RESEARCH ARTICLE

Existence of the Periodic Peaked Solitary-Wave Solutions to the Camassa–Holm–Kadomtsev–Petviashvili Equation

Byungsoo Moon[1]

Abstract

Considered in this paper is the Camassa–Holm–Kadomtsev–Petviashvili (CH–KP) equation [22], which can be obtained as a model for the propagation of shallow water waves over a flat bed. It is shown that the existence of periodic peaked solitary-wave solutions to this model equation. In addition, we show that there are a multitude of solitary waves such as smooth, peakons, cuspons, stumpons, and composite like as CH equation.

Keywords Kadomtsev–Petviashvili equation · Camassa–Holm equation · Solitary waves

Mathematics Subject Classification Primary 35Q53 · 35G25 · 76B15 · 76B25

Abbreviations

CH–KP Camassa–Holm–Kadomtsev–Petviashvili
CH Camassa–Holm
KP Kadomtsev–Petviashvili
KdV Korteweg-de Vries

1 Introduction

We consider the Camassa–Holm–Kadomtsev–Petviashvili (CH-KP) equation [22]

$$(u_t - u_{xxt} + ku_x + 3uu_x - (2u_xu_{xx} + uu_{xxx}))_x + u_{yy} = 0, \tag{1.1}$$

in which the unknown u depends upon two space variables $(x, y) \in \mathbb{R}^2$ and time $t \in \mathbb{R}$ and the parameter k is a real. This equation aries as a two-dimensional

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Camassa–Holm (CH) equation [4, 17, 21] for incompressible and irrotational three-dimensional shallow water under the CH regime [22]. We can also regard equation (1.1) as a two-dimensional version of the CH equation just as the Kadomtsev–Petviashvili (KP) equation [26] is a two-dimensional version of the well-known Korteweg-de Vries (KdV) equation [27]. It is noticed that the CH–KP equation (1.1) admits a bi-Hamiltonian structure [22] and it can be written as

$$m_t = -J_1 \frac{\delta F}{\delta m} = -J_2 \frac{\delta E}{\delta m}$$

with skew-symmetric differential operators

$$J_1 = \partial_x(1 - \partial_x^2), \quad J_2 = \partial_x\left(\left(m + \frac{k}{2}\right)\cdot\right) + \left(m + \frac{k}{2}\right)\partial_x + \partial_x^{-1}\partial_y^2$$

and the corresponding Hamiltonians

$$E = \frac{1}{2}\int_{\mathbb{S}^2}(u^2 + u_x^2)dxdy \quad \text{and} \quad F = \frac{1}{2}\int_{\mathbb{S}^2}(ku^2 + u^3 + uu_x^2 + (\partial_x^{-1}\partial_y u)^2)dxdy.$$

It is observed that the structure of CH-KP equation (1.1) is similar to that of the following KP equation [26]

$$(u_t + u_{xxx} + u_x + uu_x)_x + u_{yy} = 0, \tag{1.2}$$

which is derived as a model for propagation of the weakly transverse water waves in a long wave regime. Well-posedness, stability issue, and existence or nonexistence of solitary waves of the KP equation (1.2) were studied extensively, and many interesting results may be found in [2, 3, 23, 26, 32, 37].

If there is no y-dependence in the equation (1.1), then it becomes the Camassa–Holm (CH) equation [4, 17, 21]

$$u_t - u_{xxt} + ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.3}$$

which was proposed as a model for surface waves. CH equation has brought up much attention in many years because of its many remarkable properties, such as admitting infinitely many conservation laws and being a bi-hamiltonian system [4, 21, 36], existence of action angle variables constructed by inverse scattering [1, 9, 18, 19, 34], global existence of solutions [6, 12, 13], wave breaking [6, 12–15, 30] (i.e. the solution remains bounded, but its slope becomes unbounded in finite time) and so on. Especially, the CH equation (1.3) with $k = 0$ has non-smooth solitary waves(peakons) of the form

$$u(t,x) = ce^{-|x-ct|}, \quad x \in \mathbb{R}, \quad t \geq 0, \quad c > 0,$$

and in the periodic case

$$u(t,x) = \frac{c}{\cosh(1/2)}\cosh(1/2 - (x - ct) + [x - ct]), \quad x \in \mathbb{R}, \quad t \geq 0, \quad c > 0,$$

where the notation $[x]$ denotes the largest integer part of the real number $x \in \mathbb{R}$, which interact like soliton for integrable systems and they are stable [10, 29]. It is also pointed out that those peakons were proved to be asymptotically stable under the Camassa-Holm flow [33] (see also [24, 25] for other equations). It is also worth mentioning the recent papers [31, 35] where the H^1-stability of peaked waves in the CH equation is analyzed.

The existence of (periodic) peakons is of interest for the nonlinear integrable equations since they are relatively new solitary waves (for most models the solitary waves are quite smooth). More importantly, in the theory of water waves a number of papers have investigated the Stokes waves of greatest height, traveling waves which are smooth everywhere except at the crest where the lateral tangents differ. There is no closed form available for these waves, and the peakons capture the essential features of the extreme waves-waves of great amplitude that are exact solutions of the governing equations for irrotational water waves, see the discussion in [7, 11, 38].

The aim of the present paper is to prove the existence of periodic peaked solitary-wave solutions to the CH-KP equation (1.1) for certain cases. It should be pointed out that one of the most relevant motivations for the study of peaked waves (solitary or periodic) is the fact that the governing equations for irrotational water waves do admit peaked traveling waves (periodic, as well as solitary), namely the celebrated Stokes waves of greatest height-see the discussion in [7, 8, 16, 38]. Recently, it was found [22] that, the CH–KP equation (1.1) admits a single peaked solitary waves of the form

$$u(t, x, y) = ce^{-|x+\beta y - ct|}$$

if and only if $k + \beta^2 = 0$. As mentioned in [22], it is reasonable that CH-KP equation (1.1) also possess periodic peaked solitary waves with a choice of different parameter k and wave speed c, since CH-KP equation (1.1) could be reduced to model related to the CH equation (1.3) using the translation scaling $x \to x + \gamma y$. Our main result is in Section 2. We prove that the periodic solitary wave $\frac{c}{\cosh(1/2)} \cosh(1/2 - (x - ct) - [x - ct])), c \in \mathbb{R}$ is only periodic peaked solitary wave solution in the form of $\alpha(t, y) \cosh(1/2 - (x - ct) - [x - ct]))$ and conclusions are included in Section 3.

Notation. Throughout the paper, the norm of a Banach space Z is denoted by $\| \cdot \|_Z$, while $C([0, T); Z)$ denotes the class of continuous functions from the interval $[0, T)$ to Z. We denote $\mathbb{S}^2 = \mathbb{S} \times \mathbb{S}$, where $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ as the unit circle and regard functions on $\mathbb{S}$ as periodic on the entire line with period one. Given $T > 0$, let $C_c^\infty([0, T) \times X)$ denote the space of all smooth functions with compact support on $[0, T) \times X$, which can also be viewed as the space of smooth functions on $\mathbb{R} \times X$ having compact support contained in $[0, T) \times X$. For $1 \le p < \infty$, $L^p(X)$ denotes the space of equivalence classes of Lebesgue measurable, p^{th}-power integrable, real-valued functions defined on X. The usual modification is in effect for $p = \infty$. The norm on $L^p(X)$ is written as $\| \cdot \|_{L^p(X)}$. For $s \ge 0$, the L^2-based Sobolev space H^s is the subspace of those L^2 functions whose derivatives up to order s all lie in L^2. The associated norm is denoted as $\| \cdot \|_{H^s(X)}$. We introduce X^s for $s > 0$ as the Hilbert space: $X^s = X^s(\mathbb{S}^2) := \{u \in H^s(\mathbb{S}^2) | \partial_x^{-1} u \in H^s(\mathbb{S}^2), \partial_x u \in H^s(\mathbb{S}^2)\}$ equipped with the

norm $\|u\|_{H^s(\mathbb{S}^2)} := \left(\|u\|^2_{H^s(\mathbb{S}^2)} + \|\partial_x^{-1}u\|^2_{H^s(\mathbb{S}^2)} + \|\partial_x u\|^2_{H^s(\mathbb{S}^2)} \right)^{1/2}$ for any $u \in X^s(\mathbb{S}^2)$,

where $\partial_x^{-1}u(x,y) := \mathcal{F}^{-1}((i\xi)^{-1}\mathcal{F}(u)(\xi,\eta))$, operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse of the Fourier transform in terms of variables x, y, respectively.

2 Main Result

We study the existence of periodic peaked solitary-wave solution to the CH-KP equation (1.1) in the form

$$u(t,x,y) = \alpha \cosh\left(\frac{1}{2} - (x + \beta y - ct) + [x + \beta y - ct]\right), \quad c \in \mathbb{R}. \qquad (2.1)$$

Indeed, solution (2.1) is a special type of a weak solution in the following sense.

Definition 2.1 Given initial data $u_0 \in H^1(\mathbb{S}^2)$, the function $u \in C([0,T); H^1_{loc}(\mathbb{S}^2))$ is called to be a weak solution to equation (1.1), if it holds the following identity:

$$\int_0^T \int_{\mathbb{S}^2} \left[-\partial_t \partial_x \psi u + \partial_x \psi \left(u \partial_x u + \partial_x G * (u^2 + \frac{1}{2}(\partial_x u)^2 + ku) \right) - \partial_y^2 \psi G * u \right] dxdydt$$

$$+ \int_{\mathbb{S}^2} u_0(x,y) \partial_x \psi(0,x,y) dxdy = 0,$$

$$(2.2)$$

for any smooth test function $\psi(t,x,y) \in C_c^\infty([0,T) \times \mathbb{S}^2)$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution, where the notation $*$ denotes the convolution with respect to the spatial variable x and $G(x) = \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(\frac{1}{2})}$ is the fundamental solution of the operator $(1 - \partial_x^2)^{-1}$ on $\mathbb{S}$, which implies $(1 - \partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbb{S})$.

Our main result on existence of periodic peakons can be obtained by verifying the Definition 2.1.

Theorem 2.1 *The CH-KP equation* (1.1) *has a global weak solution in the peak form of*

$$u(t,x,y) = \frac{c}{\cosh(1/2)} \cosh\left(\frac{1}{2} - (x + \beta y - ct) + [x + \beta y - ct]\right),$$

for some $\beta \in \mathbb{R}$ if and only if there holds that $k + \beta^2 = 0$.

Remark 2.1 Theorem 2.1 does not imply on the nonexistence of other peaked solitary wave solutions.

Proof We identify $\mathbb{S}^2$ with $[0,1) \times [0,1)$ and suppose that

$$u_c(t,x,y) = \frac{c}{\cosh(1/2)} \cosh\left(\frac{1}{2} - (x + \beta y - ct) + [x + \beta y - ct]\right). \qquad (2.3)$$

Then we obtain

$$\partial_x u_c(t,x,y) = -\frac{c}{\cosh(1/2)} \sinh\xi, \quad \partial_t u_c(t,x,y) = \frac{c^2}{\cosh(1/2)} \sinh\xi, \qquad (2.4)$$

where $\xi = \frac{1}{2} - (x + \beta y - ct) + [x + \beta y - ct]$. Using (2.4) and integration by parts, for any $\psi(t,x,y) \in C_c^\infty([0,T] \times \mathbb{S}^2)$, we have

$$\int_0^\infty \int_{\mathbb{S}^2} \left[-\partial_t \partial_x \psi u_c + \partial_x \psi \left(u_c \partial_x u_c + \partial_x G * (u_c^2 + \frac{1}{2}(\partial_x u_c)^2 + ku_c) \right) - \partial_y^2 \psi G * u_c \right] dxdydt$$

$$+ \int_{\mathbb{S}^2} u_{c,0}(x,y) \partial_x \psi(0,x,y) dxdy$$

$$= \int_0^\infty \int_{\mathbb{S}^2} \partial_x \psi \left[\partial_t u_c + u_c \partial_x u_c + \partial_x G * (u_c^2 + \frac{1}{2}(\partial_x u_c)^2) \right] dxdydt$$

$$- \int_0^\infty \int_{\mathbb{S}^2} \partial_x \psi \left[(k + \beta^2) \partial_x G * (u_c) \right] dxdydt$$

$$= \frac{c^2}{\cosh^2(1/2)} \int_0^\infty \int_{\mathbb{S}^2} \partial_x \psi \left[\cosh(1/2)\sinh\xi - \sinh\xi\cosh\xi + \partial_x G * (1 + \frac{3}{2}\sinh^2\xi) \right] dxdydt$$

$$- \frac{c}{\cosh(1/2)} \int_0^\infty \int_{\mathbb{S}^2} \partial_x \psi \left[(k + \beta^2) \partial_x G * (\cosh\xi) \right] dxdydt, \qquad (2.5)$$

where we use the fact $G * \partial_y u_c = \beta \partial_x G * u_c$. Noticing from the definition of $G(x)$ for the periodic case that

$$\partial_x G(x) = -\frac{\sinh(1/2 - x + [x])}{2\sinh(1/2)}, \quad x \in \mathbb{S},$$

we obtain

$$\partial_x G(x) * \left(1 + \frac{3}{2}\sinh^2\xi\right)(t,x,y)$$

$$= -\frac{1}{2\sinh(1/2)} \int_{\mathbb{S}} \sinh\left(1/2 - (x + \beta y - z) + [x + \beta y - z]\right) \qquad (2.6)$$

$$\times \left(1 + \frac{3}{2}\sinh^2\left(1/2 - (z - ct) + [z - ct]\right)\right) dz.$$

When $x + \beta y > ct$, we split the right-hand side of (2.6) into the following three parts:

$$\partial_x G(x) * \left(1 + \frac{3}{2}\sinh^2\xi\right)(t,x,y)$$

$$= -\frac{1}{2\sinh(1/2)} \left(\int_0^{ct} + \int_{ct}^{x+\beta y} + \int_{x+\beta y}^1 \right) \sinh\left(1/2 - (x + \beta y - z) + [x + \beta y - z]\right)$$

$$\times \left(1 + \frac{3}{2}\sinh^2\left(1/2 - (z - ct) + [z - ct]\right)\right) dz$$

$$=: I_1 + I_2 + I_3. \qquad (2.7)$$

Using the identity $\sinh^2 A = \frac{1}{2}\cosh 2A - \frac{1}{2}$, a direct computation gives rise to

$$
\begin{aligned}
I_1 &= -\frac{1}{2\sinh(1/2)} \int_0^{ct} \sinh\left(1/2 - (x + \beta y - z)\right)\left(1 + \frac{3}{2}\sinh^2\left(1/2 + (z - ct)\right)\right)dz \\
&= -\frac{1}{2\sinh(1/2)} \int_0^{ct} \sinh\left(1/2 - (x + \beta y - z)\right)\left(\frac{1}{4} + \frac{3}{4}\cosh\left(1 + 2(z - ct)\right)\right)dz \\
&= -\frac{1}{2\sinh(1/2)}\Big[\frac{1}{4}\cosh\left(1/2 - (x + \beta y - ct)\right) - \frac{1}{4}\cosh\left(1/2 - (x + \beta y)\right) \\
&\quad + \frac{1}{8}\cosh\left(3/2 - (x + \beta y - ct)\right) - \frac{1}{8}\cosh\left(3/2 - (x + \beta y + 2ct)\right) \\
&\quad - \frac{3}{8}\cosh\left(1/2 + (x + \beta y - ct)\right) + \frac{3}{8}\cosh\left(1/2 + (x + \beta y - 2ct)\right)\Big].
\end{aligned}
\tag{2.8}
$$

In a similar manner,

$$
\begin{aligned}
I_2 &= -\frac{1}{2\sinh(1/2)} \int_{ct}^{x+\beta y} \sinh\left(1/2 - (x + \beta y - z)\right)\left(1 + \frac{3}{2}\sinh^2\left(1/2 - (z - ct)\right)\right)dz \\
&= -\frac{1}{2\sinh(1/2)} \int_{ct}^{x+\beta y} \sinh\left(1/2 - (x + \beta y - z)\right)\left(\frac{1}{4} + \frac{3}{4}\cosh\left(1 - 2(z - ct)\right)\right)dz \\
&= -\frac{1}{2\sinh(1/2)}\Big[\frac{1}{4}\cosh\left(1/2\right) - \frac{1}{4}\cosh\left(1/2 - (x + \beta y - ct)\right) \\
&\quad - \frac{3}{8}\cosh\left(3/2 - 2(x + \beta y - ct)\right) + \frac{3}{8}\cosh\left(3/2 - (x + \beta y - ct)\right) \\
&\quad + \frac{1}{8}\cosh\left(1/2 - 2(x + \beta y - ct)\right) - \frac{1}{8}\cosh\left(1/2 + (x + \beta y - ct)\right)\Big],
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
I_3 &= -\frac{1}{2\sinh(1/2)} \int_{x+\beta y}^{1} \sinh\left(-1/2 - (x + \beta y - z)\right)\left(1 + \frac{3}{2}\sinh^2\left(1/2 - (z - ct)\right)\right)dz \\
&= -\frac{1}{2\sinh(1/2)} \int_{x+\beta y}^{1} \sinh\left(-1/2 - (x + \beta y - z)\right)\left(\frac{1}{4} + \frac{3}{4}\cosh\left(1 - 2(z - ct)\right)\right)dz \\
&= -\frac{1}{2\sinh(1/2)}\Big[\frac{1}{4}\cosh\left(1/2 - (x + \beta y)\right) - \frac{1}{4}\cosh\left(-1/2\right) \\
&\quad - \frac{3}{8}\cosh\left(-1/2 - (x + \beta y - 2ct)\right) + \frac{3}{8}\cosh\left(1/2 - 2(x + \beta y - ct)\right) \\
&\quad + \frac{1}{8}\cosh\left(-3/2 + (x + \beta y + 2ct)\right) - \frac{1}{8}\cosh\left(3/2 - 2(x + \beta y - ct)\right)\Big].
\end{aligned}
\tag{2.10}
$$

Plugging (2.8)–(2.10) into (2.7), we deduce that for $x + \beta y > ct$,

$$\partial_x G(x) * \left(1 + \frac{3}{2}\sinh^2\xi\right)(t,x,y)$$
$$= -\frac{1}{2\sinh(1/2)}\left[\frac{1}{2}\cosh\left(3/2 - (x+\beta y - ct)\right) - \frac{1}{2}\cosh\left(1/2 + (x+\beta y - ct)\right)\right.$$
$$\left. -\frac{1}{2}\cosh\left(3/2 - 2(x+\beta y - ct)\right) + \frac{1}{2}\cosh\left(1/2 - 2(x+\beta y - ct)\right)\right]$$
$$= -\cosh(1/2)\sinh(1/2 - (x+\beta y - ct))$$
$$+ \sinh(1/2 - (x+\beta y - ct))\cosh(1/2 - (x+\beta y - ct)),$$

(2.11)

where we use the identities $\cosh(A+B) = \cosh(A)\cosh(B) + \sinh(A)\sinh(B)$ and $\sinh(2A) = 2\sinh(A)\cosh(A)$.

While for the case $x + \beta y \le ct$, we split the right hand side of (2.6) into the following three parts:

$$\partial_x G(x) * \left(1 + \frac{3}{2}\sinh^2\xi\right)(t,x,y)$$
$$= -\frac{1}{2\sinh(1/2)}\left(\int_0^{x+\beta y} + \int_{x+\beta y}^{ct} + \int_{ct}^1\right)\sinh\left(1/2 - (x+\beta y - z) + [x+\beta y - z]\right)$$
$$\times \left(1 + \frac{3}{2}\sinh^2\left(1/2 - (z - ct) + [z - ct]\right)\right)dz$$
$$=: II_1 + II_2 + II_3.$$

(2.12)

For II_1 a direct calculation gives rise to

$$II_1 = -\frac{1}{2\sinh(1/2)}\int_0^{x+\beta y}\sinh\left(1/2 - (x+\beta y - z)\right)\left(1 + \frac{3}{2}\sinh^2\left(1/2 + (z - ct)\right)\right)dz$$
$$= -\frac{1}{2\sinh(1/2)}\left[\frac{1}{4}\cosh\left(1/2\right) - \frac{1}{4}\cosh\left(1/2 - (x+\beta y)\right)\right.$$
$$+ \frac{1}{8}\cosh\left(3/2 + 2(x+\beta y - ct)\right) - \frac{1}{8}\cosh\left(3/2 - (x+\beta y + 2ct)\right)$$
$$\left. -\frac{3}{8}\cosh\left(-1/2 - 2(x+\beta y - ct)\right) + \frac{3}{8}\cosh\left(-1/2 - (x+\beta y - 2ct)\right)\right].$$

(2.13)

Similarly, one obtains

$$II_2 = -\frac{1}{2\sinh(1/2)}\int_{x+\beta y}^{ct}\sinh\left(-1/2 - (x+\beta y - z)\right)\left(1 + \frac{3}{2}\sinh^2\left(1/2 + (z - ct)\right)\right)dz$$
$$= -\frac{1}{2\sinh(1/2)}\left[\frac{1}{4}\cosh\left(-1/2 - (x+\beta y - ct)\right) - \frac{1}{4}\cosh\left(-1/2\right)\right.$$
$$+ \frac{1}{8}\cosh\left(1/2 - (x+\beta y - ct)\right) - \frac{1}{8}\cosh\left(1/2 + 2(x+\beta y - ct)\right)$$
$$\left. -\frac{3}{8}\cosh\left(-3/2 - (x+\beta y - ct)\right) + \frac{3}{8}\cosh\left(-3/2 - 2(x+\beta y - ct)\right)\right],$$

(2.14)

and

$$II_3 = -\frac{1}{2\sinh(1/2)} \int_{ct}^{1} \sinh\left(-1/2 - (x + \beta y - z)\right)\left(1 + \frac{3}{2}\sinh^2\left(1/2 - (z - ct)\right)\right)dz$$

$$= -\frac{1}{2\sinh(1/2)}\Big[\frac{1}{4}\cosh\left(1/2 - (x + \beta y)\right) - \frac{1}{4}\cosh\left(-1/2 - (x + \beta y - ct)\right)$$

$$-\frac{3}{8}\cosh\left(-1/2 - (x + \beta y - 2ct)\right) + \frac{3}{8}\cosh\left(1/2 - (x + \beta y - ct)\right)$$

$$+\frac{1}{8}\cosh\left(3/2 - (x + \beta y + 2ct)\right) - \frac{1}{8}\cosh\left(-3/2 - (x + \beta y - ct)\right)\Big].$$

$$(2.15)$$

Plugging (2.13)–(2.15) into (2.12), we deduce that for $x + \beta y \le ct$,

$$\partial_x G(x) * \left(1 + \frac{3}{2}\sinh^2\xi\right)(t, x, y)$$

$$= \cosh(1/2)\sinh(1/2 + (x + \beta y - ct))$$

$$- \sinh(1/2 + (x + \beta y - ct))\cosh(1/2 + (x + \beta y - ct)). \quad (2.16)$$

On the other hand, one can deduce

$$(\cosh(1/2)\sinh\xi - \sinh\xi\cosh\xi)(t, x, y)$$

$$= \begin{cases} \cosh(1/2)\sinh(1/2 - (x + \beta y - ct)) \\ \quad - \sinh(1/2 - (x + \beta y - ct))\cosh(1/2 - (x + \beta y - ct)), & x + \beta y > ct, \\ - \cosh(1/2)\sinh(1/2 + (x + \beta y - ct)) \\ \quad + \sinh(1/2 + (x + \beta y - ct))\cosh(1/2 + (x + \beta y - ct)), & x + \beta y \le ct. \end{cases}$$

$$(2.17)$$

In view of (2.11), (2.16) and 2.17, we deduce from (2.5)

$$\int_0^\infty \int_{\mathbb{S}^2} \partial_x \psi\left[\partial_t u_c + u_c \partial_x u_c + \partial_x G * \left(u_c^2 + \frac{1}{2}(\partial_x u_c)^2\right)\right]dxdydt = 0, \quad (2.18)$$

for every test function $\psi(t, x, y) \in C_c^\infty([0, T] \times \mathbb{S}^2)$. Now we consider

$$\partial_x G * \cosh\xi(t, x, y)$$

$$= -\frac{1}{2\sinh(1/2)} \int_{\mathbb{S}} \sinh(1/2 - (x + \beta y - z) + [x + \beta y - z]) \quad (2.19)$$

$$\times \cosh(1/2 - (z - ct) + [z - ct])dz.$$

When $x + \beta y > ct$, we split the right hand side of (2.19) into the following three parts:

$$\partial_x G * \cosh\xi(t, x, y)$$

$$= -\frac{1}{2\sinh(1/2)}\left(\int_0^{ct} + \int_{ct}^{x+\beta y} + \int_{x+\beta y}^{1}\right)\sinh(1/2 - (x + \beta y - z) + [x + \beta y - z])$$

$$\times \cosh(1/2 - (z - ct) + [z - ct])dz$$

$$:= III_1 + III_2 + III_3.$$

$$(2.20)$$

We directly compute III_1, III_2 and III_3 as follows:

$$III_1 = -\frac{1}{2\sinh(1/2)} \int_0^{ct} \sinh(1/2 - (x + \beta y - z))\cosh(1/2 + (z - ct))dz$$

$$= -\frac{1}{2\sinh(1/2)}\Big[\frac{1}{4}\cosh(1 - (x + \beta y - ct)) - \frac{1}{4}\cosh(1 - (x + \beta y + ct))$$

$$-\frac{1}{2}\sinh(x + \beta y - ct)ct\Big],$$

$$(2.21)$$

$$III_2 = -\frac{1}{2\sinh(1/2)} \int_{ct}^{x+\beta y} \sinh(1/2 - (x + \beta y - z))\cosh(1/2 - (z - ct))dz$$

$$= -\frac{1}{2\sinh(1/2)}\Big[\frac{1}{2}\sinh(1 - (x + \beta y - ct))(x + \beta y) - \frac{1}{2}\sinh(1 - (x + \beta y - ct))ct\Big],$$

$$(2.22)$$

and

$$III_3 = -\frac{1}{2\sinh(1/2)} \int_{x+\beta y}^{1} -\sinh(1/2 + (x + \beta y - z))\cosh(1/2 - (z - ct))dz$$

$$= -\frac{1}{2\sinh(1/2)}\Big[\frac{1}{4}\cosh(-1 + (x + \beta y + ct)) - \frac{1}{4}\cosh(1 - (x + \beta y - ct))$$

$$-\frac{1}{2}\sinh(x + \beta y - ct) + \frac{1}{2}\sinh(x + \beta y - ct)(x + \beta y)\Big].$$

$$(2.23)$$

Plugging (2.21)–(2.23) into (2.20), we deduce that for $x + \beta y > ct$,

$$\partial_x G * \cosh\xi(t, x, y) = \frac{1}{2}(1/2 - (x + \beta y - ct))\cosh(1/2 - (x + \beta y - ct))$$

$$-\frac{1}{4}\coth(1/2)\sinh(1/2 - (x + \beta y - ct)).$$

$$(2.24)$$

While for the case $x + \beta y \leq ct$, we split the right hand side of (2.19) into the following three parts:

$$\partial_x G * \cosh\xi(t, x, y)$$

$$= -\frac{1}{2\sinh(1/2)}\left(\int_0^{x+\beta y} + \int_{x+\beta y}^{ct} + \int_{ct}^{1}\right)\sinh(1/2 - (x + \beta y - z) + [x + \beta y - z])$$

$$\times \cosh(1/2 - (z - ct) + [z - ct])dz$$

$$:= IV_1 + IV_2 + IV_3.$$

$$(2.25)$$

For IV_1, IV_2 and IV_3, a direct computation yields

$$
\begin{aligned}
IV_1 &= -\frac{1}{2\sinh(1/2)} \int_0^{x+\beta y} \sinh(1/2 - (x+\beta y - z))\cosh(1/2 + (z - ct))dz \\
&= -\frac{1}{2\sinh(1/2)}\Big[-\frac{1}{2}\sinh(x+\beta y - ct)(x+\beta y) \\
&\qquad\qquad + \frac{1}{4}\cosh(1 + (x+\beta y - ct)) - \frac{1}{4}\cosh(1 - (x+\beta y + ct))\Big],
\end{aligned}
$$
(2.26)

$$
\begin{aligned}
IV_2 &= -\frac{1}{2\sinh(1/2)} \int_{x+\beta y}^{ct} -\sinh(1/2 + (x+\beta y - z))\cosh(1/2 + (z - ct))dz \\
&= -\frac{1}{2\sinh(1/2)}\Big[-\frac{1}{2}\sinh(1 + (x+\beta y - ct))ct + \frac{1}{2}\sinh(1 + (x+\beta y - ct))(x+\beta y)\Big],
\end{aligned}
$$
(2.27)

and

$$
\begin{aligned}
IV_3 &= -\frac{1}{2\sinh(1/2)} \int_{ct}^1 -\sinh(1/2 + (x+\beta y - z))\cosh(1/2 - (z - ct))dz \\
&= -\frac{1}{2\sinh(1/2)}\Big[\frac{1}{4}\cosh(-1 + (x+\beta y + ct)) - \frac{1}{4}\cosh(1 + (x+\beta y - ct)) \\
&\qquad\qquad - \frac{1}{2}\sinh(x+\beta y - ct) + \frac{1}{2}\sinh(x+\beta y - ct)ct\Big].
\end{aligned}
$$
(2.28)

Plugging (2.26)–(2.28) into (2.25), we obtain that for $x + \beta y \le ct$,

$$
\begin{aligned}
\partial_x G * \cosh\xi(t,x,y) &= -\frac{1}{2}(1/2 + (x+\beta y - ct))\cosh(1/2 + (x+\beta y - ct)) \\
&\quad + \frac{1}{4}\coth(1/2)\sinh(1/2 + (x+\beta y - ct)).
\end{aligned}
$$
(2.29)

In view of (2.24), (2.29) and the fact that the linear independent of the functions $\cosh\xi$ and $\sinh\xi$, we deduce from (2.5) that

$$
\begin{aligned}
&-\int_0^\infty \int_{\mathbb{S}^2} \partial_x\psi \big[(k+\beta^2)\partial_x G * (u_c)\big] dxdydt \\
&= -\frac{c}{\cosh(1/2)} \int_0^\infty \int_{\mathbb{S}^2} \partial_x\psi \Big[-\frac{(k+\beta^2)}{2}\xi\cosh\xi + \frac{(k+\beta^2)}{4}\sinh\xi \Big] dxdydt \\
&= 0
\end{aligned}
$$
(2.30)

if and only if $k + \beta^2 = 0$. Therefore, we conclude from (2.5), (2.18) and (2.30) that for every $\psi(t,x,y) \in C_c^\infty([0,\infty) \times \mathbb{S}^2)$,

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{S}^2} \Big[-\partial_t\partial_x\psi u_c + \partial_x\psi\Big(u_c\partial_x u_c + \partial_x G * (u_c^2 + \frac{1}{2}(\partial_x u_c)^2 + k u_c)\Big) - \partial_y^2\psi G * u_c \Big] dxdydt \\
&\quad + \int_{\mathbb{S}^2} u_{c,0}(x,y)\partial_x\psi(0,x,y)dxdy = 0
\end{aligned}
$$

if and only if $k + \beta^2 = 0$, which completes the proof of Theorem 2.1.   □

Note that if the initial data $u_0(x, y) = U_0(x + \gamma y)$ in (1.1), then the uniqueness of the solution (1.1) in [22] implies that the function $u(t, x, y) = U(t, x + \gamma y)$ is the solution of (1.1), where $U(t, \eta)$ solves the following Camassa-Holm equation

$$U_t - U_{t\eta\eta} + (k + \gamma^2)U_\eta + 3UU_\eta = 2U_\eta U_{\eta\eta} + UU_{\eta\eta\eta} \qquad (2.31)$$

with initial data $U(0, \eta) = U_0(\eta)$. In view of the classification of traveling-wave solution for the Camassa-Holm equation in [28], we may also obtain the following result and detailed proof is omitted.

Theorem 2.2 *Fix $k, \gamma \in \mathbb{R}$ and let $z = c - (k + \gamma^2) - M - m$. The equation (2.31) possesses a periodic peaked solitary wave solution $U(\eta - ct)$ with $m = \min_{x \in \mathbb{R}} U(\eta)$, $M = \max_{x \in \mathbb{R}} U(\eta)$ or $m = \max_{x \in \mathbb{R}} U(\eta)$, $M = \min_{x \in \mathbb{R}} U(\eta)$ if either $z < m < M = c$ or $z > m > M = c$.*

Remark 2.2 By theorem 2.2, we know that periodic peaked solitary waves exist for equation (2.31). Performing similar arguments as in [28], we can obtain an explicit expression for the periodic peakons as following:

$$U(\eta) = (m + (k + \gamma^2)/2) \cosh |\eta - \eta_0| - (k + \gamma^2)/2, \quad |\eta - \eta_0| \leq \frac{p}{2},$$

where

$$p = 4 \ln \left(\frac{\sqrt{M - m} + \sqrt{k + \gamma^2 + M + m}}{\sqrt{k + \gamma^2 + 2m}} \right),$$

or

$$U(\eta) = \frac{c + (k + \gamma^2)/2}{\cosh(p/2)} \cosh |\eta - \eta_0| - (k + \gamma^2)/2, \quad |\eta - \eta_0| \leq p/2.$$

3 Conclusions

Existence of localized travelling waves, commonly referred to as *solitary waves*, are important in general in the study of nonlinear dispersive equations. Existence and nonexistence of smooth and peaked solitary waves to the CH–KP equation (1.1) is an interesting issue, even it is not easy to deal with because of its structure with slow transverse effect. We have proven some partial results for certain cases(Theorem 2.1 and Theorem 2.2). It was observed in [5] that there is no existence of smooth localized solitary wave solution analogous to the KP case (1.2). However, we found that for the CH–KP equation (1.1) is still possible to admit line(or periodic) peaked, smooth, cuspons, stumpons, and composite solitary waves like as CH equation (1.3).

We prove the existence of periodic peaked solitary waves to the CH–KP equation (1.1) with an emphasis on the understanding of weak transverse effect. Moreover, we see that there are a multitude of solitary waves such as smooth, peakons, cuspons, stumpons, and composite waves like as CH equation (1.3). Furthermore, it is of great interest whether those solitary waves remain stable or not. To see this, it is worth noting that there are two approaches to study stability of peaked solitary waves. one approach in [20] is variational methods, that is, it should be proved that each peaked solitary wave is the unique minimum (ground state) of constrained energy. Another method is to linearize the equation around the solitary waves, and it is commonly believed that nonlinear stability is governed by the linearized equation. But, for the CH–KP equation (1.1), the nonlinearity plays the dominant role rather than being a higher-order correction to linear terms. Thus it is unclear how one can get nonlinear stability of peaked solitary waves by studying the linearized problem. Moreover, the peaked solitary waves are not differentiable, making it difficult to analyze the spectrum of the linearized operator around them.

We think that one possibility to prove the stability of the peaked solitary waves for CH–KP equation (1.1) is the simple approach in [10]. By using their method, the most difficult part to establish a suitable Lyapunov functional is that one needs to construct two functionals, which are connected to the conservation laws E and F in the introduction. On the other hand, those two functionals require to vanish at the peaked solitary waves. Also, nonlocal form of conservation law F makes difficult to construct those crucial two functionals corresponding to the conservation laws E and F. Thus, the stability issue of peaked solitary waves of CH–KP equation (1.1) is more subtle but challenging. We are planning to pursue this issue in the near future.

Declarations

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