



Existence Results for Fractional Choquard Equations with Critical or Supercritical Growth

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Abstract

In this paper, we study the following fractional Choquard equation with critical or supercritical growth

$$(-\Delta)^s u + V(x)u = f(x, u) + \lambda [|x|^{-\mu} * |u|^p] p |u|^{p-2} u, \quad x \in \mathbb{R}^N,$$

where $0 < s < 1$, $(-\Delta)^s$ denotes the fractional Laplacian of order s , $N > 2s$, $0 < \mu < 2s$ and $p \geq 2_{\mu, s}^* := \frac{2N-\mu}{N-2s}$, which is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. Under some suitable conditions, we prove that the equation admits a nontrivial solution for small $\lambda > 0$ by variational methods, which extends results in Bhattarai in J. Differ. Equ. **263**, 3197–3229 (2017).

Keywords Fractional Choquard equation · Critical or supercritical growth · Variational method

Mathematics Subject Classification 35J20 · 35J50 · 35J15 · 35J60 · 35J70

1 Introduction and main result

Consider the following fractional Choquard equation

$$(-\Delta)^s u + V(x)u = f(x, u) + \lambda [|x|^{-\mu} * |u|^p] p |u|^{p-2} u, \quad x \in \mathbb{R}^N, \quad (1.1)$$

where $0 < s < 1$, $(-\Delta)^s$ denotes the fractional Laplacian of order s , $N > 2s$, $0 < \mu < 2s$ and $p \geq 2_{\mu, s}^* := \frac{2N-\mu}{N-2s}$.

Problem (1.1) has nonlocal characteristics in the nonlinearity as well as in the (fractional) diffusion. When $s = 1$, $\mu = 1$, $\lambda = 1$, $p = 2$ and $f(x, u) = 0$, then (1.1) boils down to the so-called Choquard equation

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$$-\Delta u + V(x)u = [|x|^{-1} * |u|^2]u, \quad x \in \mathbb{R}^3, \tag{1.2}$$

which goes back to the description of the quantum theory of a polaron at rest by Pekar in 1954 [15] and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree-Fock theory of one-component plasma [6]. In some particular cases, this equation is also known as the Schrödinger-Newton equation, which was introduced by Penrose in his discussion on the selfgravitational collapse of a quantum mechanical wave function [16]. The first investigations for existence and symmetry of the solutions to (1.2) go back to the works of Lieb [6] and Lions [7]. Since then many efforts have been made to study the existence of nontrivial solutions for nonlinear Choquard equations, see for instance [3, 12, 13].

For fractional Laplacian with nonlocal Hartree-type nonlinearities, the problem has also attracted a lot of interest, we refer to Refs. [2, 4, 5, 8, 10, 11] and their references therein.

Most of the works afore mentioned are set in \mathbb{R}^N , $N > 2s$, with subcritical and critical growth nonlinearities and to the authors' best knowledge no results are available on the existence for problem (1.1) with supercritical exponent. We aim at studying the existence of nontrivial solutions for critical or supercritical problem (1.1).

In order to reduce the statements for main result, we list the assumption as follows: $(V) V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 := \inf_{x \in \mathbb{R}^N} V(x)$ and $\lim_{|x| \rightarrow +\infty} V(x) = +\infty$.

$(f_1) f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exists $q \in \left(2, \frac{2(N-\mu)}{N-2s}\right)$ such that $|f(x, t)| \leq C(1 + |t|^{q-1})$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

$(f_2) f(x, t) = o(|t|)$ uniformly in $x \in \mathbb{R}^N$ as $|t| \rightarrow 0$.

$(f_3) f(x, t)t \geq qF(x, t) := q \int_0^t f(x, \tau)d\tau$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

$(f_4) c_0 := \inf_{x \in \mathbb{R}^N, |t|=1} F(x, t) > 0$.

For any $0 < s < 1$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}},$$

where the term

$$[u]_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}$$

is the so-called Gagliardo semi-norm of u . Moreover, we can see that an alternative definition of the fractional Sobolev space $H^s(\mathbb{R}^N)$ via the Fourier transform as follows:

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi < +\infty \right\}.$$

Here we denote the Fourier transform of u by $\hat{u} := \mathcal{F}(u)$. Propositions 3.4 and 3.6 in [14] imply that

$$2C_{N,s}^{-1} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = 2C_{N,s}^{-1} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}^2 = [u]_{H^s(\mathbb{R}^N)}^2.$$

As a consequence, the norms on $H^s(\mathbb{R}^N)$,

$$\begin{aligned} u &\mapsto \|u\|_{H^s(\mathbb{R}^N)}, \\ u &\mapsto \left(\|u\|_{L^2(\mathbb{R}^N)}^2 + \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}^2 \right)^{\frac{1}{2}}, \\ u &\mapsto \left(\|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \end{aligned}$$

are all equivalent.

Set $E = \{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty\}$ with the norm

$$\|u\|_E^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} V(x)u^2 dx$$

and $\mathcal{D}^{s,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < +\infty\}$ with the norm

$$\|u\|_{\mathcal{D}^{s,2}}^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi.$$

Then $\|u\|_E^2 = \|u\|_{\mathcal{D}^{s,2}}^2 + \int_{\mathbb{R}^N} V(x)u^2 dx$.

Our main result is the following:

Theorem 1.1 *Suppose that (V) and (f₁)–(f₄) are satisfied. Then there exists some $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0]$, Eq. (1.1) admits a nontrivial solution u_λ .*

Remark 1.2 Bhattarai in [1] studied the following fractional Schrödinger equation

$$(-\Delta)^s u + \omega u = a|u|^{q-2}u + \lambda[|x|^{-\mu} * |u|^p]|u|^{p-2}u,$$

where $0 < \mu < N$, $2 < q < 2 + \frac{4s}{N} < 2^*_s$, $2 \leq p < 1 + \frac{2s+N-\mu}{N} < 2^*_{\mu,s}$. Consequently, our result extends his result to some extent.

2 Proof of Theorem 1.1

Proposition 2.1 [9] (*Hardy-Littlewood-Sobolev inequality*) Let $r, t > 1$ and $0 < \mu < N$ with $\frac{1}{r} + \frac{\mu}{N} + \frac{1}{t} = 2$. Let $g \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Then there exists a sharp constant $C_{r,N,\mu,t}$ independent of g and h such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\mu} dx dy \leq C_{r,N,\mu,t} \|g\|_r \|h\|_t.$$

Remark 2.2 In general, set $F(u) = |u|^q$ for some $q > 0$. By Hardy-Littlewood-Sobolev inequality, $\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^\mu} dx dy$ is well defined if $F(u) \in L^t(\mathbb{R}^N)$ for $t > 1$ defined by $\frac{2}{t} + \frac{\mu}{N} = 2$. Thus, for $u \in H^s(\mathbb{R}^N)$, there must hold

$$\frac{2N - \mu}{N} \leq q \leq \frac{2N - \mu}{N - 2s} = 2^*_{\mu,s}.$$

It is well known to us that a weak solution of problem (1.1) is a critical point of the following functional

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} [|x|^{-\mu} * |u|^p]|u|^p dx.$$

Clearly, we cannot apply variational methods directly because the functional I_λ is not well defined on E unless $p = 2^*_{\mu,s}$. To overcome this difficulty, we define a function

$$\phi(t) = \begin{cases} p|t|^{p-2}t, & |t| \leq M, \\ pM^{p-q}|t|^{q-2}t, & |t| > M, \end{cases}$$

where $M > 0$. Then $\phi \in C(\mathbb{R}, \mathbb{R})$, $\phi(t)t \geq q\Phi(t) := q \int_0^t \phi(s)ds \geq 0$ and $|\phi(t)| \leq pM^{p-q}|t|^{q-1}$ for all $t \in \mathbb{R}$. Moreover, there exists a constant $C > 0$ such that

$$\left| [|x|^{-\mu} * \Phi(u)] \right| \leq CM^{p-q} \tag{2.1}$$

for all $u \in H^s(\mathbb{R}^N)$. Indeed, for any $u \in H^s(\mathbb{R}^N)$, taking $t \in \left(\frac{N}{N-\mu}, \frac{2N}{q(N-2s)} \right]$, by the Hölder inequality we can calculate that

$$\begin{aligned} ||x|^{-\mu} * \Phi(u)| &= \left| \int_{\mathbb{R}^N} \frac{\Phi(u(y))}{|x-y|^\mu} dy \right| \\ &\leq CM^{p-q} \int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu} dy \\ &= CM^{p-q} \int_{|x-y|\leq 1} \frac{|u(y)|^q}{|x-y|^\mu} dy + CM^{p-q} \int_{|x-y|>1} \frac{|u(y)|^q}{|x-y|^\mu} dy \\ &\leq CM^{p-q} \left(\int_{|x-y|\leq 1} |u(y)|^{tq} dy \right)^{\frac{1}{t}} \left(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{\mu}{t-1}}} dy \right)^{\frac{t-1}{t}} + CM^{p-q} \\ &\leq CM^{p-q} \left(\int_0^1 \rho^{N-1-\frac{\mu}{t-1}} d\rho \right)^{\frac{t-1}{t}} + CM^{p-q} \leq CM^{p-q}. \end{aligned}$$

Set $h_\lambda(x, t) = \lambda[|x|^{-\mu} * \Phi(t)]\phi(t) + f(x, t)$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. Then

(h₁) $h_\lambda \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $|h_\lambda(x, t)| \leq C\lambda M^{2(p-q)}|t|^{q-1} + C(1 + |t|^{q-1})$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

(h₂) $h_\lambda(x, t) = o(|t|)$ uniformly in $x \in \mathbb{R}^N$ as $|t| \rightarrow 0$.

(h₃) $h_\lambda(x, t) \geq qH_\lambda(x, t) := q \int_0^t h_\lambda(x, \tau)d\tau \geq 0$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

(h₄) $\inf_{x \in \mathbb{R}^N, |t|=1} H_\lambda(x, t) \geq c_0 > 0$.

Let

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} H_\lambda(x, u) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx \\ &\quad - \frac{\lambda}{2} \int_{\mathbb{R}^N} [|x|^{-\mu} * \Phi(u)] \Phi(u) dx \\ &= \frac{1}{4} C_{N,s} [u]_{H^s}^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx \\ &\quad - \frac{\lambda}{2} \int_{\mathbb{R}^N} [|x|^{-\mu} * \Phi(u)] \Phi(u) dx. \end{aligned}$$

By mountain pass theorem, using a standing argument we can prove that the equation

$$(-\Delta)^s u + V(x)u = h_\lambda(x, u)$$

has a nontrivial $u_\lambda \in E$ with $J'_\lambda(u_\lambda) = 0$ and $J_\lambda(u_\lambda) = c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \sup_{t \in [0,1]} J_\lambda(\gamma(t))$, where

$$\Gamma_\lambda := \{ \gamma \in C([0, 1], E) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0 \}.$$

In the sequel, set

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx$$

and

$$\Gamma := \{ \gamma \in C([0, 1], E) : \gamma(0) = 0, J(\gamma(1)) < 0 \}$$

and $c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t))$. Then $\Gamma \subset \Gamma_\lambda$ and $c_\lambda \leq c$.

Lemma 2.3 *The solution u_λ satisfies $\|u_\lambda\|_{D^{s,2}}^2 \leq \frac{2q}{q-2} c_\lambda$ and there exists a constant $A > 0$ independent on λ such that $\|u_\lambda\|_{D^{s,2}}^2 \leq A$.*

Proof Taking into account (f_3) we can see that

$$\begin{aligned} qc_\lambda &= qJ_\lambda(u_\lambda) = qJ_\lambda(u_\lambda) - \langle J'_\lambda(u_\lambda), u_\lambda \rangle \\ &= \left(\frac{q}{2} - 1\right) \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}_\lambda(\xi)|^2 d\xi + \left(\frac{q}{2} - 1\right) \int_{\mathbb{R}^N} V(x)|u_\lambda|^2 dx + \int_{\mathbb{R}^N} [f(x, u_\lambda)u_\lambda - qF(x, u_\lambda)] dx \\ &\quad + \lambda \int_{\mathbb{R}^N} [|x|^{-\mu} * \Phi(u_\lambda)] \left[\phi(u_\lambda)u_\lambda - \frac{q}{2}\Phi(u_\lambda) \right] dx \\ &\geq \left(\frac{q}{2} - 1\right) \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}_\lambda(\xi)|^2 d\xi = \left(\frac{q}{2} - 1\right) \|u_\lambda\|_{D^{s,2}}^2, \end{aligned}$$

which implies that $\|u_\lambda\|_{D^{s,2}}^2 \leq \frac{2q}{q-2} c_\lambda \leq \frac{2q}{q-2} c := A > 0$. This completes the proof. □

Lemma 2.4 *There exist two constants $B, D > 0$ independent on λ such that $\|u_\lambda\|_{L^\infty} \leq B(1 + \lambda)^D$, where $\|u\|_{L^\infty} := \sup_{x \in \mathbb{R}^N} |u(x)|$.*

Proof For any $L > 0$ and $\beta > 1$, set

$$\gamma(u_\lambda) := \gamma_{L,\beta}(u_\lambda) = u_\lambda u_{\lambda,L}^{2(\beta-1)} \in H^s(\mathbb{R}^N),$$

where $u_{\lambda,L} := \min\{u_\lambda, L\}$. Since γ is an increasing function, one has

$$(a - b)[\gamma(a) - \gamma(b)] \geq 0$$

for all $a, b \in \mathbb{R}$. Furthermore, set $\Gamma(t) := \int_0^t (\gamma'(\tau))^{\frac{1}{2}} d\tau$ for $t \geq 0$. Then for any $a, b \in \mathbb{R}$, if $a > b$ we obtain

$$(a - b)[\gamma(a) - \gamma(b)] = (a - b) \int_b^a \gamma'(t)dt = (a - b) \int_b^a (\Gamma'(t))^2 dt \geq \left(\int_b^a \Gamma'(t)dt \right)^2 = |\Gamma(a) - \Gamma(b)|^2.$$

We can use a similar argument to obtain the above conclusion if $a \leq b$. Therefore,

$$(a - b)[\gamma(a) - \gamma(b)] \geq |\Gamma(a) - \Gamma(b)|^2$$

for all $a, b \in \mathbb{R}$. Consequently,

$$|\Gamma(u_\lambda)(x) - \Gamma(u_\lambda)(y)|^2 \leq [u_\lambda(x) - u_\lambda(y)] \left[\left(u_\lambda u_{\lambda,L}^{2(\beta-1)} \right)(x) - \left(u_\lambda u_{\lambda,L}^{2(\beta-1)} \right)(y) \right],$$

which implies that

$$\begin{aligned} & \frac{C_{N,s}}{2} [\Gamma(u_\lambda)]^2 + \int_{\mathbb{R}^N} V(x) u_\lambda^2 u_{\lambda,L}^{2(\beta-1)} dx \\ & \leq \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{u_\lambda(x) - u_\lambda(y)}{|x - y|^{N+2s}} \left[\left(u_\lambda u_{\lambda,L}^{2(\beta-1)} \right)(x) - \left(u_\lambda u_{\lambda,L}^{2(\beta-1)} \right)(y) \right] dx dy + \int_{\mathbb{R}^N} V(x) u_\lambda^2 u_{\lambda,L}^{2(\beta-1)} dx \quad (2.2) \\ & = \int_{\mathbb{R}^N} f(x, u_\lambda) u_\lambda u_{\lambda,L}^{2(\beta-1)} dx + \lambda \int_{\mathbb{R}^N} [|x|^{-\mu} * \Phi(u_\lambda)] \phi(u_\lambda) u_\lambda u_{\lambda,L}^{2(\beta-1)} dx. \end{aligned}$$

By the fact that $\Gamma(u_\lambda) \geq \frac{1}{\beta} u_\lambda u_{\lambda,L}^{\beta-1}$ we see that

$$[\Gamma(u_\lambda)]^2 \geq S_* \|\Gamma(u_\lambda)\|_{2_s^*}^2 \geq \left(\frac{1}{\beta} \right)^2 S_* \|u_\lambda u_{\lambda,L}^{\beta-1}\|_{2_s^*}^2, \quad (2.3)$$

where $S_* = S(N, s) > 0$ is a sharp constant that satisfies $S_* \|u\|_{2_s^*}^2 \leq \|u\|^2$ for any $u \in H^s(\mathbb{R}^N)$ ([14]). By the proof of (2.1) we know that there exists a constant $C_0 > 0$ such that

$$||x|^{-\mu} * \Phi(u_\lambda)| \leq C_0 M^{p-q} \quad (2.4)$$

Moreover, by virtue of (f_1) – (f_2) we know that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x, t)| \leq \varepsilon |t| + C_\varepsilon |t|^{q-1} \quad (2.5)$$

for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. For fixed $\lambda > 0$ and small $\varepsilon > 0$, by (2.5) and properties of ϕ we have

$$|f(x, t) + \lambda \phi(t)| \leq \frac{V_0}{\max\{1, C_0 M^{p-q}\}} |t| + C(1 + \lambda) |t|^{q-1} \quad (2.6)$$

for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. Therefore, in view of (2.2)–(2.4) and (2.6) one has

$$\begin{aligned}
 & \frac{C_{N,s}}{2} \left(\frac{1}{\beta}\right)^2 S_* \|u_\lambda u_{\lambda,L}^{\beta-1}\|_{2_s^*}^2 \leq \frac{C_{N,s}}{2} [\Gamma(u_\lambda)]^2 \\
 & \leq \max\{1, C_0 M^{p-q}\} \int_{\mathbb{R}^N} \left[\frac{V_0}{\max\{1, C_0 M^{p-q}\}} |u_\lambda| + C(1 + \lambda) |u_\lambda|^{q-1} \right] |u_\lambda u_{\lambda,L}^{2(\beta-1)}| dx \\
 & \quad - \int_{\mathbb{R}^N} V(x) u_\lambda^2 u_{\lambda,L}^{2(\beta-1)} dx \\
 & \leq C(1 + \lambda) \int_{\mathbb{R}^N} |u_\lambda|^q |u_{\lambda,L}^{2(\beta-1)}| dx.
 \end{aligned}
 \tag{2.7}$$

Set $w_{\lambda,L} := u_\lambda u_{\lambda,L}^{\beta-1}$. By applying the Hölder inequality and (2.7), we get

$$\begin{aligned}
 \|w_{\lambda,L}\|_{2_s^*}^2 &= \|u_\lambda u_{\lambda,L}^{\beta-1}\|_{2_s^*}^2 \\
 &\leq \beta^2 C(1 + \lambda) \int_{\mathbb{R}^N} |u_\lambda|^q |u_{\lambda,L}^{2(\beta-1)}| dx \\
 &= \beta^2 C(1 + \lambda) \int_{\mathbb{R}^N} |u_\lambda|^{q-2} w_{\lambda,L}^2 dx \\
 &\leq \beta^2 C(1 + \lambda) \left(\int_{\mathbb{R}^N} |u_\lambda|^{2_s^*} dx \right)^{\frac{q-2}{2_s^*}} \left(\int_{\mathbb{R}^N} w_{L,\lambda}^{\alpha_s^*} dx \right)^{\frac{2}{\alpha_s^*}} \\
 &\leq \beta^2 C(1 + \lambda) \|w_{L,\lambda}\|_{\alpha_s^*}^2,
 \end{aligned}
 \tag{2.8}$$

where $\alpha_s^* := \frac{22_s^*}{2_s^* - (q-2)} \in (2, 2_s^*)$. Now, we observe that if $u_\lambda^\beta \in L^{\alpha_s^*}(\mathbb{R}^N)$, from the definition of $w_{\lambda,L}$, and by using the fact that $u_{\lambda,L} \leq u_\lambda$ and (2.8) we obtain

$$\|w_{\lambda,L}\|_{2_s^*}^2 \leq C \beta^2 (1 + \lambda) \left(\int_{\mathbb{R}^N} |u_\lambda|^{\beta \alpha_s^*} \right)^{\frac{2}{\alpha_s^*}} < +\infty.$$

Using the Fatou Lemma in $L \rightarrow +\infty$ one has

$$\|u_\lambda\|_{\beta 2_s^*} \leq C^{\frac{1}{\beta}} \beta^{\frac{1}{\beta}} (1 + \lambda)^{\frac{1}{2\beta}} \|u_\lambda\|_{\beta \alpha_s^*},
 \tag{2.9}$$

where $u_\lambda^{\beta \alpha_s^*} \in L^1(\mathbb{R}^N)$.

Now, we take $\beta = \frac{2_s^*}{\alpha_s^*} > 1$. By $u_\lambda \in L^{2_s^*}(\mathbb{R}^N)$, we know that (2.9) still holds for this choice of β . Then, observing that $\beta^2 \alpha_s^* = \beta 2_s^*$, it follows that (2.9) holds with β replaced by β^2 . Therefore,

$$\begin{aligned} \|u_\lambda\|_{\beta^2 2_s^*} &\leq C^{\frac{1}{\beta^2}} \beta^{\frac{2}{\beta^2}} (1 + \lambda)^{\frac{1}{2\beta^2}} \|u_\lambda\|_{\beta^2 \alpha_s^*} \\ &= C^{\frac{1}{\beta^2}} \beta^{\frac{2}{\beta^2}} (1 + \lambda)^{\frac{1}{2\beta^2}} \|u_\lambda\|_{\beta 2_s^*} \\ &\leq C^{\frac{1}{\beta} + \frac{1}{\beta^2}} \beta^{\frac{1}{\beta} + \frac{2}{\beta^2}} (1 + \lambda)^{\frac{1}{2\beta} + \frac{1}{2\beta^2}} \|u_\lambda\|_{\beta \alpha_s^*}. \end{aligned}$$

Iterating this process and recalling that $\beta \alpha_s^* = 2_s^*$, we can infer that for every $m \in \mathbb{N}$,

$$\begin{aligned} \|u_\lambda\|_{\beta^m 2_s^*} &\leq C^{\frac{1}{\beta} + \frac{1}{\beta^2} + \dots + \frac{1}{\beta^m}} \beta^{\frac{1}{\beta} + \frac{2}{\beta^2} + \dots + \frac{m}{\beta^m}} (1 + \lambda)^{\frac{1}{2\beta} + \frac{1}{2\beta^2} + \dots + \frac{1}{2\beta^m}} \|u_\lambda\|_{2_s^*} \\ &= C^{\sum_{j=1}^m \frac{1}{\beta^j}} \beta^{\sum_{j=1}^m \frac{j}{\beta^j}} (1 + \lambda)^{\frac{1}{2} \sum_{j=1}^m \frac{1}{\beta^j}} \|u_\lambda\|_{2_s^*}. \end{aligned}$$

Let $m \rightarrow +\infty$ and recalling that $\|u_\lambda\|_{2_s^*} \leq K$ we obtain

$$\|u_\lambda\|_{L^\infty} \leq C^{\sigma_1} \beta^{\sigma_2} (1 + \lambda)^{\frac{1}{2} \sigma_1} K = C^{\sigma_1} \left(\frac{2_s^*}{\alpha_s^*}\right)^{\sigma_2} K (1 + \lambda)^{\frac{1}{2} \sigma_1} := B(1 + \lambda)^D.$$

This completes the proof. □

Proof of Theorem 1.1 For large $M > 0$, we can choose small $\lambda_0 > 0$ such that $\|u_\lambda\|_{L^\infty} \leq B(1 + \lambda)^D \leq M$ for all $\lambda \in (0, \lambda_0]$. Consequently, u_λ is a nontrivial solution of (1.1) with $\lambda \in (0, \lambda_0]$. This completes the proof. □

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