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RESEARCH ARTICLE

Dirac–Witten Operators and the Kastler–Kalau–Walze Type Theorem for Manifolds with Boundary

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Abstract

In this paper, we obtain two Lichnerowicz type formulas for the Dirac–Witten operators. And we give the proof of Kastler–Kalau–Walze type theorems for the Dirac–Witten operators on 4-dimensional and 6-dimensional compact manifolds with (resp. without) boundary.

Keywords Dirac–Witten operator · Lichnerowicz type formulas · Noncommutative residue · Kastler–Kalau–Walze type theorems

1 Introduction

Until now, many geometers have studied noncommutative residues. In [5, 16], authors found noncommutative residues are of great importance to the study of noncommutative geometry. In [2], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Connes showed us that the noncommutative residue on a compact manifold M coincided with the Dixmier's trace on pseudodifferential operators of order-dim$M$ in [3]. And Connes claimed the noncommutative residue of the square of the inverse of the Dirac operator was proportioned to the Einstein–Hilbert action. Kastler [7] gave a brute-force proof of this theorem. Kalau and Walze proved this theorem in the normal coordinates system simultaneously in [6]. Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac

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operator $\mathrm{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of D^2 in [1].

On the other hand, Wang generalized the Connes' results to the case of manifolds with boundary in [10, 11], and proved the Kastler–Kalau–Walze type theorem for the Dirac operator and the signature operator on lower-dimensional manifolds with boundary [12]. In [12, 13], Wang computed $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-1}]$ and $\widetilde{\mathrm{Wres}}[\pi^+D^{-2}\circ\pi^+D^{-2}]$, where two operators are symmetric. And in these cases, the boundary term vanished. But for $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-3}]$, Wang got a nonvanishing boundary term [14], and give a theoretical explanation for gravitational action on boundary. And then, Wang provides a kind of method to study the Kastler–Kalau–Walze type theorem for manifolds with boundary. In [8], López and his collaborators introduced an elliptic differential operator, which is called the Novikov operator. In [15], Wei and Wang proved Kastler–Kalau–Walze type theorem for modified Novikov operators on compact manifolds. In [17], in order to prove the nonsymmetric positive mass theorem, Zhang introduced the Dirac–Witten operator. The motivation of this paper is to prove the Kastler–Kalau–Walze type theorem for the Dirac–Witten operators.

The paper is organized in the following way. In Sect. 2, by using the definition of the Dirac–Witten operators, we compute the Lichnerowicz formulas for the Dirac–Witten operators. In Sects. 3 and 4, we prove the Kastler–Kalau–Walze type theorem for 4-dimensional and 6-dimensional manifolds with boundary for the Dirac–Witten operators respectively.

2 The Dirac–Witten Operators and Their Lichnerowicz Formulas

Firstly we introduce some notations about the Dirac–Witten operators. Let M be a n-dimensional ($n \geq 3$) oriented compact spin Riemannian manifold with a Riemannian metric g^M. And let ∇^L be the Levi-Civita connection about g^M. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{e_1, \cdots, e_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(e_1, \ldots, e_n) = (e_1, \ldots, e_n)(\omega_{s,t}). \tag{2.1}$$

Let $c(e_j)$ be the Clifford action. Suppose that ∂_i is a natural local frame on TM and $(g^{ij})_{1\leq i,j\leq n}$ is the inverse matrix associated to the metric matrix $(g_{ij})_{1\leq i,j\leq n}$ on M. By [12], we have the Dirac operator

$$D = \sum_{i=1}^{n} c(e_i)\left[e_i - \frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)c(e_s)c(e_t)\right] \tag{2.2}$$

Then the Dirac–Witten operators $\widetilde{D}$ and $\widetilde{D}^*$ are defined by

$$\begin{aligned}
\widetilde{D} &= D + f_1 \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + f_2 \\
&= \sum_{i=1}^{n} c(e_i) \left[e_i - \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) c(e_s) c(e_t) \right] \\
&\quad + f_1 \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + f_2, \\
\widetilde{D}^* &= D - \overline{f_1} \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + \overline{f_2} \\
&= \sum_{i=1}^{n} c(e_i) \left[e_i - \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) c(e_s) c(e_t) \right] \\
&\quad - \overline{f_1} \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + \overline{f_2}.
\end{aligned} \tag{2.3}$$

where f_1, f_2 are complex numbers, p is a (0, 2)-tensor and $p_{uv} = p(e_u, e_v)$. Then when $f_1 = \frac{\sqrt{-1}}{2}$, $f_2 = -\frac{\sqrt{-1}}{2} \sum_i p_{ii}$, $\widetilde{D}$ is the Dirac–Witten operator defined by [17].

Then, we get the following Lichnerowicz formulas,

Theorem 2.1 *The following equalities hold:*

$$\begin{aligned}
\widetilde{D}^*\widetilde{D} &= -\left[g^{ij} (\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla^L_{\partial_i} \partial_j}) \right] + \frac{1}{4} s + (f_1 \overline{f_2} - \overline{f_1} f_2) \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) \\
&\quad + \frac{1}{4} \sum_i \left[c(e_i) \left(f_1 \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + f_2 \right) \right. \\
&\quad \left. + \left(-\overline{f_1} \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + \overline{f_2} \right) c(e_i) \right]^2 \\
&\quad + \frac{1}{2} \sum_j [c(e_j) e_j \left(f_1 \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + f_2 \right) \\
&\quad - e_j \left(-\overline{f_1} \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + \overline{f_2} \right) c(e_j)] \\
&\quad - f_1 \overline{f_1} \left[\sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) \right]^2 + f_2 \overline{f_2}, \\
\widetilde{D}^2 &= -\left[g^{ij} (\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla^L_{\partial_i} \partial_j}) \right] + \frac{1}{4} s + {f_1}^2 \left[\sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) \right]^2 \\
&\quad + \frac{1}{4} \sum_i \left[c(e_i) \left(f_1 \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + f_2 \right) \right. \\
&\quad \left. + \left(f_1 \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + f_2 \right) c(e_i) \right]^2 \\
&\quad - \frac{1}{2} \sum_j \left[e_j \left(f_1 \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + f_2 \right) c(e_j) \right. \\
&\quad \left. - c(e_j) e_j \left(f_1 \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + f_2 \right) \right] \\
&\quad + 2 f_1 f_2 \sum_{u<v} (p_{uv} - p_{vu}) c(e_u) c(e_v) + {f_2}^2.
\end{aligned} \tag{2.4}$$

where s is the scalar curvature.

Proof Let M be a smooth compact oriented spin Riemannian n-dimensional manifolds without boundary and N be a vector bundle on M. If P is a differential operator of Laplace type, then it has locally the form

$$P = -(g^{ij}\partial_i\partial_j + A^i\partial_i + B), \tag{2.5}$$

where ∂_i is a natural local frame on TM and $(g^{ij})_{1\leq i,j\leq n}$ is the inverse matrix associated to the metric matrix $(g_{ij})_{1\leq i,j\leq n}$ on M, and A^i and B are smooth sections of $\mathrm{End}(N)$ on M (endomorphism). If a Laplace type operator P satisfies (2.5), then there is a unique connection ∇ on N and a unique endomorphism E such that

$$P = -\Big[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla^L_{\partial_i}\partial_j}) + E\Big], \tag{2.6}$$

where ∇^L is the Levi-Civita connection on M. Moreover (with local frames of T^*M and N), $\nabla_{\partial_i} = \partial_i + \omega_i$ and E is related to g^{ij}, A^i and B through

$$\omega_i = \frac{1}{2}g_{ij}\big(A^i + g^{kl}\Gamma^j_{kl}\mathtt{id}\big), \tag{2.7}$$

$$E = B - g^{ij}\big(\partial_i(\omega_j) + \omega_i\omega_j - \omega_k\Gamma^k_{ij}\big), \tag{2.8}$$

where Γ^j_{kl} are the Christoffel coefficients of ∇^L.

Let $g^{ij} = g(dx_i, dx_j)$, $\xi = \sum_k \xi_j dx_j$ and $\nabla^L_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k$, we denote that

$$\begin{aligned} \sigma_i &= -\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)c(e_s)c(e_t);\\ \xi^j &= g^{ij}\xi_i;\quad \Gamma^k = g^{ij}\Gamma^k_{ij};\quad \sigma^j = g^{ij}\sigma_i. \end{aligned} \tag{2.9}$$

Then the Dirac–Witten operators $\widetilde{D}$ and $\widetilde{D}^*$ can be written as

$$\begin{aligned} \widetilde{D} &= \sum_{i=1}^n c(e_i)[e_i + \sigma_i] + f_1\sum_{u<v}(p_{uv} - p_{vu})c(e_u)c(e_v) + f_2;\\ \widetilde{D}^* &= \sum_{i=1}^n c(e_i)[e_i + \sigma_i] - \overline{f_1}\sum_{u<v}(p_{uv} - p_{vu})c(e_u)c(e_v) + \overline{f_2}. \end{aligned} \tag{2.10}$$

By [7], we have

$$\begin{aligned}D^2 &= -\Delta_0 + \frac{1}{4}s\\&= -g^{ij}(\nabla_i^L\nabla_j^L - \Gamma_{ij}^k\nabla_k^L) + \frac{1}{4}s\\&= -\sum_{ij} g^{ij}[\partial_i\partial_j + 2\sigma_i\partial_j - \Gamma_{ij}^k\partial_k + \partial_i\sigma_j + \sigma_i\sigma_j - \Gamma_{ij}^k\sigma_k] + \frac{1}{4}s.\end{aligned}\tag{2.11}$$

By (2.10), we have

$$\begin{aligned}\widetilde{D}^*\widetilde{D} = D^2 &+ D[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2]\\&+\left[-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right]D\\&+\left[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right]\\&\left[-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right],\end{aligned}\tag{2.12}$$

$$\begin{aligned}&D\left[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right]+\left[-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right]D\\&=\sum_{i,j}g^{i,j}\left[c(\partial_i)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\&\quad\left.+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(\partial_i)\right]\partial_j\\&\quad-\sum_{i,j}g^{i,j}\left[\left(\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-\overline{f_2}\right)c(\partial_i)\sigma_j\right.\\&\quad-c(\partial_i)\partial_j\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\\&\quad\left.-c(\partial_i)\sigma_j\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right],\end{aligned}\tag{2.13}$$

then we obtain

$$
\begin{aligned}
\widetilde{D}^*\widetilde{D} = & -\sum_{i,j} g^{i,j}\Big[\partial_i\partial_j + 2\sigma_i\partial_j - \Gamma_{i,j}^k\partial_k + \partial_i\sigma_j + \sigma_i\sigma_j - \Gamma_{i,j}^k\sigma_k\Big] \\
& + \frac{1}{4}s + \sum_{i,j} g^{i,j}\left[c(\partial_i)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right. \\
& \left.+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(\partial_i)\right]\partial_j \\
& -\sum_{i,j} g^{i,j}\left[\left(\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-\overline{f_2}\right)c(\partial_i)\sigma_j\right. \\
& - c(\partial_i)\partial_j\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right) \\
& \left.-c(\partial_i)\sigma_j\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right] \\
& -f_1\overline{f_1}\left[\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)\right]^2 \\
& + \left(f_1\overline{f_2}-\overline{f_1}f_2\right)\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\overline{f_2}.
\end{aligned} \tag{2.14}
$$

Similarly, we have

$$
\begin{aligned}
\widetilde{D}^2 = & -\sum_{i,j} g^{i,j}\Big[\partial_i\partial_j + 2\sigma_i\partial_j - \Gamma_{i,j}^k\partial_k + \partial_i\sigma_j + \sigma_i\sigma_j - \Gamma_{i,j}^k\sigma_k\Big] \\
& + \frac{1}{4}s + \sum_{i,j} g^{i,j}\left[c(\partial_i)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right. \\
& \left.+\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)c(\partial_i)\right]\partial_j \\
& +\sum_{i,j} g^{i,j}\left[(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2)c(\partial_i)\sigma_j\right. \\
& + c(\partial_i)\partial_j\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right) \\
& \left.+c(\partial_i)\sigma_j\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right] \\
& +f_1^2\left[\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)\right]^2 \\
& + 2f_1f_2\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2^2.
\end{aligned} \tag{2.15}
$$

By (2.6), (2.7), (2.8) and (2.14), we have

$$(\omega_i)_{\widetilde{D}^*\widetilde{D}} = \sigma_i - \frac{1}{2}\left[c(\partial_i)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.$$
$$\left.+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(\partial_i)\right]. \tag{2.16}$$

$$\begin{aligned}
E_{\widetilde{D}^*\widetilde{D}} = &-c(\partial_i)\sigma^i\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\\
&-\frac{1}{4}s+\left(\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-\overline{f_2}\right)\\
&c(\partial_i)\sigma^i+c(\partial_i)\partial^i\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\\
&+\frac{1}{2}\partial^j\left[c(\partial_j)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\
&\left.+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(\partial_j)\right]\\
&-\frac{1}{2}\left[c(\partial_j)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\
&\left.+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(\partial_j)\right]\sigma^j\\
&-\frac{g^{ij}}{4}\left[c(\partial_i)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\
&\left.+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(\partial^i)\right]\\
&\cdot\left[c(\partial_j)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\
&\left.-\left(\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-\overline{f_2}\right)c(\partial_j)\right]\\
&-\frac{1}{2}\Gamma^k\left[c(\partial_k)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\
&\left.+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(\partial_k)\right]\\
&-\frac{1}{2}\sigma^j\left[c(\partial_j)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\
&+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(\partial_j)]\\
&+f_1\overline{f_1}\left[\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)\right]^2-\left(f_1\overline{f_2}-\overline{f_1}f_2\right)\\
&\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-f_2\overline{f_2}.
\end{aligned} \tag{2.17}$$

Since E is globally defined on M, taking normal coordinates at x_0, we have $\sigma^i(x_0)=0$, $\partial^j[c(\partial_j)](x_0)=0$, $\Gamma^k(x_0)=0$, $g^{ij}(x_0)=\delta_i^j$, then

$$\begin{aligned}
E_{\widetilde{D}^*\widetilde{D}}(x_0)=&-\frac{1}{4}s-\left(f_1\overline{f_2}-\overline{f_1}f_2\right)\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)\\
&+f_1\overline{f_1}\left[\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)\right]^2\\
&-f_2\overline{f_2}-\frac{1}{4}\sum_i\left[c(e_i)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\
&\left.+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(e_i)\right]^2\\
&-\frac{1}{2}\left[c(e_j)\nabla_{e_j}^{\bigwedge^* T^*M}\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\
&\left.-\nabla_{e_j}^{\bigwedge^* T^*M}\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)c(e_j)\right].
\end{aligned}\tag{2.18}$$

Similarly, we have

$$\begin{aligned}
E_{\widetilde{D}^2}(x_0)=&-\frac{1}{4}s-f_1^2\left[\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)\right]^2\\
&-2f_1f_2\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-f_2^2\\
&-\frac{1}{4}\sum_i\left[c(e_i)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right.\\
&\left.+\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)c(e_i)\right]^2\\
&+\frac{1}{2}\left[\nabla_{e_j}^{\bigwedge^* T^*M}\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)c(e_j)\right.\\
&\left.-c(e_j)\nabla_{e_j}^{\bigwedge^* T^*M}\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right],
\end{aligned}\tag{2.19}$$

by (2.5), we get Theorem 2.1. □

From [1], we know that the noncommutative residue of an appropriate power of a generalized laplacian $\overline{\Delta}$ is expressed as

$$(n-2)\Phi_2(\overline{\Delta}) = (4\pi)^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)Wres\left(\overline{\Delta}^{-\frac{n}{2}+1}\right), \tag{2.20}$$

where $\Phi_2(\overline{\Delta})$ denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of $\overline{\Delta}$. Now let $\overline{\Delta} = \widetilde{D}^*\widetilde{D}$ and $\widetilde{D}^*\widetilde{D} = \Delta - E$, then we have

$$\mathrm{Wres}(\widetilde{D}^*\widetilde{D})^{-\frac{n-2}{2}} = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{\left(\frac{n}{2}-1\right)!}\int_M \mathrm{tr}\left(\frac{1}{6}s + E_{\widetilde{D}^*\widetilde{D}}\right)d\mathrm{Vol}_{\mathrm{M}}, \tag{2.21}$$

$$\mathrm{Wres}(\widetilde{D}^2)^{-\frac{n-2}{2}} = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M \mathrm{tr}\left(\frac{1}{6}s + E_{\widetilde{D}^2}\right)d\mathrm{Vol}_{\mathrm{M}}, \tag{2.22}$$

where Wres denotes the noncommutative residue. Then,

$$\begin{aligned} tr\left[\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)\right] &= \sum_{u<v}(p_{uv}-p_{vu})tr[c(e_u)c(e_v)] \\ &= \sum_{u<v}(p_{uv}-p_{vu})tr[c(e_v)c(e_u)] \\ &= -\sum_{u<v}(p_{uv}-p_{vu})tr[c(e_u)c(e_v)] = 0. \end{aligned} \tag{2.23}$$

By

$$\begin{aligned} &tr\left[\sum_{u<v}\sum_{s<t}c(e_u)c(e_v)c(e_s)c(e_t)\right] \\ &= \begin{cases} -tr[id], & \text{if } u=s, v=t, \\ 0, & other\ cases. \end{cases} \end{aligned} \tag{2.24}$$

$$\begin{aligned} &tr\left\{\left[\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)c(e_u)c(e_v)\right]^2\right\} \\ &= tr\left[\sum_{u<v}\sum_{s<t}(p_{uv}-p_{vu})(p_{st}-p_{ts})c(e_u)c(e_v)c(e_s)c(e_t)\right] \\ &= -\sum_{u<v}(p_{uv}-p_{vu})^2 tr[id]. \end{aligned} \tag{2.25}$$

Similarly, we have

$$tr\Bigg(\sum_i\Bigg[c(e_i)\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)+\Bigg(-\bar{f}_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\bar{f}_2\Bigg)c(e_i)\Bigg]^2\Bigg)$$
$$=f_1^2(n-4)\sum_{u<v}(p_{uv}-p_{vu})^2tr[id]-nf_2^2tr[id]+\bar{f}_1^2(n-4)\sum_{u<v}(p_{uv}-p_{vu})^2tr[id]$$
$$-n\bar{f}_2^2tr[id]-2nf_1\bar{f}_1\sum_{u<v}(p_{uv}-p_{vu})^2tr[id]-2nf_2\bar{f}_2\sum_{u<v}(p_{uv}-p_{vu})^2tr[id]$$
$$=\Bigg[f_1^2(n-4)\sum_{u<v}(p_{uv}-p_{vu})^2-nf_2^2\Bigg]\mathrm{tr}[\mathtt{id}];$$
$$tr\Bigg(\sum_j c(e_j)\nabla_{e_j}^{\bigwedge^* T^*M}\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)\Bigg)$$
$$=tr\Bigg(\sum_j c(e_j)\Bigg[\sum_{u<v}\nabla_{e_j}(f_1(p_{uv}-p_{vu}))c(e_u)c(e_v)+f_1(p_{uv}-p_{vu})c(\nabla_{e_j}e_u)c(e_v)$$
$$+f_1(p_{uv}-p_{vu})c(e_u)c(\nabla_{e_j}e_v)\Bigg]+\sum_j c(e_j)\nabla_{e_j}(f_2)\Bigg)$$
$$=0. \tag{2.26}$$

Then by (2.23)–(2.26), we get

$$tr(E_{\widetilde{D}^*\widetilde{D}})=\Bigg[-\frac{s}{4}-\frac{1}{4}\{[(f_1^2+\overline{f_1}^2)(n-4)-2nf_1\overline{f_1}]\sum_{u<v}(p_{uv}-p_{vu})^2$$
$$+2nf_2\overline{f_2}-nf_2^2-n\overline{f_2}\}-f_2\overline{f_2}-f_1\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})^2\Bigg]\mathrm{tr}[\mathtt{id}],$$
$$tr(E_{\widetilde{D}^2})=\Bigg[-\frac{s}{4}+(3-n)f_1^2\sum_{u<v}(p_{uv}-p_{vu})^2+(n-1)f_2^2\Bigg]\mathrm{tr}[\mathtt{id}]. \tag{2.27}$$

By (2.21), (2.22) and (2.27), we can get the following theorem,

Theorem 2.2 *If M is a n-dimensional compact oriented spin manifolds without boundary, and n is even, then we get the following equalities:*

$$\mathrm{Wres}(\widetilde{D}^*\widetilde{D})^{-\frac{n-2}{2}}=\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M 2^{\frac{n}{2}}\Bigg(-\frac{1}{12}s-\frac{1}{4}\{[(f_1^2+\overline{f_1}^2)(n-4)-2nf_1\overline{f_1}]$$
$$\sum_{u<v}(p_{uv}-p_{vu})^2+2nf_2\overline{f_2}-nf_2^2-n\overline{f_2}^2\}-f_2\overline{f_2}-f_1\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})^2\Bigg)d\mathrm{Vol_M}. \tag{2.28}$$

$$\mathrm{Wres}(\widetilde{D}^2)^{-\frac{n-2}{2}}=\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M 2^{\frac{n}{2}}\Bigg(-\frac{1}{12}s+(3-n)f_1^2\sum_{u<v}(p_{uv}-p_{vu})^2+(n-1)f_2^2\Bigg)d\mathrm{Vol_M}. \tag{2.29}$$

where s is the scalar curvature.

3 A Kastler–Kalau–Walze Type Theorem for 4-Dimensional Manifolds with Boundary

We firstly recall some basic facts and formulas about Boutet de Monvel's calculus and the definition of the noncommutative residue for manifolds with boundary which will be used in the following. For more details (see in Sect. 2 in [12]).

Let $U \subset M$ be a collar neighborhood of ∂M which is diffeomorphic with $\partial M \times [0,1)$. By the definition of $h(x_n) \in C^\infty([0,1))$ and $h(x_n) > 0$, there exists $\widehat{h} \in C^\infty((-\varepsilon,1))$ such that $\widehat{h}|_{[0,1)} = h$ and $\widehat{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric g' on $\widetilde{M} = M \bigcup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \bigcup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$g' = \frac{1}{\widehat{h}(x_n)} g^{\partial M} + dx_n^2, \tag{3.1}$$

such that $g'|_M = g$. We fix a metric g' on the $\widetilde{M}$ such that $g'|_M = g$.

Let Fourier transformation F' be

$$F' : L^2(\mathbf{R}_t) \to L^2(\mathbf{R}_v); F'(u)(v) = \int e^{-ivt} u(t) dt \tag{3.2}$$

and let

$$r^+ : C^\infty(\mathbf{R}) \to C^\infty(\widetilde{\mathbf{R}^+}); f \to f|\widetilde{\mathbf{R}^+}; \widetilde{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}. \tag{3.3}$$

where $\Phi(\mathbf{R})$ denotes the Schwartz space and $\Phi(\widetilde{\mathbf{R}^+}) = r^+\Phi(\mathbf{R})$, $\Phi(\widetilde{\mathbf{R}^-}) = r^-\Phi(\mathbf{R})$.

We define $H^+ = F'(\Phi(\widetilde{\mathbf{R}^+}))$; $H_0^- = F'(\Phi(\widetilde{\mathbf{R}^-}))$ which satisfies $H^+ \perp H_0^-$. We have the following property: $h \in H^+$ (H_0^-) if and only if $h \in C^\infty(\mathbf{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{\mathrm{Im}\xi < 0\}$ ($\{\mathrm{Im}\xi > 0\}$) such that for all nonnegative integer l,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l}\left(\frac{c_k}{\xi^k}\right), \tag{3.4}$$

as $|\xi| \to +\infty, \mathrm{Im}\xi \leq 0$ $(\mathrm{Im}\xi \geq 0)$. Let H' be the space of all polynomials and $H^- = H_0^- \bigoplus H'$; $H = H^+ \bigoplus H^-$. Denote by π^+ (π^-) respectively the projection on H^+ (H^-). For calculations, we take $H = \widetilde{H} = \{$rational functions having no poles on the real axis$\}$ ($\tilde{H}$ is a dense set in the topology of H). Then on $\tilde{H}$,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \to 0^-} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \tag{3.5}$$

where Γ^+ is a Jordan close curve included $\mathrm{Im}(\xi) > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. Similarly, define π' on $\tilde{H}$,

$$\pi' h=\frac{1}{2\pi}\int_{\Gamma^+} h(\xi)d\xi. \tag{3.6}$$

So, $\pi'(H^-)=0$. For $h\in H\bigcap L^1(\mathbf{R})$, $\pi' h=\frac{1}{2\pi}\int_{\mathbf{R}} h(v)dv$ and for $h\in H^+\bigcap L^1(\mathbf{R})$, $\pi' h=0$.

Let M be a n-dimensional compact oriented spin manifold with boundary ∂M. Denote by $\mathcal{B}$ Boutet de Monvel's algebra, we recall the main theorem in [4, 12].

Theorem 3.1 [4] (*Fedosov–Golse–Leichtnam–Schrohe*) *Let X and ∂X be connected, $\dim X=n\geq 3$, $A=\begin{pmatrix}\pi^+P+G & K\\ T & S\end{pmatrix}\in\mathcal{B}$, and denote by p, b and s the local symbols of P, G and S respectively. Define:*

$$\begin{aligned}\widetilde{\mathrm{Wres}}(A)=&\int_X\int_{\mathbf{S}}\mathrm{tr}_E\left[p_{-n}(x,\xi)\right]\sigma(\xi)dx\\ &+2\pi\int_{\partial X}\int_{\mathbf{S}'}\left\{\mathrm{tr}_E\left[(\mathrm{tr}b_{-n})(x',\xi')\right]+\mathrm{tr}_F\left[s_{1-n}(x',\xi')\right]\right\}\sigma(\xi')dx',\end{aligned} \tag{3.7}$$

where $\widetilde{\mathrm{Wres}}$ denotes the noncommutative residue of an operator in the Boutet de Monvel's algebra.

Then a) $\widetilde{\mathrm{Wres}}([A,B])=0$, for any $A,B\in\mathcal{B}$; b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Definition 3.2 [12] Lower dimensional volumes of spin manifolds with boundary are defined by

$$\mathrm{Vol}_n^{(p_1,p_2)}M:=\widetilde{\mathrm{Wres}}[\pi^+D^{-p_1}\circ\pi^+D^{-p_2}], \tag{3.8}$$

By [12], we get

$$\begin{aligned}\widetilde{\mathrm{Wres}}[\pi^+D^{-p_1}\circ\pi^+D^{-p_2}]=&\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M\bigotimes\mathbb{C}}[\sigma_{-n}(D^{-p_1-p_2})]\sigma(\xi)dx\\ &+\int_{\partial M}\Phi,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\Phi=&\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times\mathrm{trace}_{\wedge^*T^*M\bigotimes\mathbb{C}}[\partial_{x_n}^j\partial_{\xi'}^\alpha\partial_{\xi_n}^k\sigma_r^+(D^{-p_1})(x',0,\xi',\xi_n)\\ &\times\partial_{x'}^\alpha\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l(D^{-p_2})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx',\end{aligned} \tag{3.10}$$

where the sum is taken over $r+l-k-|\alpha|-j-1=-n,\ r\leq -p_1, l\leq -p_2$ and $\widetilde{\mathrm{Wres}}$ denotes the noncommutative residue for manifolds with boundary.

Since $[\sigma_{-n}(D^{-p_1-p_2})]|_M$ has the same expression as $\sigma_{-n}(D^{-p_1-p_2})$ in the case of manifolds without boundary, so locally we can compute the first term by [6, 7, 9, 12].

For any fixed point $x_0\in\partial M$, we choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Phi(x_0)$ in the coordinates $\widetilde{U}=U\times[0,1)\subset M$ and the metric $\frac{1}{h(x_n)}g^{\partial M}+dx_n^2$. The dual metric of g^M on $\widetilde{U}$ is $h(x_n)g^{\partial M}+dx_n^2$. Write $g_{ij}^M=g^M(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})$; $g_M^{ij}=g^M(dx_i,dx_j)$, then

$$[g_{i,j}^M]=\begin{bmatrix}\frac{1}{h(x_n)}[g_{i,j}^{\partial M}] & 0\\ 0 & 1\end{bmatrix};\quad [g_M^{i,j}]=\begin{bmatrix}h(x_n)[g_{\partial M}^{i,j}] & 0\\ 0 & 1\end{bmatrix}, \tag{3.11}$$

and

$$\partial_{x_s}g_{ij}^{\partial M}(x_0)=0, 1\leq i,j\leq n-1;\quad g_{ij}^M(x_0)=\delta_{ij}. \tag{3.12}$$

From [12], we can get three lemmas.

Lemma 3.3 [12] *With the metric g^M on M near the boundary*

$$\partial_{x_j}(|\xi|_{g^M}^2)(x_0)=\begin{cases}0, & \text{if } j<n,\\ h'(0)|\xi'|_{g^{\partial M}}^2, & \text{if } j=n;\end{cases} \tag{3.13}$$

$$\partial_{x_j}[c(\xi)](x_0)=\begin{cases}0, & \text{if } j<n,\\ \partial x_n(c(\xi'))(x_0), & \text{if } j=n,\end{cases} \tag{3.14}$$

where $\xi=\xi'+\xi_n dx_n$.

Lemma 3.4 [12] *With the metric g^M on M near the boundary*

$$\omega_{s,t}(e_i)(x_0)=\begin{cases}\omega_{n,i}(e_i)(x_0)=\frac{1}{2}h'(0), & \text{if } s=n,t=i,i<n,\\ \omega_{i,n}(e_i)(x_0)=-\frac{1}{2}h'(0), & \text{if } s=i,t=n,i<n,\\ \omega_{s,t}(e_i)(x_0)=0, & \textit{other cases},\end{cases} \tag{3.15}$$

where $(\omega_{s,t})$ denotes the connection matrix of Levi-Civita connection ∇^L.

Lemma 3.5 [12]

$$\Gamma_{st}^k(x_0)=\begin{cases}\Gamma_{ii}^n(x_0)=\frac{1}{2}h'(0) & \text{if } s=t=i,k=n,i<n,\\ \Gamma_{ni}^i(x_0)=-\frac{1}{2}h'(0), & \text{if } s=n,t=i,k=i,i<n,\\ \Gamma_{in}^i(x_0)=-\frac{1}{2}h'(0), & \text{if } s=i,t=n,k=i,i<n,\\ \Gamma_{st}^i(x_0)=0, & \textit{other cases}.\end{cases} \tag{3.16}$$

By (3.6) and (3.7), we firstly compute

$$\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}^{-1}\circ\pi^+(\widetilde{D}^*)^{-1}] = \int_M\int_{|\xi|=1} \mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-4}((\widetilde{D}^*\widetilde{D})^{-1})]\sigma(\xi)dx + \int_{\partial M}\Phi, \tag{3.17}$$

where

$$\Phi = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times \mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r(\widetilde{D}^{-1})(x',0,\xi',\xi_n) \times\partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l((\widetilde{D}^*)^{-1})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx', \tag{3.18}$$

the sum is taken over $r+l-k-j-|\alpha|=-3,\ \ r\leq -1, l\leq -1$ and $\widetilde{\mathrm{Wres}}$ denotes the noncommutative residue for manifolds with boundary.

By Theorem 2.2, we can compute the interior of $\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}^{-1}\circ\pi^+(\widetilde{D}^*)^{-1}]$, so

$$\int_M\int_{|\xi|=1} \mathrm{trace}_{\wedge^*T^*M}[\sigma_{-4}((\widetilde{D}^*\widetilde{D})^{-1})]\sigma(\xi)dx = 32\pi^2\int_M\left\{-\frac{1}{3}s - 12f_2\overline{f_2} + 4f_2^2 + 4\overline{f_2}^2 + 4f_1\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})^2\right\}d\mathrm{Vol_M}. \tag{3.19}$$

Now we need to compute $\int_{\partial M}\Phi$. Since, some operators have the following symbols.

Lemma 3.6 *The following identities hold:*

$$\sigma_1(\widetilde{D}) = \sigma_1(\widetilde{D}^*) = ic(\xi);$$
$$\sigma_0(\widetilde{D}) = -\frac{1}{4}\sum_{i,s,t}\omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t) + \left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v) + f_2\right);$$
$$\sigma_0(\widetilde{D}^*) = -\frac{1}{4}\sum_{i,s,t}\omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t) + \left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v) + \overline{f_2}\right); \tag{3.20}$$

where $\xi=\Sigma^n_{i=1}\xi_i d_{x_i}$ *denotes the cotangent vector.*

Write

$$D^\alpha_x = (-i)^{|\alpha|}\partial^\alpha_x;\ \sigma(\widetilde{D}) = p_1 + p_0;\ \sigma(\widetilde{D})^{-1} = \sum_{j=1}^{\infty} q_{-j}. \tag{3.21}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned}1=\sigma(\widetilde{D}\circ\widetilde{D}^{-1})&=\sum_{\alpha}\frac{1}{\alpha!}\partial^{\alpha}_{\xi}[\sigma(\widetilde{D})]\widetilde{D}^{\alpha}_{x}[\sigma(\widetilde{D}^{-1})]\\&=(p_1+p_0)(q_{-1}+q_{-2}+q_{-3}+\cdots)\\&\quad+\sum_j(\partial_{\xi_j}p_1+\partial_{\xi_j}p_0)(D_{x_j}q_{-1}+D_{x_j}q_{-2}+D_{x_j}q_{-3}+\cdots)\\&=p_1q_{-1}+(p_1q_{-2}+p_0q_{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}q_{-1})+\cdots,\end{aligned}\tag{3.22}$$

so

$$q_{-1}=p_1^{-1};\,q_{-2}=-p_1^{-1}\left[p_0p_1^{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}(p_1^{-1})\right].\tag{3.23}$$

Lemma 3.7 *The following identities hold:*

$$\begin{aligned}\sigma_{-1}(\widetilde{D}^{-1})&=\sigma_{-1}((\widetilde{D}^*)^{-1})=\frac{ic(\xi)}{|\xi|^2};\\\sigma_{-2}(\widetilde{D}^{-1})&=\frac{c(\xi)\sigma_0(\widetilde{D})c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_jc(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)\Big];\\\sigma_{-2}((\widetilde{D}^*)^{-1})&=\frac{c(\xi)\sigma_0(\widetilde{D}^*)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_jc(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)\Big].\end{aligned}\tag{3.24}$$

When $n=4$, then $\mathrm{tr}_{\wedge^*T^*M}[\mathtt{id}]=\dim(\wedge^*(4))=4$, where tr as shorthand of trace, the sum is taken over $r+l-k-j-|\alpha|=-3,\ r\leq-1,l\leq-1$, then we have the following five cases:

Case (a) (I) $r=-1,\ l=-1,\ k=j=0,\ |\alpha|=1$

By (3.18), we get

$$\begin{aligned}\Phi_1=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}&\mathrm{tr}[\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\\&\times\partial^{\alpha}_{x'}\partial_{\xi_n}\sigma_{-1}((\widetilde{D}^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{3.25}$$

By Lemma 3.3, for $i<n$, then

$$\partial_{x_i}\left(\frac{ic(\xi)}{|\xi|^2}\right)(x_0)=\frac{i\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2}-\frac{ic(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4}=0,\tag{3.26}$$

so $\Phi_1=0$.

Case (a) (II) $r=-1,\ l=-1,\ k=|\alpha|=0,\ j=1$

By (3.18), we get

$$\Phi_2 = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.27}$$

By Lemma 3.7, we have

$$\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}^*)^{-1})(x_0) = i\left(-\frac{6\xi_n c(dx_n)+2c(\xi')}{|\xi|^4}+\frac{8\xi_n^2 c(\xi)}{|\xi|^6}\right); \tag{3.28}$$

$$\partial_{x_n}\sigma_{-1}(\widetilde{D}^{-1})(x_0) = \frac{i\partial_{x_n}c(\xi')(x_0)}{|\xi|^2}-\frac{ic(\xi)|\xi'|^2h'(0)}{|\xi|^4}. \tag{3.29}$$

By (3.5), (3.6), we get

$$\begin{aligned}\pi^+_{\xi_n}\left[\frac{c(\xi)}{|\xi|^4}\right](x_0)|_{|\xi'|=1} &= \pi^+_{\xi_n}\left[\frac{c(\xi')+\xi_n c(dx_n)}{(1+\xi_n^2)^2}\right]\\ &= \frac{1}{2\pi i}\lim_{u\to 0^-}\int_{\Gamma^+}\frac{\frac{c(\xi')+\eta_n c(dx_n)}{(\eta_n+i)^2(\xi_n+iu-\eta_n)}}{(\eta_n-i)^2}d\eta_n\\ &= -\frac{(i\xi_n+2)c(\xi')+ic(dx_n)}{4(\xi_n-i)^2}.\end{aligned} \tag{3.30}$$

Similarly we have,

$$\pi^+_{\xi_n}\left[\frac{i\partial_{x_n}c(\xi')}{|\xi|^2}\right](x_0)|_{|\xi'|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-i)}. \tag{3.31}$$

By (3.29), then

$$\pi^+_{\xi_n}\partial_{x_n}\sigma_{-1}(\widetilde{D}^{-1})|_{|\xi'|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-i)}+ih'(0)\left[\frac{(i\xi_n+2)c(\xi')+ic(dx_n)}{4(\xi_n-i)^2}\right]. \tag{3.32}$$

By the relation of the Clifford action and $\mathrm{tr}AB = \mathrm{tr}\,BA$, we have the equalities:

$$\begin{aligned}&\mathrm{tr}[c(\xi')c(dx_n)] = 0;\ \ \mathrm{tr}[c(dx_n)^2] = -4;\ \ \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -4;\\ &\mathrm{tr}[\partial_{x_n}c(\xi')c(dx_n)] = 0;\ \ \mathrm{tr}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1} = -2h'(0).\end{aligned} \tag{3.33}$$

By (3.31), we have

$$h'(0)\mathrm{tr}\bigg[\frac{(i\xi_n+2)c(\xi')+ic(dx_n)}{4(\xi_n-i)^2}\times\bigg(\frac{6\xi_nc(dx_n)+2c(\xi')}{(1+\xi_n^2)^2}-\frac{8\xi_n^2[c(\xi')+\xi_nc(dx_n)]}{(1+\xi_n^2)^3}\bigg)\bigg](x_0)|_{|\xi'|=1}$$
$$=-4h'(0)\frac{-2i\xi_n^2-\xi_n+i}{(\xi_n-i)^4(\xi_n+i)^3}. \tag{3.34}$$

Similarly, we have

$$-i\mathrm{tr}\bigg[\bigg(\frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-i)}\bigg)\times\bigg(\frac{6\xi_nc(dx_n)+2c(\xi')}{(1+\xi_n^2)^2}-\frac{8\xi_n^2[c(\xi')+\xi_nc(dx_n)]}{(1+\xi_n^2)^3}\bigg)\bigg](x_0)|_{|\xi'|=1}$$
$$=-2ih'(0)\frac{3\xi_n^2-1}{(\xi_n-i)^4(\xi_n+i)^3}. \tag{3.35}$$

Then

$$\begin{aligned}\Phi_2&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{ih'(0)(\xi_n-i)^2}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_n\sigma(\xi')dx'\\&=-ih'(0)\Omega_3\int_{\Gamma^+}\frac{1}{(\xi_n-i)^2(\xi_n+i)^3}d\xi_ndx'\\&=-ih'(0)\Omega_3 2\pi i\bigg[\frac{1}{(\xi_n+i)^3}\bigg]^{(1)}|_{\xi_n=i}dx'\\&=-\frac{3}{8}\pi h'(0)\Omega_3dx',\end{aligned} \tag{3.36}$$

where Ω_3 is the canonical volume of S^3.

Case (a) (III) $r=-1,\ l=-1,\ j=|\alpha|=0,\ k=1$

By (3.18), we get

$$\begin{aligned}\Phi_3=&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\\&\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned} \tag{3.37}$$

By Lemma 3.7, we have

$$\begin{aligned}\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}^*)^{-1})(x_0)|_{|\xi'|=1}=&-ih'(0)\bigg[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c(\xi')+\xi_nc(dx_n)}{|\xi|^6}\bigg]\\&-\frac{2\xi_ni\partial_{x_n}c(\xi')(x_0)}{|\xi|^4};\end{aligned} \tag{3.38}$$

$$\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})(x_0)|_{|\xi'|=1}=-\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}. \tag{3.39}$$

Similar to case a) II), we have

$$\begin{aligned}&\mathrm{tr}\left\{\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}\times ih'(0)\left[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c(\xi')+\xi_nc(dx_n)}{|\xi|^6}\right]\right\}\\&=2h'(0)\frac{i-3\xi_n}{(\xi_n-i)^4(\xi_n+i)^3}\end{aligned} \tag{3.40}$$

and

$$\mathrm{tr}\left[\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}\times\frac{2\xi_n i\partial_{x_n}c(\xi')(x_0)}{|\xi|^4}\right]=\frac{-2ih'(0)\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}. \tag{3.41}$$

So we have

$$\begin{aligned}\Phi_3=&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{h'(0)(i-3\xi_n)}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_n\sigma(\xi')dx'\\&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{h'(0)i\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}d\xi_n\sigma(\xi')dx'\\&=-h'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{(i-3\xi_n)}{(\xi_n+i)^3}\right]^{(3)}|_{\xi_n=i}dx'+h'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{i\xi_n}{(\xi_n+i)^2}\right]^{(3)}|_{\xi_n=i}dx'\\&=\frac{3}{8}\pi h'(0)\Omega_3dx'.\end{aligned} \tag{3.42}$$

Case (b) $r=-2,\ l=-1,\ k=j=|\alpha|=0$

By (3.18), we get

$$\Phi_4=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.43}$$

By Lemma 3.7 we have

$$\sigma_{-2}(\widetilde{D}^{-1})(x_0)=\frac{c(\xi)\sigma_0(\widetilde{D})(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)|\xi|^2_{\partial M}], \tag{3.44}$$

where

$$\begin{aligned}\sigma_0(\widetilde{D})(x_0) = -\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t)\\ +f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2.\end{aligned} \tag{3.45}$$

We denote

$$Q(x_0) = -\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t). \tag{3.46}$$

Then

$$\begin{aligned}&\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}^{-1}(x_0))|_{|\xi'|=1}\\ &= \pi^+_{\xi_n}\left[\frac{c(\xi)(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2)(x_0)c(\xi)}{(1+\xi_n^2)^2}\right]\\ &\quad+\pi^+_{\xi_n}\left[\frac{c(\xi)Q(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}-h'(0)\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^3}\right].\end{aligned} \tag{3.47}$$

And

$$\begin{aligned}&\mathrm{tr}\left[\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)c(dx_n)\right]\\ &\quad= f_1\sum_{u<v}(p_{uv}-p_{vu})\mathrm{tr}[c(e_u)c(e_v)c(dx_n)]\\ &\quad= 0;\\ &\mathrm{tr}\left[\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)c(\xi')\right]\\ &\quad= \mathrm{tr}\left[\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\sum_{j=1}^{n-1}\xi_jc(e_j)\right]\\ &\quad= 0.\end{aligned} \tag{3.48}$$

Since

$$\partial_{\xi_n}\sigma_{-1}((\widetilde{D}^*)^{-1}) = \partial_{\xi_n}q_{-1}(x_0)|_{|\xi'|=1} = i\left[\frac{c(dx_n)}{1+\xi_n^2}-\frac{2\xi_nc(\xi')+2\xi_n^2c(dx_n)}{(1+\xi_n^2)^2}\right]. \tag{3.49}$$

Then, we have

$$\pi^+_{\xi_n}\left[\frac{c(\xi)Q(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}\right]-h'(0)\pi^+_{\xi_n}\left[\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n)^3}\right] \\ := C_1 - C_2, \tag{3.50}$$

where

$$C_1=\frac{-1}{4(\xi_n-i)^2}[(2+i\xi_n)c(\xi')Q(x_0)c(\xi')+i\xi_n c(dx_n)Q(x_0)c(dx_n) \\ +(2+i\xi_n)c(\xi')c(dx_n)\partial_{x_n}c(\xi')+ic(dx_n)Q(x_0)c(\xi')+ic(\xi')Q(x_0)c(dx_n)-i\partial_{x_n}c(\xi')] \tag{3.51}$$

and

$$C_2=\frac{h'(0)}{2}\left[\frac{c(dx_n)}{4i(\xi_n-i)}+\frac{c(dx_n)-ic(\xi')}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}[ic(\xi')-c(dx_n)]\right]. \tag{3.52}$$

By (3.50) and (3.52), we have

$$\begin{aligned}\mathrm{tr}\,[C_2\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}^*)^{-1})]|_{|\xi'|=1}=&\frac{i}{2}h'(0)\frac{-i\xi_n^2-\xi_n+4i}{4(\xi_n-i)^3(\xi_n+i)^2}\mathrm{tr}[\mathtt{id}]\\ =&2ih'(0)\frac{-i\xi_n^2-\xi_n+4i}{4(\xi_n-i)^3(\xi_n+i)^2}.\end{aligned} \tag{3.53}$$

By (3.50) and (3.51), we have

$$\mathrm{tr}\,[C_1\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}^*)^{-1})]|_{|\xi'|=1}=\frac{-2ic_0}{(1+\xi_n^2)^2}+h'(0)\frac{\xi_n^2-i\xi_n-2}{2(\xi_n-i)(1+\xi_n^2)^2}, \tag{3.54}$$

where $Q=c_0c(dx_n)$ and $c_0=-\frac{3}{4}h'(0)$.

By (3.53) and (3.54), we have

$$\begin{aligned}&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[(C_1-C_2)\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\ &=-\Omega_3\int_{\Gamma^+}\frac{2c_0(\xi_n-i)+ih'(0)}{(\xi_n-i)^3(\xi_n+i)^2}d\xi_n dx'\\ &=\frac{9}{8}\pi h'(0)\Omega_3 dx'.\end{aligned} \tag{3.55}$$

Then, we have

$$
\begin{aligned}
&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\pi^+_{\xi_n}\left[\frac{c(\xi)(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2)c(\xi)}{(1+\xi_n^2)^2}\right]\right.\\
&\left.\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}^*)^{-1})\right](x_0)d\xi_n\sigma(\xi')dx'\\
&=\frac{\pi}{4}\mathrm{tr}\left[c(dx_n)\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)\right]\Omega_3dx'\\
&=0.
\end{aligned}
\tag{3.56}
$$

Then, we have

$$
\Phi_4=\frac{9}{8}\pi h'(0)\Omega_3dx'. \tag{3.57}
$$

Case (c) $r=-1,\ l=-2,\ k=j=|\alpha|=0$

By (3.18), we get

$$
\Phi_5=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-2}((\widetilde{D}^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.58}
$$

By (3.5) and (3.6), Lemma 3.7, we have

$$
\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})|_{|\xi'|=1}=\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)}. \tag{3.59}
$$

Since

$$
\begin{aligned}
\sigma_{-2}((\widetilde{D}^*)^{-1})(x_0)=&\frac{c(\xi)\sigma_0(\widetilde{D}^*)(x_0)c(\xi)}{|\xi|^4}\\
&+\frac{c(\xi)}{|\xi|^6}c(dx_n)\left[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)|\xi|^2_{\partial_M}\right],
\end{aligned}
\tag{3.60}
$$

where

$$
\begin{aligned}
\sigma_0(\widetilde{D}^*)(x_0)=&-\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t)\\
&+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)(x_0)\\
=&Q(x_0)+\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)(x_0),
\end{aligned}
\tag{3.61}
$$

then

$$
\begin{aligned}
&\partial_{\xi_n}\sigma_{-2}((\widetilde{D}^*)^{-1})(x_0)|_{|\xi'|=1}\\
&\quad=\partial_{\xi_n}\Big\{\frac{c(\xi)[Q(x_0)+(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2})(x_0)]c(\xi)}{|\xi|^4}\\
&\qquad+\frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)]\Big\}\\
&\quad=\partial_{\xi_n}\Big\{\frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)]\Big\}+\partial_{\xi_n}\frac{c(\xi)Q(x_0)c(\xi)}{|\xi|^4}\\
&\qquad+\partial_{\xi_n}\frac{c(\xi)(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2})(x_0)c(\xi)}{|\xi|^4}.
\end{aligned}
\tag{3.62}
$$

by $c(\xi)=c(\xi')+\xi_nc(dx_n), |\xi|^2=1+\xi_n^2$ and direct derivation, we get

$$
\begin{aligned}
&\partial_{\xi_n}\frac{c(\xi)(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2})(x_0)c(\xi)}{|\xi|^4}\\
&=\partial_{\xi_n}\frac{(c(\xi')+\xi_nc(dx_n))(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2})(x_0)(c(\xi')+\xi_nc(dx_n))}{|\xi|^4}\\
&=\frac{c(dx_n)(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2})(x_0)c(\xi)}{|\xi|^4}\\
&\quad+\frac{c(\xi)(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2})(x_0)c(dx_n)}{|\xi|^4}\\
&\quad-\frac{4\xi_nc(\xi)(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2})(x_0)c(\xi)}{|\xi|^4}.
\end{aligned}
\tag{3.63}
$$

We denote

$$
q_{-2}^1=\frac{c(\xi)Q(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)],
$$

then

$$
\begin{aligned}
\partial_{\xi_n}(q_{-2}^1)=&\frac{1}{(1+\xi_n^2)^3}\Big[(2\xi_n-2\xi_n^3)c(dx_n)Q(x_0)c(dx_n)+(1-3\xi_n^2)c(dx_n)Q(x_0)c(\xi')\\
&+(1-3\xi_n^2)c(\xi')Q(x_0)c(dx_n)-4\xi_nc(\xi')Q(x_0)c(\xi')+(3\xi_n^2-1)\partial_{x_n}c(\xi')\\
&-4\xi_nc(\xi')c(dx_n)\partial_{x_n}c(\xi')+2h'(0)c(\xi')+2h'(0)\xi_nc(dx_n)\Big]\\
&+6\xi_nh'(0)\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^4}.
\end{aligned}
\tag{3.64}
$$

By (3.59) and (3.64), we have

$$\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial_{\xi_n}(q^1_{-2})](x_0)|_{|\xi'|=1}=\frac{3h'(0)(i\xi_n^2+\xi_n-2i)}{(\xi-i)^3(\xi+i)^3}+\frac{12h'(0)i\xi_n}{(\xi-i)^3(\xi+i)^4}, \tag{3.65}$$

then

$$\begin{aligned}&-i\Omega_3\int_{\Gamma_+}\left[\frac{3h'(0)(i\xi_n^2+\xi_n-2i)}{(\xi_n-i)^3(\xi_n+i)^3}+\frac{12h'(0)i\xi_n}{(\xi_n-i)^3(\xi_n+i)^4}\right]d\xi_n dx'\\&=-\frac{9}{8}\pi h'(0)\Omega_3 dx'.\end{aligned} \tag{3.66}$$

By $\int_{|\xi'|=1}\xi_1\cdots\xi_{2d+1}\sigma(\xi')=0$, (3.59) and (3.62), we have

$$\begin{aligned}&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}\left[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial_{\xi_n}\frac{c(\xi)(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2})c(\xi)}{|\xi|^4}\right]\\&(x_0)d\xi_n\sigma(\xi')dx'\\&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{i}{(\xi-i)(\xi+i)^3}\mathrm{tr}\left[c(dx_n)\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)\right]\\&(x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{\pi}{4}\mathrm{tr}\left[c(dx_n)\left(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right)\right]\Omega_3 dx'\\&=0.\end{aligned} \tag{3.67}$$

Then,

$$\Phi_5=-\frac{9}{8}\pi h'(0)\Omega_3 dx'. \tag{3.68}$$

So $\Phi=\sum_{i=1}^5\Phi_i=0$.

By (3.17), (3.19) and (3.68), we can get

Theorem 3.8 *Let M be a 4-dimensional compact oriented spin manifolds with the boundary ∂M and the metric g^M as above, $\widetilde{D}$ and $\widetilde{D}^*$ be the Dirac–Witten operators on $\widetilde{M}$, then*

$$\begin{aligned}&\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}^{-1}\circ\pi^+(\widetilde{D}^*)^{-1}]\\&=32\pi^2\int_M\left\{-\frac{1}{3}s-12f_2\overline{f_2}+4f_2^2+4\overline{f_2}^2+4f_1\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})^2\right\}d\mathrm{Vol_M}.\end{aligned} \tag{3.69}$$

where s is the scalar curvature.

4 A Kastler–Kalau–Walze type theorem for 6-dimensional manifolds with boundary

Firstly, we prove the Kastler–Kalau–Walze type theorems for 6-dimensional manifolds with boundary. From [14], we know that

$$\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}^{-1}\circ\pi^+(\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1}] = \int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M\bigotimes\mathbb{C}}[\sigma_{-4}((\widetilde{D}^*\widetilde{D})^{-2})]\sigma(\xi)dx + \int_{\partial_t M}\Psi, \tag{4.1}$$

where

$$\Psi = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times\mathrm{trace}_{\wedge^*T^*M\bigotimes\mathbb{C}}[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{t\xi_n}\sigma^+_r(\widetilde{D}^{-1})(x',0,\xi',\xi_n) \times\partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx', \tag{4.2}$$

and the sum is taken over $r+\ell-k-j-|\alpha|-1=-6,\ r\leq -1,\ell\leq -3.$

By Theorem 2.2, we compute the interior term of (4.1), then

$$\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M\bigotimes\mathbb{C}}[\sigma_{-4}((\widetilde{D}^*\widetilde{D})^{-2})]\sigma(\xi)dx = 128\pi^3\int_M\left(-\frac{2}{3}s-4(f_1^2+\overline{f_1}^2-4f_1\overline{f_1})\sum_{u<v}(p_{uv}-p_{vu})^2-32f_2\overline{f_2}+12f_2^2+12\overline{f_2}^2\right)d\mathrm{Vol}_{\mathrm{M}}. \tag{4.3}$$

Next, we compute $\int_{\partial M}\Psi$. By (2.12), we get

$$\begin{aligned}\widetilde{D}^*\widetilde{D} = D^2 &+ D\left[-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right]\\ &+\left[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right]D\\ &+\left[-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\right]\\ &\left[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right],\end{aligned} \tag{4.4}$$

Then,

$$
\begin{aligned}
\widetilde{D}^*\widetilde{D}\widetilde{D}^* &= \sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle(-g^{ij}\partial_l\partial_i\partial_j)\\
&+\sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle\Big\{-(\partial_l g^{ij})\partial_i\partial_j - g^{ij}\Big(4\sigma_i\partial_j - 2\Gamma_{ij}^k\partial_k\Big)\partial_l\Big\}\\
&+\sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle\Big\{-2(\partial_l g^{ij})\sigma_i\partial_j + g^{ij}(\partial_l\Gamma_{ij}^k)\partial_k\\
&-2g^{ij}\partial_l\sigma_i\partial_j + (\partial_l g^{ij})\Gamma_{ij}^k\partial_k\\
&+\sum_{j,k}\Bigg[\partial_l\Bigg(\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(e_j)\\
&+c(e_j)\Bigg(\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-\overline{f_2}\Bigg)\Bigg)\Bigg]\langle e_j, dx^k\rangle\partial_k\\
&+\sum_{j,k}\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(e_j)\ +c(e_j)\Bigg(\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-\overline{f_2}\Bigg)\Bigg)\Big[\partial_l\langle e_j, dx^k\rangle\Big]\partial_k\Bigg\}\\
&+\sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle\partial_l\Big\{-g^{ij}\Big[(\partial_i\sigma_j)+\sigma_i\sigma_j-\Gamma_{i,j}^k\sigma_k\\
&+\sum_{i,j} g^{i,j}\Bigg[\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(\partial_i)\sigma_i\\
&+c(\partial_i)\partial_i\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\Bigg)+c(\partial_i)\sigma_i\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\Bigg)\Bigg]\\
&+\frac{1}{4}s+\Bigg(\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-\overline{f_2}\Bigg)^2\Bigg\}\\
&+\Bigg[\sigma_i-(\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)-\overline{f_2})\Bigg](-g^{ij}\partial_i\partial_j)\\
&+\sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle\Bigg\{2\sum_{j,k}\Bigg[\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(e_j)+c(e_j)\\
&\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\Bigg)\Bigg]\\
&\times\langle e_i, dx_k\rangle\Big\}_l\partial_k+\Bigg[\sigma_i+\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\Bigg)\Bigg]\\
&\Bigg\{-\sum_{i,j}g^{i,j}\Big[2\sigma_i\partial_j-\Gamma_{i,j}^k\partial_k+(\partial_i\sigma_j)+\sigma_i\sigma_j-\Gamma_{i,j}^k\sigma_k\Big]\\
&-\sum_{i,j}g^{i,j}\Bigg[c(\partial_i)\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)\\
&+\overline{f_2}\Bigg)+\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(\partial_i)\Bigg]\partial_j\\
&+\sum_{i,j}g^{i,j}\Bigg[\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(\partial_i)\,\sigma_i+c(\partial_i)\partial_i\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\Bigg)\\
&-c(\partial_i)\sigma_i\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\Bigg)+c(\partial_i)\\
&\partial_i\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\Bigg)c(\partial_i)\sigma_i\\
&\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\Bigg)\Bigg]+\frac{1}{4}s-\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)^2\Bigg\}.
\end{aligned}
\tag{4.5}
$$

Then, by (4.5), we obtain

Lemma 4.1 *The following identities hold:*

$$\begin{aligned}\sigma_2(\widetilde{D}^*\widetilde{D}\widetilde{D}^*) &= \sum_{i,j,l} c(dx_l)\partial_l(g^{i,j})\xi_i\xi_j + c(\xi)(4\sigma^k - 2\Gamma^k)\xi_k \\ &\quad + 2\Bigg[|\xi|^2\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv} - p_{vu})c(e_u)c(e_v) + \overline{f_2}\Bigg) \\ &\quad -c(\xi)\Bigg(f_1\sum_{u<v}(p_{uv} - p_{vu})c(e_u)c(e_v) + f_2\Bigg)c(\xi)\Bigg] \\ &\quad - \frac{1}{4}|\xi|^2\sum_{s,t,l}\omega_{s,t}c(e_l)c(e_s)c(e_t) \\ &\quad + |\xi|^2\Bigg(-\overline{f_1}\sum_{u<v}(p_{uv} - p_{vu})c(e_u)c(e_v) + \overline{f_2}\Bigg)^2; \\ \sigma_3(\widetilde{D}^*\widetilde{D}\widetilde{D}^*) &= ic(\xi)|\xi|^2;\end{aligned} \tag{4.6}$$

where $\xi = \Sigma_{i=1}^n \xi_i d_{x_i}$ denotes the cotangent vector.

Write

$$\sigma(\widetilde{D}^*\widetilde{D}\widetilde{D}^*) = p_3 + p_2 + p_1 + p_0;\ \sigma((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1}) = \sum_{j=3}^{\infty} q_{-j}. \tag{4.7}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned}1 = \sigma((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)\circ(\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1}) &= \sum_{\alpha}\frac{1}{\alpha!}\partial_\xi^\alpha[\sigma(\widetilde{D}^*\widetilde{D}\widetilde{D}^*)]D_x^\alpha[(\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1}] \\ &= (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \cdots) \\ &\quad + \sum_j(\partial_{\xi_j}p_3 + \partial_{\xi_j}p_2 + \partial_{\xi_j}p_1 + \partial_{\xi_j}p_0) \\ (D_{x_j}q_{-3} + D_{x_j}q_{-4} + D_{x_j}q_{-5} + \cdots) &= p_3q_{-3} + (p_3q_{-4} + p_2q_{-3} \\ &\quad + \sum_j \partial_{\xi_j}p_3D_{x_j}q_{-3}) + \cdots,\end{aligned} \tag{4.8}$$

by (4.8), we have

$$q_{-3} = p_3^{-1};\ q_{-4} = -p_3^{-1}\Bigg[p_2p_3^{-1} + \sum_j \partial_{\xi_j}p_3D_{x_j}(p_3^{-1})\Bigg]. \tag{4.9}$$

By Lemma 4.1, we have some symbols of operators.

Lemma 4.2 *The following identities hold:*

$$\begin{aligned}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1}) &= \frac{ic(\xi)}{|\xi|^4};\\ \sigma_{-4}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1}) &= \frac{c(\xi)\sigma_2(\widetilde{D}^*\widetilde{D}\widetilde{D}^*)c(\xi)}{|\xi|^8}\\ &+\frac{ic(\xi)}{|\xi|^8}\Big(|\xi|^4c(dx_n)\partial_{x_n}c(\xi')-2h'(0)c(dx_n)c(\xi)\\ &+2\xi_nc(\xi)\partial_{x_n}c(\xi')+4\xi_nh'(0)\Big).\end{aligned}\tag{4.10}$$

When $n=6$, then $\mathrm{tr}_{\wedge^*T^*M}[\mathtt{id}]=8$, where tr as shorthand of trace. Since the sum is taken over $r+\ell-k-j-|\alpha|-1=-6,\ r\leq -1,\ell\leq -3$, then we have the sum of the following five cases:

Case (a) (I) $r=-1,l=-3,j=k=0,|\alpha|=1.$

By (4.2), we get

$$\Psi_1=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})\Big](x_0)d\xi_n\sigma(\xi')dx'.\tag{4.11}$$

By Lemma 4.2, for $i<n$, we have

$$\begin{aligned}\partial_{x_i}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})(x_0)&=\partial_{x_i}\left[\frac{ic(\xi)}{|\xi|^4}\right](x_0)\\ &=i\partial_{x_i}[c(\xi)]|\xi|^{-4}(x_0)-2ic(\xi)\partial_{x_i}[|\xi|^2]|\xi|^{-6}(x_0)=0,\end{aligned}\tag{4.12}$$

so $\Psi_1=0$.

Case (a) (II) $r=-1,l=-3,|\alpha|=k=0,j=1.$

By (4.2), we have

$$\Psi_2=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial^2_{\xi_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})\Big](x_0)d\xi_n\sigma(\xi')dx'.\tag{4.13}$$

By direct derivation, we have

$$\begin{aligned}\partial^2_{\xi_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})&=\partial_{\xi_n}\left[\frac{-4i\xi_nc(\xi')}{(1+\xi_n^2)^3}+\frac{i(1-3\xi_n^2)c(\mathtt{d}x_n)}{(1+\xi_n^2)^3}\right]\\ &=i\left[\frac{(20\xi_n^2-4)c(\xi')+12(\xi_n^3-\xi_n)c(dx_n)}{(1+\xi_n^2)^4}\right].\end{aligned}\tag{4.14}$$

Since $n=6$, $\mathrm{tr}[-\mathtt{id}]=-8$. By the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, then

$$\begin{aligned}&\mathrm{tr}[c(\xi')c(dx_n)]=0;\ \mathrm{tr}[c(dx_n)^2]=-8;\ \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1}=-8;\\&\mathrm{tr}[\partial_{x_n}[c(\xi')]c(\mathrm{d}x_n)]=0;\ \mathrm{tr}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1}=-4h'(0).\end{aligned}\tag{4.15}$$

By (3.29), (4.13) and (4.15), we get

$$\begin{aligned}&\mathrm{trace}\Big[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial^2_{\xi_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})\Big](x_0)\\&=8h'(0)\frac{-1-3\xi_ni+5\xi_n^2+3i\xi_n^3}{(\xi_n-i)^6(\xi_n+i)^4}.\end{aligned}\tag{4.16}$$

Then we obtain

$$\begin{aligned}\Psi_2&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}h'(0)dimF\frac{-1-3\xi_ni+5\xi_n^2+3i\xi_n^3}{(\xi_n-i)^6(\xi_n+i)^4}d\xi_n\sigma(\xi')dx'\\&=h'(0)\Omega_4\int_{\Gamma^+}\frac{4+12\xi_ni-20\xi_n^2-122i\xi_n^3}{(\xi_n-i)^6(\xi_n+i)^4}d\xi_ndx'\\&=h'(0)\Omega_4\frac{\pi i}{5!}\Big[\frac{1+3\xi_ni-5\xi_n^2-3i\xi_n^3}{(\xi_n+i)^4}\Big]^{(5)}|_{\xi_n=i}dx'\\&=-\frac{15}{16}\pi h'(0)\Omega_4dx',\end{aligned}\tag{4.17}$$

where Ω_4 is the canonical volume of S^4.

Case (a) (III) $r=-1, l=-3, |\alpha|=j=0, k=1$.

By (4.2), we have

$$\Psi_3=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})\Big](x_0)d\xi_n\sigma(\xi')dx'.\tag{4.18}$$

By direct derivation, we have

$$\begin{aligned}\partial_{\xi_n}\partial_{x_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})&=\partial_{\xi_n}\frac{i\partial_{x_n}c(\xi')(x_0)|\xi|^2-ic(\xi)|\xi'|^2h'(0)}{|\xi|^4}\\&=-\frac{4i\xi_n\partial_{x_n}c(\xi')(x_0)}{(1+\xi_n^2)^3}+i\frac{12h'(0)\xi_nc(\xi')}{(1+\xi_n^2)^4}-i\frac{(2-10\xi_n^2)h'(0)c(dx_n)}{(1+\xi_n^2)^4}.\end{aligned}\tag{4.19}$$

Combining (3.36) and (4.19), we have

$$\begin{aligned}&\mathrm{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})\Big](x_0)|_{|\xi'|=1}\\&=h'(0)\frac{8i-32\xi_n-8i\xi_n^2}{(\xi_n-i)^5(\xi+i)^4}.\end{aligned}\tag{4.20}$$

Then

$$\begin{aligned}\Psi_3 &= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty} h'(0)\frac{8i-32\xi_n-8i\xi_n^2}{(\xi_n-i)^5(\xi+i)^4}d\xi_n\sigma(\xi')dx'\\ &= -\frac{1}{2}h'(0)\Omega_4\int_{\Gamma^+}\frac{8i-32\xi_n-8i\xi_n^2}{(\xi_n-i)^5(\xi+i)^4}d\xi_n dx'\\ &= -h'(0)\Omega_4\frac{\pi i}{4!}\Big[\frac{8i-32\xi_n-8i\xi_n^2}{(\xi+i)^4}\Big]^{(4)}|_{\xi_n=i}dx'\\ &= \frac{25}{16}\pi h'(0)\Omega_4 dx'. \end{aligned} \tag{4.21}$$

Case (b) $r=-1, l=-4, |\alpha|=j=k=0.$

By (4.2), we have

$$\begin{aligned}\Psi_4 &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-4}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})\Big](x_0)d\xi_n\sigma(\xi')dx'\\ &= i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\sigma_{-4}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \end{aligned} \tag{4.22}$$

In the normal coordinate, $g^{ij}(x_0)=\delta_i^j$ and $\partial_{x_j}(g^{\alpha\beta})(x_0)=0$, if $j<n$; $\partial_{x_j}(g^{\alpha\beta})(x_0)=h'(0)\delta^\alpha_\beta$, if $j=n$. So by [12], when $k<n$, we have $\Gamma^n(x_0)=\frac{5}{2}h'(0)$, $\Gamma^k(x_0)=0$, $\delta^n(x_0)=0$ and $\delta^k=\frac{1}{4}h'(0)c(e_k)c(e_n)$. Then, we obtain

$$\begin{aligned}&\sigma_{-4}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})(x_0)|_{|\xi'|=1}\\ &= \frac{c(\xi)\sigma_2(\widetilde{D}^*\widetilde{D}\widetilde{D}^*)(x_0)|_{|\xi'|=1}c(\xi)}{|\xi|^8}-\frac{c(\xi)}{|\xi|^4}\sum_j\partial_{\xi_j}\big(c(\xi)|\xi|^2\big)D_{x_j}\Big(\frac{ic(\xi)}{|\xi|^4}\Big)\\ &= \frac{1}{|\xi|^8}c(\xi)\Big(\frac{1}{2}h'(0)c(\xi)\sum_{k<n}\xi_k c(e_k)c(e_n)-\frac{5}{2}h'(0)\xi_n c(\xi)-\frac{1}{4}h'(0)\\ &\quad |\xi|^2c(dx_n)+2\Big[|\xi|^2\Big(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\Big)-c(\xi)\\ &\quad \Big(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Big)c(\xi)\Big]+|\xi|^2\Big(-\overline{f_1}\sum_{u<v}(p_{uv}-p_{vu})\\ &\quad c(e_u)c(e_v)+\overline{f_2})\Big)c(\xi)+\frac{ic(\xi)}{|\xi|^8}\Big(|\xi|^4c(dx_n)\partial_{x_n}c(\xi')-2h'(0)c(dx_n)c(\xi)\\ &\quad +2\xi_n c(\xi)\partial_{x_n}c(\xi')+4\xi_n h'(0)\Big). \end{aligned} \tag{4.23}$$

By (3.32) and (4.23), we have

$$\begin{aligned}&\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\sigma_{-4}(\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1}](x_0)|_{|\xi'|=1}\\&=\frac{1}{2(\xi_n-i)^2(1+\xi_n^2)^4}\left(\frac{3}{4}i+2+(3+4i)\xi_n+(-6+2i)\xi_n^2+3\xi_n^3+\frac{9i}{4}\xi_n^4\right)h'(0)\mathrm{tr}[id]\\&\quad+\frac{1}{2(\xi_n-i)^2(1+\xi_n^2)^4}\big(-1-3i\xi_n-2\xi_n^2-4i\xi_n^3-\xi_n^4-i\xi_n^5\big)\mathrm{tr}[c(\xi')\partial_{x_n}c(\xi')].\end{aligned}\tag{4.24}$$

Then by (4.24), we get

$$\begin{aligned}\Psi_4&=ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}8\times\frac{\frac{3}{4}i+2+(3+4i)\xi_n+(-6+2i)\xi_n^2+3\xi_n^3+\frac{9i}{4}\xi_n^4}{2(\xi_n-i)^5(\xi_n+i)^4}d\xi_n\sigma(\xi')dx'\\&\quad+ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}4\times\frac{1+3i\xi_n+2\xi_n^2+4i\xi_n^3+\xi_n^4+i\xi_n^5}{2(\xi_n-i)^2(1+\xi_n^2)^4}d\xi_n\sigma(\xi')dx'\\&=\Big(-\frac{41}{64}i-\frac{195}{64}\Big)\pi h'(0)\Omega_4dx'.\end{aligned}\tag{4.25}$$

Case (c) $r=-2, l=-3, |\alpha|=j=k=0.$

By (4.2), we have

$$\Psi_5=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})\Big](x_0)d\xi_n\sigma(\xi')dx'.\tag{4.26}$$

By Lemma 4.1 and Lemma 4.2, we have

$$\sigma_{-2}(\widetilde{D}^{-1})(x_0)=\frac{c(\xi)\sigma_0(\widetilde{D})c(\xi)}{|\xi|^4}(x_0)+\frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)\Big](x_0),\tag{4.27}$$

where

$$\sigma_0(\widetilde{D})=-\frac{1}{4}\sum_{i,s,t}\omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t)+\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right).\tag{4.28}$$

On the other hand,

$$\partial_{\xi_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})=\frac{-4i\xi_nc(\xi')}{(1+\xi_n^2)^3}+\frac{i(1-3\xi_n^2)c(\mathrm{d}x_n)}{(1+\xi_n^2)^3}.\tag{4.29}$$

By (4.27), (3.5) and (3.6), we have

$$\pi^+_{\xi_n}\Big(\sigma_{-2}(\widetilde{D}^{-1})\Big)(x_0)|_{|\xi'|=1}=\pi^+_{\xi_n}\Big[\frac{c(\xi)\sigma_0(\widetilde{D})(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}\Big] - h'(0)\pi^+_{\xi_n}\Big[\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^3}\Big]. \tag{4.30}$$

We denote

$$\sigma_0(\widetilde{D})(x_0)|_{\xi_n=i}=Q(x_0)+\Big(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Big). \tag{4.31}$$

Then, we obtain

$$\pi^+_{\xi_n}\Big(\sigma_{-2}(\widetilde{D}^{-1})\Big)(x_0)|_{|\xi'|=1} =\pi^+_{\xi_n}\Big[\frac{c(\xi)Q(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}-h'(0)\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^3}\Big] + \pi^+_{\xi_n}\Big[\frac{c(\xi)[(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2)]c(\xi)(x_0)}{(1+\xi_n^2)^2}\Big]. \tag{4.32}$$

Furthermore,

$$\begin{aligned}
&\pi^+_{\xi_n}\Big[\frac{c(\xi)[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2](x_0)c(\xi)}{(1+\xi_n^2)^2}\Big]\\
&=\pi^+_{\xi_n}\Big[\frac{c(\xi')[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2](x_0)c(\xi')}{(1+\xi_n^2)^2}\Big]\\
&\quad+\pi^+_{\xi_n}\Big[\frac{\xi_nc(\xi')[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2](x_0)c(dx_n)}{(1+\xi_n^2)^2}\Big]\\
&\quad+\pi^+_{\xi_n}\Big[\frac{\xi_nc(dx_n)[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2](x_0)c(\xi')}{(1+\xi_n^2)^2}\Big]\\
&\quad+\pi^+_{\xi_n}\Big[\frac{\xi_n^2c(dx_n)[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2](x_0)c(dx_n)}{(1+\xi_n^2)^2}\Big]\\
&=-\frac{c(\xi')[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2](x_0)c(\xi')(2+i\xi_n)}{4(\xi_n-i)^2}\\
&\quad+\frac{ic(\xi')[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2](x_0)c(dx_n)}{4(\xi_n-i)^2}\\
&\quad+\frac{ic(dx_n)[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2](x_0)c(\xi')}{4(\xi_n-i)^2}\\
&\quad+\frac{-i\xi_nc(dx_n)[f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2](x_0)c(dx_n)}{4(\xi_n-i)^2}.
\end{aligned} \tag{4.33}$$

By $c(\xi)=c(\xi')+\xi_nc(dx_n)$, we have

$$\begin{aligned}
&\pi^+_{\xi_n}\Big[\frac{c(\xi)Q(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}(c(\xi'))(x_0)}{(1+\xi_n^2)^2}\Big]-h'(0)\pi^+_{\xi_n}\Big[\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n)^3}\Big]\\
&=\pi^+_{\xi_n}\Big[\frac{(c(\xi')+\xi_nc(dx_n))Q(x_0)(c(\xi')+\xi_nc(dx_n))+(c(\xi')+\xi_nc(dx_n))c(dx_n)\partial_{x_n}(c(\xi'))(x_0)}{(1+\xi_n^2)^2}\Big]\\
&\quad-h'(0)\pi^+_{\xi_n}\Big[\frac{(c(\xi')+\xi_nc(dx_n))c(dx_n)(c(\xi')+\xi_nc(dx_n))}{(1+\xi_n)^3}\Big]\\
&=\frac{1}{2\pi i}\lim_{u\to 0^-}\int_{\Gamma^+}\frac{\frac{(c(\xi')+\eta_nc(dx_n))Q(x_0)(c(\xi')+\xi_nc(dx_n))+(c(\xi')+\eta_nc(dx_n))c(dx_n)\partial_{x_n}(c(\xi'))(x_0)}{(\eta_n+i)^2(\xi_n+iu-\eta_n)}}{(\eta_n-i)^2}d\eta_n\\
&\quad-\frac{1}{2\pi i}\lim_{u\to 0^-}\int_{\Gamma^+}h'(0)\frac{\frac{(c(\xi')+\eta_nc(dx_n))c(dx_n)(c(\xi')+\eta_nc(dx_n))}{(\eta_n+i)^3(\xi_n+iu-\eta_n)}}{(\eta_n-i)^3}d\eta_n\\
&:=C_1-C_2,
\end{aligned}\tag{4.34}$$

where

$$\begin{aligned}
C_1&=\frac{-1}{4(\xi_n-i)^2}\big[(2+i\xi_n)c(\xi')Qc(\xi')+i\xi_nc(dx_n)Qc(dx_n)\\
&\quad+(2+i\xi_n)c(\xi')c(dx_n)\partial_{x_n}c(\xi')+ic(dx_n)Q_0^2c(\xi')+ic(\xi')Qc(dx_n)-i\partial_{x_n}c(\xi')\big]\\
&=\frac{1}{4(\xi_n-i)^2}\Big[\frac{5}{2}h'(0)c(dx_n)-\frac{5i}{2}h'(0)c(\xi')-(2+i\xi_n)c(\xi')c(dx_n)\partial_{\xi_n}c(\xi')+i\partial_{\xi_n}c(\xi')\Big];\\
C_2&=\frac{h'(0)}{2}\Big[\frac{c(dx_n)}{4i(\xi_n-i)}+\frac{c(dx_n)-ic(\xi')}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}\big(ic(\xi')-c(dx_n)\big)\Big].
\end{aligned}\tag{4.35}$$

By (4.30) and (4.35), we have

$$\begin{aligned}
&\mathrm{tr}\,[C_2\times\partial_{\xi_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})(x_0)]|_{|\xi'|=1}\\
&\quad=\mathrm{tr}\,\Big\{\frac{h'(0)}{2}\Big[\frac{c(dx_n)}{4i(\xi_n-i)}+\frac{c(dx_n)-ic(\xi')}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}[ic(\xi')-c(dx_n)]\Big]\\
&\qquad\times\frac{-4i\xi_nc(\xi')+(i-3i\xi_n^2)c(dx_n)}{(1+\xi_n^2)^3}\Big\}\\
&\quad=h'(0)\frac{4i-11\xi_n-6i\xi_n^2+3\xi_n^3}{(\xi_n-i)^5(\xi_n+i)^3}.
\end{aligned}\tag{4.36}$$

Similarly, we have

$$
\begin{aligned}
&\operatorname{tr}[C_1 \times \partial_{\xi_n}\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})(x_0)]|_{|\xi'|=1}\\
&= \operatorname{tr}\left\{\frac{1}{4(\xi_n-i)^2}\left[\frac{5}{2}h'(0)c(dx_n)-\frac{5i}{2}h'(0)c(\xi')\right.\right.\\
&\quad\left.-(2+i\xi_n)c(\xi')c(dx_n)\partial_{\xi_n}c(\xi')+i\partial_{\xi_n}c(\xi')\right]\\
&\quad\left.\times\frac{-4i\xi_n c(\xi')+(i-3i\xi_n^2)c(dx_n)}{(1+\xi_n^2)^3}\right\}\\
&= h'(0)\frac{3+12i\xi_n+3\xi_n^2}{(\xi_n-i)^4(\xi_n+i)^3};\\
&\operatorname{tr}\left[\pi^+_{\xi_n}\left(\frac{c(\xi)[(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2)](x_0)c(\xi)}{(1+\xi_n^2)^2}\right)\right.\\
&\quad\left.\times\partial_{\xi_n}\sigma_{-3}((D^*\widetilde{D}\widetilde{D}^*)^{-1})(x_0)\right]\Bigg|_{|\xi'|=1}\\
&=\frac{2-8i\xi_n-6\xi_n^2}{4(\xi_n-i)^2(1+\xi_n^2)^3}\operatorname{tr}\left[\left(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\right)(x_0)c(\xi')\right]\\
&=0.
\end{aligned}
\tag{4.37}
$$

By $\int_{|\xi'|=1}\xi_1\ldots\xi_{2d+1}\sigma(\xi')=0$, we have

$$
\begin{aligned}
\Psi_5 &= -ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\times\frac{-7i+26\xi_n+15i\xi_n^2}{(\xi_n-i)^5(\xi_n+i)^3}d\xi_n\sigma(\xi')dx'\\
&= -ih'(0)\times\frac{2\pi i}{4!}\left[\frac{-7i+26\xi_n+15i\xi_n^2}{(\xi_n+i)^3}\right]^{(5)}|_{\xi_n=i}\Omega_4 dx'\\
&= \frac{55}{16}\pi h'(0)\Omega_4 dx'.
\end{aligned}
\tag{4.38}
$$

Now Ψ is the sum of the cases (a), (b) and (c), then

$$
\Psi=\sum_{i=1}^{5}\Psi_i=\left(\frac{65}{64}-\frac{41}{64}i\right)\pi h'(0)\Omega_4 dx'. \tag{4.39}
$$

By (4.1), (4.3) and (4.39), we can get

Theorem 4.3 *Let M be a 6-dimensional compact oriented spin manifold with the boundary ∂M and the metric g^M as above, $\widetilde{D}$ and $\widetilde{D}^*$ be the Dirac–Witten operators on $\widetilde{M}$, then*

$$
\begin{aligned}
&\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}^{-1}\circ\pi^+(\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1}]\\
&=128\pi^3\int_M\Big(-\frac{2}{3}s-4(f_1^2+\overline{f_1}^2-4f_1\overline{f_1})\sum_{u<v}(p_{uv}-p_{vu})^2-32f_2\overline{f_2}+12f_2^2+12\overline{f_2}^2\Big)d\mathrm{Vol_M}\\
&\quad+\int_{\partial M}\Big(\frac{65}{64}-\frac{41}{64}i\Big)\pi h'(0)\,\Omega_4dx'.
\end{aligned}
\tag{4.40}
$$

where s is the scalar curvature.

Next, we prove the Kastler–Kalau–Walze type theorem for 6-dimensional manifold with boundary associated to $\widetilde{D}^3$. From [14], we know that

$$
\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}^{-1}\circ\pi^+\widetilde{D}^{-3}]=\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-4}(\widetilde{D}^{-4})]\sigma(\xi)dx+\int_{\partial M}\overline{\Psi},
\tag{4.41}
$$

where $\widetilde{\mathrm{Wres}}$ denote noncommutative residue on minifolds with boundary,

$$
\begin{aligned}
\overline{\Psi}=&\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\partial_{x_n}^j\partial_{\xi'}^\alpha\partial_{\xi_n}^k\sigma_r^+(\widetilde{D}^{-1})(x',0,\xi',\xi_n)\\
&\times\partial_{x'}^\alpha\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l(\widetilde{D}^{-3})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx',
\end{aligned}
\tag{4.42}
$$

and the sum is taken over $r+\ell-k-j-|\alpha|-1=-6,\ r\leq-1,\ell\leq-3.$

By Theorem 2.2, we compute the interior term of (4.42), then

$$
\begin{aligned}
&\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-4}(\widetilde{D}^{-4})]\sigma(\xi)dx\\
&\quad=128\pi^3\int_M\Big[-\frac{2}{3}s-24f_1^2\sum_{u<v}(p_{uv}-p_{vu})^2+40f_2^2\Big]d\mathrm{Vol_M}
\end{aligned}
\tag{4.43}
$$

.

So we only need to compute $\int_{\partial M}\overline{\Psi}$. Let us now turn to compute the specification of $\widetilde{D}^3$.

$$
\begin{aligned}
\widetilde{D}^3 =& \sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle(-g^{ij}\partial_l\partial_i\partial_j) \\
&+ \sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle\left\{ -(\partial_l g^{ij})\partial_i\partial_j - g^{ij}\left(4\sigma_i\partial_j 2\Gamma_{ij}^k\partial_k\right)\partial_l \right\} \\
&+ \sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle\Big\{ -2(\partial_l g^{ij})\sigma_i\partial_j + g^{ij}(\partial_l\Gamma_{ij}^k)\partial_k - 2g^{ij}[(\partial_l\sigma_i) + (\partial_l g^{ij})\Gamma_{ij}^k\partial_k \\
&+ \sum_{j,k}\Bigg[\partial_l\Bigg(\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(e_j) \\
&+ c(e_j)\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)\Bigg)\Bigg]\langle e_j, dx^k\rangle\partial_k \\
&+ \sum_{j,k}\Bigg(\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(e_j) \\
&+ c(e_j)\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)\Bigg) \\
&\Big[\partial_l\langle e_j, dx^k\rangle\Big]\partial_k\Big\} + \sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle\partial_l\Big\{-g^{ij}\Big[(\partial_i\sigma_j) + \sigma_i\sigma_j \\
&- \Gamma_{i,j}^k\sigma_k + \sum_{i,j} g^{i,j}\Bigg[\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(\partial_i)\sigma_i \\
&+ c(\partial_i)\partial_i\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg) \\
&+ c(\partial_i)\sigma_i\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)\Bigg] + \frac{1}{4}s - \Bigg[\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)\Bigg]^2\Bigg\} \\
&+ \Big[\sigma_i + (f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2)\Big](-g^{ij}\partial_i\partial_j) \\
&+ \sum_{i=1}^{n} c(e_i)\langle e_i, dx_l\rangle\Bigg\{2\sum_{j,k}\Bigg[(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2)c(e_j) \\
&+ c(e_j)\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)\Bigg] \\
&\times\langle e_i, dx_k\rangle\Big\}_l\partial_k + \Big[\sigma_i + \Big(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Big)\Big]\Bigg\{-\sum_{i,j} g^{i,j} \\
&\Big[2\sigma_i\partial_j - \Gamma_{i,j}^k\partial_k + (\partial_i\sigma_j) + \sigma_i\sigma_j - \Gamma_{i,j}^k\sigma_k\Big] \\
&+ \sum_{i,j} g^{i,j}\Bigg[c(\partial_i)\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v) \\
&+ f_2\Bigg) + \Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)c(\partial_i)\Bigg]\partial_j \\
&+ \sum_{i,j} g^{i,j}\Bigg[\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)\ \ +f_2)c(\partial_i)\sigma_i + c(\partial_i)\partial_i\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg) \\
&+ c(\partial_i)\sigma_i\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg) + c(\partial_i)\partial_i\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg) \\
&+ c(\partial_i)\sigma_i\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)\Bigg] + \frac{1}{4}s - \Bigg[\Bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\Bigg)\Bigg]^2\Bigg\}.
\end{aligned} \tag{4.44}
$$

Then, we obtain

Lemma 4.4 *The following identities hold:*

$$\begin{aligned}\sigma_2(\widetilde{D}^3) =& \sum_{i,j,l} c(dx_l)\partial_l(g^{i,j})\xi_i\xi_j + c(\xi)(4\sigma^k - 2\Gamma^k)\xi_k - 2[c(\xi)(f_1 \sum_{u<v}(p_{uv} - p_{vu}) \\ & c(e_u)c(e_v) + f_2)c(\xi) - |\xi|^2(f_1 \sum_{u<v}(p_{uv} - p_{vu})c(e_u)c(e_v) + f_2)] \\ & - \frac{1}{4}|\xi|^2 \sum_{s,t,l} \omega_{s,t}(e_l)c(e_s)c(e_t)]; \\ \sigma_3(\widetilde{D}^3) =& ic(\xi)|\xi|^2.\end{aligned} \tag{4.45}$$

Write

$$\sigma(\widetilde{D}^3) = p_3 + p_2 + p_1 + p_0;\ \sigma(\widetilde{D}^{-3}) = \sum_{j=3}^{\infty} q_{-j}. \tag{4.46}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned}1 = \sigma(\widetilde{D}^3 \circ \widetilde{D}^{-3}) =& \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha[\sigma(\widetilde{D}^3)]\widetilde{D}_x^\alpha[\sigma(\widetilde{D}^{-3})] \\ =& (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \cdots) \\ & + \sum_j(\partial_{\xi_j}p_3 + \partial_{\xi_j}p_2 + \partial_{\xi_j}p_1 + \partial_{\xi_j}p_0) \\ & (D_{x_j}q_{-3} + D_{x_j}q_{-4} + D_{x_j}q_{-5} + \cdots) \\ =& p_3q_{-3} + (p_3q_{-4} + p_2q_{-3} + \sum_j \partial_{\xi_j}p_3D_{x_j}q_{-3}) + \cdots,\end{aligned} \tag{4.47}$$

by (4.47), we have

$$q_{-3} = p_3^{-1};\ q_{-4} = -p_3^{-1}\left[p_2p_3^{-1} + \sum_j \partial_{\xi_j}p_3D_{x_j}(p_3^{-1})\right]. \tag{4.48}$$

By (4.44)–(4.48), we have some symbols of operators.

Lemma 4.5 *The following identities hold:*

$$\begin{aligned}\sigma_{-3}(\widetilde{D}^{-3}) =& \frac{ic(\xi)}{|\xi|^4}; \\ \sigma_{-4}(\widetilde{D}^{-3}) =& \frac{c(\xi)\sigma_2(\widetilde{D}^3)c(\xi)}{|\xi|^8} + \frac{ic(\xi)}{|\xi|^8}\Big(|\xi|^4c(dx_n)\partial_{x_n}c(\xi') - 2h'(0)c(dx_n)c(\xi) \\ & + 2\xi_nc(\xi)\partial_{x_n}c(\xi') + 4\xi_nh'(0)\Big).\end{aligned} \tag{4.49}$$

When $n=6$, then $\mathrm{tr}_{\wedge^*T^*M}[\mathtt{id}]=8$, where tr as shorthand of trace. Since the sum is taken over $r+\ell-k-j-|\alpha|-1=-6,\ r\leq -1,\ell\leq -3$, then we have the following five cases:

Case (a) (I) $r=-1,l=-3,j=k=0,|\alpha|=1.$

By (4.42), we get

$$\overline{\Psi}_1=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial^{\alpha}_{x'}\partial_{\xi_n}\sigma_{-3}(\widetilde{D}^{-3})\Big](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.50}$$

Case (a) (II) $r=-1,l=-3,|\alpha|=k=0,j=1.$

By (4.42), we have

$$\overline{\Psi}_2=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial^{2}_{\xi_n}\sigma_{-3}(\widetilde{D}^{-3})\Big](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.51}$$

Case (a) (III) $r=-1,l=-3,|\alpha|=j=0,k=1.$

By (4.42), we have

$$\overline{\Psi}_3=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(\widetilde{D}^{-3})\Big](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.52}$$

By Lemma 4.2 and Lemma 4.5, we have $\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1})=\sigma_{-3}(\widetilde{D}^{-3})$, by (4.11)-(4.21), we obtain

$$\overline{\Psi}_1+\overline{\Psi}_2+\overline{\Psi}_3=\frac{5}{8}\pi h'(0)\Omega_4dx',$$

where Ω_4 is the canonical volume of S^4.

Case (b) $r=-1,l=-4,|\alpha|=j=k=0.$

By (4.42), we have

$$\begin{aligned}\overline{\Psi}_4&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-4}(\widetilde{D}^{-3})\Big](x_0)d\xi_n\sigma(\xi')dx'\\&=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\sigma_{-4}(\widetilde{D}^{-3})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned} \tag{4.53}$$

In the normal coordinate, $g^{ij}(x_0)=\delta^j_i$ and $\partial_{x_j}(g^{\alpha\beta})(x_0)=0$, if $j<n$; $\partial_{x_j}(g^{\alpha\beta})(x_0)=h'(0)\delta^{\alpha}_{\beta}$, if $j=n$. So by [12], when $k<n$, we have $\Gamma^n(x_0)=\frac{5}{2}h'(0)$, $\Gamma^k(x_0)=0$, $\delta^n(x_0)=0$ and $\delta^k=\frac{1}{4}h'(0)c(e_k)c(e_n)$. Then, we obtain

$$
\begin{aligned}
&\sigma_{-4}(\widetilde{D}^{-3})(x_0)|_{|\xi'|=1}\\
&=\frac{c(\xi)\sigma_2(\widetilde{D}^3)(x_0)|_{|\xi'|=1}c(\xi)}{|\xi|^8}-\frac{c(\xi)}{|\xi|^4}\sum_j\partial_{\xi_j}\big(c(\xi)|\xi|^2\big)D_{x_j}\left(\frac{ic(\xi)}{|\xi|^4}\right)\\
&=\frac{1}{|\xi|^8}c(\xi)\Big(\frac{1}{2}h'(0)c(\xi)\sum_{k<n}\xi_kc(e_k)c(e_n)-\frac{5}{2}h'(0)\xi_nc(\xi)-\frac{1}{4}h'(0)|\xi|^2\\
&\quad c(dx_n)-2\bigg[c(\xi)\bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\bigg)c(\xi)+|\xi|^2\bigg(-\overline{f_1}\sum_{u<v}\\
&\quad (p_{uv}-p_{vu})c(e_u)c(e_v)+\overline{f_2}\bigg)\bigg]+|\xi|^2\bigg(f_1\sum_{u<v}(p_{uv}-p_{vu})c(e_u)c(e_v)+f_2\bigg)\\
&\quad c(\xi)+\frac{ic(\xi)}{|\xi|^8}\bigg(|\xi|^4c(dx_n)\partial_{x_n}c(\xi')-2h'(0)c(dx_n)c(\xi)+2\xi_nc(\xi)\partial_{x_n}c(\xi')\\
&\quad +4\xi_nh'(0)\bigg).
\end{aligned}
\tag{4.54}
$$

By (3.29) and (4.54), we have

$$
\begin{aligned}
&\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}^{-1})\times\sigma_{-4}(\widetilde{D}^{-3})](x_0)|_{|\xi'|=1}\\
&=\frac{1}{2(\xi_n-i)^2(1+\xi_n^2)^4}\left(\frac{3}{4}i+2+(3+4i)\xi_n+(-6+2i)\xi_n^2+3\xi_n^3+\frac{9i}{4}\xi_n^4\right)h'(0)\mathrm{tr}[id]\\
&\quad+\frac{1}{2(\xi_n-i)^2(1+\xi_n^2)^4}\big(-1-3i\xi_n-2\xi_n^2-4i\xi_n^3-\xi_n^4-i\xi_n^5\big)\mathrm{tr}[c(\xi')\partial_{x_n}c(\xi')].
\end{aligned}
\tag{4.55}
$$

By (4.55), we have

$$
\begin{aligned}
\overline{\Psi}_4&=ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}8\times\frac{\frac{3}{4}i+2+(3+4i)\xi_n+(-6+2i)\xi_n^2+3\xi_n^3+\frac{9i}{4}\xi_n^4}{2(\xi_n-i)^5(\xi_n+i)^4}d\xi_n\sigma(\xi')dx'\\
&\quad+ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}4\times\frac{1+3i\xi_n+2\xi_n^2+4i\xi_n^3+\xi_n^4+i\xi_n^5}{2(\xi_n-i)^2(1+\xi_n^2)^4}d\xi_n\sigma(\xi')dx'\\
&=\Big(-\frac{41}{64}i-\frac{195}{64}\Big)\pi h'(0)\Omega_4dx'.
\end{aligned}
\tag{4.56}
$$

Case (c) $r=-2, l=-3, |\alpha|=j=k=0.$

By (4.42), we have

$$
\overline{\Psi}_5=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-3}(\widetilde{D}^{-3})\Big](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.57}
$$

By Lemma 4.2 and Lemma 4.5, we have $\sigma_{-3}((\widetilde{D}^*\widetilde{D}\widetilde{D}^*)^{-1}) = \sigma_{-3}(\widetilde{D}^{-3})$, by (4.26)-(4.38), we obtain

$$\overline{\Psi}_5 = \frac{55}{16}\pi h'(0)\Omega_4 dx'.$$

Now $\overline{\Psi}$ is the sum of the cases (a), (b) and (c), then

$$\overline{\Psi} = \sum_{i=1}^{5}\overline{\Psi}_i = \Big(\frac{65}{64} - \frac{41}{64}i\Big)\pi h'(0)\Omega_4 dx'. \tag{4.58}$$

By (4.41), (4.43) and (4.58), we can get

Theorem 4.6 *Let M be a 6-dimensional compact oriented spin manifold with the boundary ∂M and the metric g^M as above, $\widetilde{D}$ be the Dirac-Witten operator on $\widetilde{M}$, then*

$$\begin{aligned} &\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}^{-1}\circ\pi^+(\widetilde{D}^{-3})] \\ &= 128\pi^3\int_M\Big[-\frac{2}{3}s - 24f_1^2\sum_{u<v}(p_{uv} - p_{vu})^2 + 40f_2^2\Big]d\mathrm{Vol}_{\mathrm{M}} \\ &\quad + \int_{\partial M}\Big(\frac{65}{64} - \frac{41}{64}i\Big)\pi h'(0)\Omega_4 dx'. \end{aligned} \tag{4.59}$$

where s is the scalar curvature.

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Declarations

Conflict of interest The authors declare no conflict of interest.

References

1. Ackermann, T.: A note on the Wodzicki residue. J. Geom. Phys. **20**, 404–406 (1996)
2. Connes A.: Quantized calculus and applications. In: 11th International Congress of Mathematical Physics (Paris, 1994), pp. 15–36. Internat Press, Cambridge, MA (1995)
3. Connes, A.: The action functiinal in noncommutative geometry. Commun. Math. Phys. **117**, 673–683 (1998)
4. Fedosov, B.V., Golse, F., Leichtnam, E., Schrohe, E.: The noncommutative residue for manifolds with boundary. J. Funct. Anal. **142**, 1–31 (1996)
5. Guillemin, V.W.: A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. Adv. Math. **55**(2), 131–160 (1985)
6. Kalau, W., Walze, M.: Gravity, noncommutative geometry and the Wodzicki residue. J. Geom. Phys. **16**, 327–344 (1995)
7. Kastler, D.: The Dirac operator and gravitation. Commun. Math. Phys. **166**, 633–643 (1995)
8. López, J.A.Á., Kordyukov, Y.A., Leichtnam, E..: Analysis on Riemannian foliations of bounded geometry (2019). ariXiv:1905.12912
9. Ponge, R.: Noncommutative geometry and lower dimensional volumes in Riemannian geometry. Lett. Math. Phys. **83**(1), 19–32 (2008)
10. Wang, Y.: Differential forms and the Wodzicki residue for manifolds with boundary. J. Geom. Phys. **56**, 731–753 (2006)
11. Wang, Y.: Differential forms the noncommutative residue for manifolds with boundary in the non-product case. Lett. Math. Phys. **77**, 41–51 (2006)
12. Wang, Y.: Gravity and the noncommutative residue for manifolds with boundary. Lett. Math. Phys. **80**, 37–56 (2007)
13. Wang, Y.: Lower-dimensional volumes and Kastler-kalau-Walze type theorem for manifolds with boundary. Commun. Theory Phys. **54**, 38–42 (2010)
14. Wang, J., Wang, Y.: The Kastler-Kalau-Walze type theorem for six-dimensional manifolds with boundary. J. Math. Phys. **56**, 052501 (2015)
15. Wei, S., Wang, Y.: Modified Novikov Operators and the Kastler-Kalau-Walze type theorem for manifolds with boundary. Adv. Math. Phys. 9090656 (2020)
16. Wodzicki, M.: Local invariants of spectral asymmetry. Invent. Math. **75**(1), 143–178 (1995)
17. Zhang, X.: Angular momentum and positive mass theorem. Commun. Math. Phys. **206**(1), 137–155 (1999)