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RESEARCH ARTICLE

Some Results for Intuitionistic Fuzzy Inequality

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Abstract

Intuitionistic fuzzy sets, characterized by membership degree μ, non-membership degree υ and hesitation degree π, are a meaningful extension of fuzzy set. Inequalities on intuitionistic fuzzy sets/values are very important in solving real problems. In this paper, some inequalities on intuitionistic fuzzy sets are derived from operations. Moreover, three unweighted intuitionistic fuzzy aggregation operators, including unweighted intuitionistic fuzzy Square, unweighted intuitionistic fuzzy Arithmetic and unweighted intuitionistic fuzzy Geometric, are developed. Later, some corresponding inequality relations on them are deeply explored. Finally, some inequalities on intuitionistic fuzzy value are constructed by equality $\mu + \upsilon + \pi = 1$ in critical definition and proved by some existing famous inequalities, which provide a novel basis for the intuitionistic fuzzy inequalities in operations and aggregation operators.

Keywords Intuitionistic fuzzy set · Equality · Aggregation operators · Inequalities · Operations

1 Introduction

Intuitionistic fuzzy set (IFS), firstly proposed by Atanassov [1], is a resultful tool to depict vagueness. A remarkable characteristic of IFS is that it assigns membership degree (MD) and nonmembership degree (NMD) to each element of the universe, whose sum is less than or equal to one. Thus, it is a meaningful extension of fuzzy set (FS) [2] which just assigns membership degree. In the past four decades, more and more scholars are applying IFS to various fields, such as decision-making methods [3–7], cluster [8], information measure [9–12], aggregation operators [13–15], operations [1, 16–18].

Ever since the invention of IFSs [19, 20], many scholars have paid great attention to the operations on IFSs. Atanassov [1] initially defined the negation, multiplication, addition, intersection, union, necessity, possibility operations on IFSs, whose sum of MD and NMD meets the definition of IFS. In order to enrich the operations on IFS, Atanassov [18] presented some new operations, including $@, \$, \#, *$, and discussed some equalities and a part of inequalities. Xu [13] developed $\lambda-$multiplication and power operations, and deduced the subsequent aggregation operators by $\lambda-$ multiplication and multiplication. In addition to the operations mentioned above, there are two other well-known operations: subtraction and division which can be regarded as simple arithmetic operations [21]. However, the subtraction and division operations discussed in [21] have strict limitations, which seriously affect their generality. Thus, the revised subtraction and division operations are derived [22], which avoid all kinds of unnecessary constraints and have similar external structure compared with multiplication and addition operations. Furthermore, most of the existing intuitionistic fuzzy operations just discuss the equivalence relationship on them, and rarely explore their properties or theorems related to their inequality [22]. In other words, the study of related inequalities on IFSs has important value and will greatly enrich the relevant inequality theory.

Aggregation operator (AO), commonly in the form of mathematical functions, is a usual technique for fusing all the input into a single piece of data, which initially derived from operations. Xu [13] primitively developed intuitionistic fuzzy weighted averaging (IFWA) operator which

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deduces from the addition and $\lambda-$multiplication operations, and discuss some inequalities on it. However, when IFWA operator in [13] encounters special intuitionistic fuzzy value (IFV) such as (1,0) in the decision-making process, it is prone to counterintuitive phenomena, which will vastly influence the authority of the decision-making process and reduce the effectiveness of the algorithm. Therefore, Seikh and Mandal [23] introduced intuitionistic fuzzy Dombi weighted averaging (IFDWA) operator and intuitionistic fuzzy Dombi weighted geometric (IFDWG) operator for fusing job information and selecting optimal alternative. While there is nothing counterintuitive about the AOs in [23], the strict requirements for decision data are necessary and non-negotiable. In other words, data that does not meet the requirements will lead to an inability to make decisions, reducing its versatility. To sum up, it is important to develop AO without counterintuitive phenomena and insensitive to data under intuitionistic fuzzy environment. At the same time, inequalities related to AOs can also enrich the connotation of IFS.

Furthermore, it can be easily seen that the existing inequalities on IFSs/IFV arise from simple operations and AOs. That is to say, none of them depend on definition of IFS as $\mu + \upsilon + \pi = 1$. Thus, it will fill in the gaps in relevant inequality research. The inequalities on IFSs/IFV are derived by constructing operations, AOs, equality in definition, and proving them by some famous inequalities. Moreover, the developed inequalities based equality in definition may respectively prove the operations and AOs on IFSs or IFVs to some extent. The main contributions of this paper are summarized as follows:

1. The inequalities on IFSs are derived from some existing operations, where the objective is to find their relationships. The corresponding intuitionistic fuzzy inequalities are well proved.
2. Depending on the excellent peculiarity of the operation @, the unweighted intuitionistic fuzzy Square (UIFS) operator is presented, which can avert counterintuitive phenomena and insensitive to data. Some corresponding inequalities based UIFS are constructed and proved. Also, the unweighted intuitionistic fuzzy Arithmetic (UIFA) operator and unweighted intuitionistic fuzzy Geometric (UIFG) operator are explored by linking a significant inequality with UIFS.
3. The inequalities on IFV derived from the equality $\mu + \upsilon + \pi = 1$ in definition, which are proved by some existing famous inequalities. It provides a new basis for the intuitionistic fuzzy inequalities in operations and AOs.

To accomplish our ideas, the rest paper is organized as follows. The fundamental definition, score function, some operations on IFSs and some famous inequalities are reviewed in Sect. 2. Section 3 gives some inequalities on IFSs, which derived from some existing operations. Section 4 presents three unweighted intuitionistic fuzzy AOs and corresponding inequalities. Section 5 introduces a set of inequalities on IFV, which derived from equality in definition. The paper gives some conclusions in Sect. 6.

2 Preliminaries

In this section, the basic definition, score function, operations of intuitionistic fuzzy sets and some famous inequalities are reviewed to facilitate the further analysis of the paper.

2.1 Intuitionistic Fuzzy Set

Intuitionistic fuzzy set (IFS) is an effective tool to depict vagueness. The mathematical expression form of IFS can be presented as follows.

Definition 2.1 [1] Let a set $\mathbb{X}$ be fixed. An IFS $\mathcal{A}$ in $\mathbb{X}$ is an object having the form

$$\mathcal{A} = \{\langle \chi, \mu(\chi), \upsilon(\chi)\rangle \mid \chi \in \mathbb{X}\}, \tag{1}$$

where $\mu : \mathbb{X} \to [0,1]$ and $\upsilon : \mathbb{X} \to [0,1]$ signify membership degree and non-membership degree of the element $\chi \in \mathbb{X}$ to the set $\mathcal{A}$, respectively, with the condition $0 \le \mu(\chi) + \upsilon(\chi) \le 1$. The hesitation degree $\pi(\chi) = 1 - \mu(\chi) - \upsilon(\chi)$. For simplicity, Xu [13] named $\ddot{a} = (\mu, \upsilon)$ as an intuitionistic fuzzy value (IFV). Especially, we call it as crisp number when $\ddot{a} = (1, 0)$ or $\ddot{a} = (0, 1)$. In this paper, we also take the hesitation degree into consideration and signify $\ddot{a} = (\mu, \upsilon, \pi)$ as an IFV.

Definition 2.2 [1, 13, 18, 22] Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be any two IFSs, then operations on IFSs are defined as follows.

(1) $\mathcal{A}_1 \subseteq \mathcal{A}_2$ iff $\mu_1(\chi) \le \mu_2(\chi), \upsilon_1(\chi) \ge \upsilon_2(\chi)$ for $\forall \chi \in \mathbb{X}$;
(2) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \{\langle \chi, \mu_1(\chi) + \mu_2(\chi) - \mu_1(\chi)\mu_2(\chi), \upsilon_1(\chi)\upsilon_2(\chi)\rangle \mid \chi \in \mathbb{X}\}$;
(3) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \{\langle \chi, \mu_1(\chi)\mu_2(\chi), \upsilon_1(\chi) + \upsilon_2(\chi) - \upsilon_1(\chi)\upsilon_2(\chi)\rangle \mid \chi \in \mathbb{X}\}$;
(4) $\mathcal{A}_1 \ominus \mathcal{A}_2 = \{\langle \chi, \mu_1(\chi)\upsilon_2(\chi), \upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi)\rangle \mid \chi \in \mathbb{X}\}$;
(5) $\mathcal{A}_1 \varnothing \mathcal{A}_2 = \{\langle \chi, \mu_1(\chi) + \upsilon_2(\chi) - \mu_1(\chi)\upsilon_2(\chi), \upsilon_1(\chi)\mu_2(\chi)\rangle \mid \chi \in \mathbb{X}\}$;
(6) $\Box \mathcal{A}_1 = \{\langle \chi, \mu_1(\chi), 1 - \mu_1(\chi)\rangle \mid \chi \in \mathbb{X}\}$;
(7) $\Diamond \mathcal{A}_1 = \{\langle \chi, 1 - \upsilon_1(\chi), \upsilon_1(\chi)\rangle \mid \chi \in \mathbb{X}\}$;
(8) $\mathcal{A}_1 @ \mathcal{A}_2 = \left\{\langle \chi, \frac{\mu_1(\chi)+\mu_2(\chi)}{2}, \frac{\upsilon_1(\chi)+\upsilon_2(\chi)}{2}\rangle \mid \chi \in \mathbb{X}\right\}$;
(9) $\mathcal{A}_1 \$ \mathcal{A}_2 = \{\langle \chi, \sqrt{\mu_1(\chi)\mu_2(\chi)}, \sqrt{\upsilon_1(\chi)\upsilon_2(\chi)}\rangle \mid \chi \in \mathbb{X}\}$;

(10) $\mathcal{A}_1 \# \mathcal{A}_2 = \left\{ \langle \chi, \frac{2\mu_1(\chi)\mu_2(\chi)}{\mu_1(\chi)+\mu_2(\chi)}, \frac{2\upsilon_1(\chi)\upsilon_2(\chi)}{\upsilon_1(\chi)+\upsilon_2(\chi)} \rangle \mid \chi \in \mathbb{X} \right\}$;

(11) $\mathcal{A}_1 \bigstar \mathcal{A}_2 = \left\{ \langle \chi, \frac{\mu_1(\chi)+\mu_2(\chi)}{2(\mu_1(\chi)+\mu_2(\chi)+1)}, \frac{\upsilon_1(\chi)+\upsilon_2(\chi)}{2(\upsilon_1(\chi)+\upsilon_2(\chi)+1)} \rangle \mid \chi \in \mathbb{X} \right\}$;

(12) $\mathcal{A}_1 \rightarrow \mathcal{A}_2 = \{ \langle \chi, \max\{\upsilon_1(\chi), \mu_2(\chi)\}, \min\{\mu_1(\chi), \upsilon_2(\chi)\} \rangle \mid \chi \in \mathbb{X} \}$;

(13) $\lambda \mathcal{A}_1 = \left\{ \langle \chi, 1-(1-\mu_1(\chi))^{\lambda}, \upsilon_1^{\lambda}(\chi) \rangle \mid \chi \in \mathbb{X} \right\}$, $\lambda \geq 0$;

(14) $\mathcal{A}_1^{\lambda} = \left\{ \langle \chi, \mu_1^{\lambda}(\chi), 1-(1-\upsilon_1(\chi))^{\lambda} \rangle \mid \chi \in \mathbb{X} \right\}$, $\lambda \geq 0$.

Definition 2.3 [13] For any IFN $\ddot{a} = (\mu, \upsilon, \pi)$, the score function of $\ddot{a}$ having the form

$$\mathbb{S}(\ddot{a}) = \mu - \upsilon, \quad \mathbb{S}(\ddot{a}) \in [-1, 1]. \tag{2}$$

For any two IFNs $\ddot{a}_1$ and $\ddot{a}_2$,

(1) If $\mathbb{S}(\ddot{a}_1) > \mathbb{S}(\ddot{a}_2)$, then $\ddot{a}_1 > \ddot{a}_2$;
(2) If $\mathbb{S}(\ddot{a}_1) < \mathbb{S}(\ddot{a}_2)$, then $\ddot{a}_1 < \ddot{a}_2$.

2.2 Some Famous Inequalities and Theorem

The existing famous inequalities are considered as the form of Lemma, including Rearrangement inequality [24], Mean inequality (AM-GM, AM-SM, SM-GM,HM-AM,3 M) [25], Nesbitt's inequality [26], Chebyshev's inequality [27], Cauchy's inequality [28], Power-Mean inequality [29], Minkowski's inequality [28], Hölder's inequality [28], Carlson's inequality [30], Jensen's inequality [31], Wei-Wei dual inequality [32], Tangent inequality [33], Muirhead's inequality [34], Schur's inequality [35], Vasc inequality [36] and Bernoulli's inequality [37], which are given as follows. In addition, the half concave and half convex theorem is also listed.

Lemma 2.1 (*Rearrangement inequality*) [24] *For any two ordered sets* $\{x_1, x_2, \ldots, x_n\}$ *and* $\{y_1, y_2, \ldots, y_n\}$ *with* $x_1 \geq x_2 \geq \cdots \geq x_n$ *and* $y_1 \geq y_2 \geq \cdots \geq y_n$, *then*

$$\underbrace{\sum_{i=1}^{n} x_i y_{n-i+1}}_{\text{Reverse order}} \leq \underbrace{\sum_{i=1}^{n} x_{j_i} y_{j_i}}_{\text{Random order}} \leq \underbrace{\sum_{i=1}^{n} x_i y_i}_{\text{Same order}}, \tag{3}$$

where $\{j_1, j_2, \ldots, j_n\}$ is a arbitrary full permutation of $\{1, 2, \ldots, n\}$, and the equal sign occurs when $x_1 = x_2 = \cdots = x_n$ or $y_1 = y_2 = \cdots = y_n$.

Lemma 2.2 (*Mean inequality*) [25] *For any nonnegative set* $\{x_1, x_2, \ldots, x_n\}$, *then*

$$\overbrace{\underbrace{\overbrace{\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}}^{\text{Harmonicmean}} \leq \overbrace{\sqrt[n]{\prod_{i=1}^{n} x_i}}^{\text{Geometricmean}} \leq \overbrace{\frac{\sum_{i=1}^{n} x_i}{n}}^{\text{Arithmeticmean}}}_{\text{AM}-\text{GM}} \leq \overbrace{\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}}^{\text{Squaremean}}}^{\text{SM}-\text{GM}} \tag{4}$$

and

$$\underbrace{3(x_1x_2 + x_2x_3 + x_3x_1) \leq (x_1 + x_2 + x_3)^2}_{\text{3M}}, \tag{5}$$

where the equal sign occurs when $x_1 = x_2 = \cdots = x_n$.

Remark 2.1 The Mean inequality consists of Harmonic mean (HM), Geometric mean (GM), Arithmetic mean (AM) and Square mean (SM). Some common mean inequalities can be written as SM-GM, AM-GM, HM-AM, AM-SM and 3 M.

Lemma 2.3 (*Chebyshev's inequality*) [27] *For two ordered sets* $\{x_1, x_2, \ldots, x_n\}$ *and* $\{y_1, y_2, \ldots, y_n\}$ *in the same order, then*

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \geq \sum_{i=1}^{n} x_i y_{n-i+1}, \tag{6}$$

where the equal sign occurs when $x_1 = x_2 = \cdots = x_n$ or $y_1 = y_2 = \cdots = y_n$.

Lemma 2.4 (*Nesbitt's inequality) [26] Let x, y, and z be three positive numbers, then*

$$\sum_{cyc} \frac{x}{y+z} \geq \frac{3}{2}, \tag{7}$$

where the equal sign occurs when $x = y = z$.

Lemma 2.5 (*Power-Mean inequality*) [29] *For any non-negative set* $\{x_1, x_2, \ldots, x_n\}$ *and* $\varrho \geq \sigma > 0$, *then*

$$\sqrt[\varrho]{\frac{\sum_{i=1}^{n} x_i^{\varrho}}{n}} \geq \sqrt[\sigma]{\frac{\sum_{i=1}^{n} x_i^{\sigma}}{n}}, \tag{8}$$

where the equal sign occurs when $x_1 = x_2 = \cdots = x_n$.

Lemma 2.6 (*Cauchy's inequality*) [28] *For any two sets* $\{x_1, x_2, \ldots, x_n\}$ *and* $\{y_1, y_2, \ldots, y_n\}$, *then*

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \geq \left(\sum_{i=1}^{n} x_i y_i\right)^2, \tag{9}$$

where the equal sign occurs when $x_i = \kappa y_i \ (i = 1, 2, \ldots, n)$.

Lemma 2.7 (*Generalized Cauchy's inequality*) [28] *For any two nonnegative sets* $\{x_1, x_2, \ldots, x_n\}$ *and* $\{y_1, y_2, \ldots, y_n\}$, $\alpha \geq 2$, *then*

$$\sum_{i=1}^{n} \frac{x_i^{\alpha}}{y_i} \geq n^{2-\alpha} \frac{\left(\sum_{i=1}^{n} x_i\right)^{\alpha}}{\sum_{i=1}^{n} y_i}, \tag{10}$$

where the equal sign occurs when $x_1 = x_2 = \cdots = x_n$ and $y_1 = y_2 = \cdots = y_n$.

Lemma 2.8 (*Hölder's inequality*) [28] *For any two non-negative sets* $\{x_1, x_2, \ldots, x_n\}$ *and* $\{y_1, y_2, \ldots, y_n\}$ *such that for* $\alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1$, *then*

$$\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} y_i^{\beta}\right)^{\frac{1}{\beta}} \geq \sum_{i=1}^{n} x_i y_i, \tag{11}$$

where the equal sign occurs when $x_i = \kappa y_i \, (i = 1, 2, \ldots, n)$.

Lemma 2.9 (*Carlson's inequality*) *[30] Let* $x_{ij} \geq 0 \, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$, *then*

$$\left(\prod_{j=1}^{m} \sum_{i=1}^{n} x_{ij}\right)^{\frac{1}{m}} \geq \sum_{i=1}^{n} \left(\prod_{j=1}^{m} x_{ij}\right)^{\frac{1}{m}}, \tag{12}$$

where the equal sign occurs when $x_{1j} = x_{2j} = \cdots = x_{nj} = 0$ (more than one j) or $\frac{x_{1j}}{x_{2j}} = \frac{x_{2j}}{x_{3j}} = \cdots = \frac{x_{(n-1)j}}{x_{nj}}$ $(j = 1, 2, \ldots, m)$.

Lemma 2.10 (*Wei-Wei dual inequality*) [32] *For non-negative ordered set* $\{x_{i1}, x_{i2}, \ldots, x_{in}\} \, (i = 1, 2, \ldots, m)$, *and* $\{x'_{i1}, x'_{i2}, \ldots, x'_{in}\}$ *is one of its full permutations, then*

$$\begin{aligned} &\sum_{j=1}^{n} \prod_{i=1}^{m} x_{ij} \geq \sum_{j=1}^{n} \prod_{i=1}^{m} x'_{ij}, \\ &\prod_{j=1}^{n} \sum_{i=1}^{m} x_{ij} \leq \prod_{j=1}^{n} \sum_{i=1}^{m} x'_{ij}, \end{aligned} \tag{13}$$

where the equal sign occurs when $\{x'_{i1}, x'_{i2}, \ldots, x'_{in}\} \, (i = 1, 2, \ldots, m)$ is also an ordered set.

Lemma 2.11 (*Minkowski's inequality*) [28] *For any positive number* $x_{ij} \, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ *and* $\alpha > 1$, *then*

$$\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{m} x_{ij}\right)^{\alpha}\right)^{\frac{1}{\alpha}} \leq \sum_{j=1}^{m} \left(\sum_{i=1}^{n} x_{ij}^{\alpha}\right)^{\frac{1}{\alpha}}, \tag{14}$$

where the equal sign occurs when $x_{i1}, x_{i2}, \ldots, x_{im} \, (i = 1, 2, \ldots, n)$ are proportional.

Lemma 2.12 (*Jensen's inequality*) [31] *Let f(x) be a function in the open interval (a, b), then for any* $x_i \, (i = 1, 2, \ldots, n)$ *in (a, b), then*

$$\frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n)) \geq f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right), f(x) \text{ is concave function}, \tag{15}$$

$$\frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n)) \leq f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right), f(x) \text{ is convex function}, \tag{16}$$

where the equal sign occurs when $x_1 = x_2 = \cdots = x_n$.

Remark 2.2 And the converse weighted form of Jensen's inequality is

$$w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n) \leq f(w_1 x_1 + w_2 x_2 + \cdots + w_n x_n), \tag{17}$$

where $w_i > 0, \sum_{i=1} w_i = 1$ and $f(x)$ is a convex function in the open interval (a, b).

Lemma 2.13 (*Tangent inequality*) [33] *If f(x) is continuous and derivable function in the domain of definition, and is defined at* $x = x_0$, *then*

(1) If $f''(x) \geq 0$, then $f(x) \geq f'(x_0)(x - x_0) + f(x_0)$ (concave function);
(2) If $f''(x) \leq 0$, then $f(x) \leq f'(x_0)(x - x_0) + f(x_0)$ (convex function).

Lemma 2.14 (*Schur's inequality) [35] Let* $x, y, z \geq 0$ *and* $\alpha \in R$, *then*

$$\sum_{cyc} x^{\alpha}(x - y)(x - z) \geq 0, \tag{18}$$

where the equal sign occurs when two of them $\{x, y, z\}$ are equal and the other is zero, or $x = y = z$. If $\alpha = 1$, then $\left(\sum_{cyc} x\right)^3 - 4\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) + 9xyz \geq 0$; If $\alpha = 2$, then $xyz \geq \frac{\left(4\sum_{cyc} xy - 1\right)\left(1 - \sum_{cyc} xy\right)}{6}$.

Lemma 2.15 (*Muirhead's inequality*) [34] *Given* $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ *such that* $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0, \beta_1 \geq \beta_2 \geq \beta_3 \geq 0, \alpha_1 \geq \beta_1, \alpha_1 + \alpha_2 \geq \beta_1 + \beta_2, \alpha_1 + \alpha_2 + \alpha_3 \geq \beta_1 + \beta_2 + \beta_3$ *and* $x, y, z > 0$, *then*

$$\sum_{sym} x^{\alpha_1} y^{\alpha_2} \geq \sum_{sym} x^{\beta_1} y^{\beta_2}, \tag{19}$$

$$\sum_{sym} x^{\alpha_1} y^{\alpha_2} z^{\alpha_3} \geq \sum_{sym} x^{\beta_1} y^{\beta_2} z^{\beta_3}. \tag{20}$$

Lemma 2.16 (*Vasc inequality*) [36] *Let x, y, z be any real numbers, then*

$$\left(\sum_{cyc} x^2\right)^2 \geq 3\sum_{cyc} x^3 y, \tag{21}$$

$$\left(\sum_{cyc} x^2\right)^2 \geq 3\sum_{cyc} x y^3. \tag{22}$$

Lemma 2.17 *(Bernoulli's inequality) [37] Let x be any real number with* $x > -1$, *then*

$$\begin{cases} (1+x)^\alpha \geq 1 + \alpha x, & \text{if } \alpha \geq 1; \\ (1+x)^\alpha \leq 1 + \alpha x, & \text{if } 0 \leq \alpha \leq 1, \end{cases} \tag{23}$$

where the equal sign occurs when $x = 0$ or $\alpha = 0$ or $\alpha = 1$.

Remark 2.3 As can be seen from 17 famous inequalities above, there are some inherent relations among them, which are shown in the Fig. 1 [38]. The double arrows in the figure indicate that the two famous inequalities can be converted to each other to some extent, that is, there is a certain equivalence relation. Also, the single arrow means that it can only be derived from the beginning. This will give us a new idea, that is, there are many ways to prove a theorem or lemma. For instance, the AM-GM in Mean inequality is equivalent to the Bernoulli's inequality [37]. Another example, we can derive Vassilev-Missana's inequality by Young's inequality [39, 40]. Obviously, we can directly use the appropriate lemma, which will directly reduce the difficulty of global proof, but also greatly facilitate the further solution of the proof.

Theorem 2.1 (*Half concave and Half convex theorem*) [41] *Let f be a concave function on* $[a, c]$ *and convex function on* $[c, b]$. *Suppose that the variables* $x, y, z \in [a, b]$ *occur when* $x \leq y \leq z$ *and* $x + y + z = C$ (*C is constant*). *Then it occurs that*

(1) There exists $x = y$ or $z = b$ when $f(x) + f(y) + f(z)$ gets the minimum value;
(2) There exists $x = a$ or $y = z$ when $f(x) + f(y) + f(z)$ gets the maximum value.

3 Intuitionistic fuzzy inequalities derived by operations

In this section, some intuitionistic fuzzy inequalities based operations are derived. Moreover, they are correspondingly proved by some famous inequalities.

Theorem 3.1 *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be two IFSs on common* $\mathbb{X}$. *Then it holds that*:

(1) $\Diamond(\mathcal{A}_1 \oplus \mathcal{A}_2) \supseteq \Box(\mathcal{A}_1 \otimes \mathcal{A}_2)$;
(2) $\Diamond(\mathcal{A}_1 \o \mathcal{A}_2) \supseteq \Box(\mathcal{A}_1 \ominus \mathcal{A}_2)$;
(3) $\Diamond(\mathcal{A}_1 @ \mathcal{A}_2) \supseteq \Box(\mathcal{A}_1 \$ \mathcal{A}_2)$;
(4) $\Diamond(\mathcal{A}_1 \# \mathcal{A}_2) \supseteq \Box(\mathcal{A}_1 \bigstar \mathcal{A}_2)$.

Proof (1) Using the Definition 2.2, we get

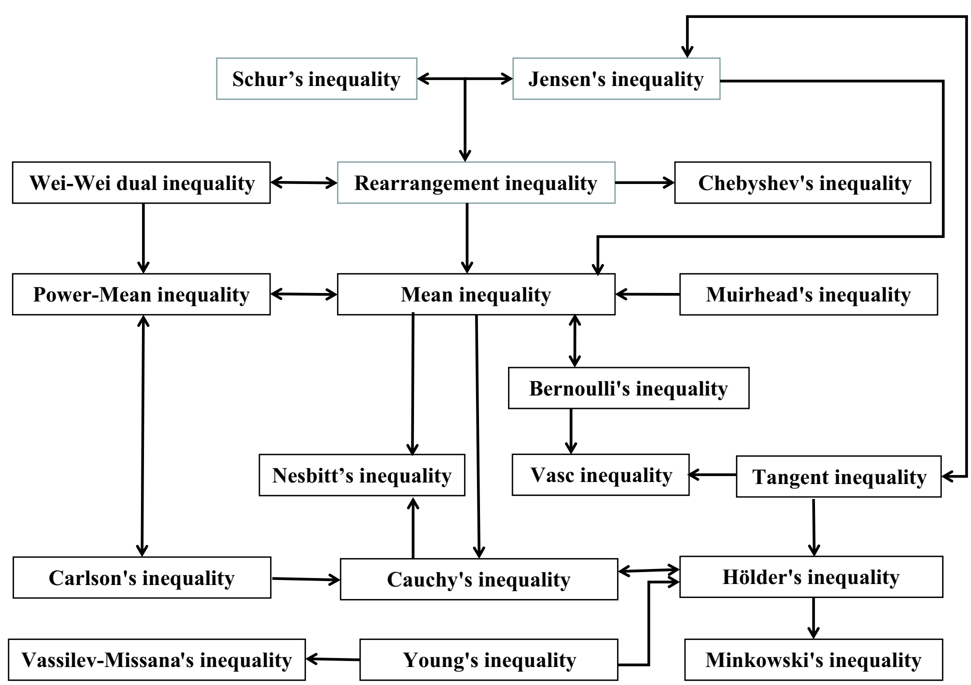

Fig. 1 The inherent relationship among the famous inequalities

$\Box(\mathcal{A}_1 \otimes \mathcal{A}_2) = \{\langle \chi, \mu_1(\chi)\mu_2(\chi), 1 - \mu_1(\chi) \quad \mu_2(\chi)\rangle \mid \chi \in \mathbb{X}\}$ and

$\Diamond(\mathcal{A}_1 \oplus \mathcal{A}_2) = \{\langle \chi, \quad 1 - \upsilon_1(\chi)\upsilon_2(\chi), \upsilon_1(\chi)\upsilon_2(\chi)\rangle \mid \chi \in \mathbb{X}\}$.

Let $f(\chi) = 1 - \upsilon_1(\chi)\upsilon_2(\chi) - \mu_1(\chi)\mu_2(\chi)$, then an immediate calculation displays

$f(\chi) \geq 1 - \sqrt{(\mu_1^2(\chi) + \upsilon_1^2(\chi)) \cdot (\mu_2^2(\chi) + \upsilon_2^2(\chi))}$ (Use Cauchy's inequality: Lemma 2.6)

$= 1 - \sqrt{\left((\mu_1(\chi) + \upsilon_1(\chi))^2 - 2\mu_1(\chi)\upsilon_1(\chi)\right) \cdot \left((\mu_2(\chi) + \upsilon_2(\chi))^2 - 2\mu_2(\chi)\upsilon_2(\chi)\right)}$

$\geq 1 - \sqrt{(\mu_1(\chi) + \upsilon_1(\chi))^2 \cdot (\mu_2(\chi) + \upsilon_2(\chi))^2}$

$= 1 - (\mu_1(\chi) + \upsilon_1(\chi)) \cdot (\mu_2(\chi) + \upsilon_2(\chi))$

$\geq 1 - (\mu_1(\chi) + \upsilon_1(\chi) + \pi_1(\chi)) \cdot (\mu_2(\chi) + \upsilon_2(\chi) + \pi_2(\chi))$

$= 1 - 1 \cdot 1$

$= 0.$

That is, $1 - \upsilon_1(\chi)\upsilon_2(\chi) \geq \mu_1(\chi)\mu_2(\chi)$.

Analogously, $1 - \mu_1(\chi)\mu_2(\chi) \geq \upsilon_1(\chi)\upsilon_2(\chi)$.

Consequently, it is fully proved by the definition (1) in Definition 2.2.

(2) It can be proved similarly to the formula (1) above.

(3) Using the Definition 2.2, we get

$\Diamond(\mathcal{A}_1 @ \mathcal{A}_2) \quad = \left\{\langle \chi, 1 - \frac{\upsilon_1(\chi)+\upsilon_2(\chi)}{2}, \frac{\upsilon_1(\chi)+\upsilon_2(\chi)}{2}\rangle \mid \chi \in \mathbb{X}\right\}$

and

$\Box(\mathcal{A}_1 \$ \mathcal{A}_2) = \{\langle \chi, \sqrt{\mu_1(\chi)\mu_2(\chi)}, 1 - \sqrt{\mu_1(\chi)\mu_2(\chi)}\rangle \mid \chi \in \mathbb{X}\}$.

Let $f(\chi) = 1 - \frac{\upsilon_1(\chi)+\upsilon_2(\chi)}{2} - \sqrt{\mu_1(\chi)\mu_2(\chi)}$, then

$f(\chi) \geq 1 - \frac{\upsilon_1(\chi)+\upsilon_2(\chi)}{2} - \frac{\mu_1(\chi)+\mu_2(\chi)}{2} \geq 0$ (Use Mean inequality (AM-GM): Lemma 2.2).

That is, $1 - \frac{\upsilon_1(\chi)+\upsilon_2(\chi)}{2} \geq \sqrt{\mu_1(\chi)\mu_2(\chi)}$.

Analogously, $1 - \sqrt{\mu_1(\chi)\mu_2(\chi)} \geq \frac{\upsilon_1(\chi)+\upsilon_2(\chi)}{2}$.

Consequently, it is fully proved by the definition (1) in Definition 2.2.

(4) Using the Definition 2.2, we derive

$\Diamond(\mathcal{A}_1 \# \mathcal{A}_2) \quad = \left\{\langle \chi, 1 - \frac{2\upsilon_1(\chi)\upsilon_2(\chi)}{\upsilon_1(\chi)+\upsilon_2(\chi)}, \frac{2\upsilon_1(\chi)\upsilon_2(\chi)}{\upsilon_1(\chi)+\upsilon_2(\chi)}\rangle \mid \chi \in \mathbb{X}\right\}$

and

$\Box(\mathcal{A}_1 \bigstar \mathcal{A}_2) = \left\{\langle \chi, \frac{\mu_1(\chi)+\mu_2(\chi)}{2(\mu_1(\chi)+\mu_2(\chi)+1)}, 1 - \frac{\mu_1(\chi)+\mu_2(\chi)}{2(\mu_1(\chi)+\mu_2(\chi)+1)}\rangle \mid \chi \in \mathbb{X}\right\}$.

Let $f(\chi) = 1 - \frac{2\upsilon_1(\chi)\upsilon_2(\chi)}{\upsilon_1(\chi)+\upsilon_2(\chi)} - \frac{\mu_1(\chi)+\mu_2(\chi)}{2(\mu_1(\chi)+\mu_2(\chi)+1)}$, then an immediate calculation displays

$f(\chi) = \frac{(\mu_1(\chi)+\mu_2(\chi)+1)(\upsilon_1(\chi)+\upsilon_2(\chi)-4\upsilon_1(\chi)\upsilon_2(\chi))+\upsilon_1(\chi)+\upsilon_2(\chi)}{2(\upsilon_1(\chi)+\upsilon_2(\chi))(\mu_1(\chi)+\mu_2(\chi)+1)}$

$\geq \frac{\frac{1}{2}(\mu_1(\chi)+\mu_2(\chi)+1+1)(2(\upsilon_1(\chi)+\upsilon_2(\chi))-4\upsilon_1(\chi)\upsilon_2(\chi))}{2(\upsilon_1(\chi)+\upsilon_2(\chi))(\mu_1(\chi)+\mu_2(\chi)+1)}$ (Use Chebyshev's inequality: Lemma 2.3)

$= \frac{(\mu_1(\chi)+\mu_2(\chi)+2)(\upsilon_1(\chi)+\upsilon_2(\chi)-2\upsilon_1(\chi)\upsilon_2(\chi))}{2(\upsilon_1(\chi)+\upsilon_2(\chi))(\mu_1(\chi)+\mu_2(\chi)+1)}$

$= \frac{(\mu_1(\chi)+\mu_2(\chi)+2)(\upsilon_1(\chi)(1-\upsilon_2(\chi))+\upsilon_2(\chi)(1-\upsilon_1(\chi)))}{2(\upsilon_1(\chi)+\upsilon_2(\chi))(\mu_1(\chi)+\mu_2(\chi)+1)}$

$\geq 0.$

That is, $1 - \frac{2\upsilon_1(\chi)\upsilon_2(\chi)}{\upsilon_1(\chi)+\upsilon_2(\chi)} \geq \frac{\mu_1(\chi)+\mu_2(\chi)}{2(\mu_1(\chi)+\mu_2(\chi)+1)}$.

Analogously, $1 - \frac{\mu_1(\chi)+\mu_2(\chi)}{2(\mu_1(\chi)+\mu_2(\chi)+1)} \geq \frac{2\upsilon_1(\chi)\upsilon_2(\chi)}{\upsilon_1(\chi)+\upsilon_2(\chi)}$.

Consequently, it is fully proved by the definition (1) in Definition 2.2. □

Remark 3.1 Four intuitionistic fuzzy inequalities in Theorem 3.1 distinctly illustrate that two intuitionistic fuzzy operations with dual form may possess a trend of potential inequalities. It provides us with a new vision, that is, when constructing inequalities, it is better to choose the dual form of two new or existing operations, so that there may produce better inequality relations under intuitionistic fuzzy environment, and even equality relations.

Theorem 3.2 *Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be three IFSs on common $\mathbb{X}$. Then it holds that*:

(1) $(\mathcal{A}_1 \ominus \mathcal{A}_2) @ \mathcal{A}_3 \supseteq (\mathcal{A}_1 @ \mathcal{A}_3) \ominus (\mathcal{A}_2 @ \mathcal{A}_3)$;
(2) $(\mathcal{A}_1 \o \mathcal{A}_2) @ \mathcal{A}_3 \subseteq (\mathcal{A}_1 @ \mathcal{A}_3) \o (\mathcal{A}_2 @ \mathcal{A}_3)$;
(3) $(\mathcal{A}_1 \ominus \mathcal{A}_2) \to \mathcal{A}_3 \supseteq (\mathcal{A}_1 \to \mathcal{A}_3) \ominus (\mathcal{A}_2 \to \mathcal{A}_3)$;
(4) $(\mathcal{A}_1 \o \mathcal{A}_2) \to \mathcal{A}_3 \subseteq (\mathcal{A}_1 \to \mathcal{A}_3) \o (\mathcal{A}_2 \to \mathcal{A}_3)$;
(5) $(\mathcal{A}_1 \ominus \mathcal{A}_2) \bigstar \mathcal{A}_3 \supseteq (\mathcal{A}_1 \bigstar \mathcal{A}_3) \ominus (\mathcal{A}_2 \bigstar \mathcal{A}_3)$;
(6) $(\mathcal{A}_1 \o \mathcal{A}_2) \bigstar \mathcal{A}_3 \subseteq (\mathcal{A}_1 \bigstar \mathcal{A}_3) \o (\mathcal{A}_2 \bigstar \mathcal{A}_3)$.

Proof We just prove the formulas (1), (3) and (5), the formulas (2), (4) and (6) can be proved similarly.

(1) Employing the Definition 2.2, we can get

$(\mathcal{A}_1 \ominus \mathcal{A}_2) @ \mathcal{A}_3 = \left\{\langle \chi, \frac{\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)}{2}, \frac{\upsilon_1(\chi)+\mu_2(\chi)-\upsilon_1(\chi)\mu_2(\chi)+\upsilon_3(\chi)}{2}\rangle \mid \chi \in \mathbb{X}\right\}$ and

$(\mathcal{A}_1 @ \mathcal{A}_3) \ominus (\mathcal{A}_2 @ \mathcal{A}_3) = \left\{\langle \chi, \frac{(\mu_1(\chi)+\mu_3(\chi))(\upsilon_2(\chi)+\upsilon_3(\chi))}{4}, \frac{\upsilon_1(\chi)}{} \quad +\upsilon_3(\chi) + \mu_2(\chi) + \mu_3(\chi)2 - (\upsilon_1(\chi) + \upsilon_3(\chi))\left(\frac{\mu_2(\chi)}{} + \mu_3(\chi))4\rangle \mid \chi \in \mathbb{X}\right\}$.

Let $f(\chi) = \frac{\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)}{2} - \frac{(\mu_1(\chi)+\mu_3(\chi))(\upsilon_2(\chi)+\upsilon_3(\chi))}{4}$, then an immediate calculation displays

$f(\chi) = \frac{\mu_1(\chi)(\upsilon_2(\chi)-\upsilon_3(\chi))+\mu_3(\chi)(2-\upsilon_2(\chi)-\upsilon_3(\chi))}{4}$

$\geq \frac{\frac{1}{2}(\mu_1(\chi)+\mu_3(\chi))(\upsilon_2(\chi)-\upsilon_3(\chi)+2-\upsilon_2(\chi)-\upsilon_3(\chi))}{4}$ (Use Chebyshev's inequality: Lemma 2.3)

$= \frac{(\mu_1(\chi)+\mu_3(\chi))(1-\upsilon_3(\chi))}{4}$

$\geq 0.$

That is, $\frac{\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)}{2} \geq \frac{(\mu_1(\chi)+\mu_3(\chi))(\upsilon_2(\chi)+\upsilon_3(\chi))}{4}$.

Analogously, $\frac{\upsilon_1(\chi)+\upsilon_3(\chi)+\mu_2(\chi)+\mu_3(\chi)}{2} - \frac{(\upsilon_1(\chi)+\upsilon_3(\chi))(\mu_2(\chi)+\mu_3(\chi))}{4} \geq \frac{\upsilon_1(\chi)+\mu_2(\chi)-\upsilon_1(\chi)\mu_2(\chi)+\upsilon_3(\chi)}{2}$.

Consequently, it is fully proved by the definition (1) in Definition 2.2.

(3) Using the Definition 2.2, we can get

$(\mathcal{A}_1 \ominus \mathcal{A}_2) \rightarrow \mathcal{A}_3 = \{\langle \chi, \max\{\mu_3(\chi), \upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi)\}, \min\{\upsilon_3(\chi), \mu_1(\chi)\upsilon_2(\chi)\}\rangle \mid \chi \in \mathbb{X}\}$ and

$(\mathcal{A}_1 \rightarrow \mathcal{A}_3) \ominus (\mathcal{A}_2 \rightarrow \mathcal{A}_3) = \{<\chi, \max\{\upsilon_1(\chi), \mu_3(\chi)\} \cdot \min\{\mu_2(\chi), \upsilon_3(\chi)\}, \max\{\mu_1(\chi), \upsilon_3(\chi)\} + \max\{\upsilon_2(\chi), \mu_3(\chi)\}.$

$- \max\{\mu_1(\chi), \upsilon_3(\chi)\} \cdot \max\{\upsilon_2(\chi), \mu_3(\chi)\} > \mid \chi \in \mathbb{X}\}.$

Let $f(\chi) = \max\{\mu_3(\chi), \upsilon_1(\chi) + \mu_2(\chi) \quad -\upsilon_1(\chi)\mu_2(\chi)\} - \max\{\upsilon_1(\chi), \mu_3(\chi)\} \cdot \min\{\mu_2(\chi), \upsilon_3(\chi)\}$, then four cases are listed and discussed as follows:

Case 1: If $\upsilon_1(\chi) \geq \mu_3(\chi)$ and $\mu_2(\chi) \geq \upsilon_3(\chi)$, then $\upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) = \upsilon_1(\chi) + \mu_2(\chi)(1 - \upsilon_1(\chi)) \geq \mu_3(\chi)$.

Thus, $f(\chi) = \upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) \quad -\upsilon_1(\chi)\upsilon_3(\chi) = \upsilon_1(\chi)(1 - \upsilon_3(\chi)) + \mu_2(\chi)(1 - \upsilon_1(\chi)) \geq 0$.

Case 2: If $\upsilon_1(\chi) \geq \mu_3(\chi)$ and $\mu_2(\chi) \leq \upsilon_3(\chi)$, then $\upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) \geq \mu_3(\chi)$.

Thus, $f(\chi) = \upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) \quad -\upsilon_1(\chi)\mu_2(\chi) = \upsilon_1(\chi)(1 - \mu_2(\chi)) + \mu_2(\chi)(1 - \upsilon_1(\chi)) \geq 0$.

Case 3: If $\upsilon_1(\chi) \leq \mu_3(\chi)$ and $\mu_2(\chi) \geq \upsilon_3(\chi)$, then

Case 3.1: If $\upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) \geq \mu_3(\chi)$, then $f(\chi) = \upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) - \mu_3(\chi)\upsilon_3(\chi) \geq \mu_3(\chi) - \mu_3(\chi)\upsilon_3(\chi) = \mu_3(\chi)(1 - \upsilon_3(\chi)) \geq 0$.

Case 3.2: If $\upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) \leq \mu_3(\chi)$, then $f(\chi) = \mu_3(\chi) - \mu_3(\chi)\upsilon_3(\chi) = \mu_3(\chi)(1 - \upsilon_3(\chi)) \geq 0$.

Case 4: If $\upsilon_1(\chi) \leq \mu_3(\chi)$ and $\mu_2(\chi) \leq \upsilon_3(\chi)$, then

Case 4.1: If $\upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) \geq \mu_3(\chi)$, then $f(\chi) = \upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi)$
$-\upsilon_1(\chi)\mu_2(\chi) = \upsilon_3(\chi)(1 - \mu_2(\chi)) + \mu_2(\chi)(1 - \upsilon_1(\chi)) \geq 0$.

Case 4.2: If $\upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) \leq \mu_3(\chi)$, then $f(\chi) = \mu_3(\chi) - \mu_3(\chi)\mu_2(\chi) = \mu_3(\chi)(1 - \mu_2(\chi)) \geq 0$.

According to the above four cases, then we get $f(\chi) \geq 0$.

That is, $\max\{\mu_3(\chi), \upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi)\} \geq \max\{\upsilon_1(\chi), \mu_3(\chi)\} \cdot \min\{\mu_2(\chi), \upsilon_3(\chi)\}$.

Analogously, $\max\{\mu_1(\chi), \upsilon_3(\chi)\} + \max\{\upsilon_2(\chi), \mu_3(\chi)\} - \max\{\mu_1(\chi), \upsilon_3(\chi)\} \cdot$
$\max\{\upsilon_2(\chi), \mu_3(\chi)\} \geq \min\{\upsilon_3(\chi), \mu_1(\chi)\, \upsilon_2(\chi)\}$.

Consequently, it is fully proved by the definition (1) in Definition 2.2.

(5) According to the Definition 2.2, we get

$(\mathcal{A}_1 \ominus \mathcal{A}_2)\bigstar\mathcal{A}_3 = \{\langle \chi, \quad \frac{\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)}{2(\mu_1(\chi)} \quad \upsilon_2(\chi) + \mu_3 + 1), \frac{\upsilon_1(\chi)+\mu_2(\chi)-\upsilon_1(\chi)\mu_2(\chi)+\upsilon_3(\chi)}{2(\upsilon_1(\chi)+\mu_2(\chi)-\upsilon_1(\chi)\mu_2(\chi)+\upsilon_3(\chi)+1)}\rangle \mid \chi \in \mathbb{X}\}$ and

$(\mathcal{A}_1\bigstar\mathcal{A}_3) \ominus (\mathcal{A}_2\bigstar\mathcal{A}_3)$

$= \Big\{\langle \chi, \frac{\mu_1(\chi)+\mu_3(\chi)}{2(\mu_1(\chi)+\mu_3(\chi)+1)} \cdot \frac{\upsilon_2(\chi)+\upsilon_3(\chi)}{2(\upsilon_2(\chi)+\upsilon_3(\chi)+1)}, \frac{\upsilon_1(\chi)+\upsilon_3(\chi)}{}$

$2(\upsilon_1(\chi) + \upsilon_3(\chi) + 1) + \frac{\mu_2(\chi)+\mu_3(\chi)}{2(\mu_2(\chi)+\mu_3(\chi)+1)} - \frac{\upsilon_1(\chi)}{}$

$+\upsilon_3(\chi)2(\upsilon_1(\chi) + \upsilon_3(\chi) + 1) \cdot \frac{\mu_2(\chi)+\mu_3(\chi)}{2(\mu_2(\chi)+\mu_3(\chi)+1)}\rangle \mid \chi \in \mathbb{X}\}$.

Let $f(\chi) = \frac{\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)}{2(\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)+1)} - \quad \frac{\mu_1(\chi)+\mu_3(\chi)}{2(\mu_1(\chi)+\mu_3(\chi)+1)} \cdot \frac{\upsilon_2(\chi)+\upsilon_3(\chi)}{2(\upsilon_2(\chi)+\upsilon_3(\chi)+1)}$, then an immediate calculation displays

$$f(\chi) \geq \frac{\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)}{2(\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)+1)}$$
$$- \frac{\mu_1(\chi)+\mu_3(\chi)}{2(\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)+1)} \cdot \frac{\upsilon_2(\chi)+\upsilon_3(\chi)}{2(\upsilon_2(\chi)+\upsilon_3(\chi)+1)}$$
$$= \frac{1}{2(\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)+1)}$$
$$\left(\mu_1(\chi)\upsilon_2(\chi) + \mu_3(\chi) - \frac{(\mu_1(\chi)+\mu_3(\chi))(\upsilon_2(\chi)+\upsilon_3(\chi))}{2(\upsilon_2(\chi)+\upsilon_3(\chi)+1)}\right)$$
$$= \frac{2\mu_1(\chi)\upsilon_2(\chi)(\upsilon_2(\chi)+\upsilon_3(\chi))+(\upsilon_2(\chi)-\upsilon_3(\chi))\mu_1(\chi)+\mu_3(\chi)(\upsilon_2(\chi)+2)}{4(\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)+1)(\upsilon_2(\chi)+\upsilon_3(\chi)+1)}$$
$$\geq \frac{(2\mu_1(\chi)\upsilon_2(\chi)+\mu_1(\chi))(\upsilon_2(\chi)+\upsilon_3(\chi)+\upsilon_2(\chi)-\upsilon_3(\chi))+\mu_3(\chi)(\upsilon_2(\chi)+2)}{4(\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)+1)(\upsilon_2(\chi)+\upsilon_3(\chi)+1)}$$

(Use Chebyshev's inequality: Lemma 2.3)

$$= \frac{2\mu_1(\chi)\upsilon_2(\chi)(2\upsilon_2(\chi)+1)+\mu_3(\chi)(\upsilon_2(\chi)+2)}{4(\mu_1(\chi)\upsilon_2(\chi)+\mu_3(\chi)+1)(\upsilon_2(\chi)+\upsilon_3(\chi)+1)}$$
$$\geq 0.$$

Similarly, we let $g(\chi) = \frac{\upsilon_1(\chi)+\upsilon_3(\chi)}{2(\upsilon_1(\chi)+\upsilon_3(\chi)+1)} + \frac{\mu_2(\chi)+\mu_3(\chi)}{2(\mu_2(\chi)}$
$+\mu_3(\chi) + 1) - \frac{\upsilon_1(\chi)+\upsilon_3(\chi)}{2(\upsilon_1(\chi)+\upsilon_3(\chi)+1)} \cdot \frac{\mu_2(\chi)}{}$
$+\mu_3(\chi)2(\mu_2(\chi) + \mu_3(\chi) + 1) - \frac{\upsilon_1(\chi)+\mu_2(\chi)-\upsilon_1(\chi)\mu_2(\chi)}{}$
$+\upsilon_3(\chi)2(\upsilon_1(\chi) + \mu_2(\chi) - \upsilon_1(\chi)\mu_2(\chi) + \upsilon_3(\chi) + 1)$, then an immediate calculation displays

$$g(\chi) = \frac{\upsilon_1(\chi)+\upsilon_3(\chi)}{2(\upsilon_1(\chi)+\upsilon_3(\chi)+1)} \quad - \frac{\upsilon_1(\chi)+\mu_2(\chi)-\upsilon_1(\chi)\mu_2(\chi)+\upsilon_3(\chi)}{2(\upsilon_1(\chi)+\mu_2(\chi)-\upsilon_1(\chi)\mu_2(\chi)+\upsilon_3(\chi)+1)} +$$
$$\frac{\mu_2(\chi)}{} + \mu_3(\chi)2(\mu_2(\chi) + \mu_3(\chi) + 1)\left(1 - \frac{\upsilon_1(\chi)+\upsilon_3(\chi)}{2(\upsilon_1(\chi)+\upsilon_3(\chi)+1)}\right)$$
$$\geq \frac{\upsilon_1(\chi)+\upsilon_3(\chi)}{2(\upsilon_1(\chi)+\mu_2(\chi)-\upsilon_1(\chi)\mu_2(\chi)+\upsilon_3(\chi)+1)}$$
$$- \frac{\upsilon_1(\chi)+\mu_2(\chi)-\upsilon_1(\chi)\mu_2(\chi)+\upsilon_3(\chi)}{2(\upsilon_1(\chi)+\mu_2(\chi)(1-\upsilon_1(\chi))}$$
$$+\upsilon_3(\chi) + 1) + \frac{\mu_2(\chi)+\mu_3(\chi)}{2(\mu_2(\chi)+\mu_3(\chi)+1)}\left(1 - \frac{\upsilon_1(\chi)+\upsilon_3(\chi)}{2(\upsilon_1(\chi)+\upsilon_3(\chi)+1)}\right)$$
$$= \frac{\mu_2(\chi)(\upsilon_1(\chi)-1)}{2(\upsilon_1(\chi)+\mu_2(\chi)(1-\upsilon_1(\chi))+\upsilon_3(\chi)+1)}$$
$$+ \frac{\mu_2(\chi)+\mu_3(\chi)}{2(\mu_2(\chi)+\mu_3(\chi)+1)} \cdot \frac{\upsilon_1(\chi)+\upsilon_3(\chi)+2}{2(\upsilon_1(\chi)+\upsilon_3(\chi)+1)}$$
$$\geq \frac{\mu_2(\chi)(\upsilon_1(\chi)-1)}{2(\upsilon_1(\chi)+\upsilon_3(\chi)+1)} + \frac{\mu_2(\chi)+\mu_3(\chi)}{2(\mu_2(\chi)+\mu_3(\chi)+1)} \cdot \frac{\upsilon_1(\chi)}{}$$
$$+\upsilon_3(\chi) + 22(\upsilon_1(\chi) + \upsilon_3(\chi) + 1)$$
$$= \frac{2\mu_2(\chi)((\mu_2(\chi)+\mu_3(\chi)+1)\upsilon_1(\chi)-\mu_2(\chi))+(\mu_2(\chi)+\mu_3(\chi))}{}$$
$$(\upsilon_1(\chi) + \upsilon_3(\chi) + 1) + 2\mu_3(\chi)(1 - \mu_2(\chi))4(\upsilon_1(\chi) + \upsilon_3(\chi) + 1)(\mu_2(\chi) + \mu_3(\chi) + 1)$$
$$\geq \frac{1}{2}(2\mu_2(\chi) + \upsilon_1(\chi) + \upsilon_3(\chi))\left(\frac{(\mu_2(\chi)+\mu_3(\chi)+1)\upsilon_1(\chi)}{}\right.$$
$$+\mu_3(\chi)) + 2\mu_3(\chi)(1 - \mu_2(\chi))4(\upsilon_1(\chi) + \upsilon_3(\chi) + 1)(\mu_2(\chi) + \mu_3(\chi) + 1)$$

(Use Chebyshev's inequality: Lemma 2.3)

$$\geq 0.$$

Consequently, it is fully proved by the definition (1) in Definition 2.2. □

Theorem 3.3 *Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be three IFSs on common $\mathbb{X}$. Then it holds that*:

(1) $(\mathcal{A}_1 \,\$\, \mathcal{A}_2) \ominus \mathcal{A}_3 \supseteq (\mathcal{A}_1 \ominus \mathcal{A}_3) \,\$\, (\mathcal{A}_2 \ominus \mathcal{A}_3)$;
(2) $(\mathcal{A}_1 \,\$\, \mathcal{A}_2)\varnothing\mathcal{A}_3 \subseteq (\mathcal{A}_1\varnothing\mathcal{A}_3) \,\$\, (\mathcal{A}_2\varnothing\mathcal{A}_3)$;
(3) $(\mathcal{A}_1 \,\#\, \mathcal{A}_2) \ominus \mathcal{A}_3 \supseteq (\mathcal{A}_1 \ominus \mathcal{A}_3) \,\#\, (\mathcal{A}_2 \ominus \mathcal{A}_3)$;
(4) $(\mathcal{A}_1 \,\#\, \mathcal{A}_2)\varnothing\mathcal{A}_3 \subseteq (\mathcal{A}_1\varnothing\mathcal{A}_3) \,\#\, (\mathcal{A}_2\varnothing\mathcal{A}_3)$;
(5) $(\mathcal{A}_1 \bigstar \mathcal{A}_2) \ominus \mathcal{A}_3 \supseteq (\mathcal{A}_1 \ominus \mathcal{A}_3) \bigstar (\mathcal{A}_2 \ominus \mathcal{A}_3)$;
(6) $(\mathcal{A}_1 \bigstar \mathcal{A}_2)\varnothing\mathcal{A}_3 \subseteq (\mathcal{A}_1\varnothing\mathcal{A}_3) \bigstar (\mathcal{A}_2\varnothing\mathcal{A}_3)$.

Proof We only prove the formula (1), and the formulas (2)-(6) can be analogously proved.

(1) Using the Definition 2.2, we can derive

$(\mathcal{A}_1 \,\$\, \mathcal{A}_2) \ominus \mathcal{A}_3 =$
$\{\langle \chi, \upsilon_3(\chi)\sqrt{\mu_1(\chi)\mu_2(\chi)}, \sqrt{\upsilon_1(\chi)\upsilon_2(\chi)} + \mu_3(\chi)$
$-\sqrt{\upsilon_1(\chi)\upsilon_2(\chi)}\mu_3(\chi)\rangle \mid \chi \in \mathbb{X}\}$ and
$(\mathcal{A}_1 \ominus \mathcal{A}_3) \,\$\, (\mathcal{A}_2 \ominus \mathcal{A}_3) = \{\langle \chi, \upsilon_3(\chi)\sqrt{\mu_1(\chi)\mu_2(\chi)},$
$\sqrt{((\upsilon_1(\chi) + \mu_3(\chi) - \upsilon_1(\chi)\mu_3(\chi)) \cdot (\upsilon_2(\chi) + \mu_3(\chi)}$
$\overline{-\upsilon_2(\chi)\mu_3(\chi)))}\rangle \mid \chi \in \mathbb{X}\}.$

Let $f(\chi) = (\sqrt{\upsilon_1(\chi)\upsilon_2(\chi)} + \mu_3(\chi) - \sqrt{\upsilon_1(\chi)\upsilon_2(\chi)}\mu_3(\chi))^2 - (\sqrt{((\upsilon_1(\chi) + \mu_3(\chi) - \upsilon_1(\chi)\mu_3(\chi)) \cdot (\upsilon_2(\chi) + \mu_3(\chi) - \upsilon_2(\chi)\mu_3(\chi)))})^2$, then an immediate calculation displays

$$f(\chi) = \left(\sqrt{\upsilon_1(\chi)\upsilon_2(\chi)} + \mu_3(\chi) - \sqrt{\upsilon_1(\chi)\upsilon_2(\chi)}\mu_3(\chi)\right)^2 - ((\upsilon_1(\chi) + \mu_3(\chi) - \upsilon_1(\chi)\mu_3(\chi)) \cdot (\upsilon_2(\chi) + \mu_3(\chi) - \upsilon_2(\chi)\mu_3(\chi)))$$
$$= \mu_3(\chi)\ (1 - \mu_3(\chi))(2\sqrt{\upsilon_1(\chi)\upsilon_2(\chi)} - (\upsilon_1(\chi) + \upsilon_2(\chi)))$$
$$\leq \mu_3(\chi)\ (1 - \mu_3(\chi))((\upsilon_1(\chi) + \upsilon_2(\chi) - (\upsilon_1(\chi) + \upsilon_2(\chi)))$$

(Use Mean inequality (AM-GM): Lemma 2.2)

$= 0.$

Consequently, it is fully proved by the definition (1) in Definition 2.2. □

Remark 3.2 Some intuitionistic fuzzy inequalities in Theorem 3.2 and Theorem 3.3 distinctly reveal that inequalities based on subtraction-division operations with other operations have a distribution law similar to that of equality. With the help of the similar structures of Theorem 3.2 and Theorem 3.3, we can prove whether it conforms to the above formula by introducing a new or developed operation, and then obtain a similar form of inequality, and further enrich the theoretical connotation based on the operation. Of course, most of them can be proved or partially proved by some existing famous inequalities.

4 Intuitionistic Fuzzy Inequalities Derived by Aggregation Operators

In this section, we develop three unweighted intuitionistic fuzzy aggregation operators (AOs), including UIFS, UIFA and UIFG. Moreover, some inequalities on them are derived and proved.

Definition 4.1 Let $\ddot{a}_i = (\mu_i, \upsilon_i)(i = 1, 2, \ldots, n)$ be a set of IFVs and let UIFS or UIFA or UIFG: $\Omega^n \to \Omega$, if

$$\mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) = \ddot{a}_1 @ \ddot{a}_2 @ \cdots @ \ddot{a}_n \tag{24}$$

or

$$\mathrm{UIFA}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) = \ddot{a}_1 \oplus \ddot{a}_2 \oplus \cdots \oplus \ddot{a}_n \tag{25}$$

or

$$\mathrm{UIFG}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) = \ddot{a}_1 \otimes \ddot{a}_2 \otimes \cdots \otimes \ddot{a}_n, \tag{26}$$

then UIFS or UIFA or UIFG is called unweighted intuitionistic fuzzy Square (UIFS) or unweighted intuitionistic fuzzy Arithmetic (UIFA) or unweighted intuitionistic fuzzy Geometric (UIFG) operators, respectively.

According to the Definition 2.2 and Definition 4.1, the following results can be deduced by using mathematical induction.

Theorem 4.1 *Let $\ddot{a}_i = (\mu_i, \upsilon_i)(i = 1, 2, \ldots, n)$ be a set of IFVs. Then their integrated value by using the UIFS operator or UIFA operator or UIFG operator is also an IFV and*

$$\mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) = \left(\frac{1}{2^{n-1}}\mu_1 + \sum_{i=2}^{n} \frac{1}{2^{n-i+1}}\mu_i, \frac{1}{2^{n-1}}\upsilon_1 + \sum_{i=2}^{n} \frac{1}{2^{n-i+1}}\upsilon_i\right) \tag{27}$$

or

$$\mathrm{UIFA}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) = \left(1 - \prod_{i=1}^{n}(1 - \mu_i), \prod_{i=1}^{n}\upsilon_i\right) \tag{28}$$

or

$$\mathrm{UIFG}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) = \left(\prod_{i=1}^{n}\mu_i, 1 - \prod_{i=1}^{n}(1 - \upsilon_i)\right). \tag{29}$$

Proof We only prove Eq. (27), the Eqs. (28) and (29) can be analogously obtained.

(1) In the following, we prove that the integrated result of some IFVs is still an IFV.

$$\frac{1}{2^{n-1}}\mu_1 + \sum_{i=2}^{n} \frac{1}{2^{n-i+1}}\mu_i + \frac{1}{2^{n-1}}\upsilon_1 + \sum_{i=2}^{n} \frac{1}{2^{n-i+1}}\upsilon_i$$
$$= \frac{1}{2^{n-1}}(\mu_1 + \upsilon_1) + \sum_{i=2}^{n} \frac{1}{2^{n-i+1}}(\mu_i + \upsilon_i)$$
$$\leq \frac{1}{2^{n-1}} \cdot 1 + \sum_{i=2}^{n} \frac{1}{2^{n-i+1}}$$
$$= \frac{1}{2^{n-1}} + \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{n-1}\right)}{1 - \frac{1}{2}}$$
$$= 1.$$

(2) Now, we prove that the integrated result is $\left(\frac{1}{2^{n-1}}\mu_1 + \sum_{i=2}^{n} \frac{1}{2^{n-i+1}}\mu_i, \frac{1}{2^{n-1}}\upsilon_1 + \sum_{i=2}^{n} \frac{1}{2^{n-i+1}}\upsilon_i\right)$.

① We first prove that Eq. (27) holds for $n = 2$.

According to the operation @ in Definition 2.2, we get $\mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2) = \left(\frac{\mu_1 + \mu_2}{2}, \frac{\upsilon_1 + \upsilon_2}{2}\right)$.

② If Eq. (27) holds for $n = \kappa$, i.e.,

$\mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_\kappa)$
$= \left(\frac{1}{2^{\kappa-1}}\mu_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+1}}\mu_i, \frac{1}{2^{\kappa-1}}\upsilon_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+1}}\upsilon_i\right).$

Thus, when $n = k + 1$, we can obtain the formula as follows:

$\mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_\kappa, \ddot{a}_{\kappa+1})$
$= \mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_\kappa) @ (\mu_{\kappa+1}, \upsilon_{\kappa+1})$
$= \left(\frac{1}{2^{\kappa-1}}\mu_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+1}}\mu_i, \frac{1}{2^{\kappa-1}}\upsilon_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+1}}\upsilon_i\right)$
$@(\mu_{\kappa+1}, \upsilon_{\kappa+1})$
$= \left(\frac{1}{2^{\kappa}}\mu_1 + \sum_{i=2}^{\kappa+1}\frac{1}{2^{\kappa-i+2}}\mu_i, \frac{1}{2^{\kappa}}\upsilon_1 + \sum_{i=2}^{\kappa+1}\frac{1}{2^{\kappa-i+2}}\upsilon_i\right).$

Accordingly, Eq. (27) holds for all n, which finishes the proof of Theorem 4.1. □

Example 4.1 For IFVs $\ddot{a}_1 = (0, 0.3), \ddot{a}_2 = (0.4, 0.2), \ddot{a}_3 = (0.6, 0.4)$, by computation, we have

$\mathrm{UIFG}(\ddot{a}_1, \ddot{a}_2, \ddot{a}_3) = (0 \cdot 0.4 \cdot 0.6, 1 - (1 - 0.3) \cdot (1 - 0.2) \cdot (1 - 0.4)) = (0, 0.664),$

$\mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ddot{a}_3) = (\frac{1}{2^2} \cdot 0 + \frac{1}{2^2} \cdot 0.4 + \frac{1}{2^1} \cdot 0.6, \frac{1}{2^2} \cdot 0.3 + \frac{1}{2^2} \cdot 0.2 + \frac{1}{2^1} \cdot 0.4) = (0.4, 0.325),$

$\mathrm{UIFA}(\ddot{a}_1, \ddot{a}_2, \ddot{a}_3) = (1 - (1 - 0) \cdot (1 - 0.4) \cdot (1 - 0.6), 0.3 \cdot 0.2 \cdot 0.4) = (0.76, 0.024).$

Remark 4.1 The unweighted intuitionistic fuzzy AOs proposed above are to perform corresponding operations on a set of IFVs, without additional weighted forms. It is not difficult to see that the UIFS operator can avoid counter-intuitive phenomena and are insensitive to data. In other words, some integrated preference information of membership or non-membership will be neglected when using some special IFVs, including (1,0), (0,1), $(0,\upsilon)$ and $(\mu,0)$. Moreover, the UIFA and UIFG operators are two particular operators without weighted forms compared with intuitionistic fuzzy weighted averaging operator [13], which also have the same issue.

Theorem 4.2 *Let $\ddot{a}_i = (\mu_i, \upsilon_i)(i = 1, 2, \ldots, n)$ be a set of IFVs. Then it holds that*

$$\mathrm{UIFG}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) \leq \mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) \leq \mathrm{UIFA}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n). \tag{30}$$

Proof We only prove the $\mathrm{UIFG}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) \leq \mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n)$, and the other one can be analogously proved.

Using the Theorem 4.1, we have

$\mathrm{UIFG}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) = \left(\prod_{i=1}^{n}\mu_i, 1 - \prod_{i=1}^{n}(1 - \upsilon_i)\right)$ and

$\mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n)$
$= \left(\frac{1}{2^{n-1}}\mu_1 + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_i, \frac{1}{2^{n-1}}\upsilon_1 + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_i\right).$

An immediate calculation displays

$\frac{1}{2^{n-1}}\mu_1 + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_i$
$\geq \frac{1}{n}\left(\frac{1}{2^{n-1}} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\right)\left(\mu_1 + \sum_{i=2}^{n}\mu_i\right)$ (Use Chebyshev's inequality: Lemma 2.3)
$= \frac{1}{n}\left(\frac{1}{2^{n-1}} + 1 - \frac{1}{2^{n-1}}\right)\sum_{i=1}^{n}\mu_i$
$= \frac{1}{n}\sum_{i=1}^{n}\mu_i$
$\geq \prod_{i=1}^{n}(\mu_i)^{\frac{1}{n}}$ (Use Mean inequality (SM-GM): Lemma 2.2)
$\geq \prod_{i=1}^{n}\mu_i.$

Assume that

$$S_n = 1 - \prod_{i=1}^{n}(1 - \upsilon_i) - \left(\frac{1}{2^{n-1}}\upsilon_1 + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_i\right) \geq 0$$

holds.

Next, we prove $S_n \geq 0$ by using mathematical induction.

(1) When $n = 2$, we can derive

$S_2 = 1 - (1 - \upsilon_1)(1 - \upsilon_2) - \frac{\upsilon_1 + \upsilon_2}{2}$
$= \frac{\upsilon_1 + \upsilon_2}{2} - \upsilon_1\upsilon_2$
$\geq \sqrt{\upsilon_1\upsilon_2} - \upsilon_1\upsilon_2$ (Use Mean inequality (AM-GM): Lemma 2.2)
$= \sqrt{\upsilon_1\upsilon_2}(1 - \sqrt{\upsilon_1\upsilon_2})$
$\geq 0.$

(2) Assuming that $n = \kappa$, $S_\kappa \geq 0$ is true, i.e.,

$$S_\kappa = 1 - \prod_{i=1}^{\kappa}(1 - \upsilon_i) - \left(\frac{1}{2^{\kappa-1}}\upsilon_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+1}}\upsilon_i\right) \geq 0.$$

So when $n = \kappa + 1$, a direct calculation shows

$$S_{\kappa+1} = 1 - \prod_{i=1}^{\kappa+1}(1 - \upsilon_i) - \left(\frac{1}{2^{\kappa}}\upsilon_1 + \sum_{i=2}^{\kappa+1}\frac{1}{2^{\kappa-i+2}}\upsilon_i\right)$$

$= 1 - (1 - \upsilon_{\kappa+1})\prod_{i=1}^{\kappa}(1 - \upsilon_i)$
$-\left(\frac{1}{2^{\kappa}}\upsilon_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+2}}\upsilon_i + \frac{1}{2}\upsilon_{\kappa+1}\right)$
$= 1 - \frac{2(1-\upsilon_{\kappa+1})}{2}\prod_{i=1}^{\kappa}$
$(1 - \upsilon_i) - \frac{1}{2}\left(\frac{1}{2^{\kappa-1}}\upsilon_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+1}}\upsilon_i\right) - \frac{1}{2}\upsilon_{\kappa+1}$

$$= \frac{1}{2} - \frac{(1-\upsilon_{\kappa+1})}{2}\prod_{i=1}^{\kappa}(1-\upsilon_i)$$

$$-\frac{1}{2}\left(\frac{1}{2^{\kappa-1}}\upsilon_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+1}}\upsilon_i\right) + \frac{1}{2} - \frac{1}{2}\upsilon_{\kappa+1} - \frac{(1-\upsilon_{\kappa+1})}{2}\prod_{i=1}^{\kappa}(1-\upsilon_i)$$

$$\geq \frac{1}{2} - \frac{1}{2}\prod_{i=1}^{\kappa}(1-\upsilon_i) - \frac{1}{2}\left(\frac{1}{2^{\kappa-1}}\upsilon_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+1}}\upsilon_i\right)$$

$$+\frac{1}{2} - \frac{1}{2}\upsilon_{\kappa+1} - \frac{(1-\upsilon_{\kappa+1})}{2}\prod_{i=1}^{\kappa}(1-\upsilon_i)$$

$$= \frac{1}{2}\left(1 - \prod_{i=1}^{\kappa}(1-\upsilon_i) - \left(\frac{1}{2^{\kappa-1}}\upsilon_1 + \sum_{i=2}^{\kappa}\frac{1}{2^{\kappa-i+1}}\upsilon_i\right)\right)$$

$$+\frac{1}{2}\left(1 - \upsilon_{\kappa+1} - (1-\upsilon_{\kappa+1})\prod_{i=1}^{\kappa}(1-\upsilon_i)\right)$$

$$\geq \frac{1}{2}\left(1 - \left(\upsilon_{\kappa+1} + (1-\upsilon_{\kappa+1})\prod_{i=1}^{\kappa}(1-\upsilon_i)\right)\right)$$

$$\geq \frac{1}{2}(1 - (\upsilon_{\kappa+1} + (1-\upsilon_{\kappa+1})\cdot 1))$$

$$= 0.$$

Further, we can see from (1) and (2) that $S_n \geq 0$ is true for all n.

Finally, employing the score function in Definition 2.3, $\mathrm{UIFG}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) \leq \mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n)$ is fully proved. □

Remark 4.2 From Theorem 4.2, it is not difficult to find that the intrinsic relationship of the inequality based on UIFG, UIFS and UIFA operators may be determined when they were used as the primitive operations ($\oplus, \otimes, @$). In other words, when we establish this type of intuitionistic fuzzy inequality, we can rely on the original operations to filter out the inequality based on the AOs that may meet the requirements at the beginning, and then use some famous inequalities with mathematical induction in the subsequent proof process. Relying on this idea and concept, it will greatly promote the process of finding inequalities and enrich the system of inequality methods. Moreover, it need to be emphasized that if the above intuitionistic fuzzy AOs appear in a weighted form which will not satisfy the above inequality relationship.

Theorem 4.3 *Let $\ddot{a}_i = (\mu_i, \upsilon_i)$ $(i = 1, 2, \ldots, n)$ be a set of IFVs and $\ddot{a} = (\mu, \upsilon)$ be an IFV. Then it holds that:*

(1) $\mathrm{UIFS}(\ddot{a}_1^{\lambda} \ominus \ddot{a}, \ddot{a}_2^{\lambda} \ominus \ddot{a}, \ldots, \ddot{a}_n^{\lambda} \ominus \ddot{a}) \leq \mathrm{UIFS}(\lambda\ddot{a}_1 \oslash \ddot{a}, \lambda\ddot{a}_2 \oslash \ddot{a}, \ldots, \lambda\ddot{a}_n \oslash \ddot{a})$, $\lambda > 0$;

(2) $\mathrm{UIFS}(\lambda\ddot{a}_1 \ominus \ddot{a}, \lambda\ddot{a}_2 \ominus \ddot{a}, \ldots, \lambda\ddot{a}_n \ominus \ddot{a}) \geq \mathrm{UIFS}(\ddot{a}_1^{\lambda} \ominus \ddot{a}, \ddot{a}_2^{\lambda} \ominus \ddot{a}, \ldots, \ddot{a}_n^{\lambda} \ominus \ddot{a})$, iff $\lambda \geq 1$;

$\mathrm{UIFS}(\lambda\ddot{a}_1 \ominus \ddot{a}, \lambda\ddot{a}_2 \ominus \ddot{a}, \ldots, \lambda\ddot{a}_n \ominus \ddot{a})$
$\leq \mathrm{UIFS}(\ddot{a}_1^{\lambda} \ominus \ddot{a}, \ddot{a}_2^{\lambda} \ominus \ddot{a}, \ldots, \ddot{a}_n^{\lambda} \ominus \ddot{a})$, *iff* $0 < \lambda \leq 1$;

(3) $\mathrm{UIFS}(\lambda\ddot{a}_1 \oslash \ddot{a}, \lambda\ddot{a}_2 \oslash \ddot{a}, \ldots, \lambda\ddot{a}_n \oslash \ddot{a}) \geq \mathrm{UIFS}(\ddot{a}_1^{\lambda} \oslash \ddot{a}, \ddot{a}_2^{\lambda} \oslash \ddot{a}, \ldots, \ddot{a}_n^{\lambda} \oslash \ddot{a})$, iff $\lambda \geq 1$;

$\mathrm{UIFS}(\lambda\ddot{a}_1 \oslash \ddot{a}, \lambda\ddot{a}_2 \oslash \ddot{a}, \ldots, \lambda\ddot{a}_n \oslash \ddot{a}) \leq \mathrm{UIFS}(\ddot{a}_1^{\lambda} \oslash \ddot{a}, \ddot{a}_2^{\lambda} \oslash \ddot{a}, \ldots, \ddot{a}_n^{\lambda} \oslash \ddot{a})$, *iff* $0 < \lambda \leq 1$.

Proof We only prove the formulas (1) and (2), and the formula (3) can be analogously proved.

(1) Using the Theorem 4.1 and Definition 2.2, then

$\mathrm{UIFS}(\ddot{a}_1^{\lambda} \ominus \ddot{a}, \ddot{a}_2^{\lambda} \ominus \ddot{a}, \ldots, \ddot{a}_n^{\lambda} \ominus \ddot{a})$

$$= \left(\frac{1}{2^{n-1}}\mu_1^{\lambda}\upsilon + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_i^{\lambda}\upsilon, \frac{1}{2^{n-1}}\left(1 - (1-\mu)(1-\upsilon_1)^{\lambda}\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1 - (1-\mu)(1-\upsilon_i)^{\lambda}\right)\right)$$ and

$\mathrm{UIFS}(\lambda\ddot{a}_1 \oslash \ddot{a}, \lambda\ddot{a}_2 \oslash \ddot{a}, \ldots, \lambda\ddot{a}_n \oslash \ddot{a})$

$$= \left(\frac{1}{2^{n-1}}\left(1 - (1-\upsilon)(1-\mu_1)^{\lambda}\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1 - (1-\upsilon)(1-\mu_i)^{\lambda}\right), \frac{1}{2^{n-1}}\upsilon_1^{\lambda}\mu + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_i^{\lambda}\mu\right).$$

Let $s_i = 1 - (1-\upsilon)(1-\mu_i)^{\lambda} - \mu_i^{\lambda}\upsilon$ $(i = 1, 2, \ldots, n)$, then we can deduce $s_i \geq 1 - (1-\upsilon)\cdot 1 - 1\cdot\upsilon = 0$.

Further, we can derive

$$\frac{1}{2^{n-1}}\left(1 - (1-\upsilon)(1-\mu_1)^{\lambda}\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1 - (1-\upsilon)(1-\mu_i)^{\lambda}\right) \geq \frac{1}{2^{n-1}}\mu_1^{\lambda}\upsilon + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_i^{\lambda}\upsilon.$$

Analogously, $\frac{1}{2^{n-1}}\left(1 - (1-\mu)(1-\upsilon_1)^{\lambda}\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1 - (1-\mu)(1-\upsilon_i)^{\lambda}\right) \geq \frac{1}{2^{n-1}}\upsilon_1^{\lambda}\mu + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_i^{\lambda}\mu.$

Therefore, employing the score function in Definition 2.3, the original formula is fully proved.

(2) Using the Theorem 4.1 and Definition 2.2, we get

$$\mathrm{UIFS}(\lambda\ddot{a}_1 \ominus \ddot{a}, \lambda\ddot{a}_2 \ominus \ddot{a}, \ldots, \lambda\ddot{a}_n \ominus \ddot{a}) = \left(\frac{1}{2^{n-1}}\left(1 - (1-\mu_1)^{\lambda}\right)\upsilon + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1 - (1-\mu_i)^{\lambda}\right)\upsilon, \frac{1}{2^{n-1}}\left(\upsilon_1^{\lambda} + \mu - \upsilon_1^{\lambda}\mu\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(\upsilon_i^{\lambda} + \mu - \upsilon_i^{\lambda}\mu\right)\right)$$ and

$$\mathrm{UIFS}(\ddot{a}_1^{\lambda} \ominus \ddot{a}, \ddot{a}_2^{\lambda} \ominus \ddot{a}, \ldots, \ddot{a}_n^{\lambda} \ominus \ddot{a}) = \left(\frac{1}{2^{n-1}}\mu_1^{\lambda}\upsilon + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_i^{\lambda}\upsilon, \frac{1}{2^{n-1}}\left(1 - (1-\mu)(1-\upsilon_1)^{\lambda}\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1 - (1-\mu)(1-\upsilon_i)^{\lambda}\right)\right).$$

Let $s = \left(1 - (1-\mu_i)^{\lambda}\right)\upsilon - \mu_i^{\lambda}\upsilon$

$$= \left(1 - (1-\mu_i)^{\lambda} - \mu_i^{\lambda}\right)\upsilon$$

$$= \left(1 - (1-\mu_i)(1-\mu_i)^{\lambda-1} - \mu_i(\mu_i)^{\lambda-1}\right)\upsilon$$

and

$$t = 1 - (1-\mu)(1-\upsilon_i)^{\lambda} - \left(\upsilon_i^{\lambda} + \mu - \upsilon_i^{\lambda}\mu\right)$$

$$= (1-\mu)\left(1 - (1-\upsilon_i)^{\lambda} - \upsilon_i^{\lambda}\right)$$

$$= (1-\mu)\left(1 - (1-\upsilon_i)(1-\upsilon_i)^{\lambda-1} - \upsilon_i(\upsilon_i)^{\lambda-1}\right).$$

① If $\lambda \geq 1$, then we have

$(\mu_i)^{\lambda-1} \leq 1, (1-\mu_i)^{\lambda-1} \leq 1, (\upsilon_i)^{\lambda-1} \leq 1$ and $(1-\upsilon_i)^{\lambda-1} \leq 1$.

Further, we get

$s = \left(1-(1-\mu_i)(1-\mu_i)^{\lambda-1} - \mu_i(\mu_i)^{\lambda-1}\right)$
$\upsilon \geq (1-(1-\mu_i)\cdot 1 - \mu_i \cdot 1)\upsilon = 0$ and

$t = (1-\mu)\left(1-(1-\upsilon_i)(1-\upsilon_i)^{\lambda-1} - \upsilon_i(\upsilon_i)^{\lambda-1}\right) \geq (1-\mu)(1-(1-\upsilon_i)\cdot 1 - \upsilon_i \cdot 1) = 0.$

Finally, we can derive

$$\frac{1}{2^{n-1}}\left(1-(1-\mu_1)^{\lambda}\right)\upsilon + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1-(1-\mu_i)^{\lambda}\right)\upsilon \geq \frac{1}{2^{n-1}}\mu_1^{\lambda}\upsilon + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_i^{\lambda}\upsilon$$ and

$$\frac{1}{2^{n-1}}\left(\upsilon_1^{\lambda} + \mu - \upsilon_1^{\lambda}\mu\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(\upsilon_i^{\lambda} + \mu - \upsilon_i^{\lambda}\mu\right) \leq \frac{1}{2^{n-1}}\left(1-(1-\mu)(1-\upsilon_1)^{\lambda}\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1-(1-\mu)(1-\upsilon_i)^{\lambda}\right).$$

Therefore, employing the score function in Definition 2.3, the original inequality holds.

② If $0 < \lambda \leq 1$, then we have

$(\mu_i)^{\lambda-1} \geq 1, (1-\mu_i)^{\lambda-1} \geq 1, (\upsilon_i)^{\lambda-1} \geq 1$ and $(1-\upsilon_i)^{\lambda-1} \geq 1$.

Further, we get

$s = \left(1-(1-\mu_i)(1-\mu_i)^{\lambda-1} - \mu_i(\mu_i)^{\lambda-1}\right)\upsilon \leq (1-(1-\mu_i)\cdot 1 - \mu_i \cdot 1)\upsilon = 0$ and

$t = (1-\mu)\left(1-(1-\upsilon_i)(1-\upsilon_i)^{\lambda-1} - \upsilon_i(\upsilon_i)^{\lambda-1}\right) \leq (1-\mu)(1-(1-\upsilon_i)\cdot 1 - \upsilon_i \cdot 1) = 0.$

Finally, we can derive

$$\frac{1}{2^{n-1}}\left(1-(1-\mu_1)^{\lambda}\right)\upsilon + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1-(1-\mu_i)^{\lambda}\right)\upsilon \leq \frac{1}{2^{n-1}}\mu_1^{\lambda}\upsilon + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_i^{\lambda}\upsilon$$ and

$$\frac{1}{2^{n-1}}\left(\upsilon_1^{\lambda} + \mu - \upsilon_1^{\lambda}\mu\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(\upsilon_i^{\lambda} + \mu - \upsilon_i^{\lambda}\mu\right) \geq \frac{1}{2^{n-1}}\left(1-(1-\mu)(1-\upsilon_1)^{\lambda}\right) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(1-(1-\mu)(1-\upsilon_i)^{\lambda}\right).$$

Therefore, utilizing the score function in Definition 2.3, the original inequality holds.

From what has been discussed above, the original formula is fully proved. □

Remark 4.3 As the data integration of a series of IFVs relying on the UIFS operator, it has some better characteristics in constructing inequality. When constructing the inequality, we first follow the preliminary formation of two special IFSs with good dual form ($\ddot{a}_i^{\lambda} \ominus \ddot{a}$ and $\lambda\ddot{a}_i ø \ddot{a}$) or partly dual form ($\lambda\ddot{a}_i \ominus \ddot{a}$ and $\ddot{a}_i^{\lambda} \ominus \ddot{a}$; $\lambda\ddot{a}_i ø \ddot{a}$ and $\ddot{a}_i^{\lambda} ø \ddot{a}$), and then compare the two magnitudes. And then extend to multiple IFVs, that is, use the idea of UIFS to further verify its feasibility. It may become a stable and feasible model for other new operations or some existing operations to construct inequalities based on UIFS or other intuitionistic fuzzy AOs.

Theorem 4.4 *Let $\ddot{a}_i = (\mu_{ai}, \upsilon_{ai})$ and $\ddot{b}_i = (\mu_{bi}, \upsilon_{bi})(i = 1, 2, \ldots, n)$ be two sets of IFVs. Then it holds that:*

(1) $\mathrm{UIFS}(\ddot{a}_1 \ominus \ddot{b}_1, \ddot{a}_2 \ominus \ddot{b}_2, \ldots, \ddot{a}_n \ominus \ddot{b}_n) \leq \mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) ø\ \mathrm{UIFS}(\ddot{b}_1, \ddot{b}_2, \ldots, \ddot{b}_n)$;

(2) $\mathrm{UIFS}(\ddot{a}_1 ø \ddot{b}_1, \ddot{a}_2 ø \ddot{b}_2, \ldots, \ddot{a}_n ø \ddot{b}_n) \geq \mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) \ominus \mathrm{UIFS}(\ddot{b}_1, \ddot{b}_2, \ldots, \ddot{b}_n)$.

Proof We only give a proof of formula (1), and the formula (2) can be analogously proved.

(1) Using the Theorem 4.1 and Definition 2.2, we can get

$$\mathrm{UIFS}(\ddot{a}_1 \ominus \ddot{b}_1, \ddot{a}_2 \ominus \ddot{b}_2, \ldots, \ddot{a}_n \ominus \ddot{b}_n) = \Big(\frac{1}{2^{n-1}}\mu_{a1}\upsilon_{b1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{ai}\upsilon_{bi}, \frac{1}{2^{n-1}}(\upsilon_{a1} + \mu_{b1} - \upsilon_{a1}\mu_{b1}) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}(\upsilon_{ai} + \mu_{bi} - \upsilon_{ai}\mu_{bi})\Big)$$ and

$$\mathrm{UIFS}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n)\, ø\, \mathrm{UIFS}(\ddot{b}_1, \ddot{b}_2, \ldots, \ddot{b}_n) = \left(\frac{1}{2^{n-1}}(\mu_{a1} + \upsilon_{b1}) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}(\mu_{ai} + \upsilon_{bi}) - \left(\frac{1}{2^{n-1}}\mu_{a1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{ai}\right)\left(\frac{1}{2^{n-1}}\upsilon_{b1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_{bi}\right). \left(\frac{1}{2^{n-1}}\upsilon_{a1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_{ai}\right)\left(\frac{1}{2^{n-1}}\mu_{b1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{bi}\right)\right).$$

Let $f = \frac{1}{2^{n-1}}\mu_{a1}\upsilon_{b1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{ai}\upsilon_{bi} - \Big(\frac{1}{2^{n-1}}(\mu_{a1} + \upsilon_{b1}) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}(\mu_{ai} + \upsilon_{bi}) - \left(\frac{1}{2^{n-1}}\mu_{a1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{ai}\right)\left(\frac{1}{2^{n-1}}\upsilon_{b1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_{bi}\right)\Big)$, then we can derive

$$f = \left(\frac{1}{2^{n-1}}\upsilon_{a1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_{bi}\right)\left(\frac{1}{2^{n-1}}\mu_{a1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{ai} - 1\right) + \frac{1}{2^{n-1}}\mu_{a1}(\upsilon_{b1} - 1) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{ai}(\upsilon_{bi} - 1) \leq 0.$$

Moreover, we let $g = \Big(\frac{1}{2^{n-1}}(\upsilon_{a1} + \mu_{b1} - \upsilon_{a1}\mu_{b1}) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}(\upsilon_{ai} + \mu_{bi} - \upsilon_{ai}\mu_{bi})\Big) - \left(\left(\frac{1}{2^{n-1}}\upsilon_{a1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_{ai}\right)\left(\frac{1}{2^{n-1}}\mu_{b1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{bi}\right)\right)$, then an immediate calculation gives

$$g = \frac{1}{2^{n-1}}(\upsilon_{a1} + \mu_{b1} - \upsilon_{a1}\mu_{b1}) + \sum_{i=2}^{n} \frac{1}{2^{n-i+1}}(\upsilon_{ai} + \mu_{bi} - \upsilon_{ai}\mu_{bi}) - \left(\frac{1}{2^{n-1}}\upsilon_{a1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_{ai}\right)\left(\frac{1}{2^{n-1}}\mu_{b1} + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{bi}\right)$$

$$= \frac{1}{2^{n-1}}(\upsilon_{a1} + \mu_{b1} - \upsilon_{a1}\mu_{b1}) + \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}(\upsilon_{ai} + \mu_{bi} - \upsilon_{ai}\mu_{bi}) - \frac{1}{2^{2n-2}}\upsilon_{a1}\mu_{b1} - \frac{1}{2^{n-1}}\upsilon_{a1}\sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{bi} - \frac{1}{2^{n-1}}\mu_{b1}\sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_{ai} - \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\upsilon_{ai}\cdot\sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{bi}$$

$$= \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}(\upsilon_{ai} + \mu_{bi} - \upsilon_{ai}\mu_{bi} \quad -\upsilon_{ai}\sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{bi} - \frac{1}{2^{n-1}}\mu_{b1}\upsilon_{ai}) + \frac{1}{2^{n-1}}(\upsilon_{a1} + \upsilon_{b1} \ -\upsilon_{a1}\upsilon_{b1} - \frac{1}{2^{n-1}}\upsilon_{a1}\mu_{b1} - \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{bi}\upsilon_{a1})$$

$$= \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\left(\upsilon_{ai}\left(1 - \frac{1}{2^{n-1}}\mu_{b1} - \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{bi}\right) + \mu_{bi}(1 - \upsilon_{ai})\right) + \frac{1}{2^{n-1}}\left(\upsilon_{a1}\left(1 - \frac{1}{2^{n-1}}\mu_{b1} - \sum_{i=2}^{n}\frac{1}{2^{n-i+1}}\mu_{bi}\right) + \upsilon_{b1}(1 - \upsilon_{a1})\right)$$

$$\geq 0.$$

Therefore, employing the score function in Definition 2.3, the original formula is fully proved. □

Remark 4.4 First of all, two operations with dual form are no longer confined to the union of some IFVs as one element in AO, but are extended to construct inequality by the connection between AO and AO. Similar to the previous Remark 4.2 and Remark 4.3, it also can become a stable and feasible model for other other new operations or some existing operations to construct new inequalities based on UIFS or other intuitionistic fuzzy AOs.

Note 4.1 To sum up, the following three points need to be specially stressed:

(1) It can be deduced that when all intuitionistic fuzzy AOs (UIFS, UIFA and UIFG) with weighted form [13], the Theorem 4.2 will not be invalid, especially for UIFS.

(2) The new operations, especially the pair of operations with dual form, can easily match the UIFS operator in Theorems 4.2, 4.3 and 4.4, and may produce better inequality relations. Finally, its correctness is verified by some famous inequalities or other classical methods.

(3) Since the UIFS operator in this paper is derived from the operation @ to obtain a series of valuable and related inequalities, whether the new intuitionistic fuzzy AO derived from the new operation maintains the consistency of the above inequalities will be the focus of our subsequent discussion.

In the following section, we will consider some special intuitionistic fuzzy inequalities derived by equality $\mu + \upsilon + \pi = 1$, which may be the foundation for proving the inequalities of operations and AOs under intuitionistic fuzzy environment.

5 Intuitionistic Fuzzy Inequalities Derived by Equality

In this section, some intuitionistic fuzzy inequalities derived by equality $\mu + \upsilon + \pi = 1$ in Definition 2.1 are cleverly constructed. Meanwhile, some necessary proofs of intuitionistic fuzzy inequalities are provided, which are employed some existing famous inequalities or their combined forms, including Rearrangement inequality [24], Mean inequality [25], Nesbitt's inequality [26], Chebyshev's inequality [27], Cauchy's inequality [28], Generalized Cauchy's inequality [28], Hölder's inequality [28], Minkowski's inequality [28], Power-Mean inequality [29], Carlson's inequality [30], Jensen's inequality [31], Wei-Wei dual inequality [32], Tangent inequality [33], Muirhead's inequality [34], Schur's inequality [35], Vasc inequality [36], and Bernoulli's inequality [37].

5.1 Intuitionistic Fuzzy Inequality Proved by Rearrangement Inequality

The Rearrangement inequality [24] is a common inequality whose intrinsic dependence can be summarized as "Reverse order ≤ Random order ≤ Same order". It depicts the intrinsic relationship between "efficiency" and "fairness" of unrestricted system, which has strong practical significance in how to allocate resources. In addition, it can derive many famous inequalities, such as: Mean inequality (AM-GM), Cauchy's inequality, Chebyshev's inequality. In the following, two intuitionistic fuzzy inequalities are developed and proved by Rearrangement inequality.

Theorem 5.1 *For any IFV $\ddot{a} = (\mu, \upsilon, \pi)$ and satisfy $\mu, \upsilon, \pi > 0$. Then it holds that*

$$\prod_{cyc} \mu^{2\mu} \geq \prod_{cyc} \mu^{1-\mu}. \tag{31}$$

Proof A direct equivalent calculation of Eq. (31) gives

$$\prod_{cyc} \mu^{2\mu} \geq \prod_{cyc} \mu^{1-\mu}$$
$$\Leftrightarrow \prod_{cyc} \mu^{2\mu} \cdot \prod_{cyc} \mu^{\mu} \geq \prod_{cyc} \mu^{1-\mu} \cdot \prod_{cyc} \mu^{\mu}$$
$$\Leftrightarrow \prod_{cyc} \mu^{3\mu} \geq \prod_{cyc} \mu$$

$\Leftrightarrow \prod_{cyc} \mu^{\mu} \geq \left(\prod_{cyc} \mu\right)^{\frac{1}{3}}.$

Thus, it suffices to prove that $\prod_{cyc} \mu^{\mu} \geq \left(\prod_{cyc} \mu\right)^{\frac{1}{3}}$.

Without loss of generality, Let us assume $\mu \geq \upsilon \geq \pi$.

Further, we can get $\lg \mu \geq \lg \upsilon \geq \lg \pi$.

It is known from the Rearrangement inequality in Lemma 2.1 that

$\mu \lg \mu + \upsilon \lg \upsilon + \pi \lg \pi \geq \mu \lg \mu + \pi \lg \upsilon + \upsilon \lg \pi$ (Same order is superior to Random order),

$\mu \lg \mu + \upsilon \lg \upsilon + \pi \lg \pi \geq \upsilon \lg \mu + \mu \lg \upsilon + \pi \lg \pi$ (Same order is superior to Random order),

$\mu \lg \mu + \upsilon \lg \upsilon + \pi \lg \pi \geq \pi \lg \mu + \upsilon \lg \upsilon + \mu \lg \pi$ (Same order is superior to Reverse order).

Summing the above three inequalities, an immediate calculation displays

$3(\mu \lg \mu + \upsilon \lg \upsilon + \pi \lg \pi)$
$\geq (\mu + \upsilon + \pi)(\lg \mu + \lg \upsilon + \lg \pi)$
$\Leftrightarrow \lg(\mu^{\mu}\upsilon^{\upsilon}\pi^{\pi}) \geq \frac{\mu+\upsilon+\pi}{3} \lg(\mu\upsilon\pi)$
$\Leftrightarrow \lg(\mu^{\mu}\upsilon^{\upsilon}\pi^{\pi}) \geq \frac{1}{3} \lg(\mu\upsilon\pi)$
$\Leftrightarrow \lg(\mu^{\mu}\upsilon^{\upsilon}\pi^{\pi}) \geq \lg(\mu\upsilon\pi)^{\frac{1}{3}}$
$\Leftrightarrow \mu^{\mu}\upsilon^{\upsilon}\pi^{\pi} \geq (\mu\upsilon\pi)^{\frac{1}{3}}$.

In other words, $\prod_{cyc} \mu^{\mu} \geq \left(\prod_{cyc} \mu\right)^{\frac{1}{3}}$ holds.

Consequently, this completes the proof of Theorem 5.1. □

Theorem 5.2 *Let $\ddot{a} = (\mu, \upsilon, \pi)$ be an IFV. Then it holds that*

$$\sum_{cyc} \frac{\mu(1+\mu)}{1+\upsilon} \geq 1. \tag{32}$$

Proof Without loss of generality, Let us assume $\mu = \min\{\mu, \upsilon, \pi\}$.

Hence, there are two cases, which are discussed as follows:

Case 1: $\mu \leq \upsilon \leq \pi$.

If $\mu \leq \upsilon \leq \pi$, then $\mu(1+\mu) \leq \upsilon(1+\upsilon) \leq \pi(1+\pi)$ and $\frac{1}{1+\pi} \leq \frac{1}{1+\upsilon} \leq \frac{1}{1+\mu}$.

It is known from the Rearrangement inequality in Lemma 2.1 that

$$\sum_{cyc} \frac{\mu(1+\mu)}{1+\upsilon} = \frac{\mu(1+\mu)}{1+\upsilon} + \frac{\upsilon(1+\upsilon)}{1+\pi} + \frac{\pi(1+\pi)}{1+\mu}$$

$= \mu(1+\mu) \cdot \frac{1}{1+\upsilon} + \upsilon(1+\upsilon) \cdot \frac{1}{1+\pi} + \pi(1+\pi) \cdot \frac{1}{1+\mu}$
$\geq \mu(1+\mu) \cdot \frac{1}{1+\mu} + \upsilon(1+\upsilon) \cdot \frac{1}{1+\upsilon} + \pi(1+\pi) \cdot \frac{1}{1+\pi}$ (Random order is superior to Reverse order)
$= \mu + \upsilon + \pi = 1$.

Case 2: $\mu \leq \pi \leq \upsilon$.

If $\mu \leq \pi \leq \upsilon$, then $\mu(1+\mu) \leq \pi(1+\pi) \leq \upsilon(1+\upsilon)$ and $\frac{1}{1+\upsilon} \leq \frac{1}{1+\pi} \leq \frac{1}{1+\mu}$.

According to the Rearrangement inequality in Lemma 2.1, a direct calculation gives

$$\sum_{cyc} \frac{\mu(1+\mu)}{1+\upsilon} = \frac{\mu(1+\mu)}{1+\upsilon} + \frac{\upsilon(1+\upsilon)}{1+\pi} + \frac{\pi(1+\pi)}{1+\mu}$$

$= \mu(1+\mu) \cdot \frac{1}{1+\upsilon} + \upsilon(1+\upsilon) \cdot \frac{1}{1+\pi} + \pi(1+\pi) \cdot \frac{1}{1+\mu}$
$\geq \mu(1+\mu) \cdot \frac{1}{1+\mu} + \upsilon(1+\upsilon) \cdot \frac{1}{1+\upsilon} + \pi(1+\pi) \cdot \frac{1}{1+\pi}$ (Random order is superior to Reverse order)
$= \mu + \upsilon + \pi = 1$.

Consequently, this completes the proof of Theorem 5.2. □

Remark 5.1 From Theorems 5.1 and 5.2, it can easily derive a conclusion that using the Rearrangement inequality assumes that each variable must be preset in the corresponding order. While there is no agreement on the order of magnitude of each variable, the optimization hypothesis can be used. Moreover, the specific use of Rearrangement inequality can be divided into two kinds, namely symmetrical (Theorem 5.1) and cyclic (Theorem 5.2). For the inequality of variable with cyclic form, we need a more classification discussion than that of the inequality of variable with symmetric form, and then prove its feasibility. It is also interesting to note that the inequality with the circular form of the variable has a preference for using Random order and Reverse order. And the inequality of the corresponding symmetric form of the variable, the preference is Same order and Random order. Of course, Same order and Reverse order can be regarded as a peculiar Random order.

5.2 Intuitionistic Fuzzy Inequality Proved by Mean Inequality

The Mean inequality is high-frequency used inequalities, which has almost become the foundation of proof of multitudinous inequalities such as Power-Mean inequality and Nesbitt's inequality. In the following, two intuitionistic fuzzy inequalities are developed and proved by Mean inequality (AM-GM and HM-AM).

Theorem 5.3 *For any IFV $\ddot{a} = (\mu, \upsilon, \pi)$ and satisfy $\mu, \upsilon, \pi > 0$. Then it holds that*

$$\sum_{cyc}\frac{\mu+1}{\mu(\mu+2)} \geq \frac{36}{7}. \tag{33}$$

Proof According to the Mean inequality (HM-AM), a direct calculation gives

$$\begin{aligned}&\sum_{cyc}\tfrac{\mu+1}{\mu(\mu+2)} \geq \tfrac{9}{\sum_{cyc}\frac{\mu(\mu+2)}{\mu+1}}\\&=\tfrac{9}{\sum_{cyc}\frac{(\mu+1)^2-1}{\mu+1}}\\&=\tfrac{9}{\sum_{cyc}(\mu+1)-\sum_{cyc}\frac{1}{\mu+1}}\\&=\tfrac{9}{3+\sum_{cyc}\mu-\sum_{cyc}\frac{1}{\mu+1}}\\&=\tfrac{9}{4-\sum_{cyc}\frac{1}{\mu+1}}\\&\geq\tfrac{9}{4-\frac{9}{\sum_{cyc}(\mu+1)}}\\&=\tfrac{36}{7}.\end{aligned}$$

Consequently, this completes the proof of Theorem 5.3. □

Theorem 5.4 *Let $\ddot{a}=(\mu,\upsilon,\pi)$ be an IFV. Then it holds that*

$$\sum_{cyc}\sqrt{\mu\upsilon}\leq 3\left(1-\sqrt[3]{\prod_{cyc}\mu}\right). \tag{34}$$

Proof A direct equivalent calculation of Eq. (34) gives

$$\begin{aligned}&\sum_{cyc}\sqrt{\mu\upsilon}\leq 3\left(1-\sqrt[3]{\prod_{cyc}\mu}\right)\\&\Leftrightarrow\sum_{cyc}\sqrt{\mu\upsilon}+3\sqrt[3]{\prod_{cyc}\mu}\leq 3.\end{aligned}$$

Thus, it suffices to prove that $\sum_{cyc}\sqrt{\mu\upsilon}+3\sqrt[3]{\prod_{cyc}\mu}\leq 3$.

According to the Mean inequality (AM-GM) in Lemma 2.2, a direct calculation shows

$$\begin{aligned}&\mu+\sqrt{\mu\upsilon}+\sqrt[3]{\mu\upsilon\pi}\\&=\mu+\sqrt{\tfrac{\mu}{2}\cdot 2\upsilon}+\sqrt[3]{\tfrac{\mu}{4}\cdot\upsilon\cdot 4\pi}\\&\leq\mu+\tfrac{1}{2}\left(\tfrac{\mu}{2}+2\upsilon\right)+\tfrac{1}{3}\left(\tfrac{\mu}{4}+\upsilon+4\pi\right)\\&=\tfrac{4}{3}(\mu+\upsilon+\pi)\\&=\tfrac{4}{3}.\end{aligned}$$

Similarity, we have $\upsilon+\sqrt{\upsilon\pi}+\sqrt[3]{\mu\upsilon\pi}\leq\frac{4}{3}$ and $\pi+\sqrt{\pi\mu}+\sqrt[3]{\mu\upsilon\pi}\leq\frac{4}{3}$.

Further, summing the three above inequalities, we have

$$\sum_{cyc}\mu+\sum_{cyc}\sqrt{\mu\upsilon}+3\sqrt[3]{\prod_{cyc}\mu}\leq 4.$$

Apparently, $\sum_{cyc}\sqrt{\mu\upsilon}+3\sqrt[3]{\prod_{cyc}\mu}\leq 3$ holds.

Consequently, this completes the proof of Theorem 5.4. □

Remark 5.2 From Theorems 5.3 and 5.4, we can find that using the right mean inequality in the right place is the heart of the proof under intuitionistic fuzzy environment. Meanwhile, using equality $\mu+\upsilon+\pi=1$ to reduce some of the terms that appear in the inequality at critical moments will also greatly reduce the complexity of the proof. In addition to the AM-GM and HM-AM listed above, other classic Mean inequalities, including SM-GM, AM-SM and 3 M, also shine in subsequent combined famous inequalities.

5.3 Intuitionistic Fuzzy Inequality Proved by Chebyshev's Inequality

Chebyshev's inequality can be easily derived by Mean inequality and Rearrangement inequality. That is, Chebyshev's inequality combines the advantages of the two, can deal with specific types of issues in a more targeted manner, and greatly improves the application scenarios of the inequality. In the following, an intuitionistic fuzzy inequality is constructed and proved by Chebyshev's inequality.

Theorem 5.5 *For any IFV $\ddot{a}=(\mu,\upsilon,\pi)$ and satisfy $\mu,\upsilon,\pi>0$. Then it holds that*

$$\sum_{cyc}\frac{1}{3\mu^2+\upsilon+\pi}\leq 3. \tag{35}$$

Proof A direct equivalent calculation of Eq. (35) gives

$$\begin{aligned}&\sum_{cyc}\tfrac{1}{3\mu^2+\upsilon+\pi}\leq 3\\&\Leftrightarrow\sum_{cyc}\tfrac{1}{3\mu^2+1-\mu}\leq 3\\&\Leftrightarrow\sum_{cyc}\tfrac{\mu(3\mu-1)}{3\mu^2-\mu+1}\geq 0\\&\Leftrightarrow\sum_{cyc}\tfrac{3\mu-1}{3\mu-1+\frac{1}{\mu}}\geq 0.\end{aligned}$$

Without loss of generality, assume that $\mu\geq\upsilon\geq\pi$. Further, we have $3\mu-1\geq 3\upsilon-1\geq 3\pi-1$.

Since $\mu\upsilon\leq\frac{(\mu+\upsilon)^2}{4}\leq\frac{1}{4}$, a direct calculation shows

$$\begin{aligned}&(3\mu\upsilon-1)(\mu-\upsilon)\leq 0\\&\Leftrightarrow 3\mu\upsilon(\mu-\upsilon)\leq\mu-\upsilon\\&\Leftrightarrow 3\mu+\tfrac{1}{\mu}\leq 3\upsilon+\tfrac{1}{\upsilon}\\&\Leftrightarrow 3\mu-1+\tfrac{1}{\mu}\leq 3\upsilon-1+\tfrac{1}{\upsilon}\\&\Leftrightarrow\tfrac{1}{3\mu-1+\frac{1}{\mu}}\geq\tfrac{1}{3\upsilon-1+\frac{1}{\upsilon}}.\end{aligned}$$

Similarly, $\frac{1}{3\upsilon-1+\frac{1}{\upsilon}}\geq\frac{1}{3\pi-1+\frac{1}{\pi}}$.

According to the Chebyshev's inequality in Lemma 2.3, an immediate calculation gives

$$\sum_{cyc}\frac{3\mu-1}{3\mu-1+\frac{1}{\mu}}$$
$$\geq \frac{1}{3}\cdot\sum_{cyc}(3\mu-1)\cdot\sum_{cyc}\frac{1}{3\mu-1+\frac{1}{\mu}}$$
$$=\frac{1}{3}\cdot\left(3\sum_{cyc}\mu-3\right)\cdot\sum_{cyc}\frac{1}{3\mu-1+\frac{1}{\mu}}$$
$$=0.$$

Consequently, this completes the proof of Theorem 5.5. □

5.4 Intuitionistic Fuzzy Inequality Proved by Cauchy's Inequality

Cauchy's inequality is a frequently used theoretical basis when proving certain inequalities. The technique is mainly to split the constant and make up the constant value. In the following, we present an intuitionistic fuzzy inequality and prove it by Cauchy's inequality.

Theorem 5.6 *Let $\ddot{a}=(\mu,\upsilon,\pi)$ be an IFV that is not a crisp number. Then it holds that*

$$\sqrt{2+\sum_{cyc}\frac{\mu}{\upsilon+\pi}}\leq\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}\leq\sqrt{3+\sum_{cyc}\frac{\mu}{\upsilon+\pi}}. \quad (36)$$

Proof The Eq. (36) can be proved in two steps, as shown below.

(1) Right inequality: $\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}\leq\sqrt{3+\sum_{cyc}\frac{\mu}{\upsilon+\pi}}$.

A direct equivalent calculation of right inequality gives

$$\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}\leq\sqrt{3+\sum_{cyc}\frac{\mu}{\upsilon+\pi}}$$
$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}\leq\sqrt{\sum_{cyc}\frac{\mu+\upsilon}{\mu+\upsilon}+\sum_{cyc}\frac{\mu}{\upsilon+\pi}}$$
$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}\leq\sqrt{\sum_{cyc}\frac{\mu+\upsilon}{\mu+\upsilon}+\sum_{cyc}\frac{\pi}{\mu+\upsilon}}$$
$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}\leq\sqrt{\sum_{cyc}\frac{\sum_{cyc}\mu}{\mu+\upsilon}}$$
$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}\leq\sqrt{\sum_{cyc}\frac{1}{\mu+\upsilon}}$$
$$\Leftrightarrow\frac{\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}}{\sqrt{\sum_{cyc}\frac{1}{\mu+\upsilon}}}\leq 1$$
$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{\mu^2\sum_{cyc}\mu+\prod_{cyc}\mu}{\sum_{cyc}(\mu+\upsilon)(\mu+\pi)}}\leq 1$$
$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{\mu^2+\prod_{cyc}\mu}{\sum_{cyc}(\mu+\upsilon)(\mu+\pi)}}\leq 1$$
$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{\mu^2+\prod_{cyc}\mu}{\left(\sum_{cyc}\mu\right)^2+\sum_{cyc}\mu\upsilon}}\leq 1$$
$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{\mu^2+\prod_{cyc}\mu}{1+\sum_{cyc}\mu\upsilon}}\leq 1$$
$$\Leftrightarrow\sum_{cyc}\sqrt{\mu^2+\prod_{cyc}\mu}\leq\sqrt{1+\sum_{cyc}\mu\upsilon}.$$

According to the Cauchy's inequality in Lemma 2.6, we have

$$\left(\sum_{cyc}\sqrt{\mu^2+\prod_{cyc}\mu}\right)^2$$
$$=\left(\sum_{cyc}\left(\sqrt{\mu}\cdot\sqrt{\mu+\upsilon\pi}\right)\right)^2$$
$$\leq\left(\sum_{cyc}\mu\right)\left(\sum_{cyc}\mu+\sum_{cyc}\mu\upsilon\right)$$
$$=1+\sum_{cyc}\mu\upsilon.$$

Hence, right inequality holds.

(2) Left inequality: $\sqrt{2+\sum_{cyc}\frac{\mu}{\upsilon+\pi}}\leq\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}$.

A direct equivalent calculation of left inequality shows

$$\sqrt{2+\sum_{cyc}\frac{\mu}{\upsilon+\pi}}\leq\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}$$
$$\Leftrightarrow\sqrt{\frac{2}{3}\sum_{cyc}\frac{1}{\upsilon+\pi}+\frac{1}{3}\sum_{cyc}\frac{\mu}{\upsilon+\pi}}\leq\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}$$
$$\Leftrightarrow\sqrt{\frac{2}{3}\sum_{cyc}\frac{1}{\mu+\upsilon}+\frac{1}{3}\sum_{cyc}\frac{\mu}{\upsilon+\pi}}\leq\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}$$
$$\Leftrightarrow\frac{\sqrt{\frac{2}{3}\sum_{cyc}\frac{1}{\mu+\upsilon}+\frac{1}{3}\sum_{cyc}\frac{\mu}{\upsilon+\pi}}}{\sqrt{\sum_{cyc}\frac{1}{\mu+\upsilon}}}\leq\frac{\sum_{cyc}\sqrt{\frac{\mu}{\upsilon+\pi}}}{\sqrt{\sum_{cyc}\frac{1}{\mu+\upsilon}}}$$
$$\Leftrightarrow\sqrt{\frac{2}{3}+\frac{\sum_{cyc}\mu^2+3\prod_{cyc}\mu}{3\left(1+\sum_{cyc}\mu\upsilon\right)}}\leq\sum_{cyc}\sqrt{\frac{\mu^2+\prod_{cyc}\mu}{1+\sum_{cyc}\mu\upsilon}}$$
$$\Leftrightarrow\sqrt{\frac{2}{3}+\frac{\sum_{cyc}\mu^2+3\prod_{cyc}\mu}{3\left(1+\sum_{cyc}\mu\upsilon\right)}}\cdot\sqrt{1+\sum_{cyc}\mu\upsilon}$$
$$\leq\sum_{cyc}\sqrt{\frac{\mu^2+\prod_{cyc}\mu}{1+\sum_{cyc}\mu\upsilon}}\cdot\sqrt{1+\sum_{cyc}\mu\upsilon}$$
$$\Leftrightarrow\sqrt{\frac{2}{3}+\frac{\left(\sum_{cyc}\mu\right)^2}{3}+\prod_{cyc}\mu}\leq\sum_{cyc}\sqrt{\mu^2+\prod_{cyc}\mu}$$

$$\Leftrightarrow \sqrt{1+\prod_{cyc}\mu} \leq \sum_{cyc}\sqrt{\mu^2+\prod_{cyc}\mu}$$

$$\Leftrightarrow \left(\sqrt{1+\prod_{cyc}\mu}\right)^2 \leq \left(\sum_{cyc}\sqrt{\mu^2+\prod_{cyc}\mu}\right)^2$$

$$\Leftrightarrow \sum_{cyc}\mu^2 + 2\sum_{cyc}\sqrt{\left(\mu^2+\prod_{cyc}\mu\right)\left(\upsilon^2+\prod_{cyc}\mu\right)} + 2\prod_{cyc}\mu - 1 \geq 0.$$

Using the Cauchy's inequality in Lemma 2.6, we have

$$\sum_{cyc}\mu^2 + 2\sum_{cyc}\sqrt{\left(\mu^2+\prod_{cyc}\mu\right)\left(\upsilon^2+\prod_{cyc}\mu\right)} + 2\prod_{cyc}\mu - 1$$

$$\geq \sum_{cyc}\mu^2 + 2\sum_{cyc}\left(\mu\upsilon+\prod_{cyc}\mu\right) + 2\prod_{cyc}\mu - 1$$

$$= \sum_{cyc}\mu^2 + 2\sum_{cyc}\mu\upsilon + 8\prod_{cyc}\mu - 1$$

$$= \left(\sum_{cyc}\mu\right)^2 + 8\prod_{cyc}\mu - 1$$

$$= 1 + 8\prod_{cyc}\mu - 1$$

$$\geq 0.$$

Hence, left inequality holds.

Consequently, this completes the proof of Theorem 5.6. □

5.5 Intuitionistic Fuzzy Inequality Proved by Hölder's Inequality

Hölder's inequality reflects the relationship between L_p spaces. Cauchy's inequality is special form of Hölder's inequality when $\alpha = \beta = 2$. In the following, we develop an intuitionistic fuzzy inequality and prove it by Hölder's inequality.

Theorem 5.7 *Let $\ddot{a} = (\mu, \upsilon, \pi)$ be an IFV. Then it holds that*

$$\sum_{cyc}\sqrt{\mu} \geq 3\sqrt{3}\sum_{cyc}\mu\upsilon, \tag{37}$$

Proof It is known from the Hölder's inequality in Lemma 2.8 that

$$\left(\sum_{cyc}(\mu^{\frac{1}{3}})^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(\sum_{cyc}(\mu^{\frac{2}{3}})^{3}\right)^{\frac{1}{3}} \geq \left(\sum_{cyc}\mu^{\frac{1}{3}}\mu^{\frac{2}{3}}\right)^{\frac{2}{3}+\frac{1}{3}}$$

$$\Leftrightarrow \left(\sum_{cyc}\sqrt{\mu}\right)^{\frac{2}{3}}\left(\sum_{cyc}\mu^2\right)^{\frac{1}{3}} \geq \sum_{cyc}\mu$$

$$\Leftrightarrow \sum_{cyc}\sqrt{\mu} \geq \sqrt{\frac{\left(\sum_{cyc}\mu\right)^3}{\sum_{cyc}\mu^2}}.$$

Hence, we just need to prove that

$$\sqrt{\frac{\left(\sum_{cyc}\mu\right)^3}{\sum_{cyc}\mu^2}} \geq 3\sqrt{3}\sum_{cyc}\mu\upsilon$$

$$\Leftrightarrow \left(\sum_{cyc}\mu\right)^3 \geq 27\left(\sum_{cyc}\mu^2\right)\left(\sum_{cyc}\mu\upsilon\right)^2$$

$$\Leftrightarrow \left(\sum_{cyc}\mu\right)^6 \geq 27\left(\sum_{cyc}\mu^2\right)\left(\sum_{cyc}\mu\upsilon\right)^2$$

$$\Leftrightarrow \left(\sum_{cyc}\mu^2+2\sum_{cyc}\mu\upsilon\right)^3 \geq 27\left(\sum_{cyc}\mu^2\right)\left(\sum_{cyc}\mu\upsilon\right)^2.$$

Let $\mathcal{P} = \sum_{cyc}\mu^2 > 0$ and $\mathcal{Q} = \sum_{cyc}\mu\upsilon \geq 0$, then a direct equivalent calculation of the above inequality gives

$$(\mathcal{P}+2\mathcal{Q})^3 \geq 27\mathcal{P}\mathcal{Q}^2$$

$$\Leftrightarrow \mathcal{P}^3 + 6\mathcal{P}^2\mathcal{Q} + 12\mathcal{P}\mathcal{Q}^2 + 8\mathcal{Q}^3 \geq 27\mathcal{P}\mathcal{Q}^2$$

$$\Leftrightarrow \mathcal{P}^3 + 6\mathcal{P}^2\mathcal{Q} - 15\mathcal{P}\mathcal{Q}^2 + 8\mathcal{Q}^3 \geq 0$$

$$\Leftrightarrow (\mathcal{P}^3 - \mathcal{P}\mathcal{Q}^2) + 6(\mathcal{P}^2\mathcal{Q} - \mathcal{P}\mathcal{Q}^2) - 8(\mathcal{P}\mathcal{Q}^2 - \mathcal{Q}^3) \geq 0$$

$$\Leftrightarrow \mathcal{P}(\mathcal{P}+\mathcal{Q})(\mathcal{P}-\mathcal{Q}) + 6\mathcal{P}\mathcal{Q}(\mathcal{P}-\mathcal{Q}) - 8\mathcal{Q}^2(\mathcal{P}-\mathcal{Q}) \geq 0$$

$$\Leftrightarrow (\mathcal{P}-\mathcal{Q})(\mathcal{P}(\mathcal{P}+\mathcal{Q}) + 6\mathcal{P}\mathcal{Q} - 8\mathcal{Q}^2) \geq 0$$

$$\Leftrightarrow (\mathcal{P}-\mathcal{Q})(\mathcal{P}^2 + 7\mathcal{P}\mathcal{Q} - 8\mathcal{Q}^2)$$

$$\Leftrightarrow (\mathcal{P}-\mathcal{Q})^2(\mathcal{P}+7\mathcal{Q}) \geq 0.$$

Consequently, this completes the proof of Theorem 5.7. □

5.6 Intuitionistic Fuzzy Inequality Proved by Minkowski's Inequality

Minkowski's inequality can be derived by the Hölder's inequality. Like the Hölder's inequality agove, Minkowski's inequality can take countable measures in particular forms of sequences or vectors. In the following, we develop an intuitionistic fuzzy inequality and prove it by Minkowski's inequality.

Theorem 5.8 *Let $\ddot{a} = (\mu, \upsilon, \pi)$ be an IFV. Then it holds that*

$$\sum_{cyc}\sqrt{\mu^2-\mu+1} \geq \sqrt{7}. \tag{38}$$

Proof It is known from the Minkowski's inequality in Lemma 2.11 that

$$\sum_{cyc}\sqrt{\mu^2-\mu+1}$$
$$=\sum_{cyc}\sqrt{\tfrac{2\mu^2-2\mu+2}{2}}$$
$$=\sum_{cyc}\sqrt{\tfrac{\mu^2+(\mu-1)^2+1}{2}}$$
$$\geq\sqrt{\frac{\left(\sum_{cyc}\mu\right)^2+\left(\sum_{cyc}(\mu-1)\right)^2+3^2}{2}}$$
$$=\sqrt{7}.$$

Consequently, this completes the proof of Theorem 5.8. □

5.7 Inutitionistic Fuzzy Inequality Proved by Jensen's Inequality

Jensen's inequality is a quite useful inequality which can easily derive some famous inequalities, including Power-Mean inequality [29], Hölder's inequality [28] and Minkowski's inequality [28]. In the following, we present a novel inutitionistic fuzzy inequality and prove it by Jensen's inequality.

Theorem 5.9 *For any IFV $\ddot{a}=(\mu,\upsilon,\pi)$ and satisfy $\mu,\upsilon,\pi\in\left[0,\frac{3}{5}\right]$. Then it holds that*

$$\sqrt{6}+\sqrt{2}\leq\sum_{cyc}\sqrt{6-10\mu}\leq2\sqrt{6}. \tag{39}$$

Proof The Eq. (39) can be proved in two steps, as shown below.

(1) Right inequality: $\sum_{cyc}\sqrt{6-10\mu}\leq2\sqrt{6}$.

Let $f(x)=\sqrt{6-10x}\ \left(x\in\left[0,\frac{3}{5}\right]\right)$, then $f''(x)=-25(6-10x)^{-\frac{3}{2}}\leq0$.

Hence, $f(x)$ is a convex function on $\left[0,\frac{3}{5}\right]$.

Further, using the Jensen's inequality in Lemma 2.12, we have

$$\sum_{cyc}\sqrt{6-10\mu}\leq3f\left(\tfrac{\mu+\upsilon+\pi}{3}\right)=3f\left(\tfrac{1}{3}\right)=2\sqrt{6}.$$

Hence, right inequality holds.

(2) Left inequality: $\sqrt{6}+\sqrt{2}\leq\sum_{cyc}\sqrt{6-10\mu}$.

Without loss of generality, assume that $\mu\leq\upsilon\leq\pi$.

Since $\mu+\upsilon+\pi=1$, it can be achieved from the drawer principle that $\mu\leq\frac{1}{3}$.

A direct calculation of left inequality shows

$$\sum_{cyc}\sqrt{6-10\mu}=\sqrt{6-10\mu}+\sqrt{6-10\upsilon}+\sqrt{6-10\pi}$$
$$\geq\sqrt{6-10\mu}+\sqrt{\left(\sqrt{6-10\upsilon}+\sqrt{6-10\pi}\right)^2}$$
$$=\sqrt{6-10\mu}$$
$$+\sqrt{12-10\upsilon-10\pi+2\sqrt{6-10\upsilon}\cdot\sqrt{6-10\pi}}$$
$$\geq\sqrt{6-10\mu}+\sqrt{12-10\upsilon-10\pi}$$
$$=\sqrt{6-10\mu}+\sqrt{2+10\mu}.$$

Let $s=\sqrt{6-10\mu}+\sqrt{2+10\mu},\ \mu\in\left[0,\frac{1}{3}\right]$, then an immediate calculation gives

$$s^2=\left(\sqrt{6-10\mu}+\sqrt{2+10\mu}\right)^2$$
$$=8+2\sqrt{(6-10\mu)(2+10\mu)}$$
$$=8+2\sqrt{-100\left(\mu-\tfrac{1}{5}\right)^2+16}$$
$$\geq8+2\sqrt{-100\left(0-\tfrac{1}{5}\right)^2+16}$$
$$=8+4\sqrt{3}=\left(\sqrt{6}+\sqrt{2}\right)^2.$$

Hence, left inequality holds.

Consequently, this completes the proof of Theorem 5.9. □

5.8 Intuitionistic fuzzy inequality proved by Tangent inequality

Tangent inequality can be derived the Jensen's inequality due to their have similar geometric meanings. In the following, combining with Theorem 2.1 (Half concave and Half convex theorem) and Tangent inequality, we construct an intuitionistic fuzzy inequality.

Theorem 5.10 *Let $\ddot{a}=(\mu,\upsilon,\pi)$ be an IFV. Then it holds that*

$$2\leq\sum_{cyc}\sqrt{\frac{1-\mu}{1+\mu}}\leq1+\frac{2\sqrt{3}}{3}. \tag{40}$$

Proof Let $f(x)=\sqrt{\frac{1-x}{1+x}}$ $(x\in[0,1])$, then we can derive its second derivative as $f''(x)=\frac{1-2x}{\sqrt{(1+x)^5(1-x)^3}}$.

According to the Tangent inequality in Lemma 2.13, we can deduce that $f(x)$ is concave on $\left[0,\frac{1}{2}\right]$ and convex on $\left[\frac{1}{2},1\right]$.

Without loss of generality, assume that $\mu\leq\upsilon\leq\pi$.

(1) Right inequality: $\sum_{cyc}\sqrt{\frac{1-\mu}{1+\mu}}\leq1+\frac{2\sqrt{3}}{3}$.

Using the Theorem 2.1, we just have to prove that:

① $f(\mu)+f(\upsilon)+f(\pi)\leq1+\frac{2\sqrt{3}}{3}$ $(\mu=0,\upsilon+\pi=1)$ and

② $f(\mu)+f(\upsilon)+f(\pi)\leq1+\frac{2\sqrt{3}}{3}$ $(\upsilon=\pi)$.

Case 1: If $\mu=0$ and $\upsilon+\pi=1$, then the Right inequality is converted to prove that

$$\sum_{cyc}\sqrt{\tfrac{1-\mu}{1+\mu}}=f(\mu)+f(\upsilon)+f(\pi)$$
$$=\sqrt{\tfrac{1-\mu}{1+\mu}}+\sqrt{\tfrac{1-\upsilon}{1+\upsilon}}+\sqrt{\tfrac{1-\pi}{1+\pi}}$$

$= 1 + \sqrt{\frac{1-\upsilon}{1+\upsilon}} + \sqrt{\frac{1-\pi}{1+\pi}}$
$= 1 + \sqrt{\frac{1-\upsilon}{1+\upsilon}} + \sqrt{\frac{\upsilon}{2-\upsilon}}$
$\leq 1 + \frac{2\sqrt{3}}{3}.$

Let $\eta = \sqrt{\frac{1-\upsilon}{1+\upsilon}}$ and $\varphi = \sqrt{\frac{\upsilon}{2-\upsilon}}$, then $\upsilon = \frac{1-\eta^2}{1+\eta^2} = \frac{2\varphi^2}{\varphi^2+1}$ and $\eta^2 + \varphi^2 = 1 - 3\eta^2\varphi^2$.

Further, we can derive

$1 + \sqrt{\frac{1-\upsilon}{1+\upsilon}} + \sqrt{\frac{\upsilon}{2-\upsilon}} = 1 + \eta + \varphi = 1 + \sqrt{(\eta+\varphi)^2} = 1 + \sqrt{\eta^2 + 2\eta\varphi + \varphi^2} = 1 + \sqrt{1 - 3\eta^2\varphi^2 + 2\eta\varphi} = 1 + \sqrt{-3\left(\eta\varphi - \frac{1}{3}\right)^2 + \frac{4}{3}} \leq 1 + \sqrt{\frac{4}{3}} = 1 + \frac{2\sqrt{3}}{3}.$

Case 2: If $\upsilon = \pi$, then $\upsilon \in \left[0, \frac{1}{2}\right]$ and the Right inequality is converted to prove that

$\sum_{cyc} \sqrt{\frac{1-\mu}{1+\mu}} = f(\mu) + f(\upsilon) + f(\pi)$
$= \sqrt{\frac{1-\mu}{1+\mu}} + 2\sqrt{\frac{1-\upsilon}{1+\upsilon}} = \sqrt{\frac{\upsilon}{1-\upsilon}} + 2\sqrt{\frac{1-\upsilon}{1+\upsilon}}.$

Let $g(\upsilon) = \sqrt{\frac{\upsilon}{1-\upsilon}} + 2\sqrt{\frac{1-\upsilon}{1+\upsilon}}$, then its derivative is

$g'(\upsilon) = \frac{1}{\sqrt{\upsilon(1-\upsilon)^3(1+\upsilon)^3}}\left[\frac{1}{2}(1+\upsilon)\sqrt{1+\upsilon} - 2\sqrt{\upsilon}(1-\upsilon)\right]$
$\geq \frac{1}{\sqrt{\upsilon(1-\upsilon)^3(1+\upsilon)^3}}\left[\frac{1}{2}\cdot 2\sqrt{\upsilon}\sqrt{1+\upsilon} - 2\sqrt{\upsilon}(1-\upsilon)\right]$
$= \frac{1}{\sqrt{(1-\upsilon)^3(1+\upsilon)^3}}\left[\sqrt{1+\upsilon} - 2(1-\upsilon)\right].$

Let $h(\upsilon) = \left(\sqrt{1+\upsilon}\right)^2 - (2(1-\upsilon))^2$, then we have $h(\upsilon) = -4\upsilon^2 + 9\upsilon - 3$.

It can be easily derived that $h(\upsilon) \geq 0$ when $\upsilon \in \left[\frac{9-\sqrt{33}}{8}, \frac{1}{2}\right]$ and $h(\upsilon) \leq 0$ when $\upsilon \in \left[0, \frac{9-\sqrt{33}}{8}\right]$.

Further, $g(\upsilon)$ is monotonically increasing on $\upsilon \in \left[\frac{9-\sqrt{33}}{8}, \frac{1}{2}\right]$ and monotonically decreasing on $\upsilon \in \left[0, \frac{9-\sqrt{33}}{8}\right]$.

Further, the $g(\upsilon)_{max} = \max\left\{g(0), g\left(\frac{1}{2}\right)\right\} = \max\left\{2, 1 + \frac{2\sqrt{3}}{3}\right\} = 1 + \frac{2\sqrt{3}}{3}.$

Hence, Right inequality holds.

(2) Left inequality: $2 \leq \sum_{cyc} \sqrt{\frac{1-\mu^2}{1+\mu^2}}.$

Using the Theorem 2.1, we just have to prove that:
① $f(\mu) + f(\upsilon) + f(\pi) \geq 2$ $(\mu + \upsilon = 0, \pi = 1)$ and
② $f(\mu) + f(\upsilon) + f(\pi) \geq 2$ $(\mu = \upsilon)$.

Case 1: If $\mu + \upsilon = 0$ and $\pi = 1$, then the Left inequality is converted to prove that

$\sum_{cyc} \sqrt{\frac{1-\mu}{1+\mu}} = f(\mu) + f(\upsilon) + f(\pi)$
$= \sqrt{\frac{1-\mu}{1+\mu}} + \sqrt{\frac{1-\upsilon}{1+\upsilon}} + \sqrt{\frac{1-\pi}{1+\pi}} = 1 + 1 = 2.$

Case 2: If $\mu = \upsilon$, then $\upsilon \in \left[0, \frac{1}{2}\right]$ and the Left inequality is converted to prove that

$\sum_{cyc} \sqrt{\frac{1-\mu}{1+\mu}} = f(\mu) + f(\upsilon) + f(\pi) = \sqrt{\frac{1-\mu}{1+\mu}} + \sqrt{\frac{1-\upsilon}{1+\upsilon}} + \sqrt{\frac{1-\pi}{1+\pi}} = 2\sqrt{\frac{1-\upsilon}{1+\upsilon}} + \sqrt{\frac{1-\pi}{1+\pi}} = 2\sqrt{\frac{1-\upsilon}{1+\upsilon}} + \sqrt{\frac{\upsilon}{1-\upsilon}}.$

Using the Case 2 in (1), we can obtain $g(\upsilon)_{min} = g\left(\frac{9-\sqrt{33}}{8}\right) > 2.$

Combining with Case 1 and Case 2, we can derive that $\sum_{cyc} \sqrt{\frac{1-\mu}{1+\mu}} \geq 2$ holds.

Hence, Left inequality holds.

Consequently, this completes the proof of Theorem 5.10. □

5.9 Intuitionistic Fuzzy Inequality Proved by Muirhead's Inequality

Muirhead's inequality is symmetric and homogeneous. Therefore, according to the number and degree characteristics of the left and right sides of the inequality, the corresponding inequality issue is often succinctly proved. In the following, we construct an intuitionistic fuzzy inequality and prove it by Muirhead's inequality.

Theorem 5.11 *For any IFV $\ddot{a} = (\mu, \upsilon, \pi)$ and satisfy $\mu, \upsilon, \pi > 0$. Then it holds that*

$$\sum_{cyc} \frac{\mu^5 + \upsilon^5}{\mu\upsilon(\mu+\upsilon)} \geq \frac{1}{3}. \tag{41}$$

Proof Using the Muirhead's inequality in Lemma 2.15, we have

$\mu^5 + \upsilon^5 \geq \mu^4\upsilon + \mu\upsilon^4 = \mu\upsilon(\mu^3 + \upsilon^3).$

Further, we can deduce

$\sum_{cyc} \frac{\mu^5+\upsilon^5}{\mu\upsilon(\mu+\upsilon)} \geq \sum_{cyc} \frac{\mu\upsilon(\mu^3+\upsilon^3)}{\mu\upsilon(\mu+\upsilon)}$
$= \sum_{cyc}(\mu^2 - \mu\upsilon + \upsilon^2)$
$= \sum_{cyc}(\mu^2 + \upsilon^2) - \sum_{cyc}\mu\upsilon$
$= 2\sum_{cyc}\mu^2 - \sum_{cyc}\mu\upsilon$
$= 2\left(\left(\sum_{cyc}\mu\right)^2 - 2\sum_{cyc}\mu\upsilon\right) - \sum_{cyc}\mu\upsilon$
$= 2 - 5\sum_{cyc}\mu\upsilon$
$\geq 2 - 5\frac{\left(\sum_{cyc}\mu\right)^2}{3}$
$= \frac{1}{3}.$

Consequently, this completes the proof of Theorem 5.11. □

5.10 Intuitionistic Fuzzy Inequality Proved by Carlson's Inequality

Carlson's inequality can be denoted as an $n \times m$ non-negative real number matrix, where the geometric mean of the sum of the elements in each column of m columns is not less than the geometric mean of the elements in each row of n rows in the matrix. In the following, we construct an intuitionistic fuzzy inequality by a same order matrix and prove it by Carlson's inequality.

Theorem 5.12 *Let $\ddot{a} = (\mu, \upsilon, \pi)$ be not a crisp number. Then it holds that*

$$\sum_{cyc} \frac{\mu^2}{\upsilon + \pi} \geq \frac{1}{2}. \tag{42}$$

Proof Construct the 3×2 matrix that relates to our conclusion:

$$\Gamma = \begin{pmatrix} \frac{\mu^2}{\upsilon+\pi} & \upsilon+\pi \\ \frac{\upsilon^2}{\pi+\mu} & \pi+\mu \\ \frac{\pi^2}{\mu+\upsilon} & \mu+\upsilon \end{pmatrix}.$$

It follows from the Carlson's inequality in Lemma 2.9 that

$$\left(\left(\frac{\mu^2}{\upsilon+\pi} + \frac{\upsilon^2}{\pi+\mu} + \frac{\upsilon^2}{\pi+\mu}\right) \cdot (\upsilon+\pi+\pi+\mu+\mu+\upsilon)\right)^{\frac{1}{2}} \geq \left(\frac{\mu^2}{\upsilon+\pi} \cdot (\upsilon+\pi)\right)^{\frac{1}{2}} + \left(\frac{\upsilon^2}{\pi+\mu} \cdot (\pi+\mu)\right)^{\frac{1}{2}} + \left(\frac{\pi^2}{\mu+\upsilon} \cdot (\mu+\upsilon)\right)^{\frac{1}{2}}$$

$$\Leftrightarrow \left(\sum_{cyc} \frac{\mu^2}{\upsilon+\pi} \cdot 2(\mu+\upsilon+\pi)\right)^{\frac{1}{2}} \geq \mu+\upsilon+\pi.$$

$$\Leftrightarrow \sum_{cyc} \frac{\mu^2}{\upsilon+\pi} \geq \frac{1}{2}.$$

Consequently, this completes the proof of Theorem 5.12. □

5.11 Intuitionistic Fuzzy Inequality Proved by Wei–Wei Dual Inequality

Wei-Wei dual inequality involves matrix that the sum of column products of the ordered matrix is greater than or equal to the sum of column products of the disordered matrix and the column sum product of the ordered matrix is less than or equal to the column sum product of the disordered matrix. In the following, we present an intuitionistic fuzzy inequality and prove it by Wei-Wei dual inequality.

Theorem 5.13 *Let $\ddot{a} = (\mu, \upsilon, \pi)$ be an IFV. Then it holds that*

$$\prod_{cyc} \mu \leq \frac{1}{27}. \tag{43}$$

Proof Without loss of generality, assume that $\mu \leq \upsilon \leq \pi$.

Construct the 3×3 similarly ordered matrix as

$$\Theta = \begin{pmatrix} \mu & \upsilon & \pi \\ \mu & \upsilon & \pi \\ \mu & \upsilon & \pi \end{pmatrix}.$$

Moreover, we also construct the 3×3 disordered matrix as

$$\Theta' = \begin{pmatrix} \mu & \upsilon & \pi \\ \upsilon & \mu & \mu \\ \pi & \pi & \upsilon \end{pmatrix}.$$

Further, we can get

$S(\Theta) = (\mu+\mu+\mu)(\upsilon+\upsilon+\upsilon)(\pi+\pi+\pi) = 27\prod_{cyc} \mu$ and $S(\Theta') = \left(\sum_{cyc} \mu\right)^3 = 1$

It follows from the Wei-Wei dual inequality in Lemma 2.10 that $S(\Theta) \leq S(\Theta')$.

Consequently, this completes the proof of Theorem 5.13. □

Remark 5.3 From Theorems 5.12 and 5.13, we can find that the clever construction of the matrix is the key to subsequent proofs. Moreover, the intuitionistic fuzzy inequality in Theorem 5.13 is special Mean inequality (AM-GM) when $n = 3$. In other words, the AM-GM inequality can be derived by Wei-Wei dual inequality.

5.12 Intuitionistic Fuzzy Inequality Proved by Some Combined Inequalities

It is no longer realistic to simply rely on a certain inequality to solve or construct an inequality under intuitionistic fuzzy environment. In fact, it is a combination of a variety of existing famous inequalities to achieve the ability to solve inequality problems in operations, AOs and so on. The combined famous inequalities in the developed Theorems and Lemma are presented in Table 1.

Lemma 5.1 *Let $\ddot{a} = (\mu, \upsilon, \pi)$ be not a crisp number. Then it holds that*

$$\sum_{cyc} \sqrt{\frac{\mu}{\mu+\upsilon}} \leq \frac{3\sqrt{2}}{2}. \tag{44}$$

Proof The Eq. (44) can be equivalent to $\sum_{cyc} \sqrt{\frac{\mu}{\mu+\upsilon}} = \sum_{cyc} (\mu+\pi)\sqrt{\frac{\mu}{(\mu+\upsilon)(\mu+\pi)^2}}$.

Let $f(x) = \sqrt{x}$, then $g''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \leq 0$.

Using the Tangent inequality in Lemma 2.13, we can conclude that $g(x)$ is convex function.

Hence, it is known from Eq. (17) that

$\sum_{cyc}\frac{\mu+\pi}{2}\sqrt{\frac{\mu}{(\mu+\upsilon)(\mu+\pi)^2}}$

$\leq\sqrt{\sum_{cyc}\frac{\mu+\pi}{2}\frac{\mu}{(\mu+\upsilon)(\mu+\pi)^2}}=\sqrt{\sum_{cyc}\frac{\mu}{2(\mu+\upsilon)(\mu+\pi)}}.$

Further, it just has to prove that

$\sum_{cyc}\frac{2\mu}{(\mu+\upsilon)(\mu+\pi)}\leq\frac{9}{2}$

$\Leftrightarrow 8\sum_{cyc}\mu\upsilon\leq 9\left(2\mu\upsilon\pi+\sum_{sym}\mu^2\upsilon\right)$

$\Leftrightarrow 8\sum_{cyc}\left(\mu\upsilon\sum_{cyc}\mu\right)\leq 9\left(2\mu\upsilon\pi+\sum_{sym}\mu^2\upsilon\right)$

$\Leftrightarrow 8\sum_{cyc}(\mu^2\upsilon+\mu\upsilon^2+\mu\upsilon\pi)\leq 9\left(2\mu\upsilon\pi+\sum_{sym}\mu^2\upsilon\right)$

$\Leftrightarrow \sum_{sym}\mu^2\upsilon\geq 6\mu\upsilon\pi.$

It follows from the Mean inequality (AM-GM) in Lemma 2.2 that above inequality holds.

Consequently, this completes the proof of Theorem 5.1. □

Theorem 5.14 *For any IFV $\ddot{a}=(\mu,\upsilon,\pi)$ and satisfy $\mu,\upsilon,\pi>0$. Then it holds that*

$$\sum_{cyc}\frac{1+\sqrt{\mu\upsilon}}{1-\sqrt{\upsilon\pi}}\leq 6. \tag{45}$$

Proof Without loss of generality, assume that $\mu\leq\upsilon\leq\pi$.

Further, we can get

$1+\sqrt{\mu\upsilon}\leq 1+\sqrt{\upsilon\pi}\leq 1+\sqrt{\mu\pi}$ and $\frac{1}{1-\sqrt{\mu\upsilon}}\leq\frac{1}{1-\sqrt{\upsilon\pi}}\leq\frac{1}{1-\sqrt{\mu\pi}}.$

Using the Rearrangement inequality in Lemma 2.1, we can have

$\sum_{cyc}\frac{1+\sqrt{\mu\upsilon}}{1-\sqrt{\upsilon\pi}}=\sum_{cyc}(1+\sqrt{\mu\upsilon})\frac{1}{1-\sqrt{\upsilon\pi}}\leq\sum_{cyc}(1+\sqrt{\mu\upsilon})\frac{1}{1-\sqrt{\mu\upsilon}}=\sum_{cyc}\frac{1+\sqrt{\mu\upsilon}}{1-\sqrt{\mu\upsilon}}.$

For this, it just has to prove that

$\sum_{cyc}\frac{1+\sqrt{\mu\upsilon}}{1-\sqrt{\mu\upsilon}}\leq 6\Leftrightarrow\sum_{cyc}\frac{1-\sqrt{\mu\upsilon}+2\sqrt{\mu\upsilon}}{1-\sqrt{\mu\upsilon}}\leq 6\Leftrightarrow\sum_{cyc}\frac{2\sqrt{\mu\upsilon}}{1-\sqrt{\mu\upsilon}}\leq 3.$

Employing the Mean inequality (AM-GM) in Lemma 2.2, we can obtain

$\sum_{cyc}\frac{2\sqrt{\mu\upsilon}}{1-\sqrt{\mu\upsilon}}=\sum_{cyc}\frac{4\sqrt{\mu\upsilon}}{2-2\sqrt{\mu\upsilon}}$

$\leq\sum_{cyc}\frac{4\sqrt{\mu\upsilon}}{2-(\mu+\upsilon)}=\sum_{cyc}\frac{4\sqrt{\mu\upsilon}}{2(\mu+\upsilon+\pi)-(\mu+\upsilon)}=\sum_{cyc}\frac{4\sqrt{\mu\upsilon}}{(\mu+\pi)+(\upsilon+\pi)}$

$\leq\sum_{cyc}\frac{4\sqrt{\mu\upsilon}}{2\sqrt{(\mu+\pi)(\upsilon+\pi)}}=\sum_{cyc}\frac{2\sqrt{\mu\upsilon}}{\sqrt{(\mu+\pi)(\upsilon+\pi)}}=\sum_{cyc}2\sqrt{\frac{\mu}{\mu+\pi}\cdot\frac{\upsilon}{\upsilon+\pi}}$

$\leq\sum_{cyc}\left(\frac{\mu}{\mu+\pi}+\frac{\upsilon}{\upsilon+\pi}\right)=\sum_{cyc}\left(\frac{\mu}{\mu+\pi}+\frac{\pi}{\mu+\pi}\right)=\sum_{cyc}\frac{\mu+\pi}{\mu+\pi}=3.$

Consequently, this completes the proof of Theorem 5.14. □

Theorem 5.15 *Let $\ddot{a}=(\mu,\upsilon,\pi)$ be not a crisp number. Then it holds that*

$$\sum_{cyc}\frac{\mu^2}{(1-\mu)^3}\geq\frac{9}{8}. \tag{46}$$

Proof A direct equivalent calculation of Eq. (46) gives

$\sum_{cyc}\frac{\mu^2}{(1-\mu)^3}\geq\frac{9}{8}$

$\Leftrightarrow\sum_{cyc}\frac{\mu^2}{(\upsilon+\pi)^3}\geq\frac{9}{8}$

$\Leftrightarrow\sum_{cyc}\mu\cdot\sum_{cyc}\frac{\mu^2}{(\upsilon+\pi)^3}\geq\frac{9}{8}$

$\Leftrightarrow\sum_{cyc}(\mu+\upsilon)\cdot\sum_{cyc}\frac{\mu^2}{(\upsilon+\pi)^3}\geq\frac{9}{4}$

$\Leftrightarrow\sum_{cyc}(\mu+\upsilon)\cdot\sum_{cyc}\left(\frac{1}{\upsilon+\pi}\cdot\frac{\mu^2}{(\upsilon+\pi)^2}\right)\geq\frac{9}{4}.$

It follows from the Cauchy's inequality in Lemma 2.6 that

$\sum_{cyc}(\mu+\upsilon)\cdot\sum_{cyc}\left(\frac{1}{\upsilon+\pi}\cdot\frac{\mu^2}{(\upsilon+\pi)^2}\right)\geq\left(\sum_{cyc}\frac{\mu}{\upsilon+\pi}\right)^2.$

It suffices to prove that $\left(\sum_{cyc}\frac{\mu}{\upsilon+\pi}\right)^2\geq\frac{9}{4}.$

According to the Nesbitt's inequality in Lemma 2.4, it holds.

Consequently, this completes the proof of Theorem 5.15. □

Theorem 5.16 *For any IFV $\ddot{a}=(\mu,\upsilon,\pi)$ and satisfy $\alpha,\beta>0$. Then it holds that*

$$\sum_{cyc}\frac{\mu^{2n}}{(\alpha\upsilon+\beta\pi)^{2n+1}}\geq\frac{9}{(\alpha+\beta)^{2n+1}}. \tag{47}$$

Proof A direct calculation gives

$\sum_{cyc}\frac{\mu^{2n}}{(\alpha\upsilon+\beta\pi)^{2n+1}}=\sum_{cyc}\frac{\left(\left(\frac{\mu}{\alpha\upsilon+\beta\pi}\right)^n\right)^2}{\alpha\upsilon+\beta\pi}\geq\frac{\left(\sum_{cyc}\left(\frac{\mu}{\alpha\upsilon+\beta\pi}\right)^n\right)^2}{\sum_{cyc}(\alpha\upsilon+\beta\pi)}$ (Use Generalized Cauchy's inequality: Lemma 2.7)

$=\left(\sum_{cyc}\left(\frac{\mu}{\alpha\upsilon+\beta\pi}\right)^n\right)^2(\alpha+\beta)\sum_{cyc}\mu=\frac{\left(\sum_{cyc}\left(\frac{\mu}{\alpha\upsilon+\beta\pi}\right)^n\right)^2}{(\alpha+\beta)}$

$\geq\frac{\left(3^{1-n}\left(\sum_{cyc}\frac{\mu}{\alpha\upsilon+\beta\pi}\right)^n\right)^2}{(\alpha+\beta)}$ (Use Power-Mean inequality: Lemma 2.5)

Table 1 The combined famous inequalities in the proposed Theorems and Lemma

Results	Used inequalities
Lemma 5.1	Mean inequality (AM-GM), Tangent inequality
Theorem 5.14	Mean inequality (AM-GM), Rearrangement inequality
Theorem 5.15	Chebyshev's inequality, Nesbitt's inequality
Theorem 5.16	Mean inequality (3 M), Generalized Cauchy's inequality, Power-Mean inequality
Theorem 5.17	Mean inequality (AM-GM and 3 M), Cauchy's inequality, Vasc inequality, Schur's inequality
Theorem 5.18	Mean inequality (AM-GM and AM-SM), Carlson's inequality
Theorem 5.19	Mean inequality (AM-GM), Cauchy's inequality, Muirhead's inequality
Theorem 5.20	Mean inequality (AM-GM and 3 M), Cauchy's inequality, Carlson's inequality
Theorem 5.21	Mean inequality (AM-GM and 3 M), Bernoulli's inequality

$$=\frac{\left(3^{1-n}\left(\sum_{cyc}\frac{\mu^2}{\alpha\mu\upsilon+\beta\mu\pi}\right)^n\right)^2}{(\alpha+\beta)} \geq \frac{\left(3^{1-n}\left(\frac{\left(\sum_{cyc}\mu\right)^2}{\sum_{cyc}(\alpha\mu\upsilon+\beta\mu\pi)}\right)^n\right)^2}{(\alpha+\beta)}$$ (Use Generalized Cauchy's inequality: Lemma 2.7)

$$=\frac{\left(3^{1-n}\left(\frac{1}{(\alpha+\beta)\sum_{cyc}\mu\upsilon}\right)^n\right)^2}{(\alpha+\beta)} \geq \frac{\left(3^{1-n}\left(\frac{1}{(\alpha+\beta)\frac{\left(\sum_{cyc}\mu\right)^2}{3}}\right)^n\right)^2}{(\alpha+\beta)}$$ (Use Mean inequality (3 M): Lemma 2.2)

$$=\frac{\left(3^{1-n}\left(\frac{1}{\frac{1}{3}(\alpha+\beta)}\right)^n\right)^2}{(\alpha+\beta)}$$
$$=\frac{9}{(\alpha+\beta)^{2n+1}}.$$

Consequently, this completes the proof of Theorem 5.16. □

Theorem 5.17 *Let $\ddot{a}=(\mu,\upsilon,\pi)$ be not a crisp number. Then it holds that*

$$\sum_{cyc}\frac{\mu^4}{\mu^3+\upsilon^3}\geq\frac{1}{2}. \tag{48}$$

Proof It follows from the Cauchy's inequality in Lemma 2.6 that $\sum_{cyc}\frac{\mu^4}{\mu^3+\upsilon^3}\cdot\sum_{cyc}\mu^2(\mu^3+\upsilon^3)\geq\left(\sum_{cyc}\mu^3\right)^2$.

Further, we just need to prove that $\frac{\left(\sum_{cyc}\mu^3\right)^2}{\sum_{cyc}\mu^2(\mu^3+\upsilon^3)}\geq\frac{1}{2}$.

It is known from the Vasc inequality in Lemma 2.16 that $\sum_{cyc}\mu\upsilon^3\leq\frac{1}{3}\left(\sum_{cyc}\mu^2\right)^2$.

Hence, a direct calculation gives

$$\sum_{cyc}\mu^2(\mu^3+\upsilon^3)=\sum_{cyc}\mu^5+\sum_{cyc}\mu^2\upsilon^3$$
$$=\sum_{cyc}\mu^5+\frac{\sum_{cyc}\mu^2\upsilon^3(\mu+\upsilon+\pi)}{\sum_{cyc}\mu}$$
$$=\sum_{cyc}\mu^5+\frac{\sum_{cyc}\mu\upsilon(\mu^2\upsilon^2+\mu\upsilon^3+\mu\upsilon^2\pi)}{\sum_{cyc}\mu}$$
$$=\sum_{cyc}\mu^5+\frac{\sum_{cyc}\mu\upsilon\left(\sum_{cyc}\mu^2\upsilon^2+\sum_{cyc}\mu\upsilon^3-\upsilon^2\pi^2-\pi^2\mu^2-\upsilon\pi^3-\pi\mu^3+\mu\upsilon^2\pi\right)}{\sum_{cyc}\mu}$$
$$=\sum_{cyc}\mu^5+\frac{\sum_{cyc}\mu\upsilon}{\sum_{cyc}\mu}\left(\sum_{cyc}\mu\upsilon^3+\sum_{cyc}\mu^2\upsilon^2\right)-\frac{\sum_{cyc}\mu\upsilon\pi\left(\upsilon^2\pi+\pi\mu^2+\upsilon\pi^2+\mu^3-\mu\upsilon\pi\right)}{\sum_{cyc}\mu}$$
$$=\sum_{cyc}\mu^5+\frac{\sum_{cyc}\mu\upsilon}{\sum_{cyc}\mu}\left(\sum_{cyc}\mu\upsilon^3+\sum_{cyc}\mu^2\upsilon^2\right)-\mu\upsilon\pi\frac{\sum_{cyc}\left(\upsilon^2\pi+\pi\mu^2+\upsilon\pi^2+\mu^3-\mu\upsilon^2\right)}{\sum_{cyc}\mu}$$
$$=\sum_{cyc}\mu^5+\frac{\sum_{cyc}\mu\upsilon}{\sum_{cyc}\mu}\left(\sum_{cyc}\mu\upsilon^3+\sum_{cyc}\mu^2\upsilon^2\right)-\mu\upsilon\pi\frac{\sum_{cyc}\left(\upsilon^2\pi+\upsilon\pi^2+\mu^3\right)}{\sum_{cyc}\mu}$$
$$=\sum_{cyc}\mu^5+\frac{\sum_{cyc}\mu\upsilon}{\sum_{cyc}\mu}\left(\sum_{cyc}\mu\upsilon^3+\sum_{cyc}\mu^2\upsilon^2\right)-\mu\upsilon\pi\left(\sum_{cyc}\mu^2\right)$$
$$\leq\sum_{cyc}\mu^5+\sum_{cyc}\mu\upsilon\left(\frac{1}{3}\left(\sum_{cyc}\mu^2\right)^2+\sum_{cyc}\mu^2\upsilon^2\right)-\mu\upsilon\pi\left(\sum_{cyc}\mu^2\right).$$

For this, it suffices to prove that

$$\left(\sum_{cyc}\mu^3\right)^2 \geq \frac{1}{2}\left(\sum_{cyc}\mu^5 + \sum_{cyc} \mu\upsilon\left(\frac{1}{3}\left(\sum_{cyc}\mu^2\right)^2 + \sum_{cyc}\mu^2\upsilon^2\right) - \mu\upsilon\pi\left(\sum_{cyc}\mu^2\right)\right). \tag{49}$$

Let $\eta = \sum_{cyc}\mu\upsilon$ and $\varphi = \prod_{cyc}\mu$, then two identities can be derived as

$$\sum_{cyc}\mu^5 + \sum_{cyc}\mu\upsilon\left(\frac{1}{3}\left(\sum_{cyc}\mu^2\right)^2 + \sum_{cyc}\mu^2\upsilon^2\right) - \mu\upsilon\pi\left(\sum_{cyc}\mu^2\right) = \frac{1}{3}\left(3(4-5\eta)\varphi + 3 - 14\eta + 11\eta^2 + 7\eta^3\right)$$

and

$$\left(\sum_{cyc}\mu^3\right)^2 = (1-3\eta+3\varphi)^2.$$

So, a direct equivalent calculation of Eq. (49) gives

$(1-3\eta+3\varphi)^2 \geq \frac{1}{2}\cdot \frac{1}{3}\left(3(4-5\eta)\varphi + 3 - 14\eta + 11\eta^2 + 7\ \eta^3\right)$

$\Leftrightarrow 54\varphi^2 + 3(8-28\eta)\varphi + 3 - 22\eta + 43\eta^2 - 7\eta^3 - 9\eta\varphi \geq 0.$

Let $g(\varphi) = 54\varphi^2 + 3(8-28\eta)\varphi$, then $g'(\varphi) = 108\varphi + 3(8-28\eta)$.

Employing $\varphi \geq \frac{4\eta-1}{9}$ from Schur's inequality in Lemma 2.14 ($p=1$) and using $\varphi = \sum_{cyc}\mu\upsilon \leq \frac{\sum_{cyc}\mu}{3} = \frac{1}{3}$ from Mean inequality (3 M) in Lemma 2.2, we can derive $g'(\varphi) \geq 108\cdot\frac{4\eta-1}{9} + 3(8-28\eta) = 12 - 36\eta \geq 0.$

That is, $g(\varphi)$ is monotonically increasing on $\varphi \geq \frac{4\eta-1}{9}$.

So we have

$g(\varphi) + 3 - 22\eta + 43\eta^2 - 7\eta^3 - 9\eta\varphi$
$\geq g\left(\frac{4\eta-1}{9}\right) + 3 - 22\eta + 43\eta^2 - 7\eta^3 - 9\eta\varphi$
$= 1 - \frac{22}{3}\eta + \frac{49}{3}\eta^2 - 7\eta^3 - 9\eta\varphi.$

Additionally, it can be easily known that $\prod_{cyc}(\mu-\upsilon)^2 \geq 0.$

Expanding it, $\prod_{cyc}\mu \leq \frac{1}{27}\left(9\sum_{cyc}\mu\upsilon - 2 + 2\left(1 - 3\sum_{cyc}\mu\upsilon\right)\sqrt{1-3\sum_{cyc}\mu\upsilon}\right)$ holds, i.e.,

$\varphi \leq \frac{1}{27}\left(9\eta - 2 + 2(1-3\eta)\sqrt{1-3\eta}\right).$

Employing the above inequality, it suffices to prove that

$1 - \frac{22}{3}\eta + \frac{49}{3}\eta^2 - 7\eta^3 - 9\eta\left(9\eta - 2 + 2(1-3\eta)\sqrt{1-3\eta}\right) \geq 0.$

It is equivalent to $\frac{1}{3}(1-3\eta)\left(7\eta^2 - 11\eta + 3 - 2\eta\sqrt{1-3\eta}\right) \geq 0.$

Since $\eta \leq \frac{1}{3}$, it suffices to prove that $7\eta^2 - 11\eta + 3 - 2\eta\sqrt{1-3\eta} \geq 0.$

It follows from the Mean inequality (AM-GM) in Lemma 2.2 that

$7\eta^2 - 11\eta + 3 - 2\eta\sqrt{1-3\eta} \geq 7\eta^2 - 11\eta + 3 - (\eta^2 + 1 - 3\eta) = 2(1-\eta)(1-3\eta) \geq 0.$

Consequently, the proof is fully completed. □

Theorem 5.18 *Let $\ddot{a} = (\mu, \upsilon, \pi)$ be not a crisp number. Then it holds that*

$$\sum_{cyc}\frac{\mu^3}{\mu^2+\upsilon^2} \geq \frac{\sqrt{3\sum_{cyc}\mu^2}}{2}. \tag{50}$$

Proof A direct equivalent calculation of Eq. (50) gives

$$\sum_{cyc}\left(\frac{\mu^3+\upsilon^3}{\mu^2+\upsilon^2} - \frac{\mu+\upsilon}{2}\right) \geq \sqrt{3\sum_{cyc}\mu^2} - \sum_{cyc}\mu + \sum_{cyc}\frac{\upsilon^3-\mu^3}{\mu^2+\upsilon^2}$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{\mu^3+\upsilon^3}{\mu^2+\upsilon^2} - \frac{\mu+\upsilon}{2}\right) \geq \sqrt{3\sum_{cyc}\mu^2} - 1 + \sum_{cyc}\frac{\upsilon^3-\mu^3}{\mu^2+\upsilon^2}$$

$$\Leftrightarrow \sum_{cyc}\frac{(\mu-\upsilon)^2(\mu+\upsilon)}{2(\mu^2+\upsilon^2)} \geq \frac{\left(\sqrt{3\sum_{cyc}\mu^2}-1\right)\left(\sqrt{3\sum_{cyc}\mu^2}+1\right)}{\sqrt{3\sum_{cyc}\mu^2}+1} + \sum_{cyc}\frac{\upsilon^3-\mu^3}{\mu^2+\upsilon^2}$$

$$\Leftrightarrow \sum_{cyc}\frac{(\mu-\upsilon)^2(\mu+\upsilon)}{2(\mu^2+\upsilon^2)} \geq \sum_{cyc}\frac{(\mu-\upsilon)^2}{\sqrt{3\sum_{cyc}\mu^2}+1} + \frac{\prod_{cyc}(\mu-\upsilon)\left(\sum_{cyc}\mu^2\upsilon^2+\mu\upsilon\pi\right)}{\prod_{cyc}(\mu^2+\upsilon^2)}.$$

According to the Mean inequality (AM-SM) in Lemma 2.2, it can derive $\sqrt{3\sum_{cyc}\mu^2} \geq \sum_{cyc}\mu = 1.$

Further, we just need to prove that

$$\sum_{cyc}\frac{(\mu-\upsilon)^2(\mu+\upsilon)}{2(\mu^2+\upsilon^2)} \geq \sum_{cyc}\frac{(\mu-\upsilon)^2}{2} + \frac{\prod_{cyc}(\mu-\upsilon)\left(\sum_{cyc}\mu^2\upsilon^2+\mu\upsilon\pi\right)}{\prod_{cyc}(\mu^2+\upsilon^2)}$$

$$\Leftrightarrow \sum_{cyc}(\mu-\upsilon)^2\left(\frac{\mu+\upsilon}{\mu^2+\upsilon^2} - 1\right) \geq \frac{2\prod_{cyc}(\mu-\upsilon)\left(\sum_{cyc}\mu^2\upsilon^2+\mu\upsilon\pi\right)}{\prod_{cyc}(\mu^2+\upsilon^2)}$$

$$\Leftrightarrow \sum_{cyc}(\mu-\upsilon)^2\cdot\frac{2\mu\upsilon+\mu\pi+\upsilon\pi}{\mu^2+\upsilon^2}\geq\frac{2\prod_{cyc}(\mu-\upsilon)\cdot\left(\sum_{cyc}\mu^2\upsilon^2+\mu\upsilon\pi\right)}{\prod_{cyc}(\mu^2+\upsilon^2)}.$$

It follows from the Mean inequality (AM-GM) in Lemma 2.2 that

$$\sum_{cyc}(\mu-\upsilon)^2\cdot\frac{2\mu\upsilon+\mu\pi+\upsilon\pi}{\mu^2+\upsilon^2}\geq 3\sqrt[3]{\frac{\prod_{cyc}(\mu-\upsilon)^2\cdot\prod_{cyc}(2\mu\upsilon+\mu\pi+\upsilon\pi)}{\prod_{cyc}(\mu^2+\upsilon^2)}}.$$

Hence, it suffices to prove that

$$3\sqrt[3]{\frac{\prod_{cyc}(\mu-\upsilon)^2\cdot\prod_{cyc}(2\mu\upsilon+\mu\pi+\upsilon\pi)}{\prod_{cyc}(\mu^2+\upsilon^2)}}\geq\frac{2\prod_{cyc}(\mu-\upsilon)\cdot\left(\sum_{cyc}\mu^2\upsilon^2+\mu\upsilon\pi\right)}{\prod_{cyc}(\mu^2+\upsilon^2)}$$

$$\Leftrightarrow 27\prod_{cyc}(2\mu\upsilon+\mu\pi+\upsilon\pi)$$

$$\cdot\prod_{cyc}(\mu^2+\upsilon^2)^2\geq 8\prod_{cyc}(\mu-\upsilon)\cdot\left(\sum_{cyc}\mu^2\upsilon^2+\mu\upsilon\pi\right)^3.$$

It is known from the Carlson's inequality in Lemma 2.9 that

$$\prod_{cyc}(2\mu\upsilon+\mu\pi+\upsilon\pi)=(2\mu\upsilon+\mu\pi+\upsilon\pi)$$

$$(\mu\upsilon+2\mu\pi+\upsilon\pi)(\mu\upsilon+\mu\pi+2\upsilon\pi)\geq 2\left(\sum_{cyc}\mu\upsilon\right)^3. \tag{51}$$

Using the Mean inequality (AM-GM) in Lemma 2.2, a direct calculation gives

$$\mu^2\upsilon+\upsilon\pi^2+\upsilon^2\pi+\pi\mu^2+\mu^2\pi+\pi\upsilon^2\geq 2\sqrt{\mu^2\upsilon\upsilon\pi^2}+2\sqrt{\upsilon^2\pi\pi\mu^2}+2\sqrt{\mu^2\pi\pi\upsilon^2}=6\mu\upsilon\pi$$

$$\Leftrightarrow 9(\mu+\upsilon)(\upsilon+\pi)(\pi+\mu)\geq 8(\mu\upsilon+\upsilon\pi+\pi\mu)$$

$$\Leftrightarrow 9\prod_{cyc}(\mu+\upsilon)\geq 8\sum_{cyc}\mu\upsilon$$

$$\Leftrightarrow 9\prod_{cyc}(\mu+\upsilon)\geq 8\sum_{cyc}\mu\upsilon\cdot\sum_{cyc}\mu$$

$$\Leftrightarrow \prod_{cyc}(\mu+\upsilon)\geq\frac{8}{9}\sum_{cyc}\mu\upsilon\cdot\sum_{cyc}\mu$$

$$\Leftrightarrow \prod_{cyc}(\mu^2+\upsilon^2)\geq\frac{8}{9}\sum_{cyc}\mu^2\upsilon^2\cdot\sum_{cyc}\mu^2$$

$$\Leftrightarrow \left(\prod_{cyc}(\mu^2+\upsilon^2)\right)^2\geq\left(\frac{8}{9}\sum_{cyc}\mu^2\upsilon^2\cdot\sum_{cyc}\mu^2\right)^2.$$

It is equivalent to

$$\prod_{cyc}(\mu^2+\upsilon^2)^2\geq\frac{64}{81}\left(\sum_{cyc}\mu^2\upsilon^2\right)^2\cdot\left(\sum_{cyc}\mu^2\right)^2. \tag{52}$$

Combining with Eqs. (51) and (52), it suffices to prove that

$$\frac{16}{3}\left(\sum_{cyc}\mu\upsilon\right)^3\left(\sum_{cyc}\mu^2\upsilon^2\right)^2\cdot\left(\sum_{cyc}\mu^2\right)^2$$

$$\geq\prod_{cyc}(\mu-\upsilon)\cdot\left(\sum_{cyc}\mu^2\upsilon^2+\mu\upsilon\pi\right)^3.$$

Meanwhile, an immediate calculation gives

$$8\left(\sum_{cyc}\mu^2\upsilon^2\right)^2\left(\sum_{cyc}\mu\upsilon\right)^2-3\left(\sum_{cyc}\mu^2\upsilon^2+\mu\upsilon\pi\right)^3$$

$$=8\left(\sum_{cyc}\mu^2\upsilon^2\right)^2\left(\sum_{cyc}\mu^2\upsilon^2+2\mu\upsilon\right)$$

$$-3\left(\sum_{cyc}\mu^2\upsilon^2+\mu\upsilon\pi\right)^3$$

$$=\left(5\left(\sum_{cyc}\mu^2\upsilon^2\right)^2+12\mu\upsilon\pi\left(\sum_{cyc}\mu^2\upsilon^2\right)+3\mu^2\upsilon^2\pi^2\right)\cdot\left(\sum_{cyc}\mu^2\upsilon^2-\mu\upsilon\pi\right)$$

$$=\left(5\left(\sum_{cyc}\mu^2\upsilon^2\right)^2+12\mu\upsilon\pi\left(\sum_{cyc}\mu^2\upsilon^2\right)+3\mu^2\upsilon^2\pi^2\right)\cdot\left(\sum_{cyc}\mu^2\upsilon^2-\mu\upsilon\pi\left(\sum_{cyc}\mu\right)\right)$$

$$=\frac{1}{2}\left(5\left(\sum_{cyc}\mu^2\upsilon^2\right)^2+12\mu\upsilon\pi\left(\sum_{cyc}\mu^2\upsilon^2\right)+3\mu^2\upsilon^2\pi^2\right)\cdot\left(\sum_{cyc}\mu^2(\upsilon-\pi)^2\right)$$

$$\geq 0.$$

Further, it just has to prove that $2\left(\sum_{cyc}\mu\upsilon\right)(\mu^2)^2\geq\prod_{cyc}(\mu-\upsilon)$.

Let $\eta=\sum_{cyc}\mu\upsilon$ and $\varphi=\mu\upsilon\pi$, then $\prod_{cyc}(\mu-\upsilon)\leq\sqrt{\prod_{cyc}(\mu-\upsilon)^2}=\sqrt{\eta^2-4\eta^3+2(9\eta-2)\varphi-27\varphi^2}$.

In other words, we just need to prove that $2\eta(1-2\eta)^2\geq\sqrt{\eta^2-4\eta^3+2(9\eta-2)\varphi-27\varphi^2}$.

Dividing it into the following two cases:

Case 1: $9\eta\leq 2$.

Since $2(1-2\eta)^2-\sqrt{1-4\eta}=\left(\sqrt{1-4\eta}-\frac{1}{2}\right)^2+\frac{1}{4}\left(2(1-4\eta)^2+1\right)\geq 0$, a direct calculation gives

$$2\eta(1-2\eta)^2-\sqrt{\eta^2-4\eta^3+2(9\eta-2)\varphi-27\varphi^2}$$

$$\geq 2\eta(1-2\eta)^2-\sqrt{\eta^2-4\eta^3}$$

$$=\eta\left(2(1-2\eta)^2-\sqrt{1-4\eta}\right)$$

$$\geq 0.$$

Case 2: $9\eta\geq 2$.

Since $\sqrt{\eta^2-4\eta^3+2(9\eta-2)\varphi-27\varphi^2}=\sqrt{\frac{4}{27}(1-3\eta)^3-\frac{1}{27}(27\varphi-9\eta+2)^2}\leq\sqrt{\frac{4}{27}(1-3\eta)^3}$, a direct calculation shows

$$2\eta(1-2\eta)^2-\sqrt{\eta^2-4\eta^3+2(9\eta-2)\varphi-27\varphi^2}$$
$$\geq 2\eta(1-2\eta)^2-\sqrt{\tfrac{4}{27}(1-3\eta)^3}$$
$$=2\eta(1-2\eta)^2-\tfrac{2}{9}(1-3\eta)\sqrt{3(1-3\eta)}$$
$$\geq 2\eta(1-2\eta)^2-\tfrac{2}{9}(1-3\eta)$$
$$=\tfrac{8}{729}(9\eta-2)\left(\left(9\eta-\tfrac{7}{2}\right)^2+\tfrac{3}{4}\right)+\tfrac{46}{729}$$
$$\geq 0.$$

Consequently, the proof is fully completed. □

Theorem 5.19 *Let $\ddot{a}=(\mu,\upsilon,\pi)$ be an IFV. Then it holds that*

$$3\prod_{cyc}\mu+1\geq 2\sum_{cyc}\mu\left(\sqrt{\upsilon\pi(1-\upsilon)(1-\pi)}+\upsilon\right). \tag{53}$$

Proof A direct equivalent calculation of Eq. (53) gives

$$3\prod_{cyc}\mu+1\geq 2\sum_{cyc}\mu\left(\sqrt{\upsilon\pi(1-\upsilon)(1-\pi)}+\upsilon\right).$$
$$\Leftrightarrow 3\prod_{cyc}\mu+\left(\sum_{cyc}\mu\right)^2-2\sum_{cyc}\mu\upsilon\geq 2\sum_{cyc}\mu\sqrt{\upsilon\pi(1-\upsilon)(1-\pi)}$$
$$\Leftrightarrow 3\prod_{cyc}\mu+\sum_{cyc}\mu^2\geq 2\sum_{cyc}\mu\sqrt{\upsilon\pi(1-\upsilon)(1-\pi)}$$
$$\Leftrightarrow \left(3\prod_{cyc}\mu+\sum_{cyc}\mu^2\right)^2\geq 4\left(\sum_{cyc}\mu\sqrt{\upsilon\pi(1-\upsilon)(1-\pi)}\right)^2.$$

According to the Cauchy's inequality in Lemma 2.6, then

$$4\left(\sum_{cyc}\mu\sqrt{\upsilon\pi(1-\upsilon)(1-\pi)}\right)^2$$
$$=4\mu\upsilon\pi\left(\sum_{cyc}\sqrt{\mu(1-\upsilon)(1-\pi)}\right)^2$$
$$\leq 4\mu\upsilon\pi\sum_{cyc}\mu\cdot\sum_{cyc}(1-\upsilon)(1-\pi)$$
$$=4\mu\upsilon\pi\sum_{cyc}(1-\upsilon)(1-\pi).$$

For this, we only need to prove that

$$\left(3\prod_{cyc}\mu+\sum_{cyc}\mu^2\right)^2-4\mu\upsilon\pi\sum_{cyc}(1-\upsilon)(1-\pi)\geq 0$$
$$\Leftrightarrow \left(\sum_{cyc}\mu^6+2\sum_{cyc}\mu^5(\upsilon+\pi)+3\sum_{cyc}\mu^4(\upsilon^2+\pi^2)-4\mu\upsilon\pi\sum_{cyc}\mu^3-6\mu\upsilon\pi\sum_{cyc}\mu^2(\upsilon+\pi)\right)+4\left(\sum_{cyc}\mu^4+2\sum_{cyc}\upsilon^2\pi^2-3\mu\upsilon\pi\right)\geq 4\left(\sum_{cyc}\mu^5+\sum_{cyc}\mu^4(\upsilon+\pi)+2\sum_{cyc}\mu^3(\upsilon^2+\pi^2)-5\mu\upsilon\pi\sum_{cyc}\mu^2-2\mu\upsilon\pi\sum_{cyc}\upsilon\pi\right).$$

According to the Muirhead's inequality in Lemma 2.15, we can get $\sum_{sym}\mu^4\upsilon^2\geq\sum_{sym}\mu^3\upsilon\pi$, $\sum_{sym}\mu^4\geq\sum_{sym}\mu^2\upsilon\pi$ and $\sum_{sym}\mu^2\upsilon^2\geq\sum_{sym}\mu^2\upsilon\pi$.

Moreover, a direct calculation gives

$$\sum_{cyc}\mu^6+2\sum_{cyc}\mu^5(\upsilon+\pi)+3\sum_{cyc}\mu^4(\upsilon^2+\pi^2)-4\mu\upsilon\pi\sum_{cyc}\mu^3-6\mu\upsilon\pi\sum_{cyc}\mu^2(\upsilon+\pi)$$
$$=\tfrac{1}{2}\sum_{sym}\mu^6+2\sum_{sym}\mu^5\upsilon+3\sum_{sym}\mu^4\upsilon^2-4\mu\upsilon\pi\sum_{cyc}\mu^3-6\mu\upsilon\pi\sum_{cyc}\mu^2(1-\mu)$$
$$=\tfrac{1}{2}\sum_{sym}\mu^6+2\sum_{sym}\mu^5\upsilon+3\sum_{sym}\mu^4\upsilon^2+2\mu\upsilon\pi\sum_{cyc}\mu^3-6\mu\upsilon\pi\sum_{cyc}\mu^2$$
$$=\tfrac{1}{2}\sum_{sym}\mu^6+2\sum_{sym}\mu^5\upsilon+3\sum_{sym}\mu^4\upsilon^2+\sum_{sym}\mu^4\upsilon\pi-3\sum_{sym}\mu^3\upsilon\pi$$
$$=\tfrac{1}{2}\sum_{sym}\mu^6+2\sum_{sym}\mu^5\upsilon+\sum_{sym}\mu^4\upsilon\pi+3\left(\sum_{sym}\mu^4\upsilon^2-\sum_{sym}\mu^3\upsilon\pi\right)$$
$$\geq 0$$

and

$$\sum_{cyc}\mu^4+2\sum_{cyc}\upsilon^2\pi^2-3\mu\upsilon\pi$$
$$=\sum_{cyc}\mu^4+2\sum_{cyc}\mu^2\upsilon^2-3\mu\upsilon\pi\sum_{cyc}\mu$$
$$=\sum_{cyc}\mu^4+2\sum_{cyc}\mu^2\upsilon^2-3\sum_{cyc}\mu^2\upsilon\pi$$
$$=\tfrac{1}{2}\left(\sum_{sym}\mu^4+2\sum_{sym}\mu^2\upsilon^2-3\sum_{sym}\mu^2\upsilon\pi\right)$$
$$=\frac{1}{2}\left(\left(\sum_{sym}\mu^4-\sum_{sym}\mu^2\upsilon\pi\right)+2\left(\sum_{sym}\mu^2\upsilon^2-\sum_{sym}\mu^2\upsilon\pi\right)\right)$$
$$\geq 0.$$

In addition, using the Mean inequality (AM-GM) in Lemma 2.2, it suffices to prove that

$$\left(\sum_{cyc}\mu^6+2\sum_{cyc}\mu^5(\upsilon+\pi)+3\sum_{cyc}\mu^4(\upsilon^2+\pi^2)-4\mu\upsilon\pi\sum_{cyc}\mu^3-6\mu\upsilon\pi\sum_{cyc}\mu^2(\upsilon+\pi)\right)\cdot\left(\sum_{cyc}\mu^4+2\sum_{cyc}\upsilon^2\pi^2-3\mu\upsilon\pi\right)\geq\left(\sum_{cyc}\mu^5+\sum_{cyc}\mu^4(\upsilon+\pi)+2\sum_{cyc}\mu^3(\upsilon^2+\pi^2)-5\mu\upsilon\pi\sum_{cyc}\mu^2-2\mu\upsilon\pi\sum_{cyc}\upsilon\pi\right)^2.$$

Let $\mathbf{G} = \Big(\sum_{cyc}\mu^6 + 2\sum_{cyc}\mu^5(\upsilon+\pi)\ +3\sum_{cyc}\mu^4(\upsilon^2+\pi^2) - 4\mu\upsilon\pi\sum_{cyc}\mu^3 - 6\mu\upsilon\pi\sum_{cyc}\mu^2(\upsilon+\pi)\Big)\cdot\ \big(\sum_{cyc}\mu^4 + 2\sum_{cyc}\upsilon^2\pi^2 - 3\mu\upsilon\pi\big) - \Big(\sum_{cyc}\mu^5 + \sum_{cyc}\mu^4(\upsilon+\pi) + 2\sum_{cyc}\mu^3(\upsilon^2 +\pi^2) - 5\mu\upsilon\pi\sum_{cyc}\mu^2 - 2\mu\upsilon\pi\sum_{cyc}\upsilon\pi\Big)^2$

$= \mu\upsilon\pi\cdot\Big(\sum_{cyc}\mu^6 - 2\sum_{cyc}\mu^5(\upsilon+\pi)\quad +3\sum_{cyc}\mu^4(\upsilon^2+\pi^2) - 4\sum_{cyc}\mu^3\upsilon^3 + 2\mu\upsilon\ \pi\sum_{cyc}\mu^2(\upsilon+\pi) - 9\mu^2\upsilon^2\pi^2\Big)$

$= \mu\upsilon\pi\cdot\mathcal{G},$ where $\mathcal{G} = \sum_{cyc}\mu^6 - 2\sum_{cyc}\mu^5(\upsilon+\pi) + 3\sum_{cyc}\mu^4(\upsilon^2+\pi^2) -4\sum_{cyc}\upsilon^3\pi^3 + 2\mu\upsilon\pi\sum_{cyc}\mu^2(\upsilon+\pi) - 9\mu^2\upsilon^2\pi^2.$

Without loss of generality, assume that $\mu\leq\upsilon\leq\pi$.

Let $\eta = \upsilon-\mu\geq 0$ and $\varphi = \pi-\upsilon\geq 0$, then

$\mathcal{G} = (\eta^2+\eta\varphi+\varphi^2)^2\mu^2 + 2\varphi^2(2\eta+\varphi)(\eta^2 +\eta\varphi+\varphi^2)\mu + \varphi^2(2\eta^2+2\eta\varphi+\varphi^2)^2 \geq 0.$

Consequently, this completes the proof of Theorem 5.19. □

Theorem 5.20 *For any IFV $\ddot{a} = (\mu,\upsilon,\pi)$ and satisfy $\mu,\upsilon,\pi > 0$. Then it holds that*

$$\sqrt[4]{\frac{27\sum_{cyc}\mu\upsilon}{4}} \leq \sum_{cyc}\frac{\mu}{\sqrt{\mu+\upsilon}} \leq 3\sqrt[4]{\frac{\sum_{cyc}\mu^2}{12}}. \tag{54}$$

Proof The Eq. (54) can be proved in two steps, as shown below.

(1) Left inequality: $\sqrt[4]{\frac{27\sum_{cyc}\mu\upsilon}{4}} \leq \sum_{cyc}\frac{\mu}{\sqrt{\mu+\upsilon}}$.

It follows from the Carlson's inequality in Lemma 2.9 that

$$\Big(\sum_{cyc}\frac{\mu}{\sqrt{\mu+\upsilon}}\Big)\Big(\sum_{cyc}\frac{\mu}{\sqrt{\mu+\upsilon}}\Big)\Big(\sum_{cyc}\mu(\mu+\upsilon)\Big) \geq \Big(\sum_{cyc}\mu\Big)^3 = 1$$

$$\Leftrightarrow \sum_{cyc}\frac{\mu}{\sqrt{\mu+\upsilon}} \geq \sqrt{\frac{1}{\sum_{cyc}\mu^2+\sum_{cyc}\mu\upsilon}}.$$

Further, it just has to prove that

$$\Big(\sqrt{\frac{1}{\sum_{cyc}\mu^2+\sum_{cyc}\mu\upsilon}}\Big)^4 \geq \frac{27\sum_{cyc}\mu\upsilon}{4}$$

$$\Leftrightarrow \left(\frac{1}{\Big(\sum_{cyc}\mu\Big)^2 - \sum_{cyc}\mu\upsilon}\right)^2 \geq \frac{27\sum_{cyc}\mu\upsilon}{4}$$

$$\Leftrightarrow \left(\frac{1}{1-\sum_{cyc}\mu\upsilon}\right)^2 \geq \frac{27\sum_{cyc}\mu\upsilon}{4}$$

$$\Leftrightarrow 2\sum_{cyc}\mu\upsilon\cdot\Big(1-\sum_{cyc}\mu\upsilon\Big)^2 \leq \frac{8}{27}.$$

According to the Mean inequality (AM-GM and 3 M) in Lemma 2.2, a direct calculation gives

$\sum_{cyc}\mu\upsilon \leq \frac{\big(\sum_{cyc}\mu\big)^2}{3} = \frac{1}{3}$ and

$$2\sum_{cyc}\mu\upsilon\cdot\Big(1-\sum_{cyc}\mu\upsilon\Big)^2 = 2\sum_{cyc}\mu\upsilon\cdot\Big(1-\sum_{cyc}\mu\upsilon\Big)\cdot\Big(1-\sum_{cyc}\mu\upsilon\Big) \leq \left(\frac{2\sum_{cyc}\mu\upsilon + \Big(1-\sum_{cyc}\mu\upsilon\Big) + \Big(1-\sum_{cyc}\mu\upsilon\Big)}{3}\right)^3 = \Big(\frac{2}{3}\Big)^3 = \frac{8}{27}.$$

Hence, $\sqrt[4]{\frac{27\sum_{cyc}\mu\upsilon}{4}} \leq \sum_{cyc}\frac{\mu}{\sqrt{\mu+\upsilon}}$ is true.

(2) Right inequality: $\sum_{cyc}\frac{\mu}{\sqrt{\mu+\upsilon}} \leq 3\sqrt[4]{\frac{\sum_{cyc}\mu^2}{12}}$.

Using the explored Lemma 5.1, it can give $\sum_{cyc}\sqrt{\frac{\mu}{\mu+\upsilon}} \leq \frac{3\sqrt{2}}{2}$.

It follows from the Cauchy's inequality in Lemma 2.6 that

$$\sum_{cyc}\frac{\mu}{\sqrt{\mu+\upsilon}} = \sum_{cyc}\Big(\sqrt[4]{\frac{\mu}{\mu+\upsilon}}\cdot\sqrt[4]{\frac{\mu^3}{\mu+\upsilon}}\Big) \leq \sqrt{\Big(\sum_{cyc}\sqrt{\frac{\mu}{\mu+\upsilon}}\Big)\cdot\Big(\sum_{cyc}\sqrt{\frac{\mu^3}{\mu+\upsilon}}\Big)} \leq \sqrt{\frac{3\sqrt{2}}{2}\sum_{cyc}\sqrt{\frac{\mu^3}{\mu+\upsilon}}}.$$

Further,it has to prove that

$$3\sqrt[4]{\frac{\sum_{cyc}\mu^2}{12}} \geq \sqrt{\frac{3\sqrt{2}}{2}\sum_{cyc}\sqrt{\frac{\mu^3}{\mu+\upsilon}}} \Leftrightarrow \frac{3}{2}\sum_{cyc}\mu^2 \geq \Big(\sum_{cyc}\sqrt{\frac{\mu^3}{\mu+\upsilon}}\Big)^2.$$

Meanwhile, a direct calculation shows

$$\Big(\sum_{cyc}\sqrt{\frac{\mu^3}{\mu+\upsilon}}\Big)^2 = \Big(\sum_{cyc}\sqrt{\frac{\mu^3(\upsilon+\pi)(\pi+\mu)}{(\mu+\upsilon)(\upsilon+\pi)(\pi+\mu)}}\Big)^2 = \Big(\sum_{cyc}\sqrt{\frac{\mu^3(\upsilon+\pi)(\pi+\mu)}{\prod_{cyc}(\mu+\upsilon)}}\Big)^2 = \left(\frac{\sum_{cyc}\sqrt{\mu^3(\upsilon+\pi)(\pi+\mu)}}{\sqrt{\prod_{cyc}(\mu+\upsilon)}}\right)^2 = \frac{\Big(\sum_{cyc}\sqrt{\mu^3(\upsilon+\pi)(\pi+\mu)}\Big)^2}{\prod_{cyc}(\mu+\upsilon)} = \frac{\Big(\sum_{cyc}\big(\mu\sqrt{\upsilon+\pi}\big)\big(\sqrt{\mu(\pi+\mu)}\big)\Big)^2}{\prod_{cyc}(\mu+\upsilon)}$$

$$\leq \frac{\left(\sum_{cyc} \mu^2(\upsilon+\pi)\right)\left(\sum_{cyc} \mu(\pi+\mu)\right)}{\prod_{cyc}(\mu+\upsilon)}$$ (Use Cauchy's inequality: Lemma 2.6)

$$= \frac{\left(\sum_{cyc} \mu \cdot \sum_{cyc} \mu\upsilon - 3\prod_{cyc} \mu\right)\left(\left(\sum_{cyc} \mu\right)^2 - \sum_{cyc} \mu\upsilon\right)}{\sum_{cyc} \mu \cdot \sum_{cyc} \mu\upsilon - \prod_{cyc} \mu}$$

$$= \frac{\left(\sum_{cyc} \mu\upsilon - 3\prod_{cyc} \mu\right)\left(1 - \sum_{cyc} \mu\upsilon\right)}{\sum_{cyc} \mu\upsilon - \prod_{cyc} \mu}.$$

Later, it just has to prove that

$$\frac{\left(\sum_{cyc} \mu\upsilon - 3\prod_{cyc} \mu\right)\left(1 - \sum_{cyc} \mu\upsilon\right)}{\sum_{cyc} \mu\upsilon - \prod_{cyc} \mu} \leq \frac{3}{2}$$

$$\sum_{cyc} \mu^2 = \frac{3}{2}\left(\left(\sum_{cyc} \mu\right)^2 - 2\sum_{cyc} \mu\upsilon\right) = \frac{3}{2}\left(1 - 2\sum_{cyc} \mu\upsilon\right)$$

$$\Leftrightarrow \sum_{cyc} \mu\upsilon + 3\prod_{cyc} \mu \geq 4\left(\sum_{cyc} \mu\upsilon\right)^2$$

$$\Leftrightarrow \left(\sum_{cyc} \mu\right)^2 \sum_{cyc} \mu\upsilon + 3\left(\sum_{cyc} \mu\right)\prod_{cyc} \mu \geq 4\left(\sum_{cyc} \mu\upsilon\right)^2$$

$$\Leftrightarrow \left(\sum_{cyc} \mu^2\right)\sum_{cyc} \mu\upsilon + 2\left(\sum_{cyc} \mu\upsilon\right)^2 + 3\left(\sum_{cyc} \mu\right)\prod_{cyc} \mu \geq 4\left(\sum_{cyc} \mu\upsilon\right)^2$$

$$\Leftrightarrow \left(\sum_{cyc} \mu^2\right)\sum_{cyc} \mu\upsilon + 3\left(\sum_{cyc} \mu\right)\prod_{cyc} \mu \geq 2\left(\sum_{cyc} \mu\upsilon\right)^2$$

$$\Leftrightarrow \left(\sum_{cyc} \mu^2\right)\sum_{cyc} \mu\upsilon \geq 2\sum_{cyc} \mu^2\upsilon^2 + \prod_{cyc} \mu \cdot \sum_{cyc} \mu$$

$$\Leftrightarrow \sum_{cyc}(\mu^3\upsilon + \mu\upsilon^3) \geq 2\sum_{cyc} \mu^2\upsilon^2$$

$$\Leftrightarrow \sum_{cyc} \mu\upsilon(\mu-\upsilon)^2 \geq 0.$$

Hence, $\sum_{cyc} \frac{\mu}{\sqrt{\mu+\upsilon}} \leq 3\sqrt[4]{\frac{\sum_{cyc} \mu}{12}}$ is true.

Finally, the proof is fully completed. □

Theorem 5.21 *Let* $\ddot{a} = (\mu, \upsilon, \pi)$ *be an IFV. Then it holds that*

$$\sum_{cyc} \mu^\lambda \upsilon \leq \begin{cases} 1 - \frac{2}{3}\lambda, & \text{if } 0 \leq \lambda \leq 1; \\ \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}, & \text{if } \lambda \geq 2. \end{cases} \quad (55)$$

Proof (1) $0 \leq \lambda \leq 1$.

Employing the Bernoulli's inequality in Lemma 2.17, a direct calculation gives

$$\sum_{cyc} \mu^\lambda \upsilon = \sum_{cyc}(1 + \mu - 1)^\lambda \upsilon$$
$$\leq \sum_{cyc}(1 + \lambda(\mu - 1))\upsilon$$
$$= (1-\lambda)\sum_{cyc} \mu + \lambda\sum_{cyc} \mu\upsilon$$
$$= 1 - \lambda + \lambda\sum_{cyc} \mu\upsilon$$
$$\leq 1 - \lambda + \frac{\left(\sum_{cyc} \mu\right)^2}{3}\lambda$$ (Use Mean inequality (3 M): Lemma 2.2)
$$= 1 - \frac{2}{3}\lambda.$$

(2) $\lambda \geq 2$.

Without loss of generality, assume that $\mu \geq \upsilon \geq \pi$.

It follows from the Bernoulli's inequality in Lemma 2.17 that

$$\left(1 + \frac{\pi}{\mu}\right)^\lambda \geq 1 + \frac{\lambda\pi}{\mu} \geq 1 + \frac{2\pi}{\mu}.$$

In addition, a direct calculation shows

$$(\mu+\pi)^\lambda \upsilon = \mu^\lambda \upsilon\left(1 + \frac{\pi}{\mu}\right)^\lambda$$
$$\geq \mu^\lambda \upsilon\left(1 + \frac{2\pi}{\mu}\right)$$
$$= \mu^\lambda \upsilon + \mu^{\lambda-1}\upsilon\pi + \mu^{\lambda-2}\mu\upsilon\pi$$
$$\geq \mu^\lambda \upsilon + \upsilon^{\lambda-1}\upsilon\pi + \pi^{\lambda-2}\upsilon\upsilon\pi$$
$$= \mu^\lambda \upsilon + \upsilon^\lambda \pi + \pi^\lambda \mu$$
$$= \sum_{cyc} \mu^\lambda \upsilon.$$

It is known from the Mean inequality (AM-GM) in Lemma 2.2 that

$$(\mu+\pi)^\lambda \upsilon$$
$$= \lambda^\lambda \left(\frac{\mu+\pi}{\lambda}\right)^\lambda \upsilon$$
$$= \lambda^\lambda \upsilon \cdot \underbrace{\frac{\mu+\pi}{\lambda} \cdot \frac{\mu+\pi}{\lambda} \cdot \cdots \cdot \frac{\mu+\pi}{\lambda}}_{\lambda}$$

$$\leq \lambda^{\lambda}\left(\frac{\upsilon+\overbrace{\frac{\mu+\pi}{\lambda}+\frac{\mu+\pi}{\lambda}+\cdots+\frac{\mu+\pi}{\lambda}}^{\lambda}}{\lambda+1}\right)^{\lambda+1}$$

$$=\lambda^{\lambda}\left(\frac{\upsilon+\mu+\pi}{\lambda+1}\right)^{\lambda+1}$$

$$=\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}.$$

Finally, the proof is fully completed. □

Remark 5.4 From the above nine intuitionistic fuzzy inequalities proved by combined well-known inequalities, it can easily derive a conclusion that the most indispensable inequality is Mean inequality, especially AM-GM. That is to say, most known or unknown inequalities can be fully or partly proved by Mean inequality (AM-GM, AM-SM, 3 M). Obviously, relying solely on a well-known inequality to solve a certain inequality problem will become extremely difficult, or even completely infeasible. How to use existing well-known inequalities in a flexible combination is the most critical part of the proof. In addition, as the information with equality $\mu+\upsilon+\pi=1$ embedded in the proof of inequality, how to add or reduce at the right time will have an unexpected and wonderful effect at the critical moment. Since the inequalities proposed based on the equality under the intuitionistic fuzzy environment are just a preliminary discussion, how to apply it to the proof of existing or developed aggregation operators (Sect. 4) and operations (Sect. 3) will be a practical discussion.

6 Conclusion

Inequality is a significant part of basic theory under intuitionistic fuzzy environment. It is usually overlooked, but it is essential for operations and AOs on IFSs/IFVs. In this paper, some inequalities are developed for IFSs based on some existing operations. Meanwhile, three unweighted intuitionistic fuzzy AOs (UIFS, UIFA and UIFG) derived by some existing operations ($\oplus, \otimes$ and @) are developed for aggregating preference information, whose their corresponding inequality relations based on UIFS are deeply explored and proved by some famous inequalities. In addition, an important inequality of UIFS, UIFA and UIFG is constructed, which reveals the nature of AOs to compare sizes. Based on the equality $\mu+\upsilon+\pi=1$, a series of inequalities on IFV are constructed and proved by some famous inequalities, which will become a new foundation of intuitionistic fuzzy inequalities in operations and AOs.

It is important to note that the developed intuitionistic fuzzy inequalities based on equality are not yet correlated effectively. In other words, each of them basically exists as an independent individual, but they may become a quick point for the next proof of inequality like a lemma or famous inequalities. In the future, we will employ the developed intuitionistic fuzzy inequalities in proving more complicated inequalities under intuitionistic fuzzy environment, and apply them in real decision-making environment to deal with real decision requirements.

7 List of Abbreviations

The list of abbreviations is shown in Table 2.

Table 2 List of abbreviations

Full name	Abbreviations
Intuitionistic Fuzzy Set	IFS
Membership Degree	MD
Nonmembership Degree	NMD
Fuzzy Set	FS
Aggregation Operator	AO
Intuitionistic fuzzy weighted averaging	IFWA
Intuitionistic fuzzy value	IFV
Intuitionistic fuzzy dombi weighted averaging	IFDWA
Intuitionistic fuzzy dombi weighted geometric	IFDWG
Unweighted intuitionistic fuzzy square	UIFS
Unweighted Intuitionistic Fuzzy Arithmetic	UIFA
Unweighted Intuitionistic Fuzzy Geometric	UIFG
Harmonic Mean	HM
Geometric Mean	GM
Arithmetic Mean	AM
Square Mean	SM

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Declarations

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