(\mathbb{R}^2)}$, $\|f\|_{\dot{H}^s}$ and $\|f\|_{H^s}$ for $\|f\|_{\dot{H}^s(\mathbb{R}^2)}$ and $\|f\|_{H^s(\mathbb{R}^2)}$

respectively, $\|f\|_{\dot{B}^s_{p,r}}$ and $\|f\|_{B^s_{p,r}}$ for $\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^2)}$ and $\|f\|_{B^s_{p,r}(\mathbb{R}^2)}$ respectively, and $L^q_t(\dot{B}^s_{p,r})$ and $\tilde{L}^q_t(\dot{B}^s_{p,r})$ for $L^q_t(\dot{B}^s_{p,r}(\mathbb{R}^2))$ and $\tilde{L}^q_t(\dot{B}^s_{p,r}(\mathbb{R}^2))$ respectively.

2 The Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We first prove the global well-posedness part (*I*) of Theorem 1. As preparation, we give the following global *a priori* estimates.

Proposition 3 *Let* $(u_0, \omega_0, b_0) \in L^2$. *Then for any* $t > 0$, *the solution* (u, ω, b) *of* (3) *satisfies*

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\right) + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + 4\chi\left(1 - \frac{2\chi}{2\chi+\mu}\right)\|\omega\|_{L^2}^2 + \nu(\|\partial_y b_1\|_{L^2}^2 + \|\partial_x b_2\|_{L^2}^2) \le 0, \tag{12}$$

$$\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \mu\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau + \frac{4\chi\mu}{\mu+2\chi}\int_0^t \|\omega(\tau)\|_{L^2}^2 d\tau + \nu\int_0^t \|\nabla b(\tau)\|_{L^2}^2 d\tau \le \|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{13}$$

Proof Taking the L^2-inner product to (3) with (u, ω, b_1, b_2) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\right) + (\mu+\chi)\|\nabla u\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2 \\ &\quad + \nu(\|\partial_2 b_1\|_{L^2}^2 + \|\partial_1 b_2\|_{L^2}^2) = 4\chi\int \nabla\times u\cdot\omega dx,\end{aligned} \tag{14}$$

where we used the facts that,

$$\begin{aligned}&\int (b\cdot\nabla u_1\cdot b_1 + b\cdot\nabla u_2\cdot b_2)dx = \int b\cdot\nabla u\cdot b dx = -\int b\cdot\nabla b\cdot u dx,\\ &\int \nabla\times u\cdot\omega dx = \int \nabla\times\omega\cdot u dx.\end{aligned}$$

By Hölder's inequality and the Young inequality, we have

$$\begin{aligned}4\chi\int \nabla\times u\cdot\omega dx &\le 4\chi\|\nabla u\|_{L^2}\|\omega\|_{L^2}\\ &\le \left(\frac{\mu}{2}+\chi\right)\|\nabla u\|_{L^2}^2 + \frac{4\chi^2}{\frac{\mu}{2}+\chi}\|\omega\|_{L^2}^2.\end{aligned} \tag{15}$$

Inserting (15) into (14), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\right)+\frac{\mu}{2}\|\nabla u\|_{L^2}^2+4\chi\left(1-\frac{2\chi}{2\chi+\mu}\right)\|\omega\|_{L^2}^2 \\ +\nu(\|\partial_2 b_1\|_{L^2}^2+\|\partial_1 b_2\|_{L^2}^2)\le 0. \tag{16}$$

Because of the divergence free condition $\nabla\cdot b=0$, we have $\|\nabla b\|_{L^2(\mathbb{R}^2)}^2=\|\nabla\times b\|_{L^2(\mathbb{R}^2)}^2\le 2\Big(\|\partial_2 b_1\|_{L^2(\mathbb{R}^2)}^2+\|\partial_1 b_2\|_{L^2(\mathbb{R}^2)}^2\Big)$. Integrating (16) in $[0,t]$, we can get

$$\|u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\mu\int_0^t\|\nabla u(\tau)\|_{L^2}^2d\tau+\frac{4\chi\mu}{\mu+2\chi}\int_0^t\|\omega(\tau)\|_{L^2}^2d\tau \\ +\nu\int_0^t\|\nabla b(\tau)\|_{L^2}^2d\tau\le\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2+\|b_0\|_{L^2}^2.$$

This completes the proof of Proposition 3 □

Next, we want to establish the global *a priori* H^s estimates. Applying $\dot{\Delta}_j$ to (3), we have

$$\partial_t\dot{\Delta}_j u+u\cdot\nabla\dot{\Delta}_j u-(\mu+\chi)\Delta\dot{\Delta}_j u=-\dot{\Delta}_j\nabla\pi-[\dot{\Delta}_j,u\cdot\nabla]u \\ +\dot{\Delta}_j(b\cdot\nabla b)+2\chi\dot{\Delta}_j\nabla\times\omega, \tag{17}$$

$$\partial_t\dot{\Delta}_j\omega+u\cdot\nabla\dot{\Delta}_j\omega+4\chi\dot{\Delta}_j\omega=-[\dot{\Delta}_j,u\cdot\nabla]\omega+2\chi\dot{\Delta}_j\nabla\times u, \tag{18}$$

$$\partial_t\dot{\Delta}_j b_1+u\cdot\nabla\dot{\Delta}_j b_1-\nu\partial_{22}\dot{\Delta}_j b_1=-[\dot{\Delta}_j,u\cdot\nabla]b_1+\dot{\Delta}_j(b\cdot\nabla u_1), \tag{19}$$

$$\partial_t\dot{\Delta}_j b_2+u\cdot\nabla\dot{\Delta}_j b_2-\nu\partial_{11}\dot{\Delta}_j b_2=-[\dot{\Delta}_j,u\cdot\nabla]b_2+\dot{\Delta}_j(b\cdot\nabla u_2), \tag{20}$$

where $[\dot{\Delta}_j,f\cdot\nabla]g=\dot{\Delta}_j(f\cdot\nabla g)-f\cdot\dot{\Delta}_j(\nabla g)$ is commutator. Dotting (17) - (20) by $\dot{\Delta}_j u$, $\dot{\Delta}_j\omega$, $\dot{\Delta}_j b_1$ and $\dot{\Delta}_j b_2$ respectively, integrating the resulting equations in $\mathbb{R}^2$, and adding them together, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\omega\|_{L^2}^2+\|\dot{\Delta}_j b_1\|_{L^2}^2+\|\dot{\Delta}_j b_2\|_{L^2}^2)+(\mu+\chi)\|\dot{\Delta}_j\nabla u\|_{L^2}^2\\&+4\chi\|\dot{\Delta}_j\omega\|_{L^2}^2+\nu(\|\dot{\Delta}_j\partial_2 b_1\|_{L^2}^2+\|\dot{\Delta}_j\partial_1 b_2\|_{L^2}^2)\\\le&-\int[\dot{\Delta}_j,u\cdot\nabla]u\cdot\dot{\Delta}_j u+\int[\dot{\Delta}_j,b\cdot\nabla]b\cdot\dot{\Delta}_j u-\int[\dot{\Delta}_j,u\cdot\nabla]\omega\cdot\dot{\Delta}_j\omega\\&+4\chi\int\dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j\omega-\int[\dot{\Delta}_j,u\cdot\nabla]b_1\cdot\dot{\Delta}_j b_1-\int[\dot{\Delta}_j,u\cdot\nabla]b_2\cdot\dot{\Delta}_j b_2\\&+\int[\dot{\Delta}_j,b\cdot\nabla]u_1\cdot\dot{\Delta}_j b_1+\int[\dot{\Delta}_j,b\cdot\nabla]u_2\cdot\dot{\Delta}_j b_2,\end{aligned}$$

where we used the facts that

$$\int b\cdot\nabla\dot{\Delta}_j u_1\cdot\dot{\Delta}_j b_1+\int b\cdot\nabla\dot{\Delta}_j u_2\cdot\dot{\Delta}_j b_2=\int b\cdot\nabla\dot{\Delta}_j u\cdot\dot{\Delta}_j b=-\int b\cdot\nabla\dot{\Delta}_j b\cdot\dot{\Delta}_j u,$$

and

$$\int\dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j\omega=\int\dot{\Delta}_j\nabla\times\omega\cdot\dot{\Delta}_j u.$$

Due to the divergence free condition $\nabla\cdot b=0$, we have

$$\begin{aligned}\|\dot{\Delta}_j\nabla b\|_{L^2}^2&=\|\dot{\Delta}_j\nabla\times b\|_{L^2}^2\le 2(\|\dot{\Delta}_j\partial_1 b_2\|_{L^2}^2+\|\dot{\Delta}_j\partial_2 b_1\|_{L^2}^2),\\ \|\dot{\Delta}_j b\|_{L^2}^2&=\|\dot{\Delta}_j b_1\|_{L^2}^2+\|\dot{\Delta}_j b_2\|_{L^2}^2.\end{aligned}$$

Then, we can derive from the above inequalities

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\omega\|_{L^2}^2+\|\dot{\Delta}_j b\|_{L^2}^2)+(\mu+\chi)\|\dot{\Delta}_j\nabla u\|_{L^2}^2\\&\quad+4\chi\|\dot{\Delta}_j\omega\|_{L^2}^2+\frac{\nu}{2}\|\dot{\Delta}_j\nabla b\|_{L^2}^2\\&\le-\int[\dot{\Delta}_j,u\cdot\nabla]u\cdot\dot{\Delta}_j u+\int[\dot{\Delta}_j,b\cdot\nabla]b\cdot\dot{\Delta}_j u-\int[\dot{\Delta}_j,u\cdot\nabla]\omega\cdot\dot{\Delta}_j\omega\\&\quad+4\chi\int\dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j\omega-\int[\dot{\Delta}_j,u\cdot\nabla]b\cdot\dot{\Delta}_j b+\int[\dot{\Delta}_j,b\cdot\nabla]u\cdot\dot{\Delta}_j b.\end{aligned}\tag{21}$$

Due to

$$\left|4\chi\int\dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j\omega\right|\le\left(\frac{\mu}{2}+\chi\right)\|\nabla u\|_{L^2}^2+\frac{8\chi^2}{\mu+2\chi}\|\omega\|_{L^2}^2,$$

then, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\omega\|_{L^2}^2+\|\dot{\Delta}_j b\|_{L^2}^2)+\frac{\mu}{2}\|\dot{\Delta}_j\nabla u\|_{L^2}^2\\&\quad+\frac{\nu}{2}\|\dot{\Delta}_j\nabla b\|_{L^2}^2+(4\chi-\frac{4\chi^2}{\frac{\mu}{2}+\chi})\|\dot{\Delta}_j\omega\|_{L^2}^2\\&\le-\int[\dot{\Delta}_j,u\cdot\nabla]u\cdot\dot{\Delta}_j u+\int[\dot{\Delta}_j,b\cdot\nabla]b\cdot\dot{\Delta}_j u-\int[\dot{\Delta}_j,u\cdot\nabla]\omega\cdot\dot{\Delta}_j\omega\\&\quad-\int[\dot{\Delta}_j,u\cdot\nabla]b\cdot\dot{\Delta}_j b+\int[\dot{\Delta}_j,b\cdot\nabla]u\cdot\dot{\Delta}_j b.\end{aligned}\tag{22}$$

Multiplying (22) by 2^{2sj}, taking the l_j^2 over $j\in\mathbb{Z}$, nothing that $\dot{B}^s_{2,2}=\dot{H}^s$ and using Hölder's inequality, we yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|_{\dot{H}^s}^2+\|\omega\|_{\dot{H}^s}^2+\|b\|_{\dot{H}^s}^2)+\frac{c_0}{2}(\|\nabla u\|_{\dot{H}^s}^2+\|\omega\|_{\dot{H}^s}^2+\|\nabla b\|_{\dot{H}^s}^2)\\&\le\|2^{sj}\|[\dot{\Delta}_j,u\cdot\nabla]u\|_{L^2}\|_{l_j^2}\|u\|_{\dot{H}^s}+\|2^{sj}\|[\dot{\Delta}_j,b\cdot\nabla]b\|_{L^2}\|_{l_j^2}\|u\|_{\dot{H}^s}\\&\quad+\|2^{sj}\|[\dot{\Delta}_j,u\cdot\nabla]\omega\|_{L^2}\|_{l_j^2}\|\omega\|_{\dot{H}^s}+\|2^{sj}\|[\dot{\Delta}_j,u\cdot\nabla]b\|_{L^2}\|_{l_j^2}\|b\|_{\dot{H}^s}\\&\quad+\|2^{sj}\|[\dot{\Delta}_j,b\cdot\nabla]u\|_{L^2}\|_{l_j^2}\|b\|_{\dot{H}^s},\end{aligned}$$

where $c_0 = \min\left\{\mu, \nu, \frac{8\chi\mu}{\mu+2\chi}\right\}$. Adding the resulting inequality and (12) together, we have

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|b\|_{H^s}^2) + c_0(\|\nabla u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|\nabla b\|_{H^s}^2) \\
\le & 2\|2^{sj}\|[\dot{\Delta}_j, u\cdot\nabla]u\|_{L^2}\|_{l_j^2}\|u\|_{H^s} + 2\|2^{sj}\|[\dot{\Delta}_j, b\cdot\nabla]b\|_{L^2}\|_{l_j^2}\|u\|_{H^s} \\
&+ 2\|2^{sj}\|[\dot{\Delta}_j, u\cdot\nabla]\omega\|_{L^2}\|_{l_j^2}\|\omega\|_{H^s} + 2\|2^{sj}\|[\dot{\Delta}_j, u\cdot\nabla]b\|_{L^2}\|_{l_j^2}\|b\|_{H^s} \\
&+ 2\|2^{sj}\|[\dot{\Delta}_j, b\cdot\nabla]u\|_{L^2}\|_{l_j^2}\|b\|_{H^s}.
\end{aligned} \tag{23}$$

Using commutator estimate (A5), and nothing that for $s > 1$,

$$\|f\|_{L^\infty} \le C\|f\|_{H^s}, \quad \|f\|_{\dot{B}^{s-1}_{2,2}} \le C\|f\|_{B^s_{2,2}} = C\|f\|_{H^s},$$

one obviously derives

$$2\|2^{sj}\|[\dot{\Delta}_j, u\cdot\nabla]u\|_{L^2}\|_{l_j^2} \le C\|\nabla u\|_{L^\infty}\|\nabla u\|_{\dot{B}^{s-1}_{2,2}} \le C\|\nabla u\|_{H^s}^2.$$

Similarly, we have

$$\begin{aligned}
&\|2^{sj}\|[\dot{\Delta}_j, b\cdot\nabla]b\|_{L^2}\|_{l_j^2} \le C\|\nabla b\|_{H^s}^2, \\
&\|2^{sj}\|[\dot{\Delta}_j, u\cdot\nabla]b\|_{L^2}\|_{l_j^2} \le C(\|\nabla u\|_{L^\infty}\|\nabla b\|_{\dot{B}^{s-1}_{2,2}} + \|\nabla b\|_{L^\infty}\|\nabla u\|_{\dot{B}^{s-1}_{2,2}}) \\
&\le C\|\nabla u\|_{H^s}\|\nabla b\|_{H^s},
\end{aligned}$$

and

$$\|2^{sj}\|[\dot{\Delta}_j, b\cdot\nabla]u\|_{L^2}\|_{l_j^2} \le C\|\nabla u\|_{H^s}\|\nabla b\|_{H^s}.$$

Taking advantage of the commutator estimate (A6), we imply that

$$\begin{aligned}
\|2^{sj}\|[\dot{\Delta}_j, u\cdot\nabla]\omega\|_{L^2}\|_{l_j^2} &\le C(\|\nabla u\|_{L^\infty}\|\omega\|_{\dot{B}^s_{2,2}} + \|\omega\|_{L^\infty}\|\nabla u\|_{\dot{B}^s_{2,2}}) \\
&\le C\|\nabla u\|_{H^s}\|\omega\|_{H^s}.
\end{aligned}$$

Combining the above estimates together, we get

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|b\|_{H^s}^2) + c_0(\|\nabla u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|\nabla b\|_{H^s}^2) \\
\le & C\|\nabla u\|_{H^s}^2\|u\|_{H^s} + C\|\nabla b\|_{H^s}^2\|u\|_{H^s} + \|\nabla u\|_{H^s}\|\omega\|_{H^s}^2 \\
&+ C\|\nabla u\|_{H^s}\|\nabla b\|_{H^s}\|b\|_{H^s}.
\end{aligned}$$

Then the Young inequality leads to

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|b\|_{H^s}^2) + \frac{c_0}{2}(\|\nabla u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|\nabla b\|_{H^s}^2) \\
\le & C(\|u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|b\|_{H^s}^2)^{\frac{1}{2}}(\|\omega\|_{H^s}^2 + \|\nabla u\|_{H^s}^2 + \|\nabla b\|_{H^s}^2).
\end{aligned} \tag{24}$$

This inequality indicates that, if the initial data (u_0, ω_0, b_0) satisfy, for $0 < \epsilon < \epsilon_0 = \left(\frac{c_0}{2C}\right)^2$,

$$\|u_0\|_{H^s}^2 + \|\omega_0\|_{H^s}^2 + \|b_0\|_{H^s}^2 < \epsilon,$$

then the corresponding solution remains for all time. Namely,

$$\|u(t)\|_{H^s}^2 + \|\omega(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 < \epsilon. \tag{25}$$

In fact, if suppose (25) is not true and T_0 is the first time such that (25) is violated, i.e.,

$$\|u(T_0)\|_{H^s}^2 + \|\omega(T_0)\|_{H^s}^2 + \|b(T_0)\|_{H^s}^2 = \epsilon,$$

and (25) holds for any $0 \le t < T_0$. We can deduce from (24) that for any $0 \le t \le T_0$,

$$\begin{aligned} &\frac{d}{dt}(\|u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|b\|_{H^s}^2) \\ &\quad + \left(\frac{c_0}{2} - C\sqrt{\epsilon})(\|\nabla u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|\nabla b\|_{H^s}^2\right) \le 0. \end{aligned} \tag{26}$$

Therefore,

$$\|u(t)\|_{H^s}^2 + \|\omega(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 \le \|u_0\|_{H^s}^2 + \|\omega_0\|_{H^s}^2 + \|b_0\|_{H^s}^2 < \epsilon. \tag{27}$$

This is a contradiction. Thus, we get the uniform bound of (25). In addition,

$$\int_0^t (\|\nabla u(\tau)\|_{H^s(\mathbb{R}^2)}^2 + \|\omega(\tau)\|_{H^s(\mathbb{R}^2)}^2 + \|\nabla b(\tau)\|_{H^s(\mathbb{R}^2)}^2) d\tau \le C\epsilon. \tag{28}$$

Therefore, the proof of (*I*) of Theorem 1 is completed.

Next, we start to prove (*II*) of Theorem 1.

Proposition 4 *Let (u, ω, b) be the global solutions of the system (3) with the initial data $(u_0, \omega_0, b_0) \in (L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2))^3$. Then (u, ω, b) satisfies the following inequality,*

$$\begin{aligned} &|\hat{u}(\xi, t)| + |\hat{\omega}(\xi, t)| + |\hat{b}_1(\xi, t)| + |\hat{b}_2(\xi, t)| \\ &\le C + C|\xi| \int_0^t \left(\|u(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{L^2}^2 + \|b(\tau)\|_{L^2}^2\right) d\tau. \end{aligned} \tag{29}$$

Proof of Proposition 4 Applying the Fourier transform to system (3), we obtain:

$$\begin{cases} \partial_t \hat{u} + (\mu + \chi)|\xi|^2 \hat{u} = -\mathcal{F}(\nabla \pi) + \mathcal{F}(b \cdot \nabla b) + 2\chi i\xi \times \hat{\omega} - \mathcal{F}(u \cdot \nabla u), \\ \partial_t \hat{\omega} + 4\chi \hat{\omega} = 2\chi i\xi \times \hat{u} - \mathcal{F}(u \cdot \nabla \omega), \\ \partial_t \hat{b}_1 + \nu|\xi_2|^2 \hat{b}_1 = \mathcal{F}[b \cdot \nabla u_1 - u \cdot \nabla b_1], \\ \partial_t \hat{b}_2 + \nu|\xi_1|^2 \hat{b}_2 = \mathcal{F}[b \cdot \nabla u_2 - u \cdot \nabla b_2]. \end{cases} \tag{30}$$

Multiplying the $(30)_1$, $(30)_2$, $(30)_3$ and $(30)_4$ by $\bar{\hat{u}}$, $\bar{\hat{\omega}}$, $\bar{\hat{b}}_1$ and $\bar{\hat{b}}_2$ respectively, and summing up, we have, noting that $|\hat{u}|^2 = \hat{u}\bar{\hat{u}}$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\hat{u}|^2 + |\hat{\omega}|^2 + |\hat{b}_1|^2 + |\hat{b}_2|^2) + (\mu+\chi)|\xi|^2|\hat{u}|^2 + \nu(|\xi_2|^2|\hat{b}_1|^2 + |\xi_1|^2|\hat{b}_2|^2) + 4\chi|\hat{\omega}|^2\\
&\quad = -\mathcal{F}(\nabla\pi)\bar{\hat{u}} + \mathcal{F}(b\cdot\nabla b)\bar{\hat{u}} - \mathcal{F}(u\cdot\nabla u)\bar{\hat{u}} - \mathcal{F}(u\cdot\nabla\omega)\bar{\hat{\omega}} + \mathcal{F}(b\cdot\nabla u_1)\bar{\hat{b}}_1\\
&\qquad - \mathcal{F}(u\cdot\nabla b_1)\bar{\hat{b}}_1 + \mathcal{F}(b\cdot\nabla u_2)\bar{\hat{b}}_2 - \mathcal{F}(u\cdot\nabla b_2)\bar{\hat{b}}_2 + 2\chi i\xi\times\hat{\omega}\bar{\hat{u}} + 2\chi i\xi\times\hat{u}\bar{\hat{\omega}}\\
&\quad = K_1 + K_2 + \cdots + K_{10}.
\end{aligned} \tag{31}$$

For K_1, taking divergence to the first equation of (3), one yields

$$\pi = (-\Delta)^{-1}(\nabla\otimes\nabla)(b\otimes b - u\otimes u).$$

And taking Fourier transformation obeys, nothing that $|\hat{u}| = |\bar{\hat{u}}|$

$$\begin{aligned}
K_1 &\le |\xi||\hat{\pi}||\bar{\hat{u}}|\\
&\le |\xi|(\|b\otimes b\|_{L^1} + \|u\otimes u\|_{L^1})|\bar{\hat{u}}|\\
&\le |\xi|(\|b\|_{L^2}^2 + \|u\|_{L^2}^2)|\hat{u}|.
\end{aligned}$$

For K_2,

$$K_2 \le |\xi||\widehat{b\otimes b}||\bar{\hat{u}}| \le |\xi|\|b\otimes b\|_{L^1}|\bar{\hat{u}}| \le |\xi|\|b\|_{L^2}^2|\hat{u}|.$$

Similarly, we obtain

$$\begin{aligned}
|K_3 + K_4| &\le 2|\xi|(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2)(|\hat{u}| + |\hat{\omega}|),\\
|K_5 + K_6| &\le |\xi|(\|b\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \|b_1\|_{L^2}^2 + \|u\|_{L^2}^2)|\hat{b}_1|\\
&\le 2|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2)|\hat{b}_1|,\\
|K_7 + K_8| &\le 2|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2)|\hat{b}_2|,\\
|K_9 + K_{10}| &\le 4\chi|\xi||\hat{\omega}||\hat{u}|\\
&\le \left(\frac{\mu}{2} + \chi\right)|\xi|^2|\hat{u}|^2 + \frac{8\chi^2}{\mu + 2\chi}|\hat{\omega}|^2.
\end{aligned}$$

Inserting K_1 - K_{10} into (31), we derive that

$$\begin{aligned}
&\frac{d}{dt}(|\hat{u}|^2 + |\hat{\omega}|^2 + |\hat{b}_1|^2 + |\hat{b}_2|^2) + \mu|\xi|^2|\hat{u}|^2\\
&\quad + 2\nu(|\xi_2|^2|\hat{b}_1|^2 + |\xi_1|^2|\hat{b}_2|^2) + \frac{8\chi\mu}{\mu + 2\chi}|\hat{\omega}|^2\\
\le & C|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\omega\|_{L^2}^2)(|\hat{u}| + |\hat{\omega}| + |\hat{b}_1| + |\hat{b}_2|),
\end{aligned}$$

which immediately yields

$$\partial_t \sqrt{|\hat{u}|^2 + |\hat{\omega}|^2 + |\hat{b}_1|^2 + |\hat{b}_2|^2} \le C|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\omega\|_{L^2}^2). \tag{32}$$

Integrating (32) in $[0, t]$, we obtain

$$\begin{aligned}
&\sqrt{|\hat{u}(t)|^2 + |\hat{\omega}(t)|^2 + |\hat{b}_1(t)|^2 + |\hat{b}_2(t)|^2} \\
&\le \sqrt{|\hat{u}(0)|^2 + |\hat{\omega}(0)|^2 + |\hat{b}_1(0)|^2 + |\hat{b}_2(0)|^2} + C|\xi| \int_0^t (\|u(\tau)\|_{L^2}^2 + \|b(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{L^2}^2) d\tau \\
&\le C + C|\xi| \int_0^t (\|u(\tau)\|_{L^2}^2 + \|b(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{L^2}^2) d\tau.
\end{aligned}$$

Thus the proof of Proposition 4 is completed. □

Next, we obtain the result of Theorem 1 by using Proposition 4 and the generalized Fourier splitting method.

Let

$$B(t) = \left\{ \xi \in \mathbb{R}^2 \,:\, |\xi|^2 \le \frac{h'(t)}{c_1 h(t)} \right\}, \qquad B^c(t) = \mathbb{R}^2 \backslash B(t),$$

where $h(t) \in C^\infty[0, +\infty)$ is a positive function with respect to t and satisfies

$$h(0) = 1, \quad h'(t) > 0, \quad and \quad \frac{h'(t)}{c_1 h(t)} \le 1, \ \ \forall\, t > t_0 > 0, \tag{33}$$

where $c_1 = \min\{\mu, \nu, \frac{4\chi\mu}{u+2\chi}\}$.

Multiplying both side of (12) by $h(t)$, we have

$$\begin{aligned}
&\frac{d}{dt}\left[h(t)(\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2)\right] \\
&\quad + c_1 h(t)(\|\nabla u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2) \\
&\le h'(t)(\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2).
\end{aligned} \tag{34}$$

By using the Plancherel Theorem for (34), we get

$$\begin{aligned}
&\frac{d}{dt}\left[h(t)\big(\|\hat{u}(t)\|_{L^2}^2 + \|\hat{b}(t)\|_{L^2}^2 + \|\hat{\omega}(t)\|_{L^2}^2\big)\right] \\
&\quad + c_1 h(t) \int_{\mathbb{R}^2} \big(|\xi|^2(|\hat{u}(\xi)|^2 + |\hat{b}(\xi)|^2) + |\hat{\omega}(\xi)|^2\big) d\xi \\
&\le h'(t) \int_{\mathbb{R}^2} \big(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\big) d\xi.
\end{aligned} \tag{35}$$

Applying (33), we can obtain

$$
\begin{aligned}
&c_1h(t)\int_{\mathbb{R}^2}\left(|\xi|^2|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2+|\xi|^2|\hat{b}(\xi)|^2\right)d\xi\\
&\quad+h'(t)\int_{B(t)}\left(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2+|\hat{b}(\xi)|^2\right)d\xi\\
&\geq c_1h(t)\int_{B^c(t)}\left(|\xi|^2|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2+|\xi|^2|\hat{b}(\xi)|^2\right)d\xi\\
&\quad+h'(t)\int_{B(t)}\left(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2+|\hat{b}(\xi)|^2\right)d\xi\\
&\geq c_1h(t)\left(\frac{h'(t)}{c_1h(t)}\right)\int_{B^c(t)}\left(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2+|\hat{b}(\xi)|^2\right)d\xi\\
&\quad+h'(t)\int_{B(t)}\left(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2+|\hat{b}(\xi)|^2\right)d\xi\\
&=h'(t)\int_{\mathbb{R}^2}\left(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2+|\hat{b}(\xi)|^2\right)d\xi.
\end{aligned}
\tag{36}
$$

Combining the result of (35) and (36), we get

$$
\begin{aligned}
&\frac{d}{dt}\left[h(t)\left(\|\hat{u}(t)\|_{L^2}^2+\|\hat{b}(t)\|_{L^2}^2+\|\hat{\omega}(t)\|_{L^2}^2\right)\right]\\
&\qquad\leq h'(t)\int_{B(t)}\left(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2+|\hat{b}(\xi)|^2\right)d\xi.
\end{aligned}
\tag{37}
$$

Employing (29), we have

$$
\begin{aligned}
&\int_{B(t)}\left(|\hat{u}(\xi)|^2+|\hat{\omega}(\xi)|^2+|\hat{b}(\xi)|^2\right)d\xi\\
&=C\int_{B(t)}\left\{|\xi|^2\left(\int_0^t(\|u(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right)^2+1\right\}d\xi\\
&\leq\frac{Ch'(t)}{h(t)}+\frac{C(h'(t))^2}{h^2(t)}\left[\int_0^t\left(\|u(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2\right)d\tau\right]^2.
\end{aligned}
\tag{38}
$$

Substituting (38) to (37), we have

$$
\begin{aligned}
&\frac{d}{dt}\left[h(t)\left(\|\hat{u}(t)\|_{L^2}^2+\|\hat{b}(t)\|_{L^2}^2+\|\hat{\omega}(t)\|_{L^2}^2\right)\right]\\
&\quad\leq\frac{C[h'(t)]^2}{h(t)}+\frac{C[h'(t)]^3}{h^2(t)}\left[\int_0^t\left(\|u(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2\right)d\tau\right]^2.
\end{aligned}
\tag{39}
$$

Next, taking $h(t)=[\ln(e+t)]^3$, then we have

$$\begin{aligned}\int_0^t \frac{[h'(\tau)]^2}{h(\tau)} d\tau &= \int_0^t \frac{3^2 \ln^4(e+\tau)}{(e+\tau)^2 \ln^3(e+\tau)} d\tau = \int_0^t \frac{9\ln(e+\tau)}{(e+\tau)^2} d\tau \\ &\leq C\int_0^t \frac{1}{e+\tau} d\tau \leq C\ln(e+t),\end{aligned} \tag{40}$$

and

$$\begin{aligned}&\int_0^t \frac{[h'(\tau)]^3}{h^2(\tau)} \left[\int_0^\tau \left(\|u(s)\|_{L^2}^2 + \|\omega(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2\right) ds\right]^2 d\tau \\ &\leq C\int_0^t \frac{\tau^2}{(e+\tau)^3} [\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \|b_0\|_{L^2}^2]^2 d\tau \\ &\leq C\int_0^t \frac{1}{(e+\tau)} d\tau \leq C\ln(e+t).\end{aligned} \tag{41}$$

Combining (39) - (41), we have

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 \\ &\leq C[\ln(e+t)]^{-3} + C[\ln(e+t)]^{-2} \\ &\leq C[\ln(e+t)]^{-2}.\end{aligned} \tag{42}$$

Now, taking $h(t) = (1+t)^2$ and inserting it into (39), together with (42) and Hölder's inequality, we have

$$\begin{aligned}&(1+t)^2\left(\|\hat{u}(t)\|_{L^2}^2 + \|\hat{b}(t)\|_{L^2}^2 + \|\hat{\omega}(t)\|_{L^2}^2\right) \\ &\leq (\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \|b_0\|_{L^2}^2) + C\int_0^t \frac{[h'(\tau)]^2}{h(\tau)} d\tau \\ &\quad + \int_0^t \frac{[h'(\tau)]^3}{h^2(\tau)} \left[\int_0^\tau \left(\|u(s)\|_{L^2}^2 + \|\omega(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2\right) ds\right]^2 d\tau,\end{aligned} \tag{43}$$

where

$$\begin{aligned}&C\int_0^t \frac{[h'(\tau)]^2}{h(\tau)} d\tau \leq C\int_0^t \frac{[2(1+\tau)]^2}{(1+\tau)^2} d\tau \leq C(t+1), \\ &\int_0^t \frac{[h'(\tau)]^3}{h^2(\tau)} \left[\int_0^\tau \left(\|u(s)\|_{L^2}^2 + \|\omega(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2\right) ds\right]^2 d\tau \\ &\leq C\int_0^t \frac{\tau}{(1+\tau)} \int_0^\tau \left(\|u(s)\|_{L^2}^2 + \|\omega(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2\right) \ln^{-2}(e+s) ds d\tau \\ &\leq C(1+t)\int_0^\tau \left(\|u(s)\|_{L^2}^2 + \|\omega(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2\right) \ln^{-2}(e+s) ds.\end{aligned}$$

From (43), we have

$$
\begin{aligned}
&(1+t)\left(\|\hat{u}(t)\|_{L^2}^2+\|\hat{b}(t)\|_{L^2}^2+\|\hat{\omega}(t)\|_{L^2}^2\right)\\
&\le C+C\int_0^t(1+s)^{-1}(1+s)\left(\|u(s)\|_{L^2}^2+\|\omega(s)\|_{L^2}^2+\|b(s)\|_{L^2}^2\right)\ln^{-2}(e+s)ds.
\end{aligned}
\tag{44}
$$

Taking $\mathcal{N}(t)=(1+t)\Big(\|u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\Big)$, then we have

$$
\mathcal{N}(t)=C+C\int_0^t(1+s)^{-1}\mathcal{N}(s)\ln^{-2}(e+s)ds.
$$

Applying Gronwall's inequality, we obtain

$$
\mathcal{N}(t)\le C\exp\left\{\int_0^\infty(1+s)^{-1}\ln^{-2}(e+s)ds\right\}<C,
$$

which implies the following decay

$$
\|u(t)\|_{L^2}+\|\omega(t)\|_{L^2}+\|b(t)\|_{L^2}\le C(1+t)^{-\frac{1}{2}}.
$$

Therefore, the proof of (6) is completed.

Next, we will prove (7). The vorticity $\Omega=\nabla\times u$, $j=\nabla\times b$ satisfies

$$
\partial_t\Omega+u\cdot\nabla\Omega-(\mu+\chi)\Delta\Omega=b\cdot\nabla j-2\chi\Delta\omega,
\tag{45}
$$

$$
\partial_t j+u\cdot\nabla j-\nu\partial_{111}b_2+\nu\partial_{222}b_1=b\cdot\nabla\Omega+T(\nabla u,\nabla b),
\tag{46}
$$

where

$$
T(\nabla u,\nabla b)=2\partial_1 b_1(\partial_1 u_2+\partial_2 u_1)-2\partial_1 u_1(\partial_1 b_2+\partial_2 b_1).
$$

Due to the lack of angular viscosity for the system (3), it is crucial to deal with $-2\chi\omega$ in (45) by introducing a new function $Z=\Omega-\frac{2\chi}{\mu+\chi}\omega$ in [11]. Subtracting $\frac{2\chi}{\mu+\chi}\times(3)_2$ from (45), we have

$$
\partial_t Z-(\mu+\chi)\Delta Z+(u\cdot\nabla)Z+\frac{4\chi^2}{\mu+\chi}Z=\left(\frac{8\chi^2}{\mu+\chi}-\frac{8\chi^3}{(\mu+\chi)^2}\right)\omega+b\cdot\nabla j.
\tag{47}
$$

Taking the L^2-inner products of (47), $(3)_2$ and (46) with Z, ω and j, respectively, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Z(t)\|_{L^2}^2+(\mu+\chi)\|\nabla Z\|_{L^2}^2+\frac{4\chi^2}{\mu+\chi}\|Z\|_{L^2}^2\\
&\quad\le\left(\frac{8\chi^2}{\mu+\chi}-\frac{8\chi^2}{(\mu+\chi)^2}\right)\|Z\|_{L^2}\|\omega\|_{L^2}+\int(b\cdot\nabla j)(\Omega-\frac{2\chi}{\mu+\chi}\omega)dx,
\end{aligned}
\tag{48}
$$

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2 \leq 2\chi\|Z\|_{L^2}\|\omega\|_{L^2} + \frac{4\chi^2}{\mu+\chi}\|\omega\|_{L^2}^2, \tag{49}$$

$$\frac{1}{2}\frac{d}{dt}\|j(t)\|_{L^2}^2 + I = \int (b\cdot\nabla\Omega j + Tj)dx, \tag{50}$$

where

$$\begin{aligned} I &= \nu\int(-\partial_{111}b_2 + \partial_{222}b_1)jdx \\ &= \nu\int(-\partial_{111}b_2 + \partial_{222}b_1)(\partial_1 b_2 - \partial_2 b_1)dx \\ &= \nu\int(\partial_{11}b_2)^2 + (\partial_{11}b_1)^2 + (\partial_{22}b_1)^2 + (\partial_{22}b_2)^2 dx \equiv H(b,t), \end{aligned}$$

due to the divergence free condition $\nabla\cdot b = \partial_1 b_1 + \partial_2 b_2$. By Hölder's inequality

$$\begin{aligned} \int Tjdx &\leq C\|\nabla u\|_{L^2}\|\nabla b\|_{L^4}\|j\|_{L^4} \\ &\leq C\|\Omega\|_{L^2}\|j\|_{L^2}\|\nabla j\|_{L^2} \\ &\leq C\|\Omega\|_{L^2}^2\|j\|_{L^2}^2 + \frac{\nu}{8}\|\nabla j\|_{L^2}^2, \end{aligned}$$

where we have used the fact that the Calderon-Zygmund operators are bounded on $L^p(1<p<+\infty)$. It is easy to verify that

$$\frac{\nu}{4}\|\nabla j\|_{L^2}^2 \leq H(b,t).$$

Indeed,

$$\begin{aligned} \nu\|\nabla j\|_{L^2}^2 &= \nu\|(\partial_1 j, \partial_2 j)\|_{L^2}^2 = \nu\|((\partial_{11}b_2 - \partial_{12}b_1), (\partial_{12}b_2 - \partial_{22}b_1))\|_{L^2}^2 \\ &= \nu\|((\partial_{11}b_2 + \partial_{22}b_2), -(\partial_{11}b_1 + \partial_{22}b_1))\|_{L^2}^2 \leq 4H(b,t). \end{aligned}$$

Then it follows from the above bounds and (50) that

$$\frac{1}{2}\frac{d}{dt}\|j\|_{L^2}^2 + \frac{1}{2}H(b,t) \leq \int b\cdot\nabla\Omega jdx + C\|\Omega\|_{L^2}^2\|j\|_{L^2}^2. \tag{51}$$

Combining (48), (49) and (51), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|Z(t)\|_{L^2}^2+\|\nabla Z\|_{L^2}^2+\|\nabla j\|_{L^2}^2)+(\mu+\chi)\|\nabla Z\|_{L^2}^2+\frac{4\chi^2}{\mu+\chi}\|Z\|_{L^2}^2\\
&\quad+4\chi\|\omega\|_{L^2}^2+\frac{\nu}{8}\|\nabla j\|_{L^2}^2\\
&\le\left(\frac{8\chi^2}{\mu+\chi}-\frac{8\chi^3}{(\mu+\chi)^2}\right)\|Z\|_{L^2}\|\omega\|_{L^2}+\int(b\cdot\nabla)j(\Omega-\frac{2\chi}{\mu+\chi}\omega)dx\\
&\quad+\int b\cdot\nabla\Omega jdx+C\|\Omega\|_{L^2}^2\|j\|_{L^2}^2+2\chi\|Z\|_{L^2}\|\omega\|_{L^2}+\frac{4\chi^2}{\mu+\chi}\|\omega\|_{L^2}^2\\
&\triangleq I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
\tag{52}
$$

Next, we consider I_1 - I_6, respectively. Applying the Young inequality

$$
\begin{aligned}
I_1&=\left(\frac{8\chi^2}{\mu+\chi}-\frac{8\chi^3}{(\mu+\chi)^2}\right)\|Z\|_{L^2}\|\omega\|_{L^2}\\
&=\frac{8\chi^2\mu}{(\mu+\chi)^2}\|Z\|_{L^2}\|\omega\|_{L^2}\\
&\le\frac{2\chi^2\mu}{(\mu+\chi)^2}\|Z\|_{L^2}^2+\frac{8\chi\mu}{(\mu+\chi)^2}\|\omega\|_{L^2}^2.
\end{aligned}
$$

By using Hölder's inequality, the Gagliardo-Nirenberg inequality, and the Young inequality,

$$
\begin{aligned}
I_2+I_3&=\int(b\cdot\nabla)j(\Omega-\frac{2\chi}{\mu+\chi})\omega dx+\int b\cdot\nabla\Omega jdx\\
&=-\int b\cdot\nabla j(\frac{2\chi}{\mu+\chi})\omega dx\\
&\le C\|b\|_{L^\infty}\|\nabla j\|_{L^2}\|\omega\|_{L^2}\\
&\le C\|b\|_{L^2}^{\frac{1}{2}}\|\nabla j\|_{L^2}^{\frac{3}{2}}\|\omega\|_{L^2}\\
&\le C\|b_0\|_{L^2}^{\frac{1}{2}}\|\nabla j\|_{L^2}^{\frac{3}{2}}\|\omega\|_{L^2}\\
&\le\frac{\nu}{16}\|\nabla j\|_{L^2}^2+C\|\omega\|_{L^2}^4,
\end{aligned}
$$

and

$$
I_5=2\chi\|Z\|_{L^2}\|\omega\|_{L^2}\le\chi\|Z\|_{L^2}^2+\chi\|\omega\|_{L^2}^2.
$$

Inserting I_1 - I_6 into (52), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|Z(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\big)+(\mu+\chi)\|\nabla Z\|_{L^2}^2\\
&\quad+\left(\frac{4\chi^2}{\mu+\chi}-\chi-\frac{2\chi^2\mu}{(\mu+\chi)^2}\right)\|Z\|_{L^2}^2+\frac{\nu}{8}\|\nabla j\|_{L^2}^2+4\chi\|\omega\|_{L^2}^2\\
&\le C\|\omega\|_{L^2}^4+C\|\Omega\|_{L^2}^2\|j\|_{L^2}^2+\left(\chi+\frac{4\chi^2}{(\mu+\chi)}-4\chi\right)\|\omega\|_{L^2}^2+C\|\omega\|_{L^2}^2\\
&\le C\|\omega\|_{L^2}^4+C\|\Omega\|_{L^2}^2\|j\|_{L^2}^2+C\|\omega\|_{L^2}^2\\
&\le C(\|Z\|_{L^2}^2+\|\Omega\|_{L^2}^2+\|j\|_{L^2}^2)(\|\Omega\|_{L^2}^2+\|\omega\|_{L^2}^2)+C\|\omega\|_{L^2}^2.
\end{aligned}
\tag{53}
$$

Due to $\mu<\sqrt{3}\chi$, then

$$
\begin{aligned}
&\frac{4\chi^2}{(\mu+\chi)}-\chi-\frac{2\chi^2\mu}{(\mu+\chi)^2}\\
&=\frac{3\chi^2\mu+3\chi^3-\mu^2\chi-\mu\chi^2-2\chi^2\mu}{(\mu+\chi)^2}=\frac{3\chi^3-\mu^2\chi}{(\mu+\chi)^2}>0.
\end{aligned}
$$

Applying Gronwall's inequality, we have

$$
\begin{aligned}
&\|Z(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\\
&\le\exp\left\{C\int_r^t(\|\Omega(s)\|_{L^2}^2+\|\omega(s)\|_{L^2}^2)ds\right\}\\
&\quad\times\left(\|Z(r)\|_{L^2}^2+\|\omega(r)\|_{L^2}^2+\|j(r)\|_{L^2}^2+\int_r^t\|\omega(s)\|_{L^2}^2ds\right)\\
&\le C(\|Z(r)\|_{L^2}^2+\|\omega(r)\|_{L^2}^2+\|j(r)\|_{L^2}^2+\|u(r)\|_{L^2}^2+\|b(r)\|_{L^2}^2),
\end{aligned}
\tag{54}
$$

where we also used (13).

Multiplying (12) by $(1+t)^n$, by (6), we have

$$
\begin{aligned}
&\frac{d}{dt}\big[(1+t)^n(\|u(t)\|_{L^2}^2+\|\omega(r)\|_{L^2}^2+\|b(r)\|_{L^2}^2)\big]\\
&\quad+c_1(1+t)^n(\|\nabla u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2)\\
&\le n(1+t)^{n-1}(\|u(t)\|_{L^2}^2+\|\omega(r)\|_{L^2}^2+\|b(r)\|_{L^2}^2)\\
&\le n(1+t)^{n-2},
\end{aligned}
\tag{55}
$$

where $c_1=\min\{\mu,\frac{4\chi\mu}{\mu+2\chi},\nu\}$. Integrating (55) in time, we have

$$
\begin{aligned}
&\int_0^T(1+r)^n(\|\nabla u(r)\|_{L^2}^2+\|\omega(r)\|_{L^2}^2+\|\nabla b(r)\|_{L^2}^2)dr\\
&\qquad\le C(1+t)^{n-1},\qquad\text{for}\quad\text{large}\quad n\ge 6.
\end{aligned}
\tag{56}
$$

Multiplying (53) by $(1+r)^n$ and integrating with respect to r over $(\frac{t}{2},t)$, we have

$$
\begin{aligned}
&\frac{t}{2}\Big(1+\frac{t}{2}\Big)^{n}\big(\|Z(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\big)\\
&\quad\le \int_{\frac{t}{2}}^{t}(1+r)^n(\|Z(r)\|_{L^2}^2+\|\omega(r)\|_{L^2}^2+\|j(r)\|_{L^2}^2+\|u(r)\|_{L^2}^2+\|b(r)\|_{L^2}^2)dr\\
&\quad\le C(1+t)^{n-1}+\int_{\frac{t}{2}}^{t}(1+r)^{n-1}dr\\
&\quad\le C(1+t)^n, \qquad \text{for some given large } n\ge 6.
\end{aligned}
\tag{57}
$$

Nothing that

$$
\begin{aligned}
&\Big(\frac{1}{2}+\frac{t}{2}\Big)^{n+1}\big(\|Z(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\big)\\
&\le \frac{t}{2}\Big(1+\frac{t}{2}\Big)^{n}\big(\|Z(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\big).
\end{aligned}
\tag{58}
$$

Combining the results of (57) and (48), for $t\ge 1$, we get

$$\|Z(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\le (1+t)^{-1},$$

then we have

$$\|\nabla u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\le (1+t)^{-1}.$$

Therefore, the proof of Theorem 1 is completed.

3 The Proof of Theorem 2

This section is devoted to the proof of Theorem 2. Firstly, we first prove the global stability part (I) of Theorem 2. As we know, the key step is to establish the global *a priori* $B^s_{2,1}(\mathbb{R}^2)$ estimates of the solution.

Proof of (I) of Theorem 2 As preparation, in the following proposition, we state that system (8) has unique global solution when the initial data (u_0,ω_0,b_0) is sufficiently small in $\dot B^0_{2,1}(\mathbb{R}^2)$.

Proposition 5 *Assume (u_0,ω_0,b_0) satisfies the conditions in (I) of Theorem 2 and (9), then system (8) has a unique global solution (u,ω,b) satisfying, for any $t>0$,*

$$
\begin{aligned}
&\|u(t)\|_{\dot B^0_{2,1}}+\|\omega(t)\|_{\dot B^0_{2,1}}+\|b(t)\|_{\dot B^0_{2,1}}\\
&\quad+\|\nabla^2u(t)\|_{L^1_t(\dot B^0_{2,1})}+\|\nabla^2\omega(t)\|_{L^1_t(\dot B^0_{2,1})}+\|\nabla^2b(t)\|_{L^1_t(\dot B^0_{2,1})}\le C\epsilon.
\end{aligned}
\tag{59}
$$

Proof Now, we turn to establish the global *a priori* $B^s_{2,1}$ estimates. Applying $\dot\Delta_j$ to (8), we have

$$\begin{aligned}\partial_t \dot{\Delta}_j u + u\cdot\nabla\dot{\Delta}_j u - (\mu+\chi)\Delta\dot{\Delta}_j u = -\dot{\Delta}_j\nabla\pi - [\dot{\Delta}_j, u\cdot\nabla]u \\ + \dot{\Delta}_j(b\cdot\nabla b) + 2\chi\dot{\Delta}_j\nabla\times\omega,\end{aligned} \tag{60}$$

$$\partial_t \dot{\Delta}_j\omega + u\cdot\nabla\dot{\Delta}_j\omega - \kappa\Delta\dot{\Delta}_j\omega + 4\chi\dot{\Delta}_j\omega = -[\dot{\Delta}_j, u\cdot\nabla]\omega + 2\chi\dot{\Delta}_j\nabla\times u, \tag{61}$$

$$\partial_t \dot{\Delta}_j b_1 + u\cdot\nabla\dot{\Delta}_j b_1 - \nu\partial_{yy}\dot{\Delta}_j b_1 = -[\dot{\Delta}_j, u\cdot\nabla]b_1 + \dot{\Delta}_j(b\cdot\nabla u_1), \tag{62}$$

$$\partial_t \dot{\Delta}_j b_2 + u\cdot\nabla\dot{\Delta}_j b_2 - \nu\partial_{xx}\dot{\Delta}_j b_2 = -[\dot{\Delta}_j, u\cdot\nabla]b_2 + \dot{\Delta}_j(b\cdot\nabla u_2), \tag{63}$$

where $[\dot{\Delta}_j, f\cdot\nabla]g = \dot{\Delta}_j(f\cdot\nabla g) - f\cdot\dot{\Delta}_j(\nabla g)$ is commutator. Dotting (60) - (63) by $\dot{\Delta}_j u$, $\dot{\Delta}_j\omega$, $\dot{\Delta}_j b_1$ and $\dot{\Delta}_j b_2$ respectively, integrating the resulting equations in $\mathbb{R}^2$, and adding them together, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\dot{\Delta}_j u\|_{L^2}^2 + \|\dot{\Delta}_j\omega\|_{L^2}^2 + \|\dot{\Delta}_j b_1\|_{L^2}^2 + \|\dot{\Delta}_j b_2\|_{L^2}^2) + (\mu+\chi)\|\dot{\Delta}_j\nabla u\|_{L^2}^2 \\ &\quad + \kappa\|\dot{\Delta}_j\nabla\omega\|_{L^2}^2 + 4\chi\|\dot{\Delta}_j\omega\|_{L^2}^2 + \nu(\|\dot{\Delta}_j\partial_y b_1\|_{L^2}^2 + \|\dot{\Delta}_j\partial_x b_2\|_{L^2}^2) \\ &\leq -\int [\dot{\Delta}_j, u\cdot\nabla]u\cdot\dot{\Delta}_j u + \int [\dot{\Delta}_j, b\cdot\nabla]b\cdot\dot{\Delta}_j u - \int [\dot{\Delta}_j, u\cdot\nabla]\omega\cdot\dot{\Delta}_j\omega \\ &\quad + 4\chi\int \dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j\omega - \int [\dot{\Delta}_j, u\cdot\nabla]b_1\cdot\dot{\Delta}_j b_1 - \int [\dot{\Delta}_j, u\cdot\nabla]b_2\cdot\dot{\Delta}_j b_2 \\ &\quad + \int [\dot{\Delta}_j, b\cdot\nabla]u_1\cdot\dot{\Delta}_j b_1 + \int [\dot{\Delta}_j, b\cdot\nabla]u_2\cdot\dot{\Delta}_j b_2,\end{aligned}$$

where we used the facts that

$$\int b\cdot\nabla\dot{\Delta}_j u_1\cdot\dot{\Delta}_j b_1 + \int b\cdot\nabla\dot{\Delta}_j u_2\cdot\dot{\Delta}_j b_2 = \int b\cdot\nabla\dot{\Delta}_j u\cdot\dot{\Delta}_j b = -\int b\cdot\nabla\dot{\Delta}_j b\cdot\dot{\Delta}_j u,$$

and

$$\int \dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j\omega = \int \dot{\Delta}_j\nabla\times\omega\cdot\dot{\Delta}_j u.$$

Due to the divergence free condition $\nabla\cdot b = 0$, we have

$$\begin{aligned}&\|\dot{\Delta}_j\nabla b\|_{L^2}^2 = \|\dot{\Delta}_j\nabla\times b\|_{L^2}^2 \leq 2(\|\dot{\Delta}_j\partial_x b_2\|_{L^2}^2 + \|\dot{\Delta}_j\partial_y b_1\|_{L^2}^2), \\ &\|\dot{\Delta}_j b\|_{L^2}^2 = \|\dot{\Delta}_j b_1\|_{L^2}^2 + \|\dot{\Delta}_j b_2\|_{L^2}^2.\end{aligned}$$

Then, we can derive from the above inequalities

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\omega\|_{L^2}^2+\|\dot{\Delta}_j b\|_{L^2}^2)+(\mu+\chi)\|\dot{\Delta}_j\nabla u\|_{L^2}^2+\kappa\|\dot{\Delta}_j\nabla\omega\|_{L^2}^2\\
&\quad+4\chi\|\dot{\Delta}_j\omega\|_{L^2}^2+\frac{\nu}{2}\|\dot{\Delta}_j\nabla b\|_{L^2}^2\\
&\le-\int[\dot{\Delta}_j,u\cdot\nabla]u\cdot\dot{\Delta}_j u+\int[\dot{\Delta}_j,b\cdot\nabla]b\cdot\dot{\Delta}_j u-\int[\dot{\Delta}_j,u\cdot\nabla]\omega\cdot\dot{\Delta}_j\omega\\
&\quad+4\chi\int\dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j\omega-\int[\dot{\Delta}_j,u\cdot\nabla]b\cdot\dot{\Delta}_j b+\int[\dot{\Delta}_j,b\cdot\nabla]u\cdot\dot{\Delta}_j b.
\end{aligned}
\tag{64}
$$

Due to

$$
\left|4\chi\int\dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j\omega\right|\le\left(\frac{\mu}{2}+\chi\right)\|\nabla u\|_{L^2}^2+\frac{8\chi^2}{\mu+2\chi}\|\omega\|_{L^2}^2,
$$

then, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\omega\|_{L^2}^2+\|\dot{\Delta}_j b\|_{L^2}^2)+\frac{\mu}{2}\|\dot{\Delta}_j\nabla u\|_{L^2}^2+\kappa\|\dot{\Delta}_j\nabla\omega\|_{L^2}^2\\
&\quad+\frac{\nu}{2}\|\dot{\Delta}_j\nabla b\|_{L^2}^2+(4\chi-\frac{4\chi^2}{\frac{\mu}{2}+\chi}\Big)\|\dot{\Delta}_j\omega\|_{L^2}^2\\
&\le-\int[\dot{\Delta}_j,u\cdot\nabla]u\cdot\dot{\Delta}_j u+\int[\dot{\Delta}_j,b\cdot\nabla]b\cdot\dot{\Delta}_j u-\int[\dot{\Delta}_j,u\cdot\nabla]\omega\cdot\dot{\Delta}_j\omega\\
&\quad-\int[\dot{\Delta}_j,u\cdot\nabla]b\cdot\dot{\Delta}_j b+\int[\dot{\Delta}_j,b\cdot\nabla]u\cdot\dot{\Delta}_j b.
\end{aligned}
\tag{65}
$$

Applying Hölder's inequality, the Young inequality, Bernstein's inequality and the divergence free condition $\nabla\cdot u=\nabla\cdot b=0$ to (65), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\omega\|_{L^2}^2+\|\dot{\Delta}_j b\|_{L^2}^2\right)+\frac{\mu}{2}\tilde{c}_2 2^{2j}\|\dot{\Delta}_j u\|_{L^2}^2\\
&\quad+\frac{\nu}{2}\tilde{c}_2 2^{2j}\|\dot{\Delta}_j b\|_{L^2}^2+\kappa\tilde{c}_2 2^{2j}\|\dot{\Delta}_j\omega\|_{L^2}^2+\frac{4\mu\chi}{\mu+2\chi}\|\dot{\Delta}_j\omega\|_{L^2}^2\\
&\le-\int\dot{\Delta}_j(u\cdot\nabla u)\cdot\dot{\Delta}_j u+\int\dot{\Delta}_j(b\cdot\nabla b)\cdot\dot{\Delta}_j u-\int\dot{\Delta}_j(u\cdot\nabla b)\cdot\dot{\Delta}_j b\\
&\quad+\int\dot{\Delta}_j(b\cdot\nabla u)\cdot\dot{\Delta}_j b-\int\dot{\Delta}_j(u\cdot\nabla\omega)\cdot\dot{\Delta}_j\omega\\
&\le C2^j\|\dot{\Delta}_j(u\otimes u)\|_{L^2}\|\dot{\Delta}_j u\|_{L^2}+C2^j\|\dot{\Delta}_j(b\otimes b)\|_{L^2}\|\dot{\Delta}_j u\|_{L^2}\\
&\quad+C2^j\|\dot{\Delta}_j(u\otimes b)\|_{L^2}\|\dot{\Delta}_j b\|_{L^2}+C2^j\|\dot{\Delta}_j(u\otimes\omega)\|_{L^2}\|\dot{\Delta}_j\omega\|_{L^2}.
\end{aligned}
\tag{66}
$$

Then, (66) implies

$$
\begin{aligned}
&\frac{d}{dt}\sqrt{\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\omega\|_{L^2}^2+\|\dot{\Delta}_j b\|_{L^2}^2}+\tilde{c}_2 c_2 2^{2j}\sqrt{\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\omega\|_{L^2}^2+\|\dot{\Delta}_j b\|_{L^2}^2}\\
&\le C2^j(\|\dot{\Delta}_j(u\otimes u)\|_{L^2}+\|\dot{\Delta}_j(b\otimes b)\|_{L^2}+\|\dot{\Delta}_j(b\otimes u)\|_{L^2}+\|\dot{\Delta}_j(u\otimes\omega)\|_{L^2}),
\end{aligned}
\tag{67}
$$

where $c_2 = \min\{\mu, \nu, 2\kappa\}$. For (67), applying Bernstein's inequality and integrating it in $[0, t]$, we obtain

$$\begin{aligned}
&\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \omega\|_{L^2} + \|\dot{\Delta}_j b\|_{L^2} + c_2\tilde{c}_2 c_2^{\star}(\|\nabla^2 u\|_{L^1_t L^2} + \|\nabla^2 \omega\|_{L^1_t L^2} + \|\nabla^2 b\|_{L^1_t L^2}) \\
\leq & 2\|\dot{\Delta}_j u_0\|_{L^2} + 2\|\dot{\Delta}_j \omega_0\|_{L^2} + 2\|\dot{\Delta}_j b_0\|_{L^2} + C2^j\|\dot{\Delta}_j(u \otimes u)\|_{L^1_t L^2} \\
& + C2^j\|\dot{\Delta}_j(b \otimes b)\|_{L^1_t L^2} + C2^j\|\dot{\Delta}_j(u \otimes \omega)\|_{L^1_t L^2} + C2^j\|\dot{\Delta}_j(u \otimes b)\|_{L^1_t L^2}.
\end{aligned}$$

Taking the l^1_j over $j \in \mathbb{Z}$, one yields

$$\begin{aligned}
&\|u\|_{\dot{B}^0_{2,1}} + \|b\|_{\dot{B}^0_{2,1}} + \|\omega\|_{\dot{B}^0_{2,1}} + c_2\Big(\|\nabla^2 u\|_{\tilde{L}^1_t \dot{B}^0_{2,1}} + \|\nabla^2 b\|_{\tilde{L}^1_t \dot{B}^0_{2,1}} + \|\nabla^2 \omega\|_{\tilde{L}^1_t \dot{B}^0_{2,1}}\Big) \\
\leq & 2\|u_0\|_{\dot{B}^0_{2,1}} + 2\|b_0\|_{\dot{B}^0_{2,1}} + 2\|\omega_0\|_{\dot{B}^0_{2,1}} \\
& + C\|2^j\|\dot{\Delta}_j(u \otimes u)\|_{L^1_t L^2}\|_{l^1_j} + C\|2^j\|\dot{\Delta}_j(b \otimes b)\|_{L^1_t L^2}\|_{l^1_j} \\
& + C\|2^j\|\dot{\Delta}_j(u \otimes b)\|_{L^1_t L^2}\|_{l^1_j} + C\|2^j\|\dot{\Delta}_j(u \otimes \omega)\|_{L^1_t L^2}\|_{l^1_j}.
\end{aligned} \tag{68}$$

Denote $c_3 = 2c_1\tilde{c}_2 c_2^{\star}$. Using Lemma 13, Lemma 14, and noting that $\|f\|_{L^\infty} \leq C\|f\|_{\dot{B}^1_{2,1}}$ and $\|f\|_{\tilde{L}^1_t(\dot{B}^s_{2,1})} \approx \|f\|_{L^1_t(\dot{B}^s_{2,1})}$, we have

$$\begin{aligned}
&\|u\|_{\dot{B}^0_{2,1}} + \|b\|_{\dot{B}^0_{2,1}} + \|\omega\|_{\dot{B}^0_{2,1}} + c_3\Big(\|\nabla^2 u\|_{L^1_t \dot{B}^0_{2,1}} + \|\nabla^2 b\|_{L^1_t \dot{B}^0_{2,1}} + \|\nabla^2 \omega\|_{L^1_t \dot{B}^0_{2,1}}\Big) \\
\leq & 2\|u_0\|_{\dot{B}^0_{2,1}} + 2\|b_0\|_{\dot{B}^0_{2,1}} + 2\|\omega_0\|_{\dot{B}^0_{2,1}} + C\int_0^t \Big(\|u(\tau)\|_{\dot{B}^0_{2,1}} + \|b(\tau)\|_{\dot{B}^0_{2,1}} + \|\omega(\tau)\|_{\dot{B}^0_{2,1}}\Big) \\
& \times \Big(\|\nabla^2 u(\tau)\|_{\dot{B}^0_{2,1}} + \|\nabla^2 b(\tau)\|_{\dot{B}^0_{2,1}} + \|\nabla^2 \omega(\tau)\|_{\dot{B}^0_{2,1}}\Big) d\tau.
\end{aligned} \tag{69}$$

This inequality indicates that, for any $0 < \epsilon < \epsilon_0$,

$$\|u_0\|_{\dot{B}^0_{2,1}} + \|b_0\|_{\dot{B}^0_{2,1}} + \|\omega_0\|_{\dot{B}^0_{2,1}} < \epsilon,$$

then bootstrap argument yields

$$\begin{aligned}
&\|u(t)\|_{\dot{B}^0_{2,1}} + \|\omega(t)\|_{\dot{B}^0_{2,1}} + \|b(t)\|_{\dot{B}^0_{2,1}} \\
&+ \|\nabla^2 u(t)\|_{L^1_t(\dot{B}^0_{2,1})} + \|\nabla^2 \omega(t)\|_{L^1_t(\dot{B}^0_{2,1})} + \|\nabla^2 b(t)\|_{L^1_t(\dot{B}^0_{2,1})} \leq C\epsilon,
\end{aligned} \tag{70}$$

which completed the Proposition 5. □

Next, we start to prove the decay estimates assertion of (II). As a tool, we first verify the following Proposition in the negative Sobolev space $\dot{H}^{-l}$, with $0 < l < 1$.

Proposition 6 *Let* $c_2 = \min\{\mu, 2\kappa, \nu\}$. *Then for* $0 < l < 1$, *we have*

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{\dot{H}^{-l}}^2 + \|b\|_{\dot{H}^{-l}}^2 + \|\omega\|_{\dot{H}^{-l}}^2) + c_1(\|\nabla u\|_{\dot{H}^{-l}}^2 + \|\nabla b\|_{\dot{H}^{-l}}^2 + \|\nabla\omega\|_{\dot{H}^{-l}}^2) \\
&\leq C(\|\nabla u\|_{L^2} + \|\nabla\omega\|_{L^2} + \|\nabla b\|_{L^2})(\|\nabla u\|_{L^2}^{1-l} + \|\nabla b\|_{L^2}^{1-l})(\|u\|_{L^2}^l + \|b\|_{L^2}^l) \\
&\quad \times (\|u\|_{\dot{H}^{-l}} + \|b\|_{\dot{H}^{-l}} + \|\omega\|_{\dot{H}^{-l}}).
\end{aligned} \tag{71}$$

Proof of Proposition 6 Due to (64) and with the divergence free condition $\nabla \cdot u = \nabla \cdot b = 0$, we derive that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\dot{\Delta}_j u\|_{L^2}^2 + \|\dot{\Delta}_j\omega\|_{L^2}^2 + \|\dot{\Delta}_j b\|_{L^2}^2) + (\mu+\chi)\|\dot{\Delta}_j\nabla u\|_{L^2}^2 + \kappa\|\dot{\Delta}_j\nabla\omega\|_{L^2}^2 \\
&\quad + 4\chi\|\dot{\Delta}_j\omega\|_{L^2}^2 + \frac{\nu}{2}\|\dot{\Delta}_j\nabla b\|_{L^2}^2 \\
&\leq -\int \dot{\Delta}_j(u\cdot\nabla u)\cdot\dot{\Delta}_j u + \int \dot{\Delta}_j(b\cdot\nabla b)\cdot\dot{\Delta}_j u - \int \dot{\Delta}_j(u\cdot\nabla b)\cdot\dot{\Delta}_j b \\
&\quad + \int \dot{\Delta}_j(b\cdot\nabla u)\cdot\dot{\Delta}_j b - \int \dot{\Delta}_j(u\cdot\nabla\omega)\cdot\dot{\Delta}_j\omega + 4\chi\int \dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j\omega.
\end{aligned}$$

Multiplying the above inequality by 2^{-2lj} and taking the l_j^2 over $j \in \mathbb{Z}$, and noting that $\dot{B}_{2,2}^{-l} = \dot{H}^{-l}$, we have

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{\dot{H}^{-l}}^2 + \|b\|_{\dot{H}^{-l}}^2 + \|\omega\|_{\dot{H}^{-l}}^2) + 2(\mu+\chi)\|\nabla u\|_{\dot{H}^{-l}}^2 \\
&\quad + 2\kappa\|\nabla\omega\|_{\dot{H}^{-l}}^2 + 8\chi\|\omega\|_{\dot{H}^{-l}}^2 + \nu\|\nabla b\|_{\dot{H}^{-l}}^2 \\
&\leq 2\|2^{-lj}\|\dot{\Delta}_j(u\cdot\nabla u)\|_{L^2}\|_{l_j^2}\|u\|_{\dot{H}^{-l}} + 2\|2^{-lj}\|\dot{\Delta}_j(b\cdot\nabla b)\|_{L^2}\|_{l_j^2}\|u\|_{\dot{H}^{-l}} \\
&\quad + 2\|2^{-lj}\|\dot{\Delta}_j(u\cdot\nabla\omega)\|_{L^2}\|_{l_j^2}\|\omega\|_{\dot{H}^{-l}} + 8\chi\|2^{-lj}\|\dot{\Delta}_j(\nabla\times u)\|_{L^2}\|_{l_j^2}\|\omega\|_{\dot{H}^{-l}} \\
&\quad + 2\|2^{-lj}\|\dot{\Delta}_j(u\cdot\nabla b)\|_{L^2}\|_{l_j^2}\|b\|_{\dot{H}^{-l}} + 2\|2^{-lj}\|\dot{\Delta}_j(b\cdot\nabla u)\|_{L^2}\|_{l_j^2}\|b\|_{\dot{H}^{-l}} \\
&\leq 2\|u\cdot\nabla u\|_{\dot{H}^{-l}}\|u\|_{\dot{H}^{-l}} + 2\|b\cdot\nabla b\|_{\dot{H}^{-l}}\|u\|_{\dot{H}^{-l}} + 2\|u\cdot\nabla\omega\|_{\dot{H}^{-l}}\|\omega\|_{\dot{H}^{-l}} \\
&\quad + (\mu+2\chi)\|\nabla u\|_{\dot{H}^{-l}}^2 + \frac{8\chi^2}{\frac{\mu}{2}+\chi}\|\omega\|_{\dot{H}^{-l}}^2 + 2(\|u\cdot\nabla b\|_{\dot{H}^{-l}} + \|b\cdot\nabla u\|_{\dot{H}^{-l}})\|b\|_{\dot{H}^{-l}}.
\end{aligned} \tag{72}$$

Applying Lemma 10, Hölder's inequality and the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}
\|u\cdot\nabla u\|_{\dot{H}^{-l}} &\leq C\|u\cdot\nabla u\|_{L^{\frac{2}{l+1}}}, \\
\|u\cdot\nabla u\|_{L^{\frac{2}{l+1}}} &\leq C\|u\|_{L^{\frac{2}{l}}}\|\nabla u\|_{L^2},
\end{aligned}$$

and

$$\|u\|_{L^{\frac{2}{l}}} \leq C\|\nabla u\|_{L^2}^{1-l}\|u\|_{L^2}^l.$$

Then

$$\|u\cdot\nabla u\|_{\dot{H}^{-l}} \leq C\|\nabla u\|_{L^2}^{2-l}\|u\|_{L^2}^l. \tag{73}$$

Similarly,

$$\|b\cdot\nabla b\|_{\dot H^{-l}}\le C\|\nabla b\|_{L^2}^{2-l}\|b\|_{L^2}^{l}, \tag{74}$$

$$\begin{aligned}&\|u\cdot\nabla b\|_{\dot H^{-l}}+\|b\cdot\nabla u\|_{\dot H^{-l}}\\&\le C(\|\nabla u\|_{L^2}^{1-l}\|u\|_{L^2}^{l}\|\nabla b\|_{L^2}+\|\nabla b\|_{L^2}^{1-l}\|b\|_{L^2}^{l}\|\nabla u\|_{L^2}),\end{aligned} \tag{75}$$

$$\|u\cdot\nabla\omega\|_{\dot H^{-l}}\le C\|\nabla\omega\|_{L^2}\|\nabla u\|_{L^2}^{1-l}\|u\|_{L^2}^{l}. \tag{76}$$

Combining (72) - (76), we obtain

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{\dot H^{-l}}^2+\|b\|_{\dot H^{-l}}^2+\|\omega\|_{\dot H^{-l}}^2)+c_2(\|\nabla u\|_{\dot H^{-l}}^2+\|\nabla b\|_{\dot H^{-l}}^2+\|\nabla\omega\|_{\dot H^{-l}}^2)\\&\le C(\|\nabla u\|_{L^2}+\|\nabla\omega\|_{L^2}+\|\nabla b\|_{L^2})(\|\nabla u\|_{L^2}^{1-l}+\|\nabla b\|_{L^2}^{1-l})(\|u\|_{L^2}^{l}+\|b\|_{L^2}^{l})\\&\quad\times(\|u\|_{\dot H^{-l}}+\|b\|_{\dot H^{-l}}+\|\omega\|_{\dot H^{-l}}),\end{aligned}$$

where $c_2=\min\{\mu,2\kappa,\nu\}$. Thus the proof of Proposition 6 is completed. □

Next, we continue to prove (10). Multiplying (67) by 2^{sj} and utilizing Bernstein's inequality and integrating it in $[0,t]$, finally, taking the l_j^1 over $j\in\mathbb{Z}$, we have

$$\begin{aligned}&\|u\|_{\dot B^s_{2,1}}+\|b\|_{\dot B^s_{2,1}}+\|\omega\|_{\dot B^s_{2,1}}+c_3\Big(\|\nabla^2u\|_{\tilde L^1_t(\dot B^s_{2,1})}+\|\nabla^2b\|_{\tilde L^1_t(\dot B^s_{2,1})}+\|\nabla^2\omega\|_{\tilde L^1_t(\dot B^s_{2,1})}\Big)\\&\le 2(\|u_0\|_{\dot B^s_{2,1}}+\|b_0\|_{\dot B^s_{2,1}}+\|\omega_0\|_{\dot B^s_{2,1}})+C\|2^{(s+1)j}\|\dot\Delta_j(u\otimes u)\|_{L^1_tL^2}\|_{l^1_j}\\&\quad+C\|2^{(s+1)j}\|\dot\Delta_j(b\otimes b)\|_{L^1_tL^2}\|_{l^1_j}+C\|2^{(s+1)j}\|\dot\Delta_j(u\otimes b)\|_{L^1_tL^2}\|_{l^1_j}\\&\quad+C\|2^{(s+1)j}\|\dot\Delta_j(u\otimes\omega)\|_{L^1_tL^2}\|_{l^1_j}\\&\triangleq A_1+A_2+A_3+A_4+A_5.\end{aligned} \tag{77}$$

Since $(u_0,\omega_0,b_0)\in B^s_{2,1}(\mathbb{R}^2)$, we have

$$A_1=2(\|u_0\|_{\dot B^s_{2,1}}+\|b_0\|_{\dot B^s_{2,1}}+\|\omega_0\|_{\dot B^s_{2,1}})\le C.$$

By Lemma 13, Lemma 14 and noting that $\|f\|_{L^1_t(\dot B^s_{2,1})}\approx\|f\|_{\tilde L^1_t(\dot B^s_{2,1})}$ and $\|f\|_{L^\infty}\le C\|f\|_{\dot B^1_{2,1}}$, yield

$$
\begin{aligned}
A_2 =& C\|2^{(s+1)j}\|\dot{\Delta}_j(u\otimes u)\|_{L^1_tL^2}\|_{l^1_j}\\
\le& C\int_0^t \|u\otimes u(\tau)\|_{\dot{B}^{s+1}_{2,1}}d\tau\\
\le& C\int_0^t \|u(\tau)\|_{\dot{B}^{s+1}_{2,1}}\|u(\tau)\|_{L^\infty}d\tau\\
\le& C\int_0^t \left(\|u(\tau)\|^{\frac{s+1}{s+2}}_{\dot{B}^{s+2}_{2,1}}\|u(\tau)\|^{\frac{1}{s+2}}_{\dot{B}^{0}_{2,1}}\|u(\tau)\|^{\frac{1}{s+2}}_{\dot{B}^{s+2}_{2,1}}\|u(\tau)\|^{\frac{s+1}{s+2}}_{\dot{B}^{0}_{2,1}}\right)d\tau\\
\le& C\int_0^t \|\nabla^2 u(\tau)\|_{\dot{B}^{s}_{2,1}}\|u(\tau)\|_{\dot{B}^{0}_{2,1}}d\tau.
\end{aligned}
$$

For A_3 - A_5, by the similar method as A_2, together with the Young inequality, we get

$$
\begin{aligned}
A_3 =& C\|2^{(s+1)j}\|\dot{\Delta}_j(b\otimes b)\|_{L^1_tL^2}\|_{l^1_j}\\
\le& C\int_0^t \|b\otimes b(\tau)\|_{\dot{B}^{s+1}_{2,1}}d\tau\\
\le& C\int_0^t \|\nabla^2 b(\tau)\|_{\dot{B}^{s}_{2,1}}\|b(\tau)\|_{\dot{B}^{0}_{2,1}}d\tau.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
A_4 \le& C\int_0^t (\|u(\tau)\|_{\dot{B}^{0}_{2,1}}+\|b(\tau)\|_{\dot{B}^{0}_{2,1}})(\|\nabla^2 u(\tau)\|_{\dot{B}^{s}_{2,1}}+\|\nabla^2 b(\tau)\|_{\dot{B}^{s}_{2,1}})d\tau,\\
A_5 \le& C\int_0^t (\|u(\tau)\|_{\dot{B}^{0}_{2,1}}+\|\omega(\tau)\|_{\dot{B}^{0}_{2,1}})(\|\nabla^2 u(\tau)\|_{\dot{B}^{s}_{2,1}}+\|\nabla^2 \omega(\tau)\|_{\dot{B}^{s}_{2,1}})d\tau.
\end{aligned}
$$

Inserting A_1 - A_5 into (77), we derive that

$$
\begin{aligned}
&\|u\|_{\dot{B}^{s}_{2,1}}+\|b\|_{\dot{B}^{s}_{2,1}}+\|\omega\|_{\dot{B}^{s}_{2,1}}+c_3\left(\|\nabla^2 u\|_{L^1_t(\dot{B}^{s}_{2,1})}+\|\nabla^2 b\|_{L^1_t(\dot{B}^{s}_{2,1})}+\|\nabla^2 \omega\|_{L^1_t(\dot{B}^{s}_{2,1})}\right)\\
&\le C+\int_0^t (\|u(\tau)\|_{\dot{B}^{0}_{2,1}}+\|\omega(\tau)\|_{\dot{B}^{0}_{2,1}}+\|b(\tau)\|_{\dot{B}^{0}_{2,1}})\\
&\quad(\|\nabla^2 u(\tau)\|_{\dot{B}^{s}_{2,1}}+\|\nabla^2 \omega(\tau)\|_{\dot{B}^{s}_{2,1}}+\|\nabla^2 b(\tau)\|_{\dot{B}^{s}_{2,1}})d\tau.
\end{aligned}
\tag{78}
$$

Combining (78) and (59) with ϵ sufficiently small, we have

$$
\begin{aligned}
&\|u(t)\|_{\dot{B}^{s}_{2,1}}+\|b(t)\|_{\dot{B}^{s}_{2,1}}+\|\omega(t)\|_{\dot{B}^{s}_{2,1}}\\
&+\frac{c_3}{2}\int_0^t \left(\|\nabla^2 u(\tau)\|_{\dot{B}^{s}_{2,1}}+\|\nabla^2 b(\tau)\|_{\dot{B}^{s}_{2,1}}+\|\nabla^2 \omega(\tau)\|_{\dot{B}^{s}_{2,1}}\right)d\tau\le C.
\end{aligned}
\tag{79}
$$

Because $\dot{B}^0_{2,1}\hookrightarrow \dot{B}^0_{2,2}$ and $\dot{B}^0_{2,2}\sim \dot{H}^0\sim L^2$, thus (59) implies

$$\|u(t)\|_{L^2} + \|b(t)\|_{L^2} + \|\omega(t)\|_{L^2} + \int_0^t \left(\|\nabla^2 u(\tau)\|_{L^2} + \|\nabla^2 b(\tau)\|_{L^2} + \|\nabla^2 \omega(\tau)\|_{L^2}\right) d\tau < C\epsilon. \tag{80}$$

Combining (79) and (80), we have

$$\|u(t)\|_{B^s_{2,1}} + \|b(t)\|_{B^s_{2,1}} + \|\omega(t)\|_{B^s_{2,1}} + \int_0^t \left(\|\nabla^2 u(\tau)\|_{B^s_{2,1}} + \|\nabla^2 b(\tau)\|_{B^s_{2,1}} + \|\nabla^2 \omega(\tau)\|_{B^s_{2,1}}\right) d\tau < C,$$

which completed the proof of (I) in Theorem 2. □

We now turn to prove the decay part (II) of Theorem 2.

Proof of (II) of Theorem 2 Multiplying (67) by 2^{mj}, and taking the l^1_j over $j \in \mathbb{Z}$, we obtain

$$\begin{aligned}&\frac{d}{dt}y(t) + c_3\left(\|u\|_{\dot B^{m+2}_{2,1}} + \|b\|_{\dot B^{m+2}_{2,1}} + \|\omega\|_{\dot B^{m+2}_{2,1}}\right)\\ \le{}& C\|2^{(m+1)j}\|\dot\Delta_j(u\otimes u)\|_{L^2}\|_{l^1_j} + C\|2^{(m+1)j}\|\dot\Delta_j(b\otimes b)\|_{L^2}\|_{l^1_j}\\ &+ C\|2^{(m+1)j}\|\dot\Delta_j(u\otimes b)\|_{L^2}\|_{l^1_j} + C\|2^{(m+1)j}\|\dot\Delta_j(u\otimes \omega)\|_{L^2}\|_{l^1_j},\end{aligned}$$

where $y(t) = \left\|2^{mj}\sqrt{\|\dot\Delta_j u\|^2 + \|\dot\Delta_j \omega\|^2 + \|\dot\Delta_j b\|^2}\right\|_{l^1_j}$. Using Lemma 13 and Lemma 15, together with $\|f\|_{L^\infty} \le C\|f\|_{\dot B^1_{2,1}}$, we have

$$\begin{aligned}&\frac{d}{dt}y(t) + c_3\left(\|u\|_{\dot B^{m+2}_{2,1}} + \|b\|_{\dot B^{m+2}_{2,1}} + \|\omega\|_{\dot B^{m+2}_{2,1}}\right)\\ \le{}& C\left(\|u\|_{\dot B^0_{2,1}} + \|b\|_{\dot B^0_{2,1}} + \|\omega\|_{\dot B^0_{2,1}}\right)\left(\|u\|_{\dot B^{m+2}_{2,1}} + \|b\|_{\dot B^{m+2}_{2,1}} + \|\omega\|_{\dot B^{m+2}_{2,1}}\right).\end{aligned}$$

Then this inequality, together with (59) with ϵ sufficiently small, we have

$$\frac{d}{dt}y(t) + \frac{c_2}{2}\left(\|u\|_{\dot B^{m+2}_{2,1}} + \|b\|_{\dot B^{m+2}_{2,1}} + \|\omega\|_{\dot B^{m+2}_{2,1}}\right) \le 0. \tag{81}$$

Applying Lemma 15 and $\dot B^s_{2,2} \hookrightarrow \dot B^s_{2,\infty}$, we infer that

$$\|u\|_{\dot B^m_{2,1}} \le C\|u\|_{\dot H^{-l}}^{\frac{2}{m+l+2}}\|\nabla^2 u\|_{\dot B^m_{2,1}}^{\frac{m+l}{m+l+2}}, \tag{82}$$

$$\|b\|_{\dot B^m_{2,1}} \le C\|b\|_{\dot H^{-l}}^{\frac{2}{m+l+2}}\|\nabla^2 b\|_{\dot B^m_{2,1}}^{\frac{m+l}{m+l+2}}, \tag{83}$$

and

$$\|\omega\|_{\dot{B}^m_{2,1}} \le C\|\omega\|^{\frac{2}{m+l+2}}_{\dot{H}^{-l}} \|\nabla^2\omega\|^{\frac{m+l}{m+l+2}}_{\dot{B}^m_{2,1}}. \tag{84}$$

Therefore, if

$$\|u\|_{\dot{H}^{-l}} + \|b\|_{\dot{H}^{-l}} + \|\omega\|_{\dot{H}^{-l}} \le C, \tag{85}$$

then we can obtain from (81) - (83), there exists a constant $a_0 > 0$ such that,

$$\frac{d}{dt}y(t) + a_0(y(t))^{\frac{m+l+2}{m+l}} \le 0. \tag{86}$$

It follows from this that

$$y(t) \le C(1+t)^{-\frac{m+l}{2}},$$

which implies

$$\|u\|_{\dot{B}^m_{2,1}} + \|b\|_{\dot{B}^m_{2,1}} + \|\omega\|_{\dot{B}^m_{2,1}} \le C(1+t)^{-\frac{m+l}{2}}, \tag{87}$$

which immediately yields (11).

Finally, to make the process more complete, we need to verify that (85) holds for $0 \le l < 1$. To this end, we divide the proof into two cases.

Case 1. ($l = 0$) Using the fact that $\dot{H}^0 = L^2$, by (13) we have

$$\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} + \|b(t)\|_{L^2} \le C.$$

Then it yields (85).

Case 2. ($0 < l < 1$) Assume that

$$\|u_0\|^2_{\dot{H}^{-l}} + \|b_0\|^2_{\dot{H}^{-l}} + \|\omega_0\|^2_{\dot{H}^{-l}} \le C_0. \tag{88}$$

Suppose that for all $t \in [0, T]$

$$\|u(t)\|^2_{\dot{H}^{-l}} + \|b(t)\|^2_{\dot{H}^{-l}} + \|\omega(t)\|^2_{\dot{H}^{-l}} \le 2C_0. \tag{89}$$

If we can derive that for all $t \in [0, T]$,

$$\|u(t)\|^2_{\dot{H}^{-l}} + \|b(t)\|^2_{\dot{H}^{-l}} + \|\omega(t)\|^2_{\dot{H}^{-l}} \le \frac{3}{2}C_0, \tag{90}$$

then an application of the bootstrapping argument would imply that the solution (u, ω, b) of system (8) satisfies (90) for all $t \in [0, T]$, which implies (85). With (86) and (87) at our disposal, we show that (90) holds.

With the help of (87) and Lemma 8, we know that

$$\|u(t)\|_{L^2} = \|u\|_{\dot{B}^0_{2,2}} \le C\|u\|_{\dot{B}^0_{2,1}}, \tag{91}$$

$$\|\nabla u\|_{L^2} \le C\|u\|_{\dot{B}^1_{2,1}} \le C\|u\|^{\frac{s-1}{s}}_{\dot{B}^0_{2,1}} \|u\|^{\frac{1}{s}}_{\dot{B}^s_{2,1}} \le C(1+\tau)^{-\frac{l+1}{2}}. \tag{92}$$

Similarly,

$$\|\omega(t)\|_{L^2} + \|b(t)\|_{L^2} = \|\omega\|_{\dot{B}^0_{2,2}} + \|b\|_{\dot{B}^0_{2,2}} \le C(\|\omega\|_{\dot{B}^0_{2,1}} + \|\omega\|_{\dot{B}^0_{2,1}}), \tag{93}$$

$$\|\nabla\omega\|_{L^2} + \|\nabla b\|_{L^2} \le C(1+t)^{-\frac{l+1}{2}}. \tag{94}$$

Integrating (71) in $[0, t]$ with $0 < t \le T$, together with (80), and (91) - (94), one infers that

$$\begin{aligned}
&\|u(t)\|^2_{\dot{H}^{-l}} + \|b(t)\|^2_{\dot{H}^{-l}} + \|\omega(t)\|^2_{\dot{H}^{-l}} \\
\le&\|u_0\|^2_{\dot{H}^{-l}} + \|b_0\|^2_{\dot{H}^{-l}} + \|\omega_0\|^2_{\dot{H}^{-l}} + C\int_0^t (\|\nabla u(\tau)\|_{L^2} + \|\nabla\omega(\tau)\|_{L^2} + \|\nabla b(\tau)\|_{L^2})^{2-l} \\
&\times (\|u(\tau)\|_{L^2} + \|\omega(\tau)\|_{L^2} + \|b(\tau)\|_{L^2})^l (\|u(\tau)\|_{\dot{H}^{-l}} + \|b(\tau)\|_{\dot{H}^{-l}} + \|\omega(\tau)\|_{\dot{H}^{-l}}) d\tau \\
\le&C_0 + C\int_0^t (\|u(\tau)\|_{\dot{B}^0_{2,1}} + \|\omega(\tau)\|_{\dot{B}^0_{2,1}} + \|b(\tau)\|_{\dot{B}^0_{2,1}})^l (1+\tau)^{-\left(\frac{l+1}{2}(2-l)\right)} \\
&\times (\|u(\tau)\|_{\dot{H}^{-l}} + \|b(\tau)\|_{\dot{H}^{-l}} + \|\omega(\tau)\|_{\dot{H}^{-l}}) d\tau \\
\le&C_0 + C\epsilon^l \sup_{0\le\tau\le t} \left(\|u\|_{\dot{H}^{-l}} + \|b\|_{\dot{H}^{-l}} + \|\omega\|_{\dot{H}^{-l}}\right) \left(\int_0^t (1+\tau)^{-\left(\frac{l+1}{2}(2-l)\right)} d\tau\right) \\
\le&C + C\epsilon^l \sup_{0\le\tau\le t} \left(\|u\|_{\dot{H}^{-l}} + \|b\|_{\dot{H}^{-l}} + \|\omega\|_{\dot{H}^{-l}}\right).
\end{aligned}$$

By choosing ϵ in (80) sufficiently small, then the above inequality yields (90) for all $t \in [0, t]$, which closes the proof. Then we have (85) and completed the proof of (11). □

Appendix A: Functional Space

This appendix provides the definition of Littlewood-Paley decomposition and the definition of Besov space. Some related inequalities used in the previous sections are also included. Materials presented in this appendix can be found in several books and many papers (see, e.g., [3, 4, 22, 27, 40]).

We start with several notations. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}\simeq$ its dual, the space of tempered distributions. To introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \{\xi \in \mathbb{R}^{\mathrm{d}} : 2^{j-1} \le |\xi| < 2^{j+1}\}.$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}$ such that

$$supp\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{A1}$$

in $\mathcal{S}\simeq$ for any $f \in \mathcal{S}\simeq$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{A2}$$

To define the homogeneous Besov space, we set

$$\dot{\Delta}_j f = \Phi_j * f, \qquad \text{if } j = 0, \pm 1, \pm 2, \cdots. \tag{A3}$$

Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k.$$

For any $f \in \mathcal{S}\simeq$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j. It is clear from (A1) that $S_j \to Id$ as $j \to \infty$ in the distributional sense.

Definition 1 (see, e.g., [4, 22])The inhomogeneous and homogeneous Besov spaces $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ consists of $f \in \mathcal{S}\simeq$ and $f \in \mathcal{S}\simeq\backslash\mathcal{P}$, respectively, satisfying

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q_j} < \infty,$$

and

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \|2^{js}\|\dot{\Delta}_j f\|_{L^p}\|_{l^q_j} < \infty,$$

respectively, where $\mathcal{P}$ represents the set of polynomials.

Many frequently used function spaces are special cases of Besov spaces. The following lemma lists some useful equivalence and embedding relations.

Lemma 7 (see, e.g., [4, 22]) *For any $s \in R$,*

$$H^s \sim B^s_{2,2}, \quad \dot{H}^s \sim \dot{B}^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$B^s_{q,min\{q,2\}} \hookrightarrow W^s_q \hookrightarrow B^s_{q,max\{q,2\}}.$$

For any non-integer $s > 0$, the Hölder space C^s is equivalent to $B^s_{\infty,\infty}$.

In the following Lemmas, we stated a Sobolev-type embedding theorem for Besov space.

Lemma 8 (see [4]) *Let $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$. Then, for any real number s, the space $\dot{B}^s_{p_1,r_1}$ is continuously emdedded in $\dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

Lemma 9 (*see* [2]) *For every $s \in \mathbb{R}, \epsilon > 0, 1 < p < +\infty$ and $1 \le q \le +\infty$, we have*

$$H^{s+\epsilon}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow H^{s-\epsilon}(\mathbb{R}^n). \tag{A4}$$

Lemma 10 (*see* [4]) *If p belongs to $(1, 2]$, then $L^p(\mathbb{R}^d)$ embeds continuously in $\dot{H}^s(\mathbb{R}^d)$ with $s = \frac{d}{2} - \frac{d}{p}$.*

We also used the space-time space defined below.

Definition 2 (see, e.g., [4, 22]) For $t > 0$, $s \in \mathbb{R}$ and $1 \le p, q, r \le \infty$, the inhomogenous and homogenous space-times spaces $L^r_t B^s_{p,q}$, $L^r_t \dot{B}^s_{p,q}$ and $\tilde{L}^r_t B^s_{p,q}$, $\tilde{L}^r_t \dot{B}^s_{p,q}$ are defined through the norms

$$\|f\|_{L^r_t B^s_{p,q}} \equiv \|\|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q_j}\|_{L^r_t}, \quad \|f\|_{L^r_t \dot{B}^s_{p,q}} \equiv \|\|2^{js}\|\dot{\Delta}_j f\|_{L^p}\|_{l^q_j}\|_{L^r_t},$$

and

$$\|f\|_{\tilde{L}^r_t B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^r_t L^p}\|_{l^q_j}, \quad \|f\|_{\tilde{L}^r_t \dot{B}^s_{p,q}} \equiv \|2^{js}\|\dot{\Delta}_j f\|_{L^r_t L^p}\|_{l^q_j},$$

respectively.

The inhomogeneous space-time space has the following properties.

$$L^r_t B^s_{p,q} \hookrightarrow \tilde{L}^r_t B^s_{p,q}, \text{ if } q \geq r, \quad \tilde{L}^r_t B^s_{p,q} \hookrightarrow L^r_t B^s_{p,q}, \text{ if } r \geq q.$$

As $q = r$,

$$\|f\|_{L^r_t B^s_{p,q}} \approx \|f\|_{\tilde{L}^r_t B^s_{p,q}}.$$

The homogeneous space-time space has similar properties.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions. These inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives. The upper bounds also hold when the fractional operators are replaced by partial derivatives.

Lemma 11 (see, e.g., [4, 22]) *Let $\alpha \geq 0$ and $1 \leq p \leq q \leq \infty$. 1) If f satisfies*

$$supp\widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq K2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

2) *If f satisfies*

$$supp\widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j\},$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1\, 2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

where C_1 and C_2 are constants depending on α, p and q.

Next, we give several useful calculus inequalities. We first give two lemmas regarding commutator estimates and product law. Lemma 12 and Lemma 13 below with $p_1 = q_2 = \infty$ and $q_1 = p_2$ have previously been obtained in [4, 22]. Here we state the following more general cases without detailed proofs since they can be proved by following the methods in [4, 22].

Lemma 12 *Let $s > -1$, $(p, r, p_1, p_2, q_1, q_2) \in [1, \infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ and u be a smooth divergence free vector field. Then for $j \in \mathbb{Z}$,*

$$\|2^{js}\|[\dot{\Delta}_j, u\cdot\nabla]v\|_{L^p}\|_{l^r_j} \leq C(\|\nabla u\|_{L^{p_1}}\|\nabla v\|_{\dot{B}^{s-1}_{q_1,r}} + \|\nabla v\|_{L^{p_2}}\|\nabla u\|_{\dot{B}^{s}_{q_2,r}}), \tag{A5}$$

$$\|2^{js}\|[\dot{\Delta}_j, u\cdot\nabla]v\|_{L^p}\|_{l^r_j} \le C(\|\nabla u\|_{L^{p_1}}\|v\|_{\dot{B}^s_{q_1,r}} + \|v\|_{L^{p_2}}\|\nabla u\|_{\dot{B}^s_{q_2,r}}), \tag{A6}$$

where $[\dot{\Delta}_j, u\cdot\nabla]v = \dot{\Delta}_j(u\cdot\nabla v) - u\cdot\dot{\Delta}_j(\nabla v)$.

Lemma 13 *Suppose that* $s>0$ *and* $(p, r, p_1, p_2, q_1, q_2) \in [1,\infty]$ *with* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. *Then the following hold true*

$$\|fg\|_{\dot{B}^s_{p,r}} \le C(\|f\|_{L^{p_1}}\|g\|_{\dot{B}^s_{q_1,r}} + \|f\|_{\dot{B}^s_{p_2,r}}\|g\|_{L^{q_2}}). \tag{A7}$$

For inhomogeneous Besov space has the similar inequality.

Finally, we recall the following Besov space interpolation estimate and the inequality for homogeneous Besov space.

Lemma 14 (see [32]) *Fixed* $m>l>k$, *and* $1\le p\le q\le r\le\infty$, *we have*

$$\|f\|_{\dot{B}^l_{q,q'}(\mathbb{R}^2)} \le \|f\|^{\theta}_{\dot{B}^k_{r,r'}(\mathbb{R}^2)}\|f\|^{1-\theta}_{\dot{B}^m_{p,p'}(\mathbb{R}^2)}. \tag{A8}$$

These parameters satisfy the following restrictions

$$l = k\theta + m(1-\theta), \quad \frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{p}, \quad \frac{1}{q'} = \frac{\theta}{r'} + \frac{1-\theta}{p'}.$$

Also $1\le p'\le q'\le r'\le\infty$ *and solving we have* $\theta = \frac{m-l}{m-k} \in (0,1]$.

Lemma 15 (*see* [4, 22]) *Let* s, s_1 *and* s_2 *be real numbers. Let* $s_1<s_2$, $0<\theta<1$, $1\le p\le\infty$ *and* $1\le r_1\le r_2\le\infty$. *Then*

$$\|f\|_{\dot{B}^s_{p,r_2}(\mathbb{R}^2)} \le \|f\|_{\dot{B}^s_{p,r_1}(\mathbb{R}^2)}. \tag{A9}$$

$$\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}(\mathbb{R}^2)} \le C\|f\|^{\theta}_{\dot{B}^{s_1}_{p,r}(\mathbb{R}^2)}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}(\mathbb{R}^2)}. \tag{A10}$$

$$\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}(\mathbb{R}^2)} \le \frac{C}{s_2-s_1}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right)\|f\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}(\mathbb{R}^2)}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}(\mathbb{R}^2)}. \tag{A11}$$

For inhomogeneous Besov space has the similar inequality.

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References

1. Ahmadi, G., Shahinpoor, M.: Universal stability of magneto-micropolar fluid motions. Int. J. Eng. Sci. **12**, 657–663 (1974)
2. Abels, H.: Pseudodifferential and Singular Integral Operators. De Gruyter, Berlin (2011)
3. Bergh, J., Löfström, J.: Interpolation Spaces. An Introduction. Springer, Berlin (1976)
4. Bahouri, H., Chemin, J., Danchin, R.: Fourier Analysis and Nonlinear Partial Differential Equations. Springer, Berlin (2011)
5. Constantin, P., Foias, C.: Navier–Stokes Equations. The University of Chicago Press, Chicago (1988)
6. Cao, C., Wu, J.: Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. **226**, 1803–1822 (2011)
7. Chen, Q., Miao, C.: Global well-posedness for the micropolar fluid system in critical Besov spaces. J. Differ. Equ. **252**, 2698–2724 (2012)
8. Chen, M.: Global well-posedness of the 2D incompressible micropolar fluid flows with partial viscosity and angular viscosity. Acta Math. Sci. Ser. B Engl. Ed. **33**, 929–935 (2013)
9. Cao, C., Wu, J., Yuan, B.: The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion. SIAM J. Math. Anal. **46**, 588–602 (2014)
10. Cheng, J., Liu, Y.: Global regularity of the 2D magnetic micropolar fluid flows with mixed partial viscosity. Comput. Math. Appl. **70**, 66–72 (2015)
11. Dong, B., Zhang, Z.: Global regularity of the 2D micropolar fluid flows with zero angular viscosity. J. Differ. Equ. **249**, 200–213 (2010)
12. Dong, B., Li, J., Wu, J.: Global well-posedness and large-time decay for the 2D micropolar equations. J. Differ. Equ. **262**, 3488–3523 (2017)
13. Dong, B., Jia, Y., Li, J., Wu, J.: Global regularity and time decay for the 2D magnetohydrodynamic equations with fractional dissipation and partial magnetic diffusion. J. Math. Fluid Mech. **20**, 1541–1565 (2018)
14. Du, L., Zhou, D.: Global well-posedness of two-dimensional magnetohydrodynamic flows with partial dissipation and magnetic diffusion. SIAM J. Math. Anal. **47**, 1562–1589 (2015)
15. Eringen, A.: Simple microfluids. Int. Eng. Sci. **2**, 205–217 (1964)
16. Eringen, A.: Theory of micropolar fluids. J. Math. Mech. **16**, 1–18 (1966)
17. Guo, Y., Shang, H.: Global well-posedness of two-dimensional magneto-micropolar equations with partial dissipation. Appl. Math. Comput. **313**, 392–407 (2017)

18. Guo, Y., Jia, Y., Dong, B.: Time decay rates of the micropolar equations with zero angular viscosity. Bull. Malays. Math. Sci. Soc. **44**(6), 3663–3675 (2021)
19. Leray, J.: Sur le mouvement d'un liquide visqueux emplissant I'espace. Acta Math. **63**, 193–248 (1934)
20. Łukaszewicz, G.: Micropolar Fluids. Theory and Applications. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston (1999)
21. Li, M.: Stability and times decay rates of the 2D magneto-micropolar equations with partial dissipation. Z. Angew. Math. Phys. **100**, 1–20 (2022)
22. Miao, C., Wu, J., Zhang, Z.: Littlewood–Paley Theory and its Applications in Partial Differential Equations of Fluid Dynamics. Science Press, Beijing (2012). (**in Chinese**)
23. Ma, L.: On two-dimensional incompressible magneto-micropolar system with mixed partial viscosity. Nonlinear Anal. Real World Appl. **40**, 95–129 (2018)
24. Niu, D., Shang, H.: Global well-posedness and optimal time decay rates of solutions to the three-dimensional magneto-micropolar fluid equations (**submitted for publication**)
25. Ortega-Torres, E., Rojas-Medar, M.: Magneto-micropolar fluid motion: global existence of strong solutions. Abstr. Appl. Anal. **4**, 109–125 (1999)
26. Priest, E., Fores, T.: Magnetic Reconnection. MHD Theory and Application. Cambridge University Press, Cambridge (2000)
27. Runst, T., Sickel, W.: Sobolev Spaces of Fractional Order. Nemytskij operators and Nonlinear Partial Differential Equations. Walter de Gruyter, Berlin (1996)
28. Rojas-Medar, M.: Magneto-micropolar fluid motion: existence and uniqueness of strong solution. Math. Nachr. **188**, 301–319 (1997)
29. Rojas-Medar, M., Boldrini, J.: Magneto-micropolar fluid motion: existence of weak solutions. Rev. Mat. Complut. **11**, 443–460 (1998)
30. Regmi, D., Wu, J.: Global regularity for the 2D magneto-micropolar equations with partial dissipation. J. Math. Study **49**, 169–194 (2016)
31. Stokes, G.: On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. Trans. Camb. Philos. Soc. **8**, 287–319 (1845)
32. Sohinger, V., Strain, R.: The Boltzmann equation, Besov spaces, and optimal time decay rates in $\mathbb{R}^n_x$. Adv. Math. **261**, 274–332 (2014)
33. Schonbek, M.: L^2 decay for weak solutions of the Navier-Stokes equations. Arch. Rational Mech. Anal. **88**, 209–222 (1985)
34. Schonbek, M.: Large time behaviour of solutions to the Navier–Stokes equations. Commun. Partial Differ. Equa. **11**, 733–763 (1986)
35. Schonbek, M., Wiegner, M.: On the decay of higher-order norms of the solutions of Navier–Stokes equations. Proc. R. Soc. Edinb. Sect. A **126**, 677–685 (1996)
36. Shang, H., Zhao, J.: Global regularity for 2D magneto-micropolar equations with only microrotational velocity dissipation and magnetic diffusion. Nonlinear Anal. **150**, 194–209 (2017)
37. Shang, H., Gu, C.: Global regularity and decay estimates for 2D magneto-micropolar equations with partial dissipation. Z. Angew. Math. Phys. **3**, 70–85 (2019)
38. Shang, H., Wu, J.: Global regularity for 2D fractional magneto-micropolar equation. Math. Z. **297**, 775–802 (2021)
39. Temam, R.: Navier–Stokes Equations. Theory and Numerical Analysis. North-Holland, Amsterdam (1977)
40. Triebel, H.: Theory of Function Spaces II. Birkhauser, Boston (1992)
41. Wang, S., Xu, w, Liu, J.: Initial-boundary value problem for 2D magneto-micropolar equations with zero angular viscosity. Z. Angew. Math. Phys. **72**, 72–103 (2021)
42. Yamazaki, K.: Global regularity of the two-dimensional magneto-micropolar fluid system with zero angular viscosity. Discret. Contin. Dyn. Syst. **35**, 2193–2207 (2015)
43. Yamazaki, K.: Global regularity of the two-dimensional magneto-micropolar fluid system with zero angular viscosity. Discret. Contin. Dyn. Syst. **35**(4), 2193–2207 (2015)