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RESEARCH ARTICLE

Perfect Fluid Spacetimes and Gradient Solitons

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Abstract

In this article, we investigate perfect fluid spacetimes equipped with concircular vector field. At first, in a perfect fluid spacetime admitting concircular vector field, we prove that the velocity vector field annihilates the conformal curvature tensor. In addition, in dimension 4, we show that a perfect fluid spacetime is a generalized Robertson–Walker spacetime with Einstein fibre. It is proved that if a perfect fluid spacetime furnished with concircular vector field admits a second order symmetric parallel tensor P, then either the equation of state of the perfect fluid spacetime is characterized by $p = \frac{3-n}{n-1}\sigma$, or the tensor P is a constant multiple of the metric tensor. Finally, The perfect fluid spacetimes with concircular vector field whose Lorentzian metrics are Ricci soliton, gradient Ricci soliton, gradient Yamabe solitons, and gradient m -quasi Einstein solitons, are characterized.

Keywords Perfect fluid spacetimes · Gradient Ricci solitons · Gradient Yamabe solitons · m-quasi Einstein solitons

Mathematics Subject Classification 53C50 · 53E20 · 53C35 · 53E40.

Abbreviations

GRW Generalized Robertson–Walker
RW Robertson–Walker

1 Introduction

We start with a Lorentzian manifold M^n whose Lorentzian metric g is of signature $(\underbrace{+, +, \ldots, +}_{(n-1)\text{times}}, -)$. In [1], Alias, Romero, and Sanchez introduced the idea of generalized Robertson–Walker (GRW) spacetimes. The M^n with $n \geq 3$ is called a *GRW*

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spacetime if it can be written as a warped product of an open interval I of $\mathbb{R}$ (set of real numbers) and an $(n-1)$–dimensional Riemannian manifold $\mathcal{M}^*$. That is, $M = -I \times \mathfrak{f}^2 \mathcal{M}^*$, where $\mathfrak{f}$ is a smooth positive function, named as warping function. The *GRW* spacetime reduces to Robertson–Walker (RW) spacetime, if $\mathcal{M}^*$ is of dimension 3 and is of constant sectional curvature. This means spontaneously, the *GRW* spacetime is an extension of *RW* spacetime. It also includes the static Einstein spacetime, the Einstein-de Sitter spacetime, the Friedman cosmological models, the de Sitter spacetime and have implementations as inhomogeneous spacetimes obeying an isotropic radiation. The reader can see the physical and geometrical features of *GRW* spacetimes in [5, 20].

The M^n is named a perfect fluid spacetime if the Ricci tensor S of M^n satisfies

$$S = ag + bA \otimes A, \tag{1}$$

in which a, b are scalar fields, ξ is a unit timelike vector field defined as $g(X, \xi) = A(X)$ for all X. We know, Every *RW* spacetime is perfect fluid [22]. For $n = 4$, the *GRW* spacetime is perfect fluid if and only if it is a *RW* spacetime. We refer [3, 23] for further details about this subject.

In [16], Hamilton introduced the notion of Ricci flow. He invented this notion from the problem of discovering a canonical metric on a smooth manifold. In a (pseudo-) Riemannian manifold M^n, the metric is called Ricci flow if it is satisfied by an evolution equation $\frac{\partial}{\partial t} g_{ij}(t) = -2S_{ij}$ [16]. The self-similar solutions to the Ricci flow yield the Ricci solitons. A metric of M^n is said to be a Ricci soliton [15] if it satisfies

$$\mathfrak{L}_W g + 2S + 2\eta g = 0, \tag{2}$$

η being the real scalar. In this case, $\mathfrak{L}_W$ denotes the Lie derivative operator. We denote (g, W, η) as a Ricci soliton on M^n. The soliton is named shrinking, expanding or steady if η is negative, positive, or zero, respectively. In special scenario, the Ricci soliton is trivial provided W is Killing or identically zero and M^n is Einstein. If the soliton vector W is the gradient of some smooth function $-f$, that is, $W = -Df$, then the forgoing (2) is in the following form

$$Hess f - S - \eta g = 0, \tag{3}$$

Hess being the Hessian and D is the gradient operator. The metric is named a gradient Ricci soliton if it obeys (3). The smooth function $-f$ is called the potential function of the gradient Ricci soliton.

Motivated by the Yamabe's conjecture ("metric of a complete Riemannian manifold is conformally related to a metric with constant scalar curvature"), Hamilton in [16], presented the idea of Yamabe flow on a complete Riemannian manifold M^n. A (pseudo-) Riemannian manifold M^n equipped with a (pseudo-) Riemannian metric g is named a Yamabe flow if it satisfies:

$$\frac{\partial}{\partial t} g(t) = -rg(t), \quad g_0 = g(t),$$

t being the time and r is the scalar curvature. A (pseudo-) Riemannian manifold furnished with a (pseudo-) Riemannian metric is named a Yamabe soliton if it satisfies

$$\frac{1}{2}\mathcal{L}_W g = (r - \eta)g. \tag{4}$$

Here, $\mathcal{L}$ denotes the Lie derivative operator and W represents a vector field, termed as the potential vector field. On a (pseudo-) Riemannian manifold M^n, Yamabe soliton with $W = Df$ reduces to the gradient Yamabe soliton. Thus, Eq. (4) becomes

$$Hessf = (r - \eta)g. \tag{5}$$

The gradient Yamabe (or Yamabe) soliton becomes trivial if f is constant (or W is Killing) on M^n. The Yamabe soliton on 3-Sasakian manifolds was studied by Sharma [23]. Wang [26] and Suh and De [24], characterized the 3-Kenmotsu manifolds and almost co-Kähler manifolds with Yamabe solitons, respectively. Properties of Riemannian manifolds with Yamabe solitons were investigated by Chen et al. [6]. In [3, 10, 11] and also by others, some important results on Yamabe solitons have been studied. In [12], the authors studied Yamabe and gradient Yamabe solitons in perfect fluid spacetimes.

A (pseudo-) Riemannian manifold M^n furnished with the semi-Riemannian metric g is named a *gradient m-quasi Einstein metric* [2] if, for a constant η and a smooth function $f : M^n \to \mathbb{R}$, we have

$$S + Hessf = \frac{1}{m}df \otimes df + \eta g, \tag{6}$$

in which $0 < m \leq \infty$ is an integer and $\otimes$ denotes the tensor product. In this setting, f indicates the m-quasi Einstein potential function [2]. In this case, the Bakry–Emery Ricci tensor $S + Hessf - \frac{1}{m}df \otimes df$ is proportional to the metric g and η is a constant [27].

If $m = \infty$, Eq. (6) represents a gradient Ricci soliton. If $m = \infty$ and η is a smooth function, then the metric represents almost gradient Ricci soliton. Some basic classifications of m-quasi Einstein metrics was characterized by He et al. [17] on Einstein product manifold with non-empty base. In [18], characterization of m -quasi Einstein solitons have been presented (in details) .

Motivated by the above studies, properties of perfect fluid spacetimes are investigated if the Lorentzian metrics are Ricci, gradient Ricci, gradient Yamabe, and m-quasi Einstein solitons.

This article is organised as: in Sect. 2, perfect fluid spacetime with concircular vector field is investigated. After that in a perfect fluid spacetime with concircular vector field, we study the properties of second order symmetric parallel tensor. Section 4 is devoted to study Ricci soliton and gradient Ricci soliton on a perfect fluid spacetime with concircular vector field. In Sect. 5, gradient yamabe soliton on perfect fluid spacetime is considered. As a final point, Sect. 6 focuses on studying gradient m-quasi Einstein solitons on perfect fluid spacetime.

2 Perfect Fluid Spacetime

We know that in a perfect fluid spacetime, the unit timelike vector field ξ , also called the velocity vector field of the fluid, satisfies

$$g(U,\xi) = A(U), \quad g(\xi,\xi) = A(\xi) = -1, \tag{7}$$

where $U \in \mathfrak{X}(M)$. $\mathfrak{X}(M)$ denotes the collection of all C^∞ vector fields of M and A is a non-zero 1−form. Now, applying the covariant derivative on (7), we infer that

$$g(\nabla_U \xi, \xi) = 0 \ \text{ and } \ (\nabla_U A)(\xi) = 0, \tag{8}$$

∇ is the Levi-Civita connection. Einstein's field equations (EFE) is given as

$$S - \frac{r}{2} g = \kappa T, \tag{9}$$

where T is the energy momentum tensor and the constant κ denotes the gravitational constant. In a perfect fluid spacetime, the *energy momentum tensor* is in the following form

$$T = (\sigma + p)A \otimes A + pg, \tag{10}$$

where σ and p are the energy density and the isotropic pressure of the perfect fluid, respectively.

Contracting Eq. (1) over U and V, one easily gets

$$r = na - b. \tag{11}$$

For a perfect fluid spacetime, the necessary and sufficient condition for the constant scalar curvature is that $nU(a) = U(b)$. The combination of Eqs. (1), (9) and (10) give

$$b = \kappa(p + \sigma), \ a = \frac{\kappa(p - \sigma)}{2 - n}. \tag{12}$$

Further, p and σ are connected by the following state equation $p = p(\sigma)$. The perfect fluid in this setting is called isentropic. On the other hand, the perfect fluid is named stiff matter if $p = \sigma$. In [29], a stiff matter state equation was publicized by Zeldovich. It is well-known that the stiff matter era preceded the dust matter era with $p = 0$, the radiation era with $p - \frac{\sigma}{3} = 0$ and the dark energy era with $p + \sigma = 0$ [4].

Failkow in [14], introduced the idea of concircular vector field. On a (pseudo-) Riemannian manifold, a vector field ξ is named concircular if, for a smooth function ω (termed as potential function of ξ), it obeys

$$\nabla_U \xi = \omega U, \ \forall \ U \in \mathfrak{X}(M).$$

It is to be noted that in [25], the world lines of receding or colliding galaxies in de Sitter's model of general relativity are trajectories of timelike concircular vector fields. In this case, ξ is called non-trivial if ξ is non-constant. The vector ξ becomes concurrent if ω is non-zero constant. Concircular vector fields and their applications to Ricci solitons were investigated in [7] by Chen. Spheres and Euclidean spaces with the Concircular vector fields have been studied by Deshmukh et al. [13]. For further details, see [8, 9].

The use of Eqs. (1) and (7) imply

$$S(U,\xi) = (a-b)A(U) \tag{13}$$

and conclude that corresponding to the eigenvector ξ, $a-b$ is an eigenvalue of S.

Agreement: In a perfect fluid spacetime, we consider that ξ is of concircular type throughout this paper.

If ξ of the perfect fluid spacetime is of concircular type, then

$$\nabla_U \xi = \omega U,$$

for all U. The use of the above expression with $R(U,V)\xi = \nabla_U \nabla_V \xi - \nabla_V \nabla_U \xi - \nabla_{[U,V]}\xi$ yield

$$R(U,V)\xi = (U\omega)V - (V\omega)U. \tag{14}$$

Contracting over U implies

$$S(V,\xi) = (1-n)(V\omega). \tag{15}$$

Combining (13) and (15), we acquire

$$(U\omega) = \frac{a-b}{1-n}A(U). \tag{16}$$

Utilizing (16) into (14), we infer

$$R(U,V)\xi = \frac{a-b}{1-n}\{A(U)V - A(V)U\}. \tag{17}$$

Applying the foregoing equation and from the expression of Weyl conformal curvature tensor [28]

$$\begin{aligned} C(U,V)X = {} & R(U,V)X - \frac{1}{n-2}[g(V,X)QU - g(U,X)QV + S(V,X)U \\ & - S(U,X)V - \frac{r}{n-1}\{g(V,X)U - g(U,X)V\}], \end{aligned}$$

where the symbol R is the curvature tensor whereas Q is the Ricci operator. The Ricci operator is defined by $g(QU,V) = S(U,V)$. We find that

$$C(U,V)\xi = 0, \;\; \forall \; U,V \in \mathfrak{X}(M).$$

Theorem 2.1 *Let M be a perfect fluid spacetime with concircular vector field. Then the velocity vector field ξ annihilates the conformal curvature tensor.*

For $n = 4$, $C(U, V)\xi = 0$ is equivalent to $C(U, V)W = 0$ [19] . Also, from $\nabla_U \xi = \omega U$, we find

$$(\nabla_U A)(V) = \omega g(U, V) = (\nabla_V A)(U).$$

Thus, the 1-form A is closed.

For $n = 4$, Mantica et al. proved that a perfect fluid spacetime obeying $div\mathbf{C} = 0$ is a *GRW* spacetime with Einstein fibre, provided ξ is closed [21] .

Since $C(U, V)W = 0 \Rightarrow div\mathbf{C} = 0$. Therefore, in view of the the above discussion, we can conclude that a perfect fluid spacetime with concircular vector field is a *GRW* spacetime with Einstein fibre. We thus have the following result:

Theorem 2.2 *A perfect fluid spacetime admitting concircular vector field is a GRW spacetime with Einstein fibre.*

3 Perfect Fluid Spacetime with Second Order Symmetric Parallel Tensor

In a perfect fluid spacetime, let P be a (0, 2) symmetric tensor which is parallel with respect to ∇, that is $\nabla P = 0$. Therefore, by $\nabla P = 0$, one finds

$$P(R(U, V)X, Y) + P(X, R(U, V)Y) = 0, \tag{18}$$

in which U, V, X, and Y are arbitrary vectors fields. Since P is symmetric, putting $X = Y = \xi$ in Eq. (18), one gets

$$P(R(U, V)\xi, \xi) = 0. \tag{19}$$

Using Eq. (17), we find

$$\frac{a-b}{1-n} P(A(U)V - A(V)U, \xi) = 0. \tag{20}$$

Putting ξ instead of U in the aforementioned equation and utilizing Eq. (7), one infers

$$\frac{a-b}{1-n} \{-P(V, \xi) - A(V)P(\xi, \xi)\} = 0, \tag{21}$$

which implies that either $a = b$, or

$$P(V, \xi) = -A(V)P(\xi, \xi). \tag{22}$$

P is parallel as mentioned earlier, thus we have

$$\begin{aligned} 0 =& (\nabla_X P)(V,\xi) = \nabla_X P(V,\xi) - P(\nabla_X V,\xi) - P(V,\nabla_X \xi) \\ =& - \nabla_X A(V) P(\xi,\xi) + A(\nabla_X V) P(\xi,\xi) - P(V,\omega X) \\ =& - \nabla_X A(V) P(\xi,\xi) - \omega P(V,X) \\ =& - \omega g(X,V) P(\xi,\xi) - \omega P(V,X). \end{aligned}$$

Since $\omega \neq 0$,

$$g(X,V)P(\xi,\xi) = P(V,X),$$

which entails that

$$g(X,V)(\nabla_Y P)(\xi,\xi) = (\nabla_Y P)(V,X).$$

Since $\nabla P = 0$, thus we easily conclude that $P(\xi,\xi) = constant$.

From Eq. (17) we can derive

$$R(U,\xi)V = \frac{a-b}{1-n}\{g(U,V)\xi - A(V)U\}. \tag{23}$$

Putting $V = Y = \xi$ in Eq. (18) and utilizing (23), it leads us either $a = b$, or

$$P(X,U) = P(\xi,\xi)g(X,U).$$

Since $a = b$, from(12), one infers

$$p = \frac{3-n}{n-1}\sigma,$$

which gives the form of the state equation in a perfect fluid spacetime. Hence, we have the following:

Theorem 3.1 *If a perfect fluid spacetime possesses a second order symmetric parallel tensor. Then either*

(1) *the equation of state of a perfect fluid spacetime is given as* $p = \frac{3-n}{n-1}\sigma$, *or*
(2) P *is constant multiple of* g.

Remark 3.1 For $n = 4$, we get the state equation as $\sigma + 3p = 0$, which implies the radiation and it characterizes the early universe.

Corollary 3.1 *If a perfect fluid spacetime is Ricci symmetric, then the spacetime is Einstein, provided* $a \neq b$.

4 Perfect Fluid Spacetime Admitting Ricci Soliton and Gradient Ricci Soliton

Now assume that a perfect fluid spacetime with concircular vector field admits a Ricci soliton given by Eq. (2). Since η in (2) is constant; therefore, we have $\nabla g = 0$ and $\nabla \eta g = 0$. Therefore, $\pounds_W g + 2S$ is parallel. The previous theorem implies that $\pounds_W g + 2S$ is constant multiple of g. That is, $\pounds_W g + 2S = a_1 g$, being a_1 is constant, provided $a \neq b$. Therefore, $\pounds_V g + 2S + 2\eta g$ is in the form $(a_1 + 2\eta)g$, which implies $\eta = -a_1/2$. Hence, we have the following result:

Theorem 4.1 *Let M be a perfect fluid spacetime endowed with concircular vector field. Then the Ricci soliton (g, V, η) is expanding or shrinking according as a_1 is negative or positive, provided $a \neq b$.*

In particular case, let us consider the following case $W = \xi$. Hence, Eq. (2) is of the form

$$(\pounds_\xi g)(U, V) + 2S(U, V) + 2\eta g(U, V) = 0. \tag{24}$$

The use of $\nabla_U \xi = \omega U$ implies

$$S(U, V) = -(\omega + \eta) g(U, V). \tag{25}$$

This equation means that the spacetime is Einstein. That is, trivial Ricci soliton. We thus have:

Theorem 4.2 *Let M be a perfect fluid spacetime equipped with concircular vector field. Then the Ricci soliton (g, ξ, η) is both trivial and Einstein, provided $a \neq b$.*

Now, moving on to the next part of this section. This part focus on the investigation of gradient Ricci solitons in perfect fluid spacetimes equipped with concircular vector field.

Hence,

$$\begin{aligned}(\nabla_V A)(U) =& \nabla_V A(U) - A(\nabla_V U)\\ =& \nabla_V g(U, \xi) - g(\nabla_V U, \xi)\\ =& g(U, \nabla_V \xi) = \omega g(U, V),\end{aligned}$$

$\forall\, U,\ V \in \mathfrak{X}(M)$ and (1) imply

$$QU = aU + bA(U)\xi,\ \ \forall\ U \in \mathfrak{X}(M). \tag{26}$$

In a perfect fluid spacetime with concircular vector field, consider that the soliton vector field W of (g, W, η) is a gradient of some smooth function $-f$. Thus Eq. (2) is reduced to be in the following form

$$\nabla_U Df = QU + \eta U \tag{27}$$

for all $U \in \mathfrak{X}(M)$. Equation (27) along with the subsequent relation

$$R(U, V)Df = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]} Df \tag{28}$$

yield

$$R(U, V)Df = (\nabla_U Q)(V) - (\nabla_V Q)(U). \tag{29}$$

Applying the covariant derivative on (26) and utilizing (25), we have

$$(\nabla_U Q)(V) = (Ua)V + (Ub)A(V)\xi + b(\omega g(U, V)\xi + \omega A(V)U). \tag{30}$$

In view of (29) and (30), we find

$$\begin{aligned} R(U, V)Df =& (Ua)V - (Va)U + \{(Ub)A(V) - (Vb)A(U)\}\xi \\ &+ \omega b\{A(V)U - A(U)V\}. \end{aligned} \tag{31}$$

Taking a set of orthonormal frame field and hence, executing contraction of (31), it is

$$S(U, Df) = (1 - n)(Ua) + (Ub) + (\xi b)A(U) + \omega b(n - 1)A(U). \tag{32}$$

From Eq. (1) we can easily get

$$S(V, Df) = a(Vf) + bA(V)(\xi f). \tag{33}$$

Putting$V = \xi$ in (32) and (33). We then can equate the values of $S(\xi, Df)$, one infers

$$(a - b)(\xi f) = (1 - n)\{(\xi a) - \omega b\}. \tag{34}$$

Suppose that

$$(\xi a) = \omega b.$$

Thus, from Eqs. (34) we get

$$(a - b)(\xi f) = 0. \tag{35}$$

On a perfect fluid spacetime equipped with the gradient Ricci soliton, the last equation illustrates that either $a = b$ or $(\xi f) = 0$.

Case I. If $a = b$ and $(\xi f) \neq 0$ and hence from(12), we easily infer that

$$p = \frac{3 - n}{n - 1}\sigma,$$

which demonstrates the form of the state equation in a perfect fluid spacetime. Moreover, $\eta = b - a = 0$ and thus the gradient Ricci soliton is steady.

Case II. If $(\xi f) = 0$ and $a \neq b$. Thus, we conclude that f is invariant under the velocity vector field ξ.

In consequence of the above, our following result can be stated:

Theorem 4.3 *Let M be a perfect fluid spacetime equipped with concircular vector field possesses a gradient Ricci soliton with* $(\xi a) = \omega b$. *Then either*

(1) *the equation of state of M is governed by* $p = \frac{3-n}{n-1}\sigma$ *and the soliton is steady, or*
(2) *under the velocity vector* ξ*,* f *is invariant.*

5 Perfect Fluid Spacetimes Admitting Gradient Yamabe Soliton

From Eq. (5), we find

$$\nabla_V Df = (r - \eta)V. \tag{36}$$

Covariantly, differentiating (36) along V, we easily obtain the following

$$\nabla_U \nabla_V Df = (Ur)V + (r - \eta)\nabla_U V. \tag{37}$$

U and V are interchanged in the the previous equation and hence using Eqs. (36) and (37) in $R(U, V)Df = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]} Df$, one infers

$$R(U, V)Df = (Ur)V - (Vr)U. \tag{38}$$

Now, consider an orthonormal frame field and making a contraction over U, we find that

$$S(V, Df) = -(n-1)(Vr).$$

From Eq. (1) we infer

$$S(V, Df) = a(Vf) + b(\xi f)A(V).$$

Combining the last two equations, we infer

$$a(Vf) + b(\xi f)A(V) = -(n-1)(Vr). \tag{39}$$

Putting $V = \xi$ in the preceding equation, we get

$$(a - b)(\xi f) = -(n-1)(\xi r). \tag{40}$$

Now, from (38) we infer that

$$g(R(U, V)Df, \xi) = (Ur)A(V) - (Vr)A(U). \tag{41}$$

Again (17) implies that

$$g(R(U, V)\xi, Df) = \frac{a-b}{1-n}\{A(U)(Vf) - A(V)(Uf)\}. \tag{42}$$

Combining Eqs. (41) and (42), we have

$$(Ur)A(V) - (Vr)A(U) = -\frac{a-b}{1-n}\{A(U)(Vf) - A(V)(Uf)\}. \tag{43}$$

Setting $V = \xi$ in the previous equation gives

$$(Ur) = \frac{a-b}{1-n}(Uf). \tag{44}$$

Utilizing (44) in (39) we infer that

$$b\{(Uf) + (\xi f)A(U)\} = 0, \tag{45}$$

this equation implies that either $b = 0$ or $(Uf) + (\xi f)A(U) = 0$.

It is easy to find $\sigma + p = 0$ if $b = 0$. This represents a dark energy.

Now, assume that $b \neq 0$ and $(Uf) + (\xi f)A(U) = 0$, which gives $Df = -(\xi f)\xi$. Hence, we conclude the result as:

Theorem 5.1 *If the Lorentzian metric of a perfect fluid spacetime M with concircular vector field is a gradient Yamabe soliton. Then either*

(1) *M denotes a dark energy, or*
(2) *the gradient of Yamabe soliton potential function is pointwise collinear with the velocity vector field of the perfect fluid spacetime.*

Now, assume that $b \neq 0$ on a perfect fluid spacetime with concircular vector field admitting a gradient Yamabe soliton.

The covariant derivative of $Df = -(\xi f)\xi$ implies

$$-(r - \eta) = -\xi(\xi f) + \omega(\xi f),$$

where (36) is used. If f is invariant ξ, then we get

$$\eta = r. \tag{46}$$

In view of the above, we can conclude that r is constant. Therefore, from Eq. (44), we have either $a = b$ or $Df = 0$.

Case I If $a = b$ and $(Df) \neq 0$, then from (12), we infer that

$$p = \frac{3-n}{n-1}\sigma.$$

which gives the state equation in such spacetime.

Case II If $(Df) = 0$ and $a \neq b$. Then, $Df = 0$ means in other words that f is constant. It follows that the gradient Yamabe soliton is trivial. Thus, we can write:

Corollary 5.1 *Let the Lorentzian metric of a perfect fluid spacetime endowed with concircular vector field admits a gradient Yamabe soliton with $b \neq 0$. If f is invariant under ξ, Then either*

(1) *the equation of state of a perfect fluid spacetime is governed by $p = \frac{3-n}{n-1}\sigma$ or,*
(2) *the gradient Yamabe soliton is trivial.*

Now, consider the next setting, if f is invariant under the velocity vector field ξ, then (46) is fulfilled. Therefore, we can easily find that the nature of the flow varies according to r. We thus can write the subsequent corollaries.

Corollary 5.2 *Let a perfect fluid spacetime with concircular vector field admits a gradient Yamabe soliton with $b \neq 0$. If f is invariant under the velocity vector field ξ. Then the gradient Yamabe soliton is expanding, shrinking or, steady according as r is positive, negative or zero, respectively.*

Corollary 5.3 *Assume that the metric of a perfect fluid spacetime M equipped with concircular vector field admits a gradient Yamabe soliton with $b \neq 0$. Then it admits the constant scalar curvature r, provided f is invariant under ξ.*

Remark 5.1 Gradient Yamabe soliton in perfect fluid spacetimes was studied in [12]. In this article, under extra conditions, we study gradient Yamabe soliton in perfect fluid spacetime. That is, in a perfect fluid spacetime M, we assume that the velocity vector field ξ is of concircular type which was first coined by Failkow [14]. In this setting, we discover some interesting results. These results are different from the results of the paper [12].

6 Gradient *m*-quasi Einstein Solitons on Perfect Fluid Spacetimes

In this section, perfect fluid spacetimes equipped with concircular vector field with m-quasi Einstein metric are investigated. At first, let us prove the following result:

Lemma 6.1 *For a perfect fluid spacetime with concircular vector field obeys the following relation*:

$$\begin{aligned} R(U,V)Df =& (\nabla_V Q)U - (\nabla_U Q)V + \frac{\eta}{m}\{(Vf)U - (Uf)V\} \\ &+ \frac{1}{m}\{(Uf)QV - (Vf)QU\}, \end{aligned} \tag{47}$$

for all U, $V \in \mathfrak{X}(M)$.

Proof At first, consider that a perfect fluid spacetime (with concircular vector field) with m-quasi Einstein metric. Hence, Eq. (6) may be expressed as

$$\nabla_U Df + QU = \frac{1}{m} g(U, Df)Df + \eta U. \tag{48}$$

After executing covariant derivative of (48) along V, one infers

$$\begin{aligned} \nabla_V \nabla_U Df = & - \nabla_V QU + \frac{1}{m} \nabla_V g(U, Df)Df \\ & + \frac{1}{m} g(U, Df)\nabla_V Df + \eta \nabla_V U. \end{aligned} \tag{49}$$

Exchanging U and V in (49), it is easy to get

$$\begin{aligned} \nabla_U \nabla_V Df = & - \nabla_U QV + \frac{1}{m} \nabla_U g(V, Df)Df \\ & + \frac{1}{m} g(V, Df)\nabla_U Df + \eta \nabla_U V, \end{aligned} \tag{50}$$

and

$$\nabla_{[U,V]} Df = -Q[U, V] + \frac{1}{m} g([U, V], Df)Df + \eta[U, V]. \tag{51}$$

Utilizing (48)–(51) and the symmetric property of ∇. Both together with $R(U, V)Df = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]} Df$, one easily finds

$$\begin{aligned} R(U, V)Df = & (\nabla_V Q)U - (\nabla_U Q)V + \frac{\eta}{m}\{(Vf)U - (Uf)V\} \\ & + \frac{1}{m}\{(Uf)QV - (Vf)QU\}. \end{aligned}$$

□

In view of the Eqs. (26), (30) and the foregoing Lemma, one can get

$$\begin{aligned} R(U, V)Df = & (Ua)V - (Va)U + \{(Ub)A(V) - (Vb)A(U)\}\xi \\ & + \omega b\{A(V)U - A(U)V\} + \frac{\eta}{m}\{(Vf)U - (Uf)V\} \\ & + \frac{1}{m}\{a(Uf)V + b(Uf)A(V)\xi - a(Vf)U - b(Vf)A(U)\xi\}. \end{aligned} \tag{52}$$

Taking a set of orthonormal frame field. And hence executing contraction of the Eq. (52), we conclude that

$$\begin{aligned} S(U, Df) = & (1 - n)(Ua) + (Ub) + (\xi b)A(U) \\ & + \omega b(n - 1)A(U) + \frac{\eta}{m}(n - 1)(Uf) \\ & + \frac{1}{m}\{a(Uf) + b(\xi f)A(U) - na(Uf) + b(Uf)\}. \end{aligned} \tag{53}$$

Setting $V = \xi$ in Eqs. (53) and (33) and then equating the values of $S(\xi, Df)$, we find

$$\left(\frac{m}{1-n} + \eta - a\right)(\xi f) = m\{(\xi a) - \omega b\}. \tag{54}$$

Assume that f and a are invariant under the velocity vector field ξ. Then, we get from the previous equation that $b = 0$, since $m \neq 0$. Thus, following the proof of Theorem 5.1, we conclude our result as:

Theorem 6.1 *Let M be a perfect fluid spacetime equipped with concircular vector field admits a gradient m-quasi Einstein soliton. Then the spacetime M denotes a dark energy if f and a are invariant under* ξ.

7 Conclusion

In true sense, solitons are nothing but the waves which is physically propagate with some loss of energy and hold their speed and shape after colliding with one more such wave. Solitons play an essential role in the treatment of initial-value problems. This occurs in nonlinear partial differential equations describing wave propagation.

The investigation in this present article is established that a perfect fluid spacetime with concircular vector field is a GRW spacetime with Einstein fibre. Furthermore, it is proved that if a perfect fluid spacetime endowed with concircular vector field possesses a second order symmetric parallel tensor, then either the equation of state is distinguished by $p = \frac{3-n}{n-1}\sigma$ or the tensor is a constant multiple of g. Also, in the perfect fluid spacetimes with concircular vector field different metrics like Ricci soliton, gradient Ricci soliton, gradient Yamabe solitons and gradient m-quasi Einstein solitons are studied. In particular case, we find the condition for which the vector field ξ is steady, expanding and shrinking. It is noted that the spacetime denotes a dark matter era under certain restriction on ξ.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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