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RESEARCH ARTICLE

Existence and Decay Estimates of Solution for a Fourth Order Quasi-Geostrophic Equation

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Abstract

This paper considers a single-layer fourth order quasi-geostrophic equation in two-dimensional case. We prove the existence and uniqueness of global smooth solution to the Cauchy problem of this equation by using energy estimate. We also establish a new estimate for the nonlinear term and obtain decay estimates of the solution in L^2.

Keywords Quasi-geostrophic model · Decay estimate · Smooth solution

Mathematics Subject Classification 35Q86 · 35A01 · 35B40

1 Introduction

A hierarchy of ocean models occur in the literature of wind-driven circulation, starting from the most complex model and ending with a very elementary model (see e.g. [1, 7]). One of them is the so-called quasi-geostrophic β-plane model, which is considered as a simplification of the shallow-water equations when the Rossby number is small and the magnitude of bottom topography variations is comparable to the Rossby number. This paper studies the homogeneous quasi-geostrophic model by ignoring the effect of the bottom friction and the wind-stress effect. In this case, the model takes the form

$$\frac{\partial}{\partial t}[\Delta\psi - F\psi] + J(\psi, \Delta\psi) + \beta\frac{\partial\psi}{\partial x} = \frac{1}{R_e}\Delta^2\psi, \tag{1.1}$$

where $\psi = \psi(x, y, t)$ is the geostrophic pressure (or the geostrophic stream function), and the nonlinear term J is defined by

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$$J(f,g) := \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial x}.$$

The coefficients in equation (1.1) are: the rotational Froude number F, the Coriolis parameter β and the Reynolds number R_e.

Equation (1.1) is a single-layer quasi-geostrophic model. A direct extension of this model is a two-layer quasi-geostrophic model, where the densities are constant in each layer and the motion of fluid in both layers is coupled through the continuity of pressure and vertical velocity (see [1]). Due to wide applications in meteorology and oceanography, these models have been intensively studied in the past years. In [8], the authors proved global existence of weak solutions for the fractional quasi-geostrophic equation with $\Delta^2\psi$ replaced by $(-\Delta)^{1+\alpha}\psi$ ($\alpha \in (0,1]$) and they also obtained long-time behavior of the solution when $\alpha \in (0, \frac{1}{2}]$. The authors of [3] discussed the existence theory and decay estimates for two-layer quasi-geostrophic model with fractional dissipative term. Decay estimates were also studied in [2] for a type of two-layer quasi-geostrophic model with both viscosity and friction. Medjo in [6] investigated the existence of strong solutions and maximal attractor for the multi-layer quasi-geostrophic equations.

In this work, we study the existence and large time behavior of smooth solution for the initial-value problem equipped with the initial data

$$\psi(x,y,0) = \psi_0(x,y). \tag{1.2}$$

Throughout the paper, for $1 \le p < +\infty$, we denote by $L^p(\mathbb{R}^2)$ the Lebesgue space equipped the norm

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}^2} |u(x)|^p dx\right)^{\frac{1}{p}}.$$

For $s \in \mathbb{R}$, $H^s(\mathbb{R}^2)$ denotes the nonhomogeneous Sobolev space whose norm is defined by

$$H^s(\mathbb{R}^2) = \{u \in \mathcal{S}' \mid \|u\|^2_{H^s} = \int_{\mathbb{R}^2} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi < +\infty\},$$

where $\widehat{u}(\xi)$ is the Fourier transform of u.

Now we state the main results of the paper.

Theorem 1.1 *Assume that $\psi_0 \in H^m(\mathbb{R}^2)$ with $m \ge 4$ be an integer, then system (1.1)–(1.2) admits a unique global solution $\psi \in C(\mathbb{R}^+; H^m(\mathbb{R}^2))$.*

Theorem 1.2 *Let $\psi_0 \in H^m(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with $m \ge 4$ be an integer, and ψ is the solution obtained by Theorem 1.1. Then for any multi-index α we have the decay estimates*

$$\|D^{\alpha}\psi\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{4}}, \ \ |\alpha| = 0, \ 1, \ \cdots, \ m-1, \tag{1.3}$$

and

$$\|D^{\alpha}\psi\|_{L^2} \leq C(1+t)^{-\frac{m}{4}}, \ \ |\alpha| = m. \tag{1.4}$$

Theorem 1.1 is proved via a-priori energy estimates, and the proof is given in the next section. In Section 3, we present the proof of Theorem 1.2. We remark that the decay estimates of system (1.1)–(1.2) are not obtained in the previous works due to the effect of the nonlinear term $J(\psi, \Delta\psi)$. In this work, a new estimate is established for this nonlinear term and we apply this estimate to get the large time behavior for all the derivatives of the solution.

2 Global Existence of the Solution

In this section, we give the proof of Theorem 1.1. Indeed, the proof consists of two crucial steps. The first step is to obtain local existence of the solution to system (1.1)–(1.2), and the second step is to extend the local solution globally in time by establishing the *a-priori* estimates. For the first step, we can apply the regularized strategy of [5, Chapter 3] to study the approximated system

$$\frac{\partial}{\partial t}[\Delta\psi^{\epsilon} - F\psi^{\epsilon}] + \mathcal{J}_{\epsilon}J(\mathcal{J}_{\epsilon}\psi^{\epsilon}, \mathcal{J}_{\epsilon}\Delta\psi^{\epsilon}) + \beta\mathcal{J}_{\epsilon}^{2}\frac{\partial\psi}{\partial x} = \frac{1}{R_e}\mathcal{J}_{\epsilon}^{2}\Delta^{2}\psi, \tag{2.1}$$

$$\psi(x, y, 0) = \mathcal{J}_{\epsilon}\psi_0, \tag{2.2}$$

where $\mathcal{J}_{\epsilon}f$ denotes the mollification of function f defined by

$$\mathcal{J}_{\epsilon}f(x) = \frac{1}{\epsilon^2}\int_{\mathbb{R}^2}\rho\Big(\frac{x-y}{\epsilon}\Big)f(y)dy$$

with ρ be a positive and radial C_0^{∞} function whose mass is equal to one. By a limiting argument for the regularized system (2.1)–(2.2), it is not hard to obtain local existence and uniqueness of solution to system (1.1)–(1.2). Moreover, if $T^* < +\infty$ is the maximal existence time of the solution, then there holds

$$\lim_{t\to T^*}\|\psi(t)\|_{H^m} = +\infty.$$

Since the argument for the local existence part is standard, we omit further details. Hence, in order to complete the proof of Theorem 1.1, it is sufficient to establish the following three propositions which give the *a-priori* estimates.

Proposition 2.1 *Let ψ be a sufficiently smooth solution to system (1.1)–(1.2). Assume ψ and its derivatives decay at infinity, then there hold*

$$\iint_{\mathbb{R}^2} (|\nabla\psi|^2 + F|\psi|^2)dxdy + \frac{2}{R_e}\int_0^t \iint_{\mathbb{R}^2} |\Delta\psi|^2 dxdyds \le C_1, \tag{2.3}$$

$$\iint_{\mathbb{R}^2} (|\Delta\psi|^2 + F|\nabla\psi|^2)dxdy + \frac{2}{R_e}\int_0^t \iint_{\mathbb{R}^2} |\nabla\Delta\psi|^2 dxdyds \leqslant C_2, \tag{2.4}$$

where C_1, C_2 depend only on $\|\psi_0\|_{H^1},\ \|\nabla\psi_0\|_{H^1}$, respectively.

Proof We multiply equation (1.1) with 2ψ to get

$$-\frac{d}{dt}\iint_{\mathbb{R}^2} (|\nabla\psi|^2 + F|\psi|^2)dxdy + 2\iint_{\mathbb{R}^2} J(\psi,\Delta\psi)\psi dxdy = \frac{2}{R_e}\iint_{\mathbb{R}^2} |\Delta\psi|^2 dxdy. \tag{2.5}$$

Since

$$\iint_{\mathbb{R}^2} J(\psi,\Delta\psi)\psi dxdy = 0,$$

integrating (2.5) in time gives

$$\iint_{\mathbb{R}^2} (|\nabla\psi|^2 + F|\psi|^2)dxdy + \frac{2}{R_e}\int_0^t \iint_{\mathbb{R}^2} |\Delta\psi|^2 dxdyds \leqslant \iint_{\mathbb{R}^2} (|\nabla\psi_0|^2 + F|\psi_0|^2)dxdy.$$

Hence, the bound (2.3) follows.

To obtain (2.4), we multiply Eq. (1.1) with $2\Delta\psi$ to get

$$\frac{d}{dt}\iint_{\mathbb{R}^2} (|\Delta\psi|^2 + F|\nabla\psi|^2)dxdy + 2\iint_{\mathbb{R}^2} J(\psi,\Delta\psi)\Delta\psi dxdy + \frac{2}{R_e}\iint_{\mathbb{R}^2} |\nabla\Delta\psi|^2 dxdy = 0.$$

Note that

$$\iint_{\mathbb{R}^2} J(\psi,\Delta\psi)\Delta\psi dxdy = 0,$$

thus the bound (2.4) follows immediately. □

Proposition 2.2 *With the same assumptions as Proposition 2.1, we have*

$$\iint_{\mathbb{R}^2} (|\nabla\Delta\psi|^2 + F|\Delta\psi|^2)dxdy + \frac{1}{R_e}\int_0^t \iint_{\mathbb{R}^2} |\Delta^2\psi|^2 dxdyds \le C_3, \tag{2.6}$$

$$\iint\limits_{\mathbb{R}^2}(|\Delta^2\psi|^2+F|\nabla\Delta\psi|^2)dxdy+\frac{1}{R_e}\int_0^t\iint\limits_{\mathbb{R}^2}|\nabla\Delta^2\psi|^2dxdyds\leqslant C_4, \tag{2.7}$$

where C_3, C_4 depend on $\|\nabla\psi_0\|_{H^3}$ and t.

Proof From Eq. (1.1), we can get the following energy identity

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^2}(|\nabla\Delta\psi|^2+F|\Delta\psi|^2)dxdy+\frac{2}{R_e}\iint\limits_{\mathbb{R}^2}|\Delta^2\psi|^2dxdy=-2\iint\limits_{\mathbb{R}^2}\nabla J(\psi,\Delta\psi)\nabla\Delta\psi dxdy.$$

Note that

$$\iint\limits_{\mathbb{R}^2}J(f,g)gdxdy=0,$$

the nonlinear integral term of the above identity is estimated by

$$\begin{aligned}\Big|\iint\limits_{\mathbb{R}^2}\nabla J(\psi,\Delta\psi)\nabla\Delta\psi dxdy\Big|&=\Big|\iint\limits_{\mathbb{R}^2}J(\nabla\psi,\Delta\psi)\nabla\Delta\psi dxdy\Big|\\&\le 2\|\Delta\psi\|_{L^2}\|\nabla\Delta\psi\|_{L^4}^2\\&\le C\|\Delta\psi\|_{L^2}\left(\|\Delta\psi\|_{L^2}^{\frac{1}{2}}\|\Delta^2\psi\|_{L^2}^{\frac{3}{2}}\right)\\&\le\delta\|\Delta^2\psi\|_{L^2}^2+C,\end{aligned}$$

where we have used Young's inequality and the fact $\|\Delta\psi\|_{L^2}\le C$ in the last step. Hence, the bound (2.6) follows by choosing $\delta=\frac{1}{R_e}$.

Similarly, taking energy estimate at the level of fourth order derivative gives

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^2}(|\Delta^2\psi|^2+F|\nabla\Delta\psi|^2)dxdy+\frac{2}{R_e}\iint\limits_{\mathbb{R}^2}|\nabla\Delta^2\psi|^2dxdy=-2\iint\limits_{\mathbb{R}^2}\Delta J(\psi,\Delta\psi)\Delta^2\psi dxdy.$$

For the nonlinear term, we have

$$\begin{aligned}\Big|\iint\limits_{\mathbb{R}^2}\Delta J(\psi,\Delta\psi)\Delta^2\psi dxdy\Big|&=2\Big|\iint\limits_{\mathbb{R}^2}J(\nabla\psi,\nabla\Delta\psi)\Delta^2\psi dxdy\Big|\\&\le 2\|\Delta\psi\|_{L^2}\|\Delta^2\psi\|_{L^4}^2\\&\le C\|\Delta\psi\|_{L^2}(\|\nabla\Delta\psi\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta^2\psi\|_{L^2}^{\frac{3}{2}})\\&\le\delta\|\nabla\Delta^2\psi\|_{L^2}^2+C.\end{aligned}$$

Then choosing $\delta=\frac{1}{R_e}$ yields the desired bound (2.7). □

Proposition 2.3 *With the same assumptions as Proposition 2.2, there exists $C_m > 0$ depending on $\|\psi_0\|_{H^m}$ and t such that*

$$\|\psi(t)\|_{H^m}^2 + \int_0^t \|\nabla\psi(s)\|_{H^m}^2 ds \le C_m.$$

This proposition can be proved with an induction on m. We omit further details for simplicity. With these propositions, Theorem 1.1 thus follows.

3 Decay Estimates of the Solution

In this section we study the large time behaviour of solution to the Cauchy problem for the nonlinear quasi-geostrophic model (1.1)–(1.2). We first derive the integral identity of the solution. Applying Fourier transform to Eq. (1.1), we get

$$(-|\xi|^2\hat{\psi} - F\hat{\psi})_t + \hat{J}(\psi, \Delta\psi) + i\beta\xi_1\hat{\psi} = \frac{1}{R_e}|\xi|^4\hat{\psi}, \tag{3.1}$$

which implies that

$$\hat{\psi}(\xi_1, \xi_2, t) = e^{\frac{i\beta\xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F}t}\hat{\psi}_0 + \int_0^t e^{(t-s)\frac{i\beta\xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F}}\frac{\hat{J}(\psi, \Delta\psi)}{|\xi|^2 + F}ds. \tag{3.2}$$

In the succeeding arguments, we need to estimate the nonlinear term in (3.2) which is presented in Lemma 3.1 below.

Lemma 3.1 *For any $\psi \in H^3(\mathbb{R}^2)$, there holds that*

$$|\hat{J}(\psi, \Delta\psi)| \leqslant 2(\xi_1^2 + \xi_2^2)\|\psi\|_{L^2}\|\Delta\psi\|_{L^2}. \tag{3.3}$$

Proof Recall that

$$J(f, g) = f_x g_y - f_y g_x,$$

we use integration by parts to rewrite $\hat{J}(\psi, \Delta\psi)$ as

$$
\begin{aligned}
\hat{J}(\psi, \Delta\psi) &= \iint\limits_{\mathbb{R}^2} \left(\psi_x \Delta\psi_y - \psi_y \Delta\psi_x\right) e^{-i(x\xi_1 + y\xi_2)} dxdy \\
&= i\xi_2 \iint\limits_{\mathbb{R}^2} \Delta\psi \psi_x e^{-i(x\xi_1+y\xi_2)} dxdy - i\xi_1 \iint\limits_{\mathbb{R}^2} \Delta\psi \psi_y e^{-i(x\xi_1+y\xi_2)} dxdy \\
&= i\xi_2 \Bigg(\iint\limits_{\mathbb{R}^2} \psi_{xx}\psi_x e^{-i(x\xi_1+y\xi_2)} dxdy + i\xi_2 \iint\limits_{\mathbb{R}^2} \psi_x \psi_y e^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad - \iint\limits_{\mathbb{R}^2} \psi_{xy}\psi_y e^{-i(x\xi_1+y\xi_2)} dxdy \Bigg) - i\xi_1 \Bigg(i\xi_1 \iint\limits_{\mathbb{R}^2} \psi_x \psi_y e^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad - \iint\limits_{\mathbb{R}^2} \psi_x \psi_{xy} e^{-i(x\xi_1+y\xi_2)} dxdy + \iint\limits_{\mathbb{R}^2} \psi_{yy}\psi_y e^{-i(x\xi_1+y\xi_2)} dxdy \Bigg) \\
&= \left(\xi_1^2 - \xi_2^2\right) \iint\limits_{\mathbb{R}^2} \psi_x \psi_y \mathrm{e}^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad - \frac{\xi_1\xi_2}{2} \iint\limits_{\mathbb{R}^2} \left|\psi_x\right|^2 e^{-i(x\xi_1+y\xi_2)} dxdy + \frac{\xi_1\xi_2}{2} \iint\limits_{\mathbb{R}^2} \left|\psi_y\right|^2 e^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad - \frac{\xi_1\xi_2}{2} \int_{\mathbb{R}^2} \left|\psi_x\right|^2 e^{-i(x\xi_1+y\xi_2)} dxdy + \frac{\xi_1\xi_2}{2} \iint\limits_{\mathbb{R}^2} \left|\psi_y\right|^2 e^{-i(x\xi_1+y\xi_2)} dxdy \\
&= \left(\xi_1^2 - \xi_2^2\right) \iint\limits_{\mathbb{R}^2} \psi_x \psi_y e^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad - \xi_1\xi_2 \iint\limits_{\mathbb{R}^2} \left|\psi_x\right|^2 e^{-i(x\xi_1+y\xi_2)} dxdy + \xi_1\xi_2 \int_{\mathbb{R}^2} \left|\psi_y\right|^2 e^{-i(x\xi_1+y\xi_2)} dxdy \\
&= -\left(\xi_1^2 - \xi_2^2\right) \iint\limits_{\mathbb{R}^2} \psi\psi_{xy} e^{-i(x\xi_1+y\xi_2)} dxdy - \left(\xi_1^2 - \xi_2^2\right)(-i\xi_2) \iint\limits_{\mathbb{R}^2} \psi\psi_x e^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad + \xi_1\xi_2 \iint\limits_{\mathbb{R}^2} \psi\psi_{xx} e^{-i(x\xi_1+y\xi_2)} dxdy + (\xi_1\xi_2)(-i\xi_1) \iint\limits_{\mathbb{R}^2} \psi\psi_x e^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad - \xi_1\xi_2 \iint\limits_{\mathbb{R}^2} \psi\psi_{yy} e^{-i(x\xi_1+y\xi_2)} dxdy - (\xi_1\xi_2)(-i\xi_2) \iint\limits_{\mathbb{R}^2} \psi\psi_y e^{-i(x\xi_1+y\xi_2)} dxdy \\
&= -\left(\xi_1^2 - \xi_2^2\right) \iint\limits_{\mathbb{R}^2} \psi\psi_{xy} e^{-i(x\xi_1+y\xi_2)} dxdy - \frac{1}{2}\left(\xi_1^2 - \xi_2^2\right)(\xi_1\xi_2) \iint\limits_{\mathbb{R}^2} |\psi|^2 e^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad + \xi_1\xi_2 \iint\limits_{\mathbb{R}^2} \psi\psi_{xx} e^{-i(x\xi_1+y\xi_2)} dxdy + \frac{1}{2}\xi_1^3\xi_2 \iint\limits_{\mathbb{R}^2} |\psi|^2 e^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad - \xi_1\xi_2 \iint\limits_{\mathbb{R}^2} \psi\psi_{yy} e^{-i(x\xi_1+y\xi_2)} dxdy - \frac{1}{2}\xi_1\xi_2^3 \iint\limits_{\mathbb{R}^2} |\psi|^2 e^{-i(x\xi_1+y\xi_2)} dxdy \\
&= -\left(\xi_1^2 - \xi_2^2\right) \iint\limits_{\mathbb{R}^2} \psi\psi_{xy} e^{-i(x\xi_1+y\xi_2)} dxdy + \xi_1\xi_2 \iint\limits_{\mathbb{R}^2} \psi\psi_{xx} e^{-i(x\xi_1+y\xi_2)} dxdy \\
&\quad - \xi_1\xi_2 \iint\limits_{\mathbb{R}^2} \psi\psi_{yy} e^{-i(x\xi_1+y\xi_2)} dxdy.
\end{aligned}
$$

Thus there holds

$$|\hat{J}(\psi, \Delta\psi)| \leqslant 2(\xi_1^2 + \xi_2^2) \iint\limits_{\mathbb{R}^2} |\psi||\Delta\psi| dxdy \leqslant 2(\xi_1^2 + \xi_2^2)\|\psi\|_{L^2}\|\Delta\psi\|_{L^2}.$$

□

We remark that for the nonlinear estimate $\hat{J}(\psi, \Delta\psi)$, in the works [2, 8], the authors used the following bound (due to [4])

$$|\hat{J}(\psi, \Delta\psi)| \leqslant (\xi_1^2 + \xi_2^2)\|\nabla\psi\|_{L^2}^2.$$

However, we observe that it is not sufficient to prove Theorem 1.2 by using this bound. Therefore, the decay argument in [2, 8] can not cover our fourth-order quasi-geostrophic equation. Hence, the bound (3.3) is new and crucial in the following decay estimates. In particular, the step of establishing logarithmic decay bound is not needed in our proof by using this new bound (3.3). Now we can prove Theorem 1.2 in the framework of Fourier splitting method which is originally due to Schonbek [9, 10] and improved by Zhang [11].

Proof of Theorem 1.2 We first show

$$\|\nabla\psi\|_{L^2} + \|\psi\|_{L^2} \leqslant C(1+t)^{-\frac{1}{4}}. \tag{3.4}$$

From the basic energy estimate (2.5), namely,

$$\frac{d}{dt} \iint\limits_{\mathbb{R}^2} (|\nabla\psi|^2 + F|\psi|^2) dxdy + \frac{2}{R_e} \iint\limits_{\mathbb{R}^2} |\Delta\psi|^2 dxdy = 0, \tag{3.5}$$

we have

$$\|\nabla\psi\|_{L^2}^2 + F\|\psi\|_{L^2}^2 \leqslant C, \quad \int_0^t \iint\limits_{\mathbb{R}^2} |\Delta\psi|^2 dxdyds \leqslant C. \tag{3.6}$$

Applying Plancherel's theorem (that is, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ for any $f \in L^2$) to (3.5), we see

$$\frac{d}{dt} \iint\limits_{\mathbb{R}^2} [(\xi_1^2 + \xi_2^2)|\hat{\psi}|^2 + F|\hat{\psi}|^2] d\xi_1 d\xi_2 + \frac{2}{R_e} \iint\limits_{\mathbb{R}^2} (\xi_1^2 + \xi_2^2)^2 |\hat{\psi}|^2 d\xi_1 d\xi_2 = 0. \tag{3.7}$$

Define

$$S_1(t) = \left\{ (\xi_1, \xi_2); (\xi_1, \xi_2) \in \mathbb{R}^2, \frac{\xi_1^2 + \xi_2^2}{\sqrt{F + \xi_1^2 + \xi_2^2}} \leqslant \frac{\sqrt{3R_e}}{\sqrt{4(1+t)}} \right\},$$

then

$$\begin{aligned}\iint\limits_{\mathbb{R}^2} |\xi|^4|\hat{\psi}|^2 d\xi_1 d\xi_2 &= \iint\limits_{\mathbb{R}^2\setminus S_1(t)} |\xi|^4|\hat{\psi}|^2 d\xi_1 d\xi_2 + \iint\limits_{S_1(t)} |\xi|^4|\hat{\psi}|^2 d\xi_1 d\xi_2 \\ &\geqslant \iint\limits_{\mathbb{R}^2\setminus S_1(t)} \frac{3R_e(F+|\xi|^2)}{4(1+t)}|\hat{\psi}|^2 d\xi_1 d\xi_2 + \iint\limits_{S_1(t)} |\xi|^4|\hat{\psi}|^2 d\xi_1 d\xi_2 \\ &\geqslant \iint\limits_{\mathbb{R}^2} \frac{3R_e(F+|\xi|^2)}{4(1+t)}|\hat{\psi}|^2 d\xi_1 d\xi_2 - \iint\limits_{S_1(t)} \frac{3R_e(F+|\xi|^2)}{4(1+t)}|\hat{\psi}|^2 d\xi_1 d\xi_2.\end{aligned} \tag{3.8}$$

Inserting (3.8) into (3.7), we obtain

$$\begin{aligned}&\frac{d}{dt}\iint\limits_{\mathbb{R}^2}(|\nabla\psi|^2+F|\psi|^2)dxdy + \frac{3}{2}\iint\limits_{\mathbb{R}^2}\frac{F+|\xi|^2}{1+t}|\hat{\psi}|^2 d\xi_1 d\xi_2 \\ &\leqslant C\iint\limits_{S_1(t)}\frac{F+|\xi|^2}{1+t}|\hat{\psi}|^2 d\xi_1 d\xi_2.\end{aligned} \tag{3.9}$$

From (3.2), Lemma 3.1 and (3.6), we can get

$$\begin{aligned}|\hat{\psi}| &\leqslant C\|\psi_0\|_{L^1} + C\int_0^t e^{\frac{-|\xi|^4(t-s)}{R_e(|\xi|^2+F)}}\frac{|\xi|^2\|\psi\|_{L^2}\|\Delta\psi\|_{L^2}}{|\xi|^2+F}ds \\ &\leqslant C\|\psi_0\|_{L^1} + C|\xi|^2\int_0^t \|\psi\|_{L^2}\|\Delta\psi\|_{L^2}ds \\ &\leqslant C + C|\xi|^2\left(\int_0^t\|\psi\|_{L^2}^2 ds\right)^{\frac{1}{2}}\left(\int_0^t\|\Delta\psi\|_{L^2}^2 ds\right)^{\frac{1}{2}} \\ &\leqslant C + C|\xi|^2 t^{\frac{1}{2}}.\end{aligned} \tag{3.10}$$

Using this estimate, we have

$$\begin{aligned}\iint\limits_{S_1(t)}\frac{F+|\xi|^2}{(1+t)}|\hat{\psi}|^2 d\xi_1 d\xi_2 &\leqslant \frac{C}{(1+t)}\int_0^{2\pi}\int_0^{\left(\frac{C}{1+t}\right)^{\frac{1}{4}}}(F+r^2)(1+r^4 t)rdrd\theta \\ &\leqslant \frac{C}{(1+t)}\cdot\left(\frac{1}{\sqrt{1+t}}+\frac{t}{(1+t)^{\frac{3}{2}}}\right) \\ &\leqslant C(1+t)^{-\frac{3}{2}}.\end{aligned} \tag{3.11}$$

Now it follows from (3.10) and (3.11) that

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^2}(|\nabla\psi|^2+F|\psi|^2)dxdy + \frac{3}{2}\iint\limits_{\mathbb{R}^2}\frac{F+|\xi|^2}{1+t}|\hat{\psi}|^2 d\xi_1 d\xi_2 \leqslant \frac{C}{(1+t)^{\frac{3}{2}}},$$

then we have

$$\frac{d}{dt}\left[(1+t)^{\frac{3}{2}}\iint_{\mathbb{R}^2}(|\nabla\psi|^2+F|\psi|^2)dxdy\right]\leqslant C.$$

Integrating the above inequality over interval $[0, t]$, we get

$$\iint_{\mathbb{R}^2}(|\nabla\psi|^2+F|\psi|^2)dxdy\leqslant C(1+t)^{-\frac{1}{2}}.$$

Thus, we get the decay bound (3.4).

Next, we want to prove

$$\|\Delta\psi\|_{L^2}+\|\nabla\psi\|_{L^2}\leqslant C(1+t)^{-\frac{1}{2}}. \tag{3.12}$$

From the proof of Proposition 2.1, we get the energy estimate

$$\frac{d}{dt}\iint_{\mathbb{R}^2}(|\Delta\psi|^2+F|\nabla\psi|^2)dxdy+\frac{2}{R_e}\iint_{\mathbb{R}^2}|\nabla\Delta\psi|^2dxdy=0,$$

which can be rewritten in the Fourier space as

$$\frac{d}{dt}\iint_{\mathbb{R}^2}[|\xi|^4|\hat{\psi}|^2+F|\xi|^2|\hat{\psi}|^2]d\xi_1d\xi_2+\frac{2}{R_e}\iint_{\mathbb{R}^2}|\xi|^6|\hat{\psi}|^2d\xi_1d\xi_2=0. \tag{3.13}$$

Define

$$S_2(t)=\left\{(\xi_1,\xi_2);(\xi_1,\xi_2)\in\mathbb{R}^2,\frac{\xi_1^2+\xi_2^2}{\sqrt{F+\xi_1^2+\xi_2^2}}\leqslant\frac{\sqrt{R_e}}{\sqrt{1+t}}\right\}.$$

We now treat the dissipative term as

$$\iint_{\mathbb{R}^2}|\xi|^6|\hat{\psi}|^2d\xi_1d\xi_2\geqslant\iint_{\mathbb{R}^2}\frac{R_e(F|\xi|^2+|\xi|^4)}{1+t}|\hat{\psi}|^2d\xi_1d\xi_2-\iint_{S_2(t)}\frac{R_e(F|\xi|^2+|\xi|^4)}{1+t}|\hat{\psi}|^2d\xi_1d\xi_2,$$

and we use (3.10) to get

$$\begin{aligned}\iint_{S_2(t)}\frac{F|\xi|^2+|\xi|^4}{1+t}|\hat{\psi}|^2d\xi_1d\xi_2&\leqslant\frac{C}{1+t}\int_0^{2\pi}\int_0^{\left(\frac{C}{1+t}\right)^{\frac{1}{4}}}(Fr^2+r^4)(1+r^4t)rdrd\theta\\&\leqslant C(1+t)^{-2}.\end{aligned}$$

Inserting the above two estimates into (3.13), we can get

$$\frac{d}{dt}[(1+t)^2 \iint\limits_{\mathbb{R}^2} (|\Delta\psi|^2 + F|\nabla\psi|^2)dxdy] \leqslant C,$$

and integrating this inequality gives (3.12).

Then we will prove

$$\|\nabla\Delta\psi\|_{L^2} + F\|\Delta\psi\|_{L^2} \leqslant C(1+t)^{-\frac{3}{4}}. \tag{3.14}$$

From the proof of Proposition 2.2, we actually obtain the estimate

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^2} (|\nabla\Delta\psi|^2 + F|\Delta\psi|^2)dxdy + \frac{2}{R_e}\iint\limits_{\mathbb{R}^2} |\Delta^2\psi|^2 dxdy \le \frac{1}{R_e}\|\Delta^2\psi\|_{L^2}^2 + C\|\Delta\psi\|_{L^2}^6,$$

and by (3.12), we have

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^2} (|\nabla\Delta\psi|^2 + F|\Delta\psi|^2)dxdy + \frac{1}{R_e}\iint\limits_{\mathbb{R}^2} |\xi|^8|\hat{\psi}|^2 dxdy \le \frac{C}{(1+t)^3}.$$

Denote

$$S_3(t) = \left\{(\xi_1,\xi_2);(\xi_1,\xi_2) \in \mathbb{R}^2, \frac{\xi_1^2+\xi_2^2}{\sqrt{F+\xi_1^2+\xi_2^2}} \leqslant \frac{\sqrt{5R_e}}{\sqrt{4(1+t)}}\right\},$$

then there holds

$$\iint\limits_{\mathbb{R}^2} |\xi|^8|\hat{\psi}|^2 d\xi_1 d\xi_2 \geqslant \iint\limits_{\mathbb{R}^2} \frac{5R_e(F|\xi|^4+|\xi|^6)}{4(1+t)}|\hat{\psi}|^2 d\xi_1 d\xi_2 - \iint\limits_{S_3(t)} \frac{5R_e(F|\xi|^4+|\xi|^6)}{4(1+t)}|\hat{\psi}|^2 d\xi_1 d\xi_2,$$

where

$$\begin{aligned}\frac{5R_e}{4}\iint\limits_{S_3(t)} \frac{F|\xi|^4+|\xi|^6}{1+t}|\hat{\psi}|^2 d\xi_1 d\xi_2 &\leqslant \frac{C}{1+t}\int_0^{2\pi}\int_0^{\left(\frac{C}{1+t}\right)^{\frac{1}{4}}} (Fr^4+r^6)(1+r^4t)rdrd\theta \\ &\leqslant C(1+t)^{-\frac{5}{2}}.\end{aligned}$$

Combining these estimates yields that

$$\frac{d}{dt}[(1+t)^{\frac{5}{2}} \iint\limits_{\mathbb{R}^2} (|\nabla\Delta\psi|^2 + F|\Delta\psi|^2)dxdy] \leqslant C,$$

so we obtain (3.14) as desired.

Finally, applying the same treatment as above, we can get

$$\|\Delta^k\psi\|_{L^2} + F\|\nabla\Delta^{k-1}\psi\|_{L^2} \leqslant C(1+t)^{-\frac{m}{4}},\ \ m=2k,\ \ k=1,\ 2,\ \ldots, \tag{3.15}$$

or

$$\|\nabla\Delta^k\psi\|_{L^2}+F\|\Delta^k\psi\|_{L^2}\leqslant C(1+t)^{-\frac{m}{4}},\ \ m=2k+1,\quad k=0,\ 1,\ 2,\ \dots. \tag{3.16}$$

As the idea of the proof is similar to (3.14), so it is omitted here. By the bounds (3.4), (3.12), (3.14), (3.15) and (3.16), we thus obtain the decay bounds (1.3) and (1.4) in Theorem 1.2. □

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Declarations

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