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RESEARCH ARTICLE

Investigation of the Main Energies of Picture Fuzzy Graph and its Applications

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Abstract
Picture fuzzy graph, belonging to fuzzy graphs family, has good capabilities at times when we are faced with problems that cannot be expressed by fuzzy graphs and intuitionistic fuzzy graphs. When an element membership is not clear, neutrality is a good option that can be well-supported by a picture fuzzy graph. The previous definitions limitations in fuzzy graph energy have led us to offer new definitions in picture fuzzy graphs. In this article, we expanded the energy concept on the picture fuzzy graph and sought to use this concept in modeling issues related to this graph and solving some problems including the neutrality state. We were able to show that neutrality, as part of total energy, is effective in energy-based decisions. This is noticeable in some types of energy and is more pronounced. We were looking for a way to rank the available options using the picture fuzzy graph and its Laplacian energy/energy in decision making. We studied some types of energy including Laplacian and skew Laplacian in both picture fuzzy graphs and picture fuzzy digraphs, and discussed some of its properties. We discussed some energy boundaries in this graph, and finally, the applications of energy were presented.

Keywords Picture fuzzy graph · Laplacian energy · Picture fuzzy digraph · Skew Laplacian energy

Abbreviations

Notation	Meaning
FS	Fuzzy set
FG	Fuzzy graph
PFS	Picture fuzzy set
PFG	Picture fuzzy graph
PFDG	Picture fuzzy digraph
AM	Adjacency matrix
LM	Laplacian matrix
SAM	Skew adjacency matrix
LE	Laplacian energy
SLE	Skew Laplacian energy

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1 Introduction

Graphs have long been used to describe objects and the relationships between them. Many of the issues and phenomena around us are associated with complexities and ambiguities that make it difficult to express certainty. These difficulties were alleviated by the introduction of fuzzy sets by Zadeh [48]. This concept established well-grounded allocation membership degree to elements of a set. The existence of a single degree for a membership could not resolve the ambiguity on uncertain issues, so the need for a degree of membership was felt. Afterward, to overcome the existing ambiguities, Atanassov [7] introduced non-membership degrees and defined an intuitionistic fuzzy set as the sum of degrees not greater than 1. This set is used in image processing [11], robotic system [17], decision making [23], medical diagnosis [43] and etc. Fuzzy graphs were presented in different types, and researchers conducted many studies about their properties and characteristics [3132, 34, 35, 37, 38, 40, 44–46]. Liu et al. [24] presented t-spherical fuzzy 2-tuple linguistic muirhead mean aggregation operators. Naz et al. [26] explained decision analysis under the hesitant dual fuzzy environment q-rung orthopair. An extension of social network group decision-making based on trust rank and personas was proposed by Cai et al. [10]. Pal et al. [25] investigated new concepts in neutrosophic graphs. Certain properties of single-valued neutrosophic graph, presented by Zeng et al. [49]. Rao et al. [33], studied intuitionistic fuzzy tree.

Intuitionistic fuzzy graphs have only two modes of membership and non-membership, but in some issues such as voting, medical diagnosis, and etc., we also face a state of neutrality. When it comes to the climate impact on the human environment in different climatic conditions, ineffectiveness, in addition to being effective or not, can be proposed. To resolve this case, Cuong and Kreinevich [14] presented the concept of picture fuzzy set (PFS) as a combination of the fuzzy set (FS) and intuitionistic fuzzy set (IFS). Any element in this set is inclusive of three degrees of being a member, neutral, and non-member, where the sum of degrees is not greater than 1. Some PFS properties were studied by Cuong [13]. Phong et al. [30] investigated some picture fuzzy relations combinations. A fuzzy inference system on PFSs was proposed by Son et al. [42]. Akram et al. [3] examined the q-Rung PFG. Zuo et al. [51] gave some operations on PFG, i.e., union, joint, and Cartesian products. Meanwhile, Xiao et al. [47] studied regular PFGs and Garg [19] studied some picture fuzzy aggregation operations. Khan et al. [22] introduced bipolar PFGs. Amanathulla et al. [5] initiated the concept of balanced PFGs. An approach to decision-making via picture fuzzy soft graphs was introduced by Chellamani et al. [12]. Certain operations on picture fuzzy graph were studied by Shoaib et al. [41]. Picture fuzzy incidence graphs were introduced by Nazeer and Rashid [28].

On the advent of graph theory over algebraic graph theory, algebraic methods were used to study graphs. The main branch of algebraic graph theory is the spectral graph theory, which studies the characteristics of polynomial properties, eigenvalues, and eigenvectors related to graph matrices. Conceptual energy is related to the spectrum of a graph that plays an important role in recognizing patterns, modeling virus spread in computer networks, and securing personal data in databases. The energy of a graph is a useful tool in deciding which option to choose. When searching for the best option in decisions, the application of energy, because of the expression of the strength of the graph obtained from the selected vertices, is a great help in determining the desired option. This was a good incentive for researchers to study graph energies. Gutman [20] introduced the concept of a graph energy. He found this concept in the study of the electrons energy of specific molecules. In fact, the energy of a given molecular graph compared to the total energy of the electrons of a molecule is of interest to chemists. Obviously, the energy of a graph with all isolated vertices is zero, but for a complete graph with n vertices, it is $2(n-1)$. The energy of a graph is used for entropy [16], properties of proteins [18], Alzheimer's disease [15], etc. The Laplacian energy (LE) of a graph is defined by Gutman and Zhou [21]. This was obtained from the sum of the absolute values of the differences of the mean degree vertices of a graph with its Laplacian eigenvalues. The LE is applied in image analysis [50], brain activity [9], etc. Pena and Rada analyzed the digraph energy [29]. The digraph skew energy was introduced by Adiga and Balakrishnan [2]. The skew Laplacian energy (SLE) of a digraph was defined by Adiga and Smitha [1]. Other types of energy were later introduced by researchers. The energy of pythagorean fuzzy graphs was proposed by Akram and Naz [4]. Certain notions of energy in single-valued neutrosophic graphs were introduced by Naz et al. [27].

The fuzzy graph (FG) energy was introduced by Anjali and Mathew [6]. The LE of an FG was defined by Sharbaf and Fayazi [39]. Basha and Kartheek [8] generalized the LE concept of an FG to the Laplacian energy (LE) of an IFG. Simultaneously, with the variety of FGs, different energies of fuzzy graphs were introduced. The limitation of previous definitions of energy in FGs and IFGs led us to define energy in a PFG. Since PFG is a good tool in fuzzy modeling of uncertain problems, for its degree of neutrality, it is a good incentive to examine the energy in it, especially in decisions based on the degree of neutrality. This shows that neutrality, as part of total energy, is effective in energy-based decisions.

In this research, with the aim of developing energy on a PFG and examining its properties, we sought to use the energy applications of this graph to solve real problems. We studied the PFG energy, PFG Laplacian energy and picture fuzzy digraph (PFDG) Laplacian energy. We also introduced the PFDG skew Laplacian energy. We compared three types of energy in varying degrees with examples. In addition, considering the decision making, a method was proposed to rank the available options using the PFG and its Laplacian energy/energy. Some of the energy boundaries were examined, and finally, the applications of energy were presented.

2 Preliminaries

In this section, we have an overview of the concepts we need in this article.

A graph $G = (V, E)$ is a mathematical model consisting of a set of vertices V and a set of edges E, where each is an unordered pair of distinct vertices. If G is a graph with n vertices and m edges, Its adjacency matrix (AM) M is the $n \times n$ matrix whose ij-th entry is the number of edges joining vertices i and j. The eigenvalues λ_i, $i = 1, 2, \dots, n$, of the AM of G are the eigenvalues of G. The spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of the AM of

G is the spec(G). The eigenvalues of a graph satisfy the following relations:

$$\sum_{i=1}^{n} \lambda_i = 0, \quad \sum_{i=1}^{n} \lambda_i^2 = 2m.$$

The energy of a graph G, denoted by $E(G)$, is defined as the sum of the absolute values of the eigenvalues of G, i.e., $E(G) = \sum_{i=1}^{n} \mid \lambda_i \mid$. The energy of a digraph D, denoted by $E(D)$, is defined as the sum of the absolute values of the real part of eigenvalues of D, i.e., $E(D) = \sum_{i=1}^{n} \mid Re(z_i) \mid$. A graph with all isolated vertices K_n^c has zero energy while the complete graph K_n with n vertices has $2(n-1)$ energy. The LE of a graph G with n vertex and m edge is $\text{LE}(G) = \sum_{i=1}^{n} \mid \lambda_i - \frac{2m}{n} \mid$ where λ_i relates to the eigenvalues of LM of G. The skew energy of G is some of the absolute values [8] of eigenvalues. The skew adjacency matrix (SAM) of the digraph G is a matrix so that the (i, j)-entry of the SAM is $+1$ if an edge is directed from the i-th vertex to the j-th vertex, vice versa the (j, i)-entry is -1. If there is no directed edge between the nodes i and j, then, the respective matrix element is zero.

The SLE of the digraph G is defined as:

$$\text{SLE}(G) = \sum_{i=1}^{n} \lambda_i^2,$$

where λ_i refers to the eigenvalues of the LM of the digraph G.

Definition 1 [48] An FS ν on a set V is defined through its membership function $\nu : V \to [0, 1]$, where $\nu(x)$ represents the degree to which point $x \in V$ belongs to the FS. The smallest and largest elements are the functions constantly equal to 0 and 1, respectively.

A fuzzy relation on a set V is a mapping $\eta : V \times V \to [0, 1]$ so that $\eta(x, y) \leq \min\{\nu(x), \nu(y)\}$ for all $x, y \in V$. A fuzzy relation η is symmetric if $\eta(x, y) = \eta(y, x)$, for all $x, y \in V$.

Definition 2 [36] An FG $Z = (V, \nu, \eta)$ is a non-empty set V together with a pair of functions $\nu : V \to [0, 1]$ and $\eta : V \times V \to [0, 1]$ so that $\eta(xy) \leq \min\{\nu(x), \nu(y)\}$, for all $x, y \in V$. Here η is a symmetric fuzzy relation on $V \times V$.

Definition 3 [14] A PFS A on X is specified as follow

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\},$$

so that

$$\mu_A, \eta_A, \nu_A : X \to [0, 1],$$

and

$$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1,$$

where $\mu_A(x)$, $\eta_A(x)$ and $\nu_A(x)$ are called the degrees of positive, neutral, and negative membership of x in A.

Definition 4 [51] A picture fuzzy relation B is a PFS of $X \times Y$ as shown by

$$B = \{(xy, \mu_A(xy), \eta_A(xy), \nu_A(xy)) \mid xy \in X \times Y\},$$

so that

$$\mu_B, \eta_B, \nu_B : X \times Y \to [0, 1],$$

and

$$0 \leq \mu_B(xy) + \eta_B(xy) + \nu_B(xy) \leq 1 \quad xy \in X \times Y.$$

Definition 5 [51] A pair $G = (A, B)$ is called a PFG on $G^* = (V, \mathcal{E})$ where A is a PFS on V and B is a picture fuzzy relation on $\mathcal{E} \subseteq V \times V$ so that for each $xy \in \mathcal{E}$

$$\mu_B(xy) \leq \mu_A(x) \wedge \mu_A(y),$$
$$\eta_B(xy) \leq \eta_A(x) \wedge \eta_A(y),$$
$$\nu_B(xy) \geq \nu_A(x) \vee \nu_A(y).$$

Definition 6 [51] The degree of a vertex x on PFG $G = (A, B)$ is specified as follow:

$$d(x) = (d_\mu(x), d_\eta(x), d_\nu(x)),$$

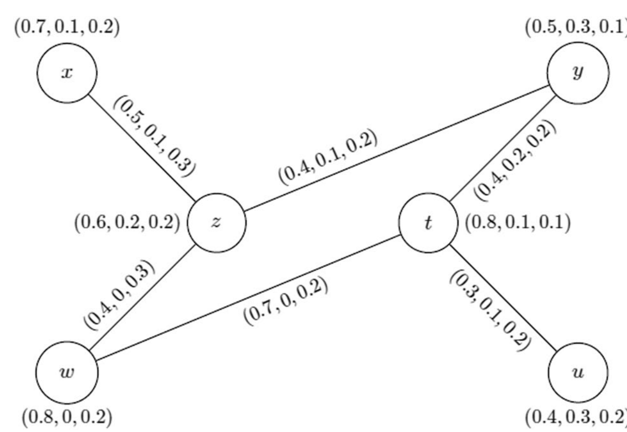


Fig. 1 A PFG

where for every $xy \in \mathcal{E}$

$$d_\mu(x) = \sum_{x \neq y} \mu_B(xy), \quad d_\eta(x) = \sum_{x \neq y} \eta_B(xy), \quad d_\nu(x) = \sum_{x \neq y} \nu_B(xy).$$

Example 1 Consider a PFG as shown in Fig. 1. The vertex degree z is (1, 3, 0.2, 0.8) while the vertex degree y is (0.8, 0.3, 0.4) .

Some notations are listed in the table of abbreviations.

3 Energy of Picture Fuzzy Graph

In this section, some of the most important PFG energies will be studied and their properties will be examined.

Definition 7 The AM $M(G)$ of a PFG $G = (A, B)$ is specified as a square matrix $M(G) = [m_{ij}], m_{ij} = (\mu_B(x_ix_j), \eta_B(x_ix_j), \nu_B(x_ix_j))$, and this can be written as three matrices $M(\mu_B(x_ix_j))$, $M(\eta_B(x_ix_j))$, and $M(\nu_B(x_ix_j))$, thus,

$$M(G) = (M(\mu_B(x_ix_j)), M(\eta_B(x_ix_j)), M(\nu_B(x_ix_j))).$$

Definition 8 The energy of a PFG $G = (A, B)$ is specified as the follow:

$$E(G) = (E(\mu_B(x_ix_j)), E(\eta_B(x_ix_j)), E(\nu_B(x_ix_j))),$$

in other words,

$$E(G) = \left(\sum_{i=1}^{n} \mid \alpha_i \mid, \sum_{i=1}^{n} \mid \beta_i \mid, \sum_{i=1}^{n} \mid \gamma_i \mid \right),$$

where α_i, β_i, and γ_i are eigenvalues of $M(\mu_B(x_ix_j))$, $M(\eta_B(x_ix_j))$, and $M(\nu_B(x_ix_j))$, respectively.

Example 2 Consider a PFG $G = (A, B)$ given in Fig. 2. The AM M is as follows:

$$M(G) = \begin{bmatrix} (0,0,0) & (0.2,0.2,0.5) & (0,0,0) & (0.2,0.4,0.3) & (0.3,0.2,0.3) \\ (0.2,0.2,0.5) & (0,0,0) & (0.3,0.2,0.4) & (0.3,0.2,0.4) & (0,0,0) \\ (0,0,0) & (0.3,0.2,0.4) & (0,0,0) & (0.2,0.3,0.3) & (0,0,0) \\ (0.2,0.4,0.3) & (0.3,0.2,0.4) & (0.2,0.3,0.3) & (0,0,0) & (0.2,0.3,0.4) \\ (0.3,0.2,0.3) & (0,0,0) & (0,0,0) & (0.2,0.3,0.4) & (0,0,0) \end{bmatrix}.$$

The AMs and eigenvalues of each degree of G are obtained as follows:

$$M(\mu_B(x_ix_j)) = \begin{bmatrix} 0 & 0.2 & 0 & 0.2 & 0.3 \\ 0.2 & 0 & 0.3 & 0.3 & 0 \\ 0 & 0.3 & 0 & 0.2 & 0 \\ 0.2 & 0.3 & 0.2 & 0 & 0.2 \\ 0.3 & 0 & 0 & 0.3 & 0 \end{bmatrix}.$$

$$\mathrm{Spec}(\mu_B(x_ix_j)) = \{-0.434, -0.281, -0.211, 0.218, 0.708\},$$

$$M(\eta_B(x_ix_j)) = \begin{bmatrix} 0 & 0.2 & 0 & 0.4 & 0.2 \\ 0.2 & 0 & 0.2 & 0.2 & 0 \\ 0 & 02 & 0 & 0.3 & 0 \\ 0.4 & 0.2 & 0.3 & 0 & 0.3 \\ 0.2 & 0 & 0 & 0.3 & 0 \end{bmatrix}.$$

$$\mathrm{Spec}(\eta_B(x_ix_j)) = \{-0.527, -0.287, -0.096, 0.127, 0.784\},$$

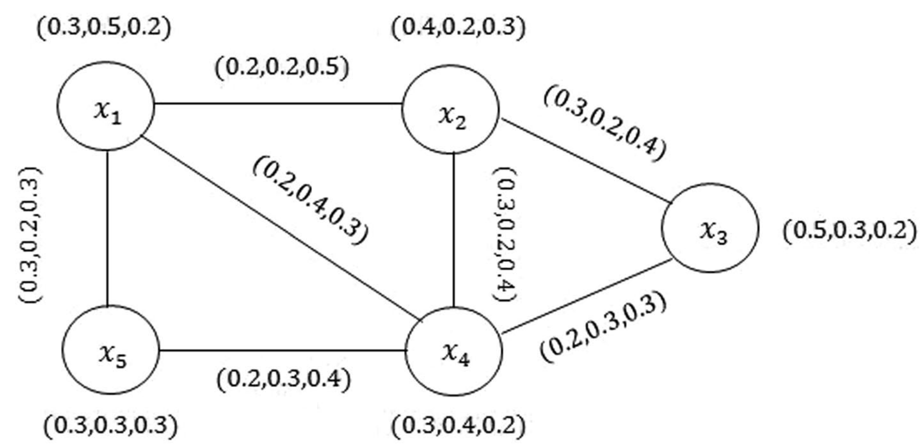
$$M(\nu_B(x_ix_j)) = \begin{bmatrix} 0 & 0.5 & 0 & 0.2 & 0.3 \\ 0.5 & 0 & 0.4 & 0.4 & 0 \\ 0 & 0.4 & 0 & 0.3 & 0 \\ 0.3 & 0.4 & 0.3 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0.4 & 0 \end{bmatrix}.$$

$$\mathrm{Spec}(\nu_B(x_ix_j)) = \{-0.716, -0.485, -0.056, 0.182, 1.075\}.$$

So, the energy of a PFG $G = (A, B)$ is equal to $E(G) = (1.852, 1.821, 2.514)$.

Theorem 1 *Let $G = (A, B)$ be a PFG and $M(G)$ be its AM. If $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, are the eigenvalues of $M(\mu_B(x_ix_j))$, $M(\eta_B(x_ix_j))$, and $M(\nu_B(x_ix_j))$, respectively, then,*

Fig. 2 PFG $G = (A, B)$



(i) $\sum_{i=1}^{n} \alpha_i = 0, \quad \sum_{i=1}^{n} \beta_i = 0, \quad \sum_{i=1}^{n} \gamma_i = 0$

(ii) $\sum_{i=1}^{n} \alpha_i^2 = 2 \sum_{1 \le i<j \le n} (\mu_B(x_i x_j))^2, \quad \sum_{i=1}^{n} \beta_i^2 = 2 \sum_{1 \le i<j \le n} (\eta_B(x_i x_j))^2,$
$\sum_{i=1}^{n} \gamma_i^2 = 2 \sum_{1 \le i<j \le n} (\nu_B(x_i x_j))^2.$

Proof (i) is held because $M(G)$ is a symmetric matrix with zero trace.

(ii) According to the trace properties of a matrix, we have

$$\operatorname{tr}((M(\mu_\beta(x_i x_j)))^2) = \sum_{i=1}^{n} \alpha_i^2.$$

On the other hand,

$$\begin{aligned}\operatorname{tr}((M(\mu_\beta(x_i x_j)))^2) &= \big(0 + (\mu_\beta(x_1 x_2))^2 + \cdots + (\mu_\beta(x_1 x_n))^2\big) \\ &\quad + \big((\mu_\beta(x_2 x_1))^2 + 0 + \cdots + (\mu_\beta(x_2 x_n))^2\big) \\ &\quad \vdots \\ &\quad + \big((\mu_\beta(x_n x_1))^2 + (\mu_\beta(x_n x_2))^2 + \cdots + 0\big) \\ &= 2 \sum_{1 \le i<j \le n} (\mu_\beta(x_i x_j))^2.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} \alpha_i^2 = 2 \sum_{1 \le i<j \le n} (\mu_\beta(x_i x_j))^2.$$

Similarly, $\sum_{i=1}^{n} \beta_i^2 = 2 \sum_{1 \le i<j \le n} (\eta_B(x_i x_j))^2$, and $\sum_{i=1}^{n} \gamma_i^2 = 2 \sum_{1 \le i<j \le n} (\nu_B(x_i x_j))^2$. □

Theorem 2 *If $G = (A, B)$ is a PFG on n vertices with the AM $M(G)$, then,*

$$\text{(i)} \sqrt{2 \sum_{1 \le i<j \le n} (\mu_B(x_i x_j))^2 + n(n-1) \mid \det(M(\mu_B(x_i x_j))) \mid^{\frac{2}{n}}} \le E(\mu_B(x_i x_j)) \le \sqrt{2n \sum_{1 \le i<j \le n} (\mu_B(x_i x_j))^2},$$

$$\text{(ii)} \sqrt{2 \sum_{1 \le i<j \le n} (\eta_B(x_i x_j))^2 + n(n-1) \mid \det(M(\eta_B(x_i x_j))) \mid^{\frac{2}{n}}} \le E(\eta_B(x_i x_j))$$

$$lesqrt2n \sum_{1 \le i<j \le n} (\eta_B(x_i x_j))^2,$$

$$\text{(iii)} \sqrt{2 \sum_{1 \le i<j \le n} (\nu_B(x_i x_j))^2 + n(n-1) \mid \det(M(\nu_B(x_i x_j))) \mid^{\frac{2}{n}}} \le E(\nu_B(x_i x_j)) \le \sqrt{2n \sum_{1 \le i<j \le n} (\nu_B(x_i x_j))^2}.$$

Proof According to the Cauchy–Schwarz inequality to the vectors $(1, 1, \ldots, 1)$ and $(|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|)$ with n entries, we have

$$\sum_{i=1}^{n} |\alpha_i| \leq \sqrt{n}\sqrt{\sum_{i=1}^{n} |\alpha_i|^2}, \tag{1}$$

$$\left(\sum_{i=1}^{n} \alpha_i\right)^2 = \sum_{i=1}^{n} |\alpha_i|^2 + 2\sum_{1\leq i<j\leq n} \alpha_i\alpha_j. \tag{2}$$

By comparing the coefficients of α^{n-2} in the characteristic

$$\begin{aligned}&\prod_{i=1}^{n}(\alpha-\alpha_i) = |M(G)-\alpha I|,\\ &\sum_{1\leq i<j\leq n} \alpha_i\alpha_j = -\sum_{1\leq i<j\leq n} (\mu_\beta(x_ix_j))^2.\end{aligned} \tag{3}$$

By placing (3) in (2), we have

$$\sum_{i=1}^{n} |\alpha_i|^2 = 2\sum_{1\leq i<j\leq n} (\mu_\beta(x_ix_j))^2. \tag{4}$$

By placing (4) in (1), we have

$$\sum_{i=1}^{n} |\alpha_i| \leq \sqrt{n}\sqrt{2\sum_{1\leq i<j\leq n} (\mu_\beta(x_ix_j))^2} = \sqrt{2n\sum_{1\leq i<j\leq n} (\mu_\beta(x_ix_j))^2}.$$

Hence,

$$E(\mu_\beta(x_i, x_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n} (\mu_\beta(x_ix_j))^2}.$$

For lower bound

$$\begin{aligned}\left(E(\mu_\beta(x_ix_j))\right)^2 &= \left(\sum_{i=1}^{n} |\alpha_i|\right)^2 = \sum_{i=1}^{n} |\alpha_i|^2 + 2\sum_{1\leq i<j\leq n} |\alpha_i\alpha_j|\\ &= 2\sum_{1\leq i<j\leq n} \left(\mu_\beta(x_ix_j)\right)^2 + \frac{2n(n-1)}{2}\mathrm{AM}\{|\alpha_i\alpha_j|\}.\end{aligned}$$

Since $AM\{|\alpha_i\alpha_j|\} \geq \mathrm{GM}\{|\alpha_i\alpha_j|\}, 1 \leq i < j \leq n$, therefore,

$$E(\mu_\beta(x_ix_j)) \geq \sqrt{2\sum_{1\leq i<j\leq n} \left(\mu_\beta(x_ix_j)\right)^2 + n(n-1)\mathrm{GM}\{|\alpha_i\alpha_j|\}}.$$

Also,

$$\begin{aligned}\mathrm{GM}\{|\alpha_i\alpha_j|\} &= \left(\prod_{1\leq i<j\leq n} |\alpha_i\alpha_j|\right)^{\frac{2}{n(n-1)}} = \left(\prod_{i=1}^{n} |\alpha_i|^{n-1}\right)^{\frac{2}{n(n-1)}}\\ &= \left(\prod_{i=1}^{n} |\alpha_i|\right)^{\frac{2}{n}} = |\det(M(\mu_\beta(x_ix_j)))|^{\frac{2}{n}}.\end{aligned}$$

Therefore,

$$E(\mu_\beta(x_ix_j)) \geq \sqrt{2\sum(\mu_\beta(x_ix_j))^2 + n(n-1)\,|\det(M(\mu_\beta(x_ix_j)))|^{\frac{2}{n}}}.$$

Hence,

$$\begin{aligned}&\sqrt{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2 + n(n-1)\,|\det(M(\mu_\beta(x_ix_j)))|^{\frac{2}{n}}}\\ &\leq E(\mu_\beta(x_ix_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2}.\end{aligned}$$

Similarly (ii) and (iii) are also established, of course, the upper bound is subjected to the following conditions $n \leq 2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2$, $n \leq 2\sum_{1\leq i<j\leq n}(\eta_\beta(x_ix_j))^2$, and $n \leq 2\sum_{1\leq i<j\leq n}(\gamma_\beta(x_ix_j))^2$. □

Theorem 3 *Let $G = (A, B)$ be a PFG on n vertices. If $n \leq 2\sum_{1\leq i<j\leq n}(\mu_B(x_ix_j))^2$, $n \leq 2\sum_{1\leq i<j\leq n}(\eta_B(x_ix_j))^2$, and $n \leq 2\sum_{1\leq i<j\leq n}(\nu_B(x_ix_j))^2$, then*

(i) $$E(\mu_B(x_ix_j)) \leq \frac{2\sum_{1\leq i<j\leq n}(\mu_B(x_ix_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{1\leq i<j\leq n}(\mu_B(x_ix_j))^2 - \left(\frac{2\sum_{1\leq i<j\leq n}(\mu_B(x_ix_j))^2}{n}\right)^2\right\}},$$

(ii) $$E(\eta_B(x_ix_j)) \leq \frac{2\sum_{1\leq i<j\leq n}(\eta_B(x_ix_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{1\leq i<j\leq n}(\eta_B(x_ix_j))^2 - \left(\frac{2\sum_{1\leq i<j\leq n}(\eta_B(x_ix_j))^2}{n}\right)^2\right\}},$$

(iii) $$E(\nu_B(x_ix_j)) \leq \frac{2\sum_{1\leq i<j\leq n}(\nu_B(x_ix_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{1\leq i<j\leq n}(\nu_B(x_ix_j))^2 - \left(\frac{2\sum_{1\leq i<j\leq n}(\nu_B(x_ix_j))^2}{n}\right)^2\right\}}.$$

Proof If $M = [m_{ij}]_{n\times n}$ is a symmetric matrix with zero trace, then, $\alpha_{\max} \geq \dfrac{2\sum_{1\leq i<j\leq n} m_{ij}}{n}$, where $\alpha_{\max}$ is the maximum eigenvalue of $M(G)$ and is the AM of a PFG G, then,

$\alpha_1 \geq \frac{2\sum_{1\leq i<j\leq n}\mu_\beta(x_ix_j)}{n}$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. On the other hand, since

$$\sum_{i=1}^{n}\alpha_i^2 = 2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2$$
$$\sum_{i=2}^{n}\alpha_i^2 = 2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2 - \alpha_1^2. \tag{5}$$

According to the Cauchy–Schwarz inequality to the vectors $(1, 1, \dots, 1)$ and $(|\alpha_1|, |\alpha_2|, \dots, |\alpha_n|)$ with $n-1$ entries, we have

$$E(\mu_\beta(x_ix_j)) - \alpha_1 = \sum_{i=2}^{n}|\alpha_i| \leq \sqrt{(n-1)\sum_{i=2}^{n}|\alpha_i|^2}. \tag{6}$$

By placing (5) in (6), we get

$$E(\mu_\beta(x_ix_j)) - \alpha_1 \leq \sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2 - \alpha_1^2\right)},$$
$$E(\mu_\beta(x_ix_j)) \leq \alpha_1 + \sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2 - \alpha_1^2\right)}. \tag{7}$$

Since the least value of the function $F(x) = x + \sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2 - x^2\right)}$ is in the interval $\left(\sqrt{\frac{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2}{n}}, \sqrt{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2}\right)$, $n \leq 2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2$, $1 \leq \frac{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2}{n}$. So,

$$\begin{aligned}\sqrt{\frac{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2}{n}} &\leq \frac{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2}{n}\\ &\leq \frac{2\sum_{1\leq i<j\leq n}\mu_\beta(x_ix_j)}{n}\\ &\leq \alpha_1 \leq \sqrt{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2}.\end{aligned}$$

Therefore, the inequality (7) holds

$$E(\mu_\beta(x_ix_j)) \leq \frac{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2 - \left(\frac{2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2}{n}\right)^2\right\}}.$$

Similarly, we can show (ii) and (iii). □

Theorem 4 *If $G = (A, B)$ is a PFG on n vertices, then $E(G) \leq \frac{n}{2}(1 + \sqrt{n})$.*

Proof Suppose that $G = (A, B)$ is a PFG on n nodes and $n \leq 2\sum_{1\leq i<j\leq n}(\mu_\beta(x_ix_j))^2 = 2z$. It can be easily shown that $f(z) = \frac{2z}{n} + \sqrt{(n-1)\left(2z - \left(\frac{2z}{n}\right)^2\right)}$ is maximized when $z = \frac{n^2 + n\sqrt{n}}{4}$. By placing this value of z in Theorem 3; we have

$$E(\mu_\beta(x_ix_j)) \leq \frac{n}{2}(1 + \sqrt{n}).$$

Similarly,

$$E(\eta_\beta(x_ix_j)) \leq \frac{n}{2}(1 + \sqrt{n}),$$
$$E(\nu_\beta(x_ix_j)) \leq \frac{n}{2}(1 + \sqrt{n}).$$

Therefore, $E(G) \leq \frac{n}{2}(1 + \sqrt{n})$. □

Definition 9 Let $G = (A, B)$ be a PFG on n nodes. The degree matrix $K(G) = [k_{ij}]$ of G is a $n \times n$ diagonal matrix which is defined as:

$$k_{ij} = \begin{cases} d_G(x_i) & i = j,\\ 0 & i \neq j.\end{cases}$$

Definition 10 The LM of a PFG $G = (A, B)$ is defined as $L(G) = K(G) - M(G)$, where $K(G)$ and $M(G)$ are the degrees matrix and AM of a PFG, respectively.

Definition 11 The LE of PFG $G = (A, B)$ is specified as the following:

$$\mathrm{LE}(G) = \big(\mathrm{LE}(\mu_B(x_ix_j)), \mathrm{LE}(\eta_B(x_ix_j)), \mathrm{LE}(\nu_B(x_ix_j))\big),$$
$$\mathrm{LE}(G) = \left(\sum_{i=1}^{n}|\varphi_i|, \sum_{i=1}^{n}|\psi_i|, \sum_{i=1}^{n}|\omega_i|\right),$$

where

$$\varphi_i = \varphi_i^* - \frac{2\sum_{1\leq i<j\leq n}\mu_B(x_ix_j)}{n},$$
$$\psi_i = \psi_i^* - \frac{2\sum_{1\leq i<j\leq n}\eta_B(x_ix_j)}{n},$$
$$\omega_i = \omega_i^* - \frac{2\sum_{1\leq i<j\leq n}\nu_B(x_ix_j)}{n}.$$

φ_i^*, ψ_i^* and ω_i^*, $i = 1, 2, \dots, n$ are the eigenvalues of $L(\mu_B(x_ix_j))$, $L(\eta_B(x_ix_j))$ and $L(\nu_B(x_ix_j))$, respectively.

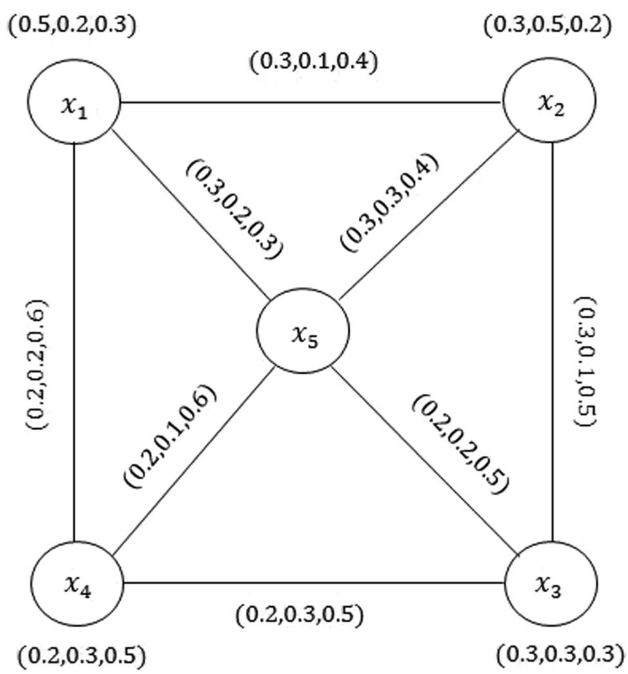


Fig. 3 PFG for LE

Example 3 Consider a PFG G given in Fig. 3.

The AM, degree matrix, and LM are as follows, respectively.

$$M(G)=\begin{bmatrix} (0,0,0) & (0.3,0.1,0.4) & (0,0,0) & (0.2,0.2,0.6) & (0.3,0.2,0.3) \\ (0.3,0.1,0.4) & (0,0,0) & (0.3,0.1,0.5) & (0,0,0) & (0.3,0.3,0.4) \\ (0,0,0) & (0.3,0.1,0.5) & (0,0,0) & (0.2,0.3,0.5) & (0.2,0.2,0.5) \\ (0.2,0.2,0.6) & (0,0,0) & (0.2,0.3,0.5) & (0,0,0) & (0.2,0.1,0.6) \\ (0.3,0.2,0.3) & (0.3,0.3,0.4) & (0.2,0.2,0.5) & (0.2,0.1,0.6) & (0,0,0) \end{bmatrix}.$$

$$K(G)=\begin{bmatrix} (1.2,0.5,0.9) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0.9,0.5,1.3) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0.7,0.6,1.5) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0.6,0.6,1.7) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (1,0.8,1.8) \end{bmatrix}.$$

$$L(G)=\begin{bmatrix} (1.2,0.5,0.9) & (-0.3,-0.1,-0.4) & (0,0,0) & (-0.2,-0.2,-0.6) & (-0.3,-0.2,-0.3) \\ (-0.3,-0.1,-0.4) & (0.9,0.5,1.3) & (-0.3,-0.1,-0.5) & (0,0,0) & (-0.3,-0.3,-0.4) \\ (0,0,0) & (-0.3,-0.1,-0.5) & (0,7,0,6,1.5) & (-0.2,-0.3,-0.5) & (-0.2,-0.2,-0.5) \\ (-0.2,-0.2,-0.6) & (0,0,0) & (-0.2,-0.3,-0.5) & (0.6,0.6,1.7) & (-0.2,-0.1,-0.6) \\ (-0.3,-0.2,-0.3) & (-0.3,-0.3,-0.4) & (-0.2,-0.2,-0.5) & (-0.2,-0.1,-0.6) & (1,0.8,1.8) \end{bmatrix}.$$

After computing we have $\mathrm{LE}(\mu_B(x_ix_j))=1.681$, $\mathrm{LE}(\eta_B(x_ix_j))=1.516$ and $\mathrm{LE}(\nu_B(x_ix_j))=3.806$.

So, $\mathrm{LE}(G)=(1.681, 1.516.3.806)$.

Theorem 5 *Let $G=(A,B)$ be a PFG and $L(G)$ be the LM of G. If $\varphi_1^*\geq\varphi_2^*\geq\cdots\geq\varphi_n^*$, $\psi_1^*\geq\psi_2^*\geq\cdots\geq\psi_n^*$ and $\omega_1^*\geq\omega_2^*\geq\cdots\geq\omega_n^*$ are the eigenvalues of $L(\mu_B(x_ix_j))$, $L(\eta_B(x_ix_j))$ and $L(\nu_B(x_ix_j))$, respectively, then,*

$$(i)\ \sum_{i=1}^{n}\varphi_i^*=2\sum_{1\leq i<j\leq n}\mu_B(x_ix_j),$$

$$\sum_{i=1}^{n}\psi_i^*=2\sum_{1\leq i<j\leq n}\eta_B(x_ix_j),$$

$$\sum_{i=1}^{n}\omega_i^*=2\sum_{1\leq i<j\leq n}\nu_B(x_ix_j).$$

$$\text{(ii)}\ \sum_{i=1}^{n}\varphi_i^{*2}=2\sum_{1\leq i<j\leq n}(\mu_B(x_ix_j))^2+\sum_{i=1}^{n}d_\mu^2(x_i),$$

$$\sum_{i=1}^{n}\psi_i^{*2}=2\sum_{1\leq i<j\leq n}(\eta_B(x_ix_j))^2+\sum_{i=1}^{n}d_\eta^2(x_i),$$

$$\sum_{i=1}^{n}\omega_i^{*2}=2\sum_{1\leq i<j\leq n}(\nu_B(x_ix_j))^2+\sum_{i=1}^{n}d_\nu^2(x_i).$$

Proof (i) Since $L(G)$ is a symmetric matrix with non-negative Laplacian eigenvalues, therefore,

$$\sum_{i=1}^{n}\varphi_i^*=tr(L(G))=\sum_{i=1}^{n}d_\mu(x_i)=2\sum_{1\leq i<j\leq n}\mu_\beta(x_ix_j).$$

Then, $\sum_{i=1}^{n}\varphi_i^*=2\sum_{1\leq i<j\leq n}\mu_\beta(x_ix_j)$. Similarly, $\sum_{i=1}^{n}\psi_i^*=2\sum_{1\leq i<j\leq n}\eta_\beta(x_ix_j)$ and $\sum_{i=1}^{n}\omega_i^*=2\sum_{1\leq i<j\leq n}\nu_\beta(x_ix_j)$.

(ii) According to the trace properties of a matrix, we have

$$\mathrm{tr}((L(\mu_\beta(x_ix_j)))^2)=\sum_{i=1}^{n}\varphi_i^*,$$

where

$$\mathrm{tr}((L(\mu_\beta(x_ix_j)))^2) = \left(d^2\mu(x_1) + \mu_\beta^2(x_1x_2) + \cdots + \mu_\beta^2(x_1x_n)\right) + \left(\mu_\beta^2(x_2x_1) + d^2\mu(x_2) + \cdots + \mu_\beta^2(x_2x_n)\right) + \cdots + \left(\mu_\beta^2(x_nx_1) + \mu_\beta^2(x_nx_2) + \cdots + d^2\mu(x_n)\right) = 2\sum_{1\le i<j\le n}(\mu_\beta(x_ix_j))^2 + \sum_{i=1}^{n} d_\mu^2(x_i).$$

Therefore,

$$\sum_{i=1}^{n}\varphi_i^{*2} = 2\sum_{1\le i<j\le n}(\mu_\beta(x_ix_j))^2 + \sum_{i=1}^{n} d_\mu^2(x_i).$$

Similarly, the other equation are fixed. □

Proposition 6 *Let $G=(A,B)$ be a PFG and $L(G)$ be the LM of G. If $\varphi_1^* \ge \varphi_2^* \ge \cdots \ge \varphi_n^*$, $\psi_1^* \ge \psi_2^* \ge \cdots \ge \psi_n^*$, and $\omega_1^* \ge \omega_2^* \ge \cdots \ge \omega_n^*$ are the eigenvalues of $L(\mu_B(x_ix_j))$, $L(\eta_B(x_ix_j))$, and $L(\nu_B(x_ix_j))$, respectively, and*

$$\varphi_i = \varphi_i^* - \frac{2\sum_{1\le i<j\le n}\mu_B(x_ix_j)}{n},$$

$$\psi_i = \psi_i^* - \frac{2\sum_{1\le i<j\le n}\mu_B(x_ix_j)}{n},$$

$$\omega_i = \omega_i^* - \frac{2\sum_{1\le i<j\le n}\mu_B(x_ix_j)}{n},$$

then,

(i) $\sum_{i=1}^{n}\varphi_i = 0, \quad \sum_{i=1}^{n}\psi_i = 0, \quad \sum_{i=1}^{n}\omega_i = 0,$

(ii) $\sum_{i=1}^{n}\varphi_i^2 = 2M_\mu, \quad \sum_{i=1}^{n}\psi_i^2 = 2M_\eta, \quad \sum_{i=1}^{n}\omega_i^2 = 2M_\nu,$

where

$$M_\mu = \sum_{1\le i<j\le n}\left(\mu_B(x_ix_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_\mu(x_i) - \frac{2\sum_{1\le i<j\le n}\mu_B(x_ix_j)}{n}\right)^2,$$

$$M_\eta = \sum_{1\le i<j\le n}\left(\eta_B(x_ix_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_\eta(x_i) - \frac{2\sum_{1\le i<j\le n}\eta_B(x_ix_j)}{n}\right)^2,$$

$$M_\nu = \sum_{1\le i<j\le n}\left(\nu_B(x_ix_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_\nu(x_i) - \frac{2\sum_{1\le i<j\le n}\nu_B(x_ix_j)}{n}\right)^2.$$

Theorem 7 *Let G be a PFG on n vertices and $L(G)$ be the LM of G, then*

(i) $\mathrm{LE}(\mu_B(x_ix_j))$

$$\le \sqrt{2n\sum_{1\le i<j\le n}\left(\mu_B(x_ix_j)\right)^2 + n\sum_{i=1}^{n}\left(d_\mu(x_i) - \frac{2\sum_{1\le i<j\le n}\mu_B(x_ix_j)}{n}\right)^2},$$

(ii) $\mathrm{LE}(\eta_B(x_ix_j))$

$$\le \sqrt{2n\sum_{1\le i<j\le n}\left(\eta_B(x_ix_j)\right)^2 + n\sum_{i=1}^{n}\left(d_\eta(x_i) - \frac{2\sum_{1\le i<j\le n}\eta_B(x_ix_j)}{n}\right)^2},$$

(iii) $\mathrm{LE}(\nu_B(x_ix_j))$

$$\le \sqrt{2n\sum_{1\le i<j\le n}\left(\nu_B(x_ix_j)\right)^2 + n\sum_{i=1}^{n}\left(d_\nu(x_i) - \frac{2\sum_{1\le i<j\le n}\nu_B(x_ix_j)}{n}\right)^2}.$$

Proof According to the Cauchy–Schwarz inequality to the n numbers $1, 1, \ldots, 1$ and $|\varphi_1|, |\varphi_2|, \ldots, |\varphi_n|$, we have

$$\sum_{i=1}^{n}|\varphi_i| \le \sqrt{n}\sqrt{\sum_{i=1}^{n}|\varphi_i|^2},$$

$$\mathrm{LE}(\mu_B(x_ix_j)) \le \sqrt{n}\sqrt{2M_\mu} = \sqrt{2nM_\mu},$$

because

$$M_\mu = 2n\sum_{1\le i<j\le n}\left(\mu_B(x_ix_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_\mu(x_i) - \frac{2\sum_{1\le i<j\le n}\mu_B(x_ix_j)}{n}\right)^2.$$

So, we have

$$\mathrm{LE}(\mu_B(x_ix_j)) \le \sqrt{2n\sum_{1\le i<j\le n}\left(\mu_B(x_ix_j)\right)^2 + n\sum_{i=1}^{n}\left(d_\mu(x_i) - \frac{2\sum_{1\le i<j\le n}\mu_B(x_ix_j)}{n}\right)^2}.$$

Similarly, we can show other inequalities. □

Theorem 8 *Let G be a PFG on n vertices and $L(G)$ be the LM of G. Then,*

(i) $\mathrm{LE}(\mu_B(x_ix_j))$

$$\ge 2\sqrt{\sum_{1\le i<j\le n}\left(\mu_B(x_ix_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_\mu(x_i) - \frac{2\sum_{1\le i<j\le n}\mu_B(x_ix_j)}{n}\right)^2},$$

(ii) $\mathrm{LE}(\eta_B(x_ix_j))$

$$\ge 2\sqrt{\sum_{1\le i<j\le n}\left(\eta_B(x_ix_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_\eta(x_i) - \frac{2\sum_{1\le i<j\le n}\eta_B(x_ix_j)}{n}\right)^2},$$

(iii) $\mathrm{LE}(\nu_B(x_ix_j))$

$$\ge 2\sqrt{\sum_{1\le i<j\le n}\left(\nu_B(x_ix_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_\nu(x_i) - \frac{2\sum_{1\le i<j\le n}\nu_B(x_ix_j)}{n}\right)^2}.$$

Proof According to the assumptions we have

$$\Big(\sum_{i=1}^{n} \mid \varphi_i \mid\Big)^2 = \sum_{i=1}^{n} \mid \varphi_i \mid^2 + 2\sum_{1\le i<j\le j} \mid \varphi_i\varphi_j \mid \ge 4M_\mu,$$

$$\mathrm{LE}(\mu_B(x_ix_j)) \ge 2\sqrt{M_\mu},$$

which will be demonstrated with situation (i). Similarly, we can prove another inequality. □

Theorem 9 *Let G be a PFG on n vertices and $L(G)$ be the LM of G. Then,*

$$\text{(i) } \mathrm{LE}(\mu_\beta(x_ix_j)) \le \mid \varphi_i \mid + \sqrt{(n-1)\left(2\sum_{1\le i<j\le n}(\mu_\beta(x_ix_j))^2 + \sum_{i=1}^{n}\left(d_\mu(x_i) - \frac{2\sum_{1\le i<j\le n}\mu_\beta(x_ix_j)}{n}\right)^2 - \varphi_i^2\right)},$$

$$\text{(ii) } \mathrm{LE}(\eta_\beta(x_ix_j)) \le \mid \psi_i \mid + \sqrt{(n-1)\left(2\sum_{1\le i<j\le n}(\eta_\beta(x_ix_j))^2 + \sum_{i=1}^{n}\left(d_\eta(x_i) - \frac{2\sum_{1\le i<j\le n}\eta_\beta(x_ix_j)}{n}\right)^2 - \psi_i^2\right)},$$

$$\text{(iii) } \mathrm{LE}(\nu_\beta(x_ix_j)) \le \mid \omega_i \mid + \sqrt{(n-1)\left(2\sum_{1\le i<j\le n}(\nu_\beta(x_ix_j))^2 + \sum_{i=1}^{n}\left(d_\nu(x_i) - \frac{2\sum_{1\le i<j\le n}\nu_\beta(x_ix_j)}{n}\right)^2 - \omega_i^2\right)}.$$

Proof Using the Caushy–Schwarz inequality, we get

$$\sum_{i=1}^{n} \mid \varphi_i \mid \le \sqrt{n\sum_{i=1}^{n} \mid \varphi_i \mid^2},$$

$$\sum_{i=2}^{n} \mid \varphi_i \mid \le \sqrt{(n-1)\sum_{i=2}^{n} \mid \varphi_i \mid^2},$$

$$\mathrm{LE}(\mu_\beta(x_ix_j)) - \mid \varphi_1 \mid \le \sqrt{(n-1)(2M_\mu - \varphi_1^2)},$$

$$\mathrm{LE}(\mu_\beta(x_ix_j)) \le \mid \varphi_1 \mid + \sqrt{(n-1)(2M_\mu - \varphi_1^2)}.$$

Since $M_\mu = \sum_{1\le i<j\le n}(\mu_\beta(x_ix_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_\mu(x_i) - \frac{2\sum_{1\le i<j\le n}\mu_\beta(x_ix_j)}{n}\right)^2$, so

$$\mathrm{LE}(\mu_\beta(x_ix_j)) \le \mid \varphi_1 \mid + \sqrt{(n-1)\left(2\sum_{1\le i<j\le n}(\mu_\beta(x_ix_j))^2 + \sum_{i=1}^{n}\left(d_\mu(x_i) - \frac{2\sum_{1\le i<j\le n}\mu_\beta(x_ix_j)}{n}\right)^2 - \varphi_1^2\right)}.$$

Similarly, other inequalities are proved. □

Remark All results from the energy of a PFG are confirmed for a PFDG while eigenvalues are complex numbers we spot real part therefrom. Herein, suffice it to say the following definition.

Definition 12 Let $D = (A, \vec{B})$ be a PFDG. The energy of D is defined as follow:

$$E(D) = \left(E(\mu_{\vec{B}}(x_ix_j)), E(\eta_{\vec{B}}(x_ix_j)), E(\nu_{\vec{B}}(x_ix_j))\right)$$

Here, we survey the LE of the PFDGs.

Definition 13 The out-degree matrix of a PFG is shown with symbol $K^+(D) = [k_{ij}]$ and it is a $n \times n$ diagonal matrix defined as

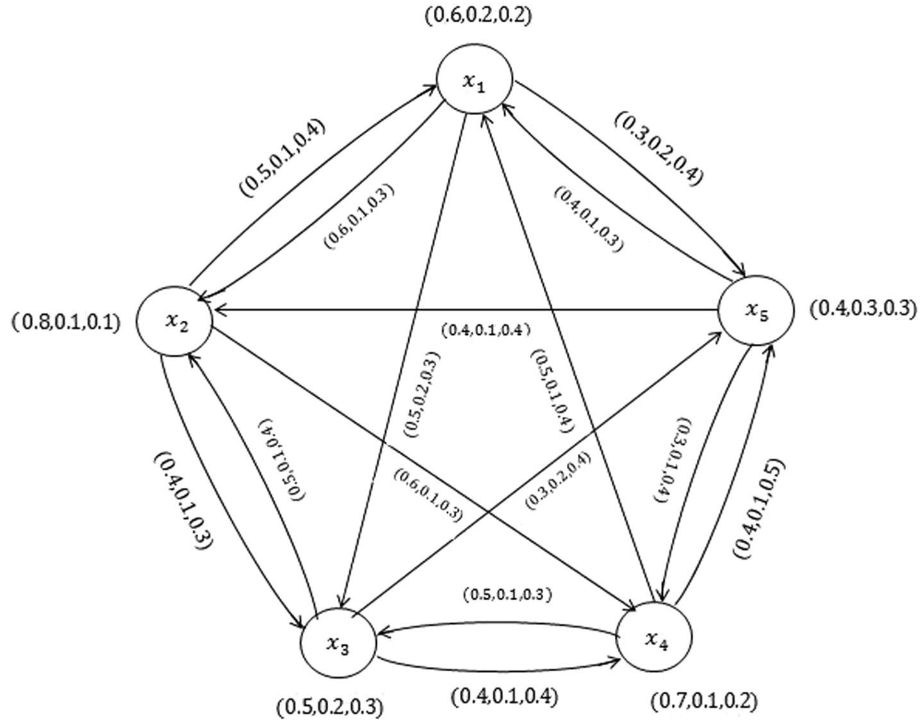
$$k_{ij} = \begin{cases} d_D^+(x_i) & i = j, \\ 0 & i \ne j. \end{cases}$$

Definition 14 The LM of PFDG $D = (A, \vec{B})$ is defined as follow:

$$L(D) = \left(L(\mu_{\vec{B}}(x_ix_j)), L(\eta_{\vec{B}}(x_ix_j)), L(\nu_{\vec{B}}(x_ix_j))\right)$$

$$L(D) = K^+(D) - M(D)$$

Fig. 4 A PFDG



where $K^{+}(D)$ and $M(D)$ are the out-degree matrix and AM of D, respectively.

Definition 15 The LE of a PFDG $D = (A, \overrightarrow{B})$ is specified as follow:

$$\mathrm{LE}(D) = \left(\mathrm{LE}(\mu_{\overrightarrow{B}}(x_i x_j)), \mathrm{LE}(\eta_{\overrightarrow{B}}(x_i x_j)), \mathrm{LE}(\nu_{\overrightarrow{B}}(x_i x_j)) \right),$$

$$\mathrm{LE}(D) = \left(\sum_{i=1}^{n} \mid t_i \mid, \sum_{i=1}^{n} \mid w_i \mid, \sum_{i=1}^{n} \mid z_i \mid \right),$$

where

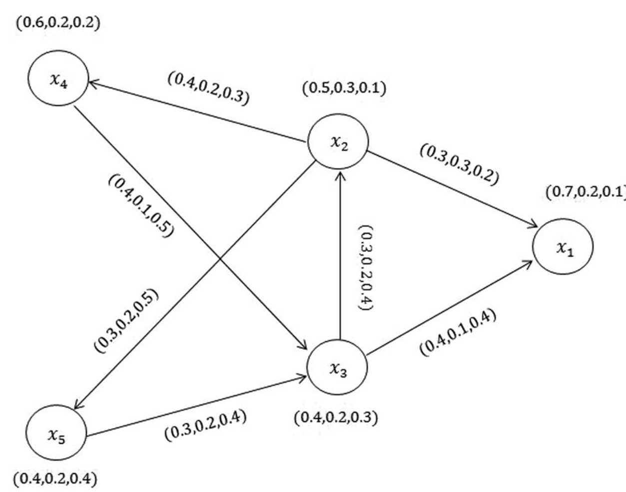


Fig. 5 A PFDG $G = (A, \mathbf{B})$

$$t_i = t_i^* - \frac{2\sum_{1\le i<j\le n} \mu_{\vec{B}}(x_ix_j)}{n},$$
$$w_i = w_i^* - \frac{2\sum_{1\le i<j\le n} \eta_{\vec{B}}(x_ix_j)}{n},$$
$$z_i = z_i^* - \frac{2\sum_{1\le i<j\le n} \nu_{\vec{B}}(x_ix_j)}{n},$$

t_i^*, w_i^*, and z_i^*, are real part the eigenvalues of $L\big(\mu_{\vec{B}}(x_ix_j)\big)$, $L\big(\eta_{\vec{B}}(x_ix_j)\big)$ and $L\big(\nu_{\vec{B}}(x_ix_j)\big)$, respectively.

Example 4 Consider a PFDG $D = (A, \vec{B})$ which is given in Fig. 4.

$$M(D) = \begin{bmatrix} (0,0,0) & (0.6,0.1,0.3) & (0.5,0.2,0.3) & (0,0,0) & (0.3,0.2,0.4) \\ (0.5,0.1,0.4) & (0,0,0) & (0.4,0.1,0.3) & (0.6,0.1,0.3) & (0,0,0) \\ (0,0,0) & (0.5,0.1,0.4) & (0,0,0) & (0.4,0.1,0.4) & (0.3,0.2,0.4) \\ (0.5,0.1,0.4) & (0,0,0) & (0.5,0.1,0.3) & (0,0,0) & (0.4,0.1,0.5) \\ (0.4,0.1,0.3) & (0.4,0.1,0.4) & (0,0,0) & (0.3,0.1,0.4) & (0,0,0) \end{bmatrix}.$$

$$K^+(D) = \begin{bmatrix} (1.4,0.5,1) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (1.5,0.3,1) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (1.2,0.4,1.2) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (1.4,0.3,1.2) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (1.1,0.3,1.1) \end{bmatrix}.$$

$$L(D) = \begin{bmatrix} (1.4,0.5,1) & (-0.6,-0.1,-0.3) & (-0.5,-0.2,-0.3) & (0,0,0) & (-0.3,-0.2,-0.4) \\ (-0.5,-0.1,-0.4) & (1.5,0.3,1) & (-0.4,-0.1,-0.3) & (-0.6,-0.1,-0.3) & (0,0,0) \\ (0,0,0) & (-0.5,-0.1,-0.4) & (1.2,0.4,1.2) & (-0.4,-0.1,-0.4) & (-0.3,-0.2,-0.4) \\ (-0.5,-0.1,-0.4) & (0,0,0) & (-0.5,-0.1,-0.3) & (1.4,0.3,1.2) & (-0.4,-0.1,-0.5) \\ (-0.4,-0.1,-0.3) & (-0.4,-0.1,-0.4) & (0,0,0) & (-0.3,-0.1,-0.4) & (1.1,0.3,1.1) \end{bmatrix}.$$

$$\mathrm{spec}(\mu_{\vec{B}}(x_ix_j)) = \{1.332, -0.565 + 0.449i, -0.565 - 0.449i, -0.101 + 0.274i, -0.101 - 0.274i\},$$
$$\mathrm{spec}(\eta_{\vec{B}}(x_ix_j)) = \{0.355, -0.138 + 0.142i, -0.138 - 0.1421i, -0.039 + 0.059i, -0.039 - 0.059i\},$$
$$\mathrm{spec}(\nu_{\vec{B}}(x_ix_j)) = \{1.095, -0.487 + 0.354i, -0.487 - 0.354i, -0.610 + 0.188i, -0.610 - 0.188i\}.$$

So, the energy of PFDG is as follow:

$$E(D) = \Big(E(\mu_{\vec{B}}(x_ix_j)), E(\eta_{\vec{B}}(x_ix_j)), E(\nu_{\vec{B}}(x_ix_j))\Big)$$
$$E(D) = (2.664, 0.709, 4.144).$$

On the other hand,

$$\text{Laplacian spec}(\mu_{\vec{B}}(x_ix_j)) = \{1.933 + 0.446i, 1.933 - 0.446i, 1.367 + 0.237i, 1.367 - 0.237i\},$$
$$\text{Laplacian spec}(\eta_{\vec{B}}(x_ix_j)) = \{0.528 + 0.123i, 0.528 - 0123i, 0.371 + 0.048i, 0.371 - 0.048i\},$$
$$\text{Laplacian spec}(\nu_{\vec{B}}(x_ix_j)) = \{1.581 + 0.333i, 1.581 - 0.333i, 1.168 + 0.175i, 1.168 - 0.175i\}.$$

We have

$$\mathrm{LE}(\mu_{\vec{B}}(x_ix_j)) = 2.532,$$
$$\mathrm{LE}(\eta_{\vec{B}}(x_ix_j)) = 0.754,$$
$$\mathrm{LE}(\nu_{\vec{B}}(x_ix_j)) = 2.018.$$

So, $\mathrm{LE}(D) = (2.532, 0.754, 2.018)$.

One of the energies of a digraph is skew energy. Here, we define the SLE of a PFDG.

Definition 16 Let $D^* = (V, \vec{\mathcal{E}})$ be a simple digraph without any loop and multiple arcs and $D = (A, \vec{B})$ be a PFDG on D^* .The SAM of D is the matrix in the form of

$$S(D) = \left(S(\mu_{\vec{B}}(x_ix_j)), S(\eta_{\vec{B}}(x_ix_j)), S(\nu_{\vec{B}}(x_ix_j) \right)$$

where

$$S(D) = [s_{ij}] = \begin{cases} (\mu_{\vec{B}}(x_ix_j), \eta_{\vec{B}}(x_ix_j), \nu_{\vec{B}}(x_ix_j)) & x_ix_j \in \vec{\mathcal{E}}, \\ (-\mu_{\vec{B}}(x_ix_j), -\eta_{\vec{B}}(x_ix_j), -\nu_{\vec{B}}(x_ix_j)) & x_jx_i \in \vec{\mathcal{E}}, \\ (0,0,0) & otherwise. \end{cases}$$

Definition 17 The SLE of the PFDG $D = (A, \vec{B})$ is specified as

$$\mathrm{SLE}(D) = \left(\sum_{i=1}^{n} p_i^2, \sum_{i=1}^{n} q_i^2, \sum_{i=1}^{n} r_i^2 \right)$$

where p_i, q_i and r_i are eigenvalues of the LM $L(D) = K^+(D) - S(D)$ of D.

Example 5 Consider a PFDG which is given in Fig. 5.

The SAM of D is as the follow:

$$S(D) = \begin{bmatrix} (0,0,0) & (-0.3,-0.3,-0.2) & (-0.4,-0.1,-0.4) & (0,0,0) & (0.3,0.2,0.4) \\ (0.3,0.3,0.2) & (0,0,0) & (-0.3,-0.2,-0.4) & (0.4,0.2,0.3) & (0.3,0.2,0.5) \\ (0.4,0.1,0.4) & (0.3,0.2,0.4) & (0,0,0) & (-0.4,-0.1,-0.5) & (-0.3,-0.2,-0.4) \\ (0,0,0) & (-0.4,-0.2,-0.3) & (0.4,0.1,0.5) & (0,0,0) & (0,0,0) \\ (0,0,0) & (-0.3,-0.2,-0.5) & (0.3,0.2,0.4) & (0,0,0) & (0,0,0) \end{bmatrix}.$$

$$K^+(D) = \begin{bmatrix} (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (1,0.7,1) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0.7,0.3,0.8) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0.4,0.1,0.5) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0.3,0.2,0.4) \end{bmatrix}.$$

$$L(D) = \begin{bmatrix} (0,0,0) & (0.3,0.3,0.2) & (0.4,0.1,0.4) & (0,0,0) & (0,0,0) \\ (-0.3,-0.3,-0.2) & (1,0.7,1) & (0.3,0.2,0.4) & (-0.4,-0.2,-0.3) & (-0.3,-0.2,-0.5) \\ (-0.4,-0.1,-0.4) & (-0.3,-0.2,-0.4) & (0.7,0.3,0.8) & (0.4,0.1,0.5) & (0.3,0.2,0.4) \\ (0,0,0) & (0.4,0.2,0.3) & (-0.4,-0.1,-0.5) & (0.4,0.1,0.5) & (0,0,0) \\ (0,0,0) & (0.3,0.2,0.5) & (-0.3,-0.2,-0.4) & (0,0,0) & (0.3,0.2,0.4) \end{bmatrix}.$$

The skew Laplacian spectrum of a PFDG D, given in Fig. 5, is

$$\begin{aligned} &\text{skew Laplacian spec}(\mu_{\vec{B}}(x_ix_j)) \\ &= \{0.625 + 0.726i, 0.625 - 0.726i, 0.408 \\ &\quad + 0.234i, 0.408 - 0.234i, 0.334\}, \\ &\text{skew Laplacian spec}(\eta_{\vec{B}}(x_ix_j)) \\ &= \{0.396 + 0.377i, 0.396 - 0.377i, 0.197 \\ &\quad + 0.123i, 0.197 - 0123i, 0.112\}, \\ &\text{skew Laplacian spec}(\nu_{\vec{B}}(x_ix_j)) \\ &= \{0.694 + 0.929i, 0.694 - 0929i, 0.541 \\ &\quad + 0.147i, 0.541 - 0.147i, 0.229\}. \end{aligned}$$

So, the SLE of D is

$$\mathrm{SLE}(D) = (1.225, 0.405, 1.602).$$

Theorem 10 *If D is a simple PFDG with vertex degrees of $d(x_1), d(x_2), \ldots, d(x_n)$ where $d(x_i) = (d_\mu(x_i), d_\eta(x_i), d_\nu(x_i))$, then,*

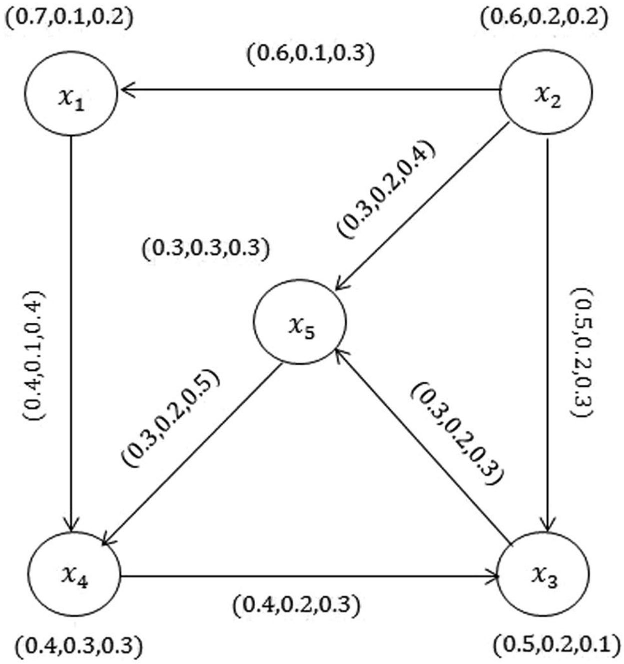


Fig. 6 PFDG D_1

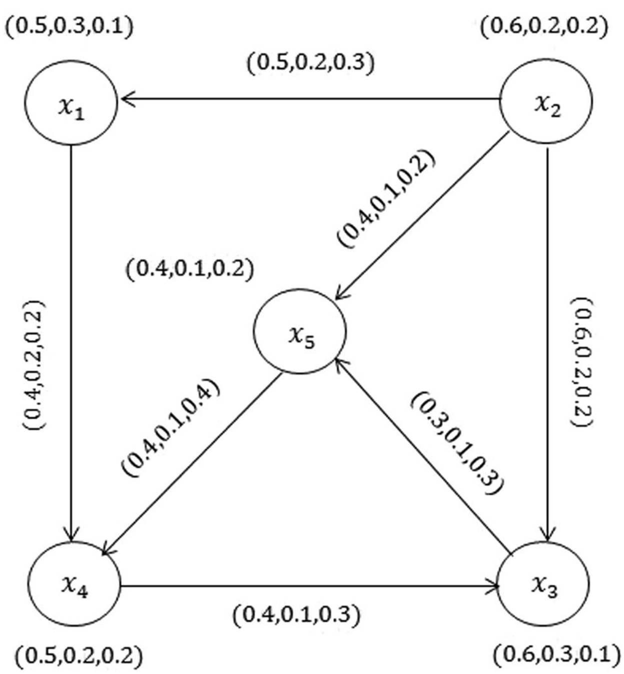


Fig. 7 PFDG D_2

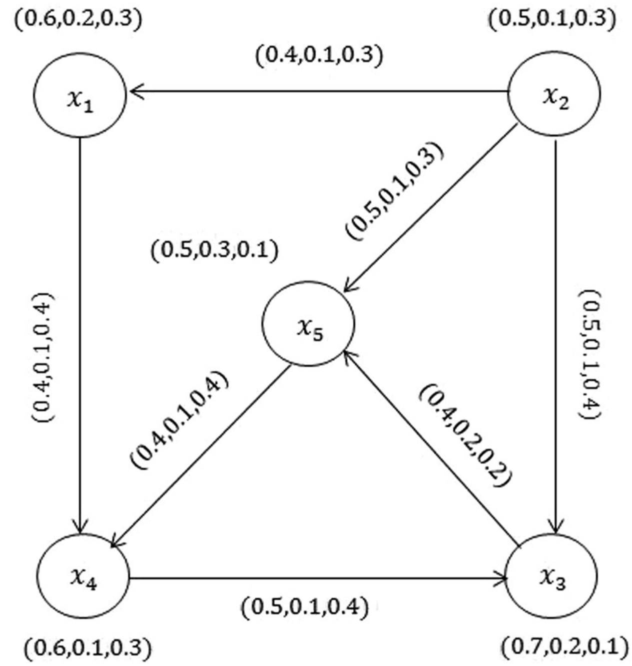


Fig. 8 PFDG D_3

$$\mathrm{SLE}(D) = \Bigg(\sum_{i=1}^{n} d_\mu(x_i)(d_\mu(x_i)-1), \sum_{i=1}^{n} d_\eta(x_i)(d_\eta(x_i)-1), \times \sum_{i=1}^{n} d_\nu(x_i)(d_\nu(x_i)-1) \Bigg).$$

Proof Let $p_1, p_2, \dots, p_n$ be the eigenvalues of the LM $L(\mu_{\vec{B}}(x_ix_j)) = K^+(\mu_{\vec{B}}(x_ix_j)) - S(\mu_{\vec{B}}(x_ix_j))$ where $K^+(\mu_{\vec{B}}(x_ix_j))$ and $S(\mu_{\vec{B}}(x_ix_j))$ are out-degree matrix and SAM of membership element of D. We have

$$\begin{aligned}\sum_{i=1}^{n} p_i &= \text{sum of determinants of all } 1 \\ &\quad \times 1 \text{ principal submatrices of } L(\mu_{\vec{B}}(x_ix_j)) \\ &= \text{trace of } L(\mu_{\vec{B}}(x_ix_j)) \\ &= \sum_{i=1}^{n} d_\mu(x_i)\end{aligned}$$

$$\begin{aligned}\sum_{i<j} p_ip_j &= \text{sum of determinants of all } 2 \\ &\quad \times 2 \text{ principal submtrices of } L(\mu_{\vec{B}}(x_ix_j)) \\ &= \sum_{i<j} \left| \begin{matrix} d_\mu(x_i) & -\mu_{ij} \\ -\mu_{ji} & d_\mu(x_j) \end{matrix} \right| = \sum_{i<j} d_\mu(x_i)d_\mu(x_j) - \mu_{ij}\mu_{ji} \\ &= \sum_{i<j} \left(d_\mu(x_i)d_\mu(x_j) + \mu_{ij}^2 \right).\end{aligned}$$

Since $\mu_{ij}^2 =| \mu_{ij} |$ for $i < j$, so,

$$\sum_{i\neq j} p_i p_j = 2\sum_{i<j} p_i p_j = \sum_{i\neq j} d_\mu(x_i) d_\mu(x_j) + \sum_{i\neq j} | a_{ij} |$$
$$= \sum_{i\neq j} d_\mu(x_i) d_\mu(x_j) + \sum_{i=1}^{n} d_\mu(x_i).$$

Thus,

$$\begin{aligned}
\mathrm{SLE}(\mu_{\vec{B}}(x_i x_j)) &= \sum_{i=1}^{n} p_i^2 = \left(\sum_{i=1}^{n} p_i \right)^2 - \sum_{i\neq j} p_i p_j \\
&= \left(\sum_{i=1}^{n} d_\mu(x_i) \right)^2 \\
&\quad - \left[\sum_{i\neq j} d_\mu(x_i) d_\mu(x_j) + \sum_{i=1}^{n} d_\mu(x_i) \right] \\
&= \sum_{i=1}^{n} \left(d_\mu(x_i) \right)^2 - \sum_{i=1}^{n} d_\mu(x_i) \\
&= \sum_{i=1}^{n} d_\mu(x_i) \left(d_\mu(x_i) - 1 \right).
\end{aligned}$$

Similarly, we can show that

$$\mathrm{SLE}(\eta_{\vec{B}}(x_i x_j)) = \sum_{i=1}^{n} d_\eta(x_i) \left(d_\eta(x_i) - 1 \right),$$
$$\mathrm{SLE}(\nu_{\vec{B}}(x_i x_j)) = \sum_{i=1}^{n} d_\nu(x_i) \left(d_\nu(x_i) - 1 \right).$$

This completes the proof. □

4 Applications

4.1 Selecting a Suitable Construction Plan in Different Climatic Zone

In studying the characteristics of housing and architecture, not only historical and economic factors should be carefully studied, but also the impact of natural geographical factors on housing architecture in different climatic regions should be considered. Architects have traditionally paid attention to nature and the surrounding environment and have maintained a peaceful relationship with nature in the traditional way, and in architecture, they have always paid attention to the environment and nature around them. The house in the warm and dry climate had dense, introverted plans facing the central courtyard. These houses showed all their beauties to the family. Traditional houses in the cold climate were also built as a central courtyard and had dense and compact plans. In temperate and humid climates, due to the high humidity in traditional houses, the roofs are sloping and the porches around the building have been considered as an important space in the house. The shape of the building is also unlike the hot and dry climate houses which have a central courtyard. This area has been designed extroverted and the view of the windows of the house and the yard has been around it. A construction holding company is considering new plans for housing based on advisers' opinions. These plans include:

(1) Access to public services.
(2) Architectural style.
(3) Infrastructure networks.
(4) Auxiliary facilities.
(5) Smart systems.

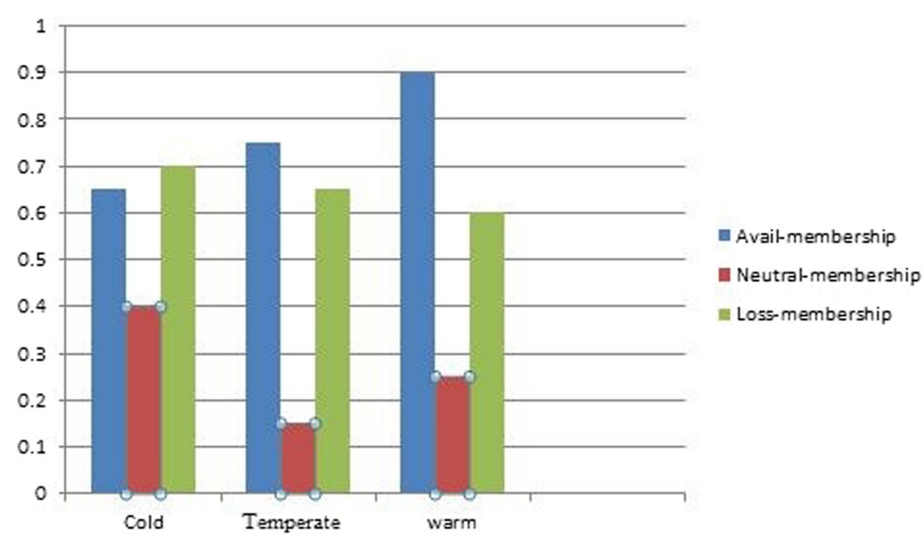


Fig. 9 Comparison of the energy of PFDGs

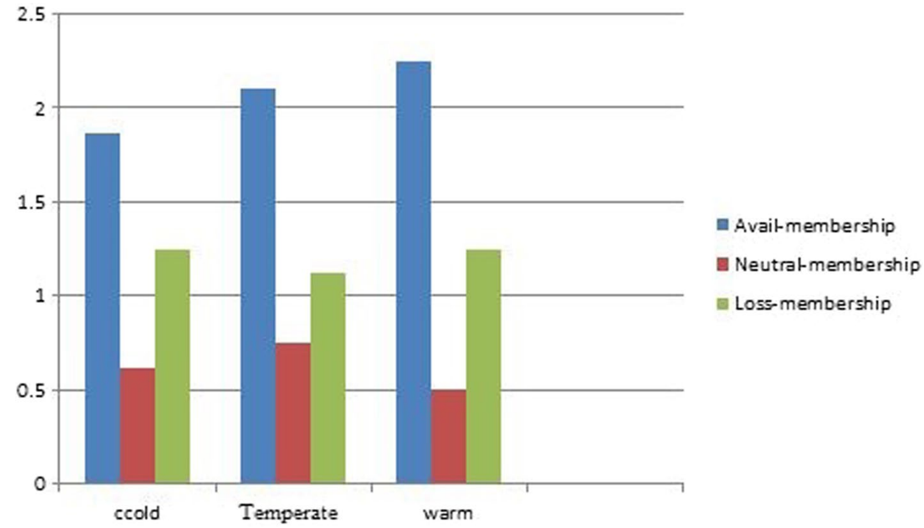


Fig. 10 Comparison of LEs of PFDGs

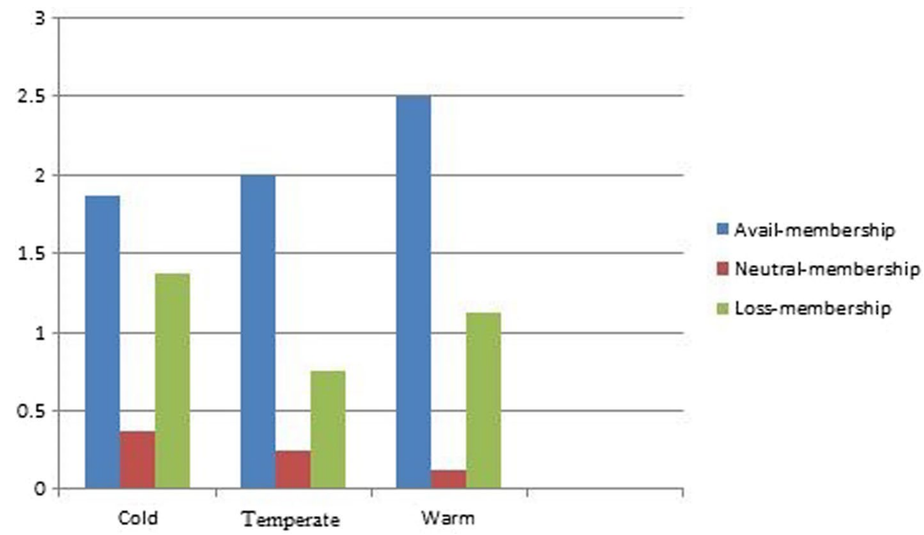


Fig. 11 Comparison of SLE of PFDGs

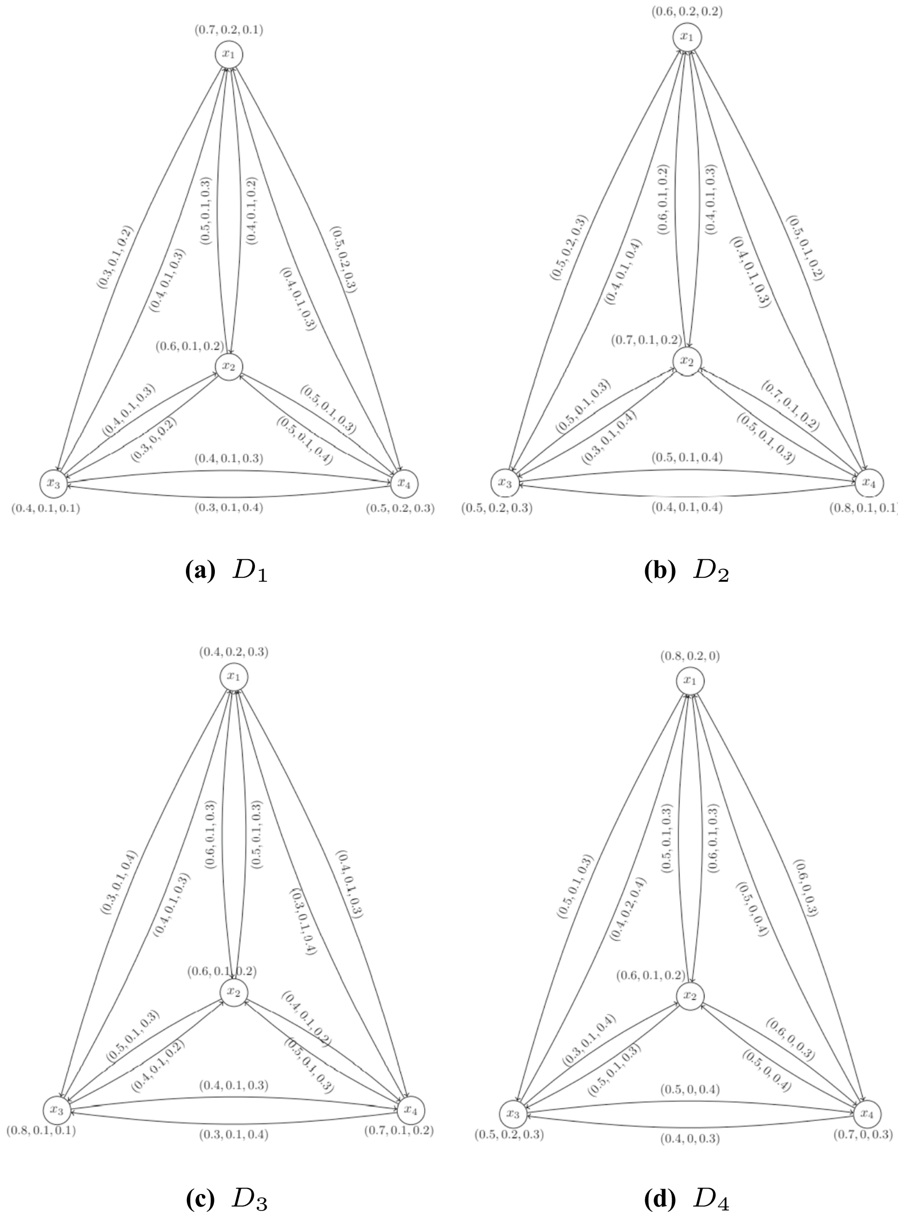
These plans and relations are more important for housing and employment in climate regions. In relation to different climatic regions, carrying out the above plans for the construction company is associated with avail, loss, or neutral states. Doing any of the above projects is directly or indirectly related to other projects. We showed the interactions of the above designs in terms of their location in different climatic points in the form of a PFDG. Therefore, we consider a PFDG at three climate regions of cold, temperate, and warm. Afterward, the energy, LE, and SLE of a PFDG are calculated for each of these regions.

The above plans are considered for the cold regions as a PFDG D_1 shown in Fig. 6. The picture fuzzy numbers for the vertices are the percentage of the impact of each of the construction company's plans for construction in different climatic zones. These percentages are determined by the opinion of consultants and based on the experiences and statistics available in the company. The relationships between the vertices are the edges that represent the most connection between the plans of the company, and therefore the edges are all strong.

In this digraph, the plan degree x_3 is (0.5, 0.2, 0.1). i.e. in cold regions in which for applying this plan 50% is avail, 20% is neutral and 10% is loss, too, and the degree of impact of infrastructure networks on intelligent systems (x_3x_5) is (0.3, 0.2, 0.3). i.e. 30% is avail, 20% is neutral and 40% is loss.

We calculated energy, LE, and SLE.

Fig. 12 PFDG_s



$$\mathrm{spec}(\mu_{\vec{B}}(x_ix_j)) = \{-0.1651 + 0.2859i, -0.1651 - 0.2859i, 0.3301, 0, 0\},$$
$$\mathrm{spec}(\eta_{\vec{B}}(x_ix_j)) = \{0.2, -0.1 + 0.1732i, -0.1 - 1732i, 0, 0\},$$
$$\mathrm{spec}(\nu_{\vec{B}}(x_ix_j)) = \{-0.1778 + 0.3080i, -0.1778 - 0.3080i, 0.3557, 0, 0\}.$$

So, $E(D_1) = (0.6603, 0.4, 0.7113)$. Also,

$$\text{Laplacian spec}(\mu_{\vec{B}}(x_ix_j)) = \{1.4, 0.5 + 0.28282i, 0.5 - 0.2828i, 0.4, 0\},$$
$$\text{Laplacian spec}(\eta_{\vec{B}}(x_ix_j)) = \{0.5, 0.3 + 0.1732i, 0.3 - 0.1732i, 0.1, 0\},$$
$$\text{Laplacian spec}(\nu_{\vec{B}}(x_ix_j)) = \{1, 0.55 + 0.2958i, 0.55 - 0.2958i, 0.4, 0\}.$$

So, $LE(D_1) = (1.8, 0.72, 1.18)$.

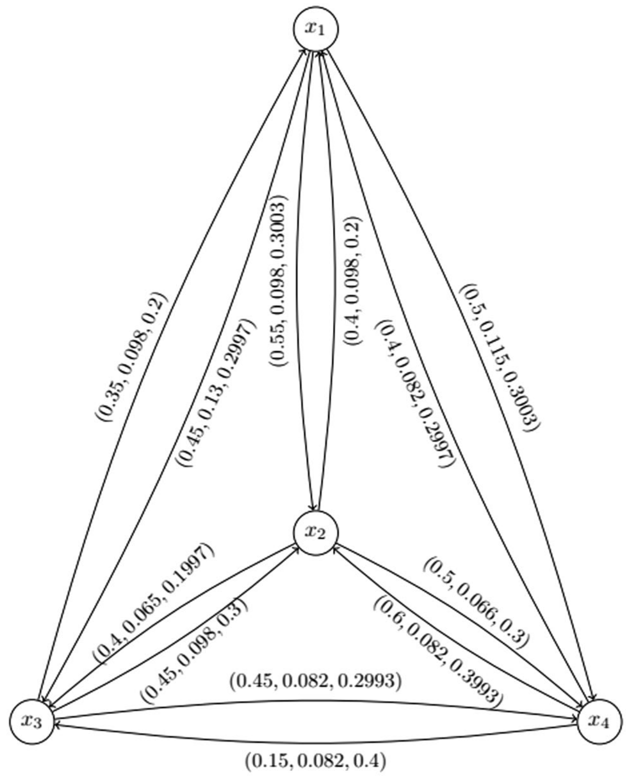


Fig. 13 PFDG corresponding to PFPRA

skew *Lap*Laplacian *lacian* spec($\mu_{\vec{B}}(x_i x_j)$)

$= \{0.8239 + 0.6484i, 0.8239 - 0.6434i, 0.3829 + 0.6706i, 0.3829 - 0.6706i, 0.3862\}$,

skew Laplacian spec($\eta_{\vec{B}}(x_i x_j)$)

$= \{0.2378 + 0.3792i, 0.2378 - 0.3792i, 0.2853 + 0.2145i, 0.2853 - 0.2145i, 0.1534\}$,

skew Laplacian spec($\nu_{\vec{B}}(x_i x_j)$)

$= \{0.4378 + 0.7807i, 0.4378 - 0.7807i, 0.5729 + 0.3604i, 0.5729 - 0.3604i, 0.4786\}$.

So, the SLE D_1 is equal to

SLE(D_1) = (1.7990, 0.2994, 1.2688).

The PFDG D_2 shown in Fig. 7 is considered for temperate regions.

After calculation, we have

spec($\mu_{\vec{B}}(x_i x_j)$)

$= \{-0.1817 + 0.3147i, -0.1817 - 0.3147i, 0.3624, 0, 0\}$,

spec($\eta_{\vec{B}}(x_i x_j)$)

$= \{0.1, -0.05 + 0.0866i, -0.05 - 0.0866i, 0, 0\}$,

spec($\nu_{\vec{B}}(x_i x_j)$)

$= \{0.3302, -0.1651 + 0.2859i, -0.1651 - 0.2859i, 0, 0\}$.

So, $E(D_2) = (0.7258, 0.2, 0.6604)$.

Laplacian spec($\mu_{\vec{B}}(x_i x_j)$)

$= \{1.5, 0.55 + 0.3122i, 1.5 - 0.3122i, 0.4, 0\}$,

Laplacian spec($\eta_{\vec{B}}(x_i x_j)$)

$= \{0.5, 0.2, 0.15 + 0.0866i, 0.15 - 0.0866i, 0\}$,

Laplacian spec($\nu_{\vec{B}}(x_i x_j)$)

$= \{0.7, 0.5 + 0.2828i, 0.5 - 0.2828i, 0.2, 0\}$.

So, $LE(D_2) = (2.04, 0.72, 1.14)$.

skew Laplacian spec($\mu_{\vec{B}}(x_i x_j)$)

$= \{0.8636 + 0.6533i, 0.8636 - 0.6533i, 0.4128 + 0.7207i, 0.4128 - 0.7207i, 0.4470\}$

skew Laplacian spec($\eta_{\vec{B}}(x_i x_j)$)

$= \{0.2888 + 0.2496i, 0.2888 - 0.2496i, 0.1371 + 2285i, 0.1371 - 2285i, 0.1483\}$

skew Laplacian spec($\nu_{\vec{B}}(x_i x_j)$)

$= \{0.3445 + 0.6250i, 0.3445 - 0.6250i, 0.4612 + 0.2595i, 0.4612 - 0.2595i, 0.2885\}$.

So, the SLE D_2 is equal to

SLE(D_2) = (2.0322, 0.2263, 0.7460).

For warm regions, a PFDG D_3 shown in Fig. 8 is considered. After calculation, we have

spec($\mu_{\vec{B}}(x_i x_j)$)

$= \{0.4309, -0.2154 + 0.3731i, -0.2154 - 0.3731i, 0, 0\}$,

spec($\eta_{\vec{B}}(x_i x_j)$)

$= \{-0.0629 + 0.1091i, -0.0629 - 0.1091i, 0.1260, 0, 0\}$,

spec($\nu_{\vec{B}}(x_i x_j)$)

$= \{0.3175, -0.1587 + 0.2749i, -0.1587 - 0.2749i, 0, 0\}$.

So, $E(D_3) = (0.8617, 0.2518, 0.6349)$.

$$\text{Laplacian spec}(\mu_{\vec{B}}(x_ix_j)) = \{1.4, 0.8670, 0.2165 + 0.3710i, 0.2165 - 0.3710i, 0.4\},$$
$$\text{Laplacian spec}(\eta_{\vec{B}}(x_ix_j)) = \{0.3, 0.2 + 0.1i, 0.2 - 0.1i, 0.1, 0\},$$
$$\text{Laplacian spec}(\nu_{\vec{B}}(x_ix_j)) = \{1, 0.5 + 0.2646i, 0.5 - 0.2646i, 0.4, 0\}.$$

So, $LE(D_3) = (2.154, 0.4, 1.16)$.

environmental regulations, and growth potential. In addition to physical businesses, this also affects Internet businesses.

Four different locations were considered for starting a new business, and a group of four consultants in finance, business, real estate and law were invited to decide on the appropriate location. Each expert made separate judgments based on their experiences and comparisons between the two different locations. These views were based on three factors: appropriate, neutral and inappropriate. The AMs

$$\begin{aligned}
&\text{skew Laplacian spec}(\mu_{\vec{B}}(x_ix_j))\\
&\quad = \{0.9801 + 2.1042i, 0.9801 - 2.1042i, 0.3650 + 0.6963i,\\
&\quad 0.3650 - 0.6963i, 0.4097\},\\
\text{Laplacian}&\quad = \{0.1540 + 0.2661i, 0.1540 - 0.2661i, 0.1649 + 0.1151i\\
&\quad 0.1649 - 0.1151i, 0.1621\},\\
&\text{skew Laplacian spec}(\nu_{\vec{B}}(x_ix_j))\\
&\quad = \{0.3842 + 0.7014i, 0.3842 - 0.7014i, 0.6001 + 0.4569i,\\
&\quad 0.6001 - 0.4569i, 0.4313\}.
\end{aligned}$$

So, the SLE D_3 is equal to

$$\text{SLE}(D_3) = (2.3555, 0.1281, 1.2015).$$

The figures 9, 10 and 11 represent a comparison between the energy, LE, and SLE of applied plans in three regions corresponding to the avail, neutral, and loss memberships, respectively. From the above bar graphs, the energy, LE and SLE of avail membership for warm regions is high as compared to other regions. This shows that the above designs are more efficient in hot areas and more coherence between the designs can be considered. The energy, LE, and SLE of neutral membership for cold regions is high and the energy, LE, and SLE of loss membership for cold regions is high.

4.2 Choosing the Best Place to Start a Business

Everyone who works in business has heard that one of the most important factors in ensuring the profitability of a business is its location. Basic decision making in this field can transform a startup and lead to growing success. The size, layout, physical location and appearance of the workplace should all serve to develop your operations. The deeper you go into your business plan, the more realistic and tangible your budget for your property and business expenses will be, given the infrastructure, tax laws, facilities, regional laws,

related to picture fuzzy relations to each of the experts are as follows:

$$M_1 = \begin{bmatrix} (0,0,0) & (0.4,0.1,0.3) & (0.4,0.1,0.3) & (0.3,0.1,0.4)\\ (0.3,0.1,0.4) & (0,0,0) & (0.5,0.1,0.2) & (0.5,0.1,0.3)\\ (0.3,0.1,0.3) & (0.4,0.1,0.4) & (0,0,0) & (0.6,0.1,0.2)\\ (0.4,0.1,0.3) & (0.4,0.1,0.2) & (0.5,0.1,0.3) & (0,0,0) \end{bmatrix},$$
$$M_2 = \begin{bmatrix} (0,0,0) & (0.4,0.1,0.3) & (0.4,0.1,0.3) & (0.5,0.1,0.2)\\ (0.6,0.1,0.2) & (0,0,0) & (0.5,0.1,0.3) & (0.5,0.1,0.3)\\ (0.5,0.2,0.3) & (0.5,0.1,0.3) & (0,0,0) & (0.5,0.1,0.4)\\ (0.4,0.1,0.3) & (0.7,0.1,0.2) & (0.4,0.1,0.4) & (0,0,0) \end{bmatrix},$$
$$M_3 = \begin{bmatrix} (0,0,0) & (0.4,0.1,0.2) & (0.3,0.1,0.2) & (0.5,0.2,0.3)\\ (0.5,0.1,0.3) & (0,0,0) & (0.3,0,0.2) & (0.5,0.1,0.3)\\ (0.4,0.1,0.3) & (0.4,0.1,0.3) & (0,0,0) & (0.4,0.1,0.3)\\ (0.4,0.1,0.3) & (0.5,0.1,0.4) & (0.3,0.1,0.4) & (0,0,0) \end{bmatrix},$$
$$M_4 = \begin{bmatrix} (0,0,0) & (0.6,0.1,0.3) & (0.5,0.1,0.3) & (0.6,0,0.3)\\ (0.5,0.1,0.3) & (0,0,0) & (0.5,0.1,0.3) & (0.6,0,0.3)\\ (0.4,0.2,0.4) & (0.3,0.1,0.4) & (0,0,0) & (0.5,0,0.4)\\ (0.5,0,0.4) & (0.5,0,0.4) & (0.4,0,0.3) & (0,0,0) \end{bmatrix}.$$

The PFDGs D_i corresponding to picture fuzzy preference relation (PFPR) given in matrices M_i are shown in Fig. 12 ($i = 1, 2, 3, 4$). The energy of each PFDG is calculated as:

$$E(D_1) = (0.001, 0.2, 0.3), \quad E(D_2) = (0.001, 0.1, 0),$$
$$E(D_3) = (0, 0.2, 0.001), \quad E(D_4) = (0, 0.1, -0.001).$$

Then, the weight of each expert can be calculated as:

$$w_i = ((w_\mu)_i, (w_\eta)_i, (w_r)_i) = \left(\frac{E(M_\mu)_i}{\sum_{j=1}^{4} E(M_\mu)_j}, \frac{E(M_\eta)_i}{\sum_{j=1}^{4} E(M_\eta)_j}, \frac{E(M_r)_i}{\sum_{j=1}^{4} E(M_r)_j} \right)$$

$$w_1 = (0.5, 0.33, 1), \quad w_2 = (0.5, 0.16, 0)$$

$$w_3 = (0, 0.33, 0.0033), \quad w_4 = (0, 0.16, -0.0033).$$

The collective picture fuzzy preference relation (PFPR), aggregated from the four PFPRs, is determined as:

$$M = \sum_{i=1}^{4} w_i M_i = \begin{bmatrix} (0,0,0) & (0.4, 0.098, 0.2) & (0.35, 0.098, 0.2) & (0.5, 0.115, 0.3003) \\ (0.55, 0.098, 0.3003) & (0,0,0) & (0.4, 0.065, 0.1997) & (0.5, 0.066, 0.3) \\ (0.45, 0.13, 0.2997) & (0.45, 0.098, 0.3) & (0,0,0) & (0.45, 0.082, 0.2993) \\ (0.4, 0.082, 0.2997) & (0.6, 0.082, 0.3993) & (0.15, 0.082, 0.4) & (0,0,0) \end{bmatrix}.$$

A PFDG corresponding to a collective PFPRA above is drawn, as shown in Fig. 13. Then, under the $\mu_{j_k} \geq 0.5$ $(j, k = 1, 2, 3, 4)$ condition, a partial diagram is drawn, as shown in Fig. 14.

Calculate the out-degrees $d^+(x_i)$ $(i = 1, 2, 3, 4)$ of all criteria in a partial PFDG as follow:

$$d^+(x_1) = (1, 05, 0.213, 0.6006), \quad d^+(x_2) = (0.5, 0.066, 0.3)$$

$$d^+(x_3) = (0, 0, 0), \quad d^+(x_4) = (0.6, 0.082, 0.3993).$$

According to membership degrees of $d^+(x_i)\,(i = 1, 2, 3, 4)$, we get the ranking of the places x_i as:

$$x_1 \succ x_4 \succ x_2 \succ x_3.$$

Therefore, the best choice is x_1.

The flowchart for the selection of the best choice in business is as follow:

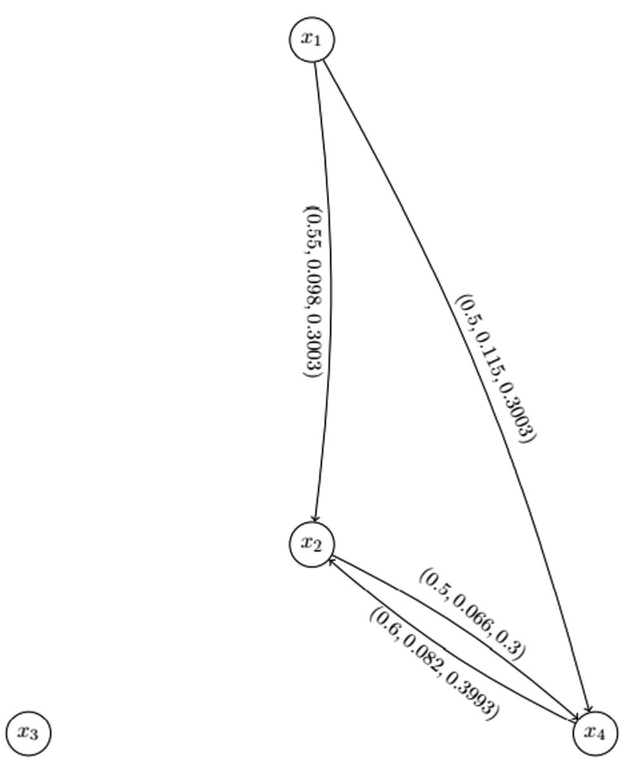
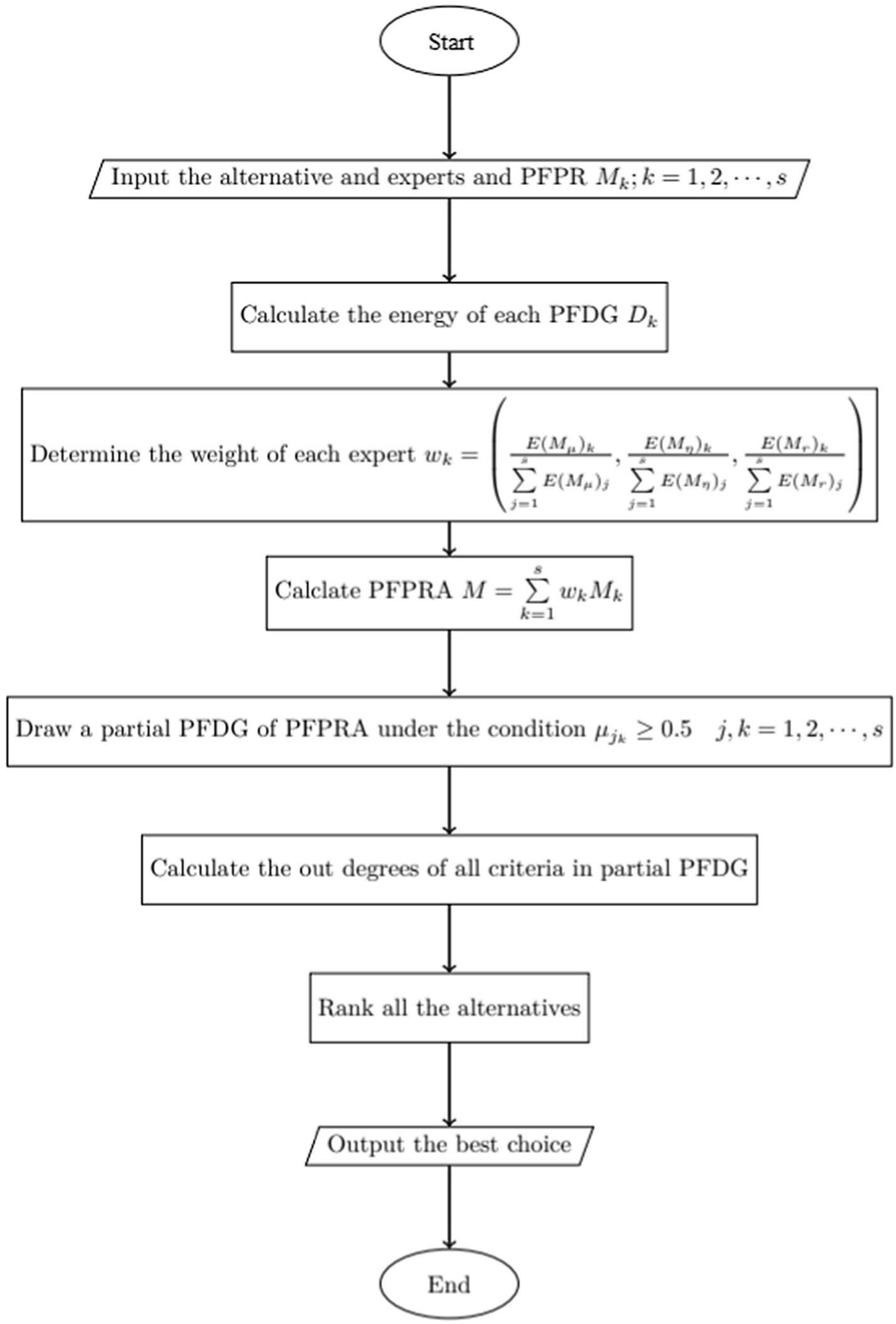


Fig. 14 Partial PFDG of PFPRA



5 Conclusions

The PFG can amplify flexibility and precision to model complex real-time problems better than an FG and IFG. They have several applications in many decision-making processes among solution choice, weather forecasting, prognosis risks in business, etc. In recent years, graph energy has been used in many fields. It is clear that the neutrality degree is associated with more flexibility for the graph energy. In this research, aiming at expanding the concept of energy on the PFG and using its results in modeling and solving the problems ahead, we studied some types of energy and their results in the PFG. We were able to show that neutrality, as part of total energy, is effective in energy-based decisions. Neutral energy is more pronounced in some types of energy and is not always very small compared to membership and non-membership degrees. We have obtained some properties and relations of lower and upper bounds of energy, LE, and SLE of PFG. Despite the energy fluctuations in different degrees of the PFG, no significant relationship was found between the degrees of energy. Finally, we presented applications of energy in decision-making based on superior choice. Examining the energy in the interval-valued intuitionistic fuzzy graph is an interval of the amount of our future work plans.

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Declarations

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Consent for publication All authors are agree for possible publication of the manuscript.

Competing interests Not applicable.

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