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RESEARCH ARTICLE

Some Integral Inequalities for Generalized Convex Fuzzy-Interval-Valued Functions via Fuzzy Riemann Integrals

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Abstract

In this study, we introduce the new concept of *h*-convex fuzzy-interval-valued functions. Under the new concept, we present new versions of Hermite–Hadamard inequalities (H–H inequalities) are called fuzzy-interval Hermite–Hadamard type inequalities for *h*-convex fuzzy-interval-valued functions (*h*-convex FIVF) by means of fuzzy order relation. This fuzzy order relation is defined level wise through Kulisch–Miranker order relation defined on fuzzy-interval space. Fuzzy order relation and inclusion relation are two different concepts. With the help of fuzzy order relation, we also present some H–H type inequalities for the product of *h*-convex FIVFs. Moreover, we have also established strong relationship between Hermite–Hadamard–Fej´er (H–H–Fej´er) type inequality and *h*-convex FIVF. There are also some special cases presented that can be considered applications. There are useful examples provided to demonstrate the applicability of the concepts proposed in this study. This paper's thoughts and methodologies could serve as a springboard for more research in this field.

Keywords Convex fuzzy-interval-valued function · Fuzzy Riemann integral · Fuzzy-interval Hermite · Hadamard inequality · Fuzzy-interval Hermite · Hadamard · Fej´er inequality

1 Introduction

The significance and supreme applications of convex functions are well known in different fields, especially in the study of integral inequalities, variational inequalities and optimization. Therefore, much attention has been given in studying and characterizing different directions of classical idea of convexity. Recently, many extensions and generalizations of convex functions have been studied. For more useful details, see [1–4, 6, 7, 14, 18–21, 25] and the references are therein. In classical approach, a real valued function $\Psi : K \to \mathbb{R}$ is called convex if

$$\Psi(\xi z + (1-\xi)y) \leq \xi\Psi(z) + (1-\xi)\Psi(y), \quad (1)$$

for all $z, \epsilon y \in K, \xi \in [0, 1]$.

The concept of convexity with integral problem is an interesting area for research. Therefore, many inequalities have been introduced as applications of convex functions. Among those, the H–H inequality is an interesting outcome in convex analysis. The H–H inequality [16, 17] for convex function $\Psi : K \to \mathbb{R}$ on an interval $K = [u, v]$

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$$\Psi\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v \Psi(z)\mathrm{d}z \le \frac{\Psi(u)+\Psi(v)}{2}, \tag{2}$$

for all $z \in K$.

On the other hand, the concept of interval analysis was proposed and investigated by Moore [24], and Kulish and Miranker [23] to improve the reliability of the calculation results and automatically perform error analysis. It is a discipline in which an uncertain variable is represented by an interval of real numbers and real operations are replaced by interval operation. In last 5 decades, the concept of interval was used as a tool to handle uncertain problems. Recently, many authors have contributed their role in this theory and introduced new concepts. Zhao et al. [30] introduced h-convex interval-valued functions and proved that the following H–H type inequality for h-convex interval-valued functions:

Theorem 1.1. *Let $\Psi : [u, v] \subset \mathbb{R} \to \mathcal{K}_C^+$ be a h-convex interval-valued function given by $\Psi(z) = \left[\Psi_*(z), \Psi^*(z)\right]$ for all $z \in [u, v]$, where $\Psi_*(z)$ is a h-convex function and $\Psi^*(z)$ is a h-concave function. If Ψ is Riemann integrable, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi\left(\frac{u+v}{2}\right) \supseteq \frac{1}{v-u}\int_u^v \Psi(z)\mathrm{d}z$$

$$\supseteq \frac{\Psi(u)+\Psi(v)}{2}\int_0^1 h(\xi)\mathrm{d}\xi. \tag{3}$$

If $\Psi_*(z), = \Psi^*(z)$, then integral inequality (3) reduces for h-convex function, see [29]. We refer to the readers for further analysis of literature on the applications and properties of generalized convex functions and H–H integral inequalities, see [5, 8, 9, 15, 17, 22, 27, 28, 31–48] and the references therein.

There are some integrals to deal with fuzzy-interval-valued functions (FIVF), where the integrands are FIVFs. For instance, Oseuna-Gomez et al. [26], and Costa [11] constructed H–H and Jensen's integral inequality for FIVFs using fuzzy Aumann integrals. Using same approach Costa and Floures [10] and Costa et al. [12] also presented Minkowski and Beckenbach's inequalities, where the integrands are interval-valued and FIVFs. Motivated by [10, 11, 26, 29] and [30], we generalize integral inequality (2) by constructing fuzzy-interval integral inequality for h-convex FIVF, where the integrands are h-convex FIVFs.

This study is organized as follows: Sect. 2 presents preliminary notions, new concepts and results in interval space, in the space of fuzzy-intervals and for h-convex FIVFs. Section 3 obtains fuzzy-interval H–H inequalities via h-convex FIVFs. In addition, some interesting examples are also given to verify our results. Section 4 gives conclusions.

2 Preliminaries

In this section, we recall some basic preliminary notions, definitions and results. With the help of these results, some new basic definitions and results are also discussed.

We begin by recalling the basic notations and definitions. We define interval as, $\left[\omega_*, \omega^*\right] = \left\{z \in \mathbb{R} : \omega_* \le z \le \omega^* \text{and} \omega_*, \omega^* \in \mathbb{R}\right\}$, where $\omega_* \le \omega^*$.

We write len $\left[\omega_*, \omega^*\right] = \omega^* - \omega_*$, If len $\left[\omega_*, \omega^*\right] = 0$ then, $\left[\omega_*, \omega^*\right]$ is called degenerate. In this article, all intervals will be non-degenerate intervals. The collection of all closed and bounded intervals of $\mathbb{R}$ is denoted and defined as $\mathcal{K}_C = \left\{\left[\omega_*, \omega^*\right] : \omega_*, \omega^* \in \mathbb{R} \text{and} \omega_* \le \omega^*\right\}$. If $\omega_* \ge 0$ then, $\left[\omega_*, \omega^*\right]$ is called positive interval. The set of all positive interval is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \left\{\left[\omega_*, \omega^*\right] : \left[\omega_*, \omega^*\right] \in \mathcal{K}_C \text{and} \omega_* \ge 0\right\}$.

We'll now look at some of the properties of intervals using arithmetic operations. Let $\left[\varrho_*, \varrho^*\right], \left[\&_*, \&^*\right] \in \mathcal{K}_C$ and $\rho \in \mathbb{R}$, then we have

$$\left[\varrho_*, \varrho^*\right] + \left[\&_*, \&^*\right] = \left[\varrho_* + \&_*, \varrho^* + \&^*\right],$$

$$\left[\varrho_*, \varrho^*\right] \times \left[\&_*, \&^*\right] = \begin{bmatrix} min\left\{\varrho_*\&_*, \varrho^*\&_*, \varrho_*\&^*, \varrho^*\&^*\right\}, \\ max\left\{\varrho_*\&_*, \varrho^*\&_*, \varrho_*\&^*, \varrho^*\&^*\right\} \end{bmatrix}$$

$$\rho.\left[\varrho_*, \varrho^*\right] = \begin{cases} \left[\rho\varrho_*, \rho\varrho^*\right] if\, \rho \ge 0, \\ \left[\rho\varrho^*, \rho\varrho_*\right] if\, \rho < 0. \end{cases}$$

For $\left[\varrho_*, \varrho^*\right], \left[\&_*, \&^*\right] \in \mathcal{K}_C$, the inclusion "$\subseteq$" is defined by $\left[\varrho_*, \varrho^*\right] \subseteq \left[\&_*, \&^*\right]$, if and only if $\&_* \le \varrho_*, \varrho^* \le \&^*$.

Remark 2.1. The relation "$\le_I$" defined on $\mathcal{K}_C$ by $\left[\varrho_*, \varrho^*\right] \le_I \left[\&_*, \&^*\right]$ if and only if $\varrho_* \le \&_*, \varrho^* \le \&^*$, for all $\left[\varrho_*, \varrho^*\right], \left[\&_*, \&^*\right] \in \mathcal{K}_C$, it is an order relation, see [23]. For given $\left[\varrho_*, \varrho^*\right], \left[\&_*, \&^*\right] \in \mathcal{K}_C$, we say that $\left[\varrho_*, \varrho^*\right] \le_I \left[\&_*, \&^*\right]$ if and only if $\varrho_* \le \&_*, \varrho^* \le \&^*$ or $\varrho_* \le \&_*, \varrho^* < \&^*$.

Moore [24] initially proposed the concept of Riemann integral for IVF, which is defined as follows:

Theorem 2.2. [24] *If $\Psi : [u, v] \subset \mathbb{R} \to \mathcal{K}_C$ is an IVF on such that $\Psi(z) = \left[\Psi_*, \Psi^*\right]$. Then Ψ is Riemann integrable over $[u, v]$ if and only if, Ψ_* and Ψ^* both are Riemann integrable over $[u, v]$ such that*

$$(IR)\int_u^v \Psi(z)\mathrm{d}z$$

$$=\left[(R)\int_u^v \Psi_*(z)\mathrm{d}z,(R)\int_u^v \Psi^*(z)\mathrm{d}z\right]$$

Let $\mathbb{R}$ be the set of real numbers. A mapping $\zeta:\mathbb{R}\to[0,1]$ called the membership function distinguishes a fuzzy subset set $\mathcal{A}$ of $\mathbb{R}$. This representation is found to be acceptable in this study. $\mathbb{F}(\mathbb{R})$ also stand for the collection of all fuzzy subsets of $\mathbb{R}$.

A real fuzzy interval ζ is a fuzzy set in $\mathbb{R}$ with the following properties:

(1) ζ is normal i.e. there exists $z\in\mathbb{R}$ such that $\zeta(z)=1$;
(2) ζ is upper semi continuous i.e., for given $z\in\mathbb{R}$, for every $z\in\mathbb{R}$ there exist $\varepsilon>0$ there exist $\delta>0$ such that $\zeta(z)-\zeta(y)<\varepsilon$ for all $y\in\mathbb{R}$ with $|z-y|<\delta$.
(3) ζ is fuzzy convex i.e., $\zeta((1-\xi)z+\xi y)\geq\min(\zeta(z),\zeta(y))$, $\forall\, z,y\in\mathbb{R}$ and $\xi\in[0,1]$;
(4) ζ is compactly supported i.e., $cl\{z\in\mathbb{R}|\zeta(z)\rangle 0\}$ is compact.

The collection of all real fuzzy intervals is denoted by $\mathbb{F}_0$.

Let $\zeta\in\ \mathbb{F}_0$ be real fuzzy interval, if and only if, β-levels $[\zeta]^\beta$ is a nonempty compact convex set of $\mathbb{R}$. This is represented by

$$[\zeta]^\beta=\{z\in\mathbb{R}|\zeta(z)\geq\beta\},$$

from these definitions, we have

$$[\zeta]^\beta=\left[\zeta_*(\beta),\zeta^*(\beta)\right],$$

where

$$\zeta_*(\beta)=\inf\{z\in\mathbb{R}|\zeta(z)\geq\beta\},$$

$$\zeta^*(\beta)=\sup\{z\in\mathbb{R}|\zeta(z)\geq\beta\}.$$

Thus a real fuzzy interval ζ can be identified by a parametrized triples

$$\left\{\left(\zeta_*(\beta),\zeta^*(\beta),\beta\right):\beta\in[0,1]\right\}.$$

These two end point functions $\zeta_*(\beta)$ and $\zeta^*(\beta)$ are used to characterize a real fuzzy interval as a result.

Proposition 2.3. [10] *Let* $\zeta,\Theta\in\mathbb{F}_0$. *Then fuzzy order relation* "$\preccurlyeq$" *given on* $\mathbb{F}_0$ *by*

$$\zeta\preccurlyeq\Theta \text{ if and only if, } [\zeta]^\beta\leq_I[\Theta]^\beta \text{ for all } \beta\in(0,1],$$

it is partial order relation.

We will now look at some of the properties of fuzzy intervals using arithmetic operations. Let $\zeta,\Theta\in\mathbb{F}_0$ and $\rho\in\mathbb{R}$, then we have

$$\left[\zeta\tilde{+}\Theta\right]^\beta=[\zeta]^\beta+[\Theta]^\beta, \tag{4}$$

$$[\zeta\tilde{\times}\Theta]^\beta=[\zeta]^\beta\times[\Theta]^\beta, \tag{5}$$

$$[\rho.\zeta]^\beta=\rho.[\zeta]^\beta. \tag{6}$$

For $\psi\in\mathbb{F}_0$ such that $\zeta=\Theta\tilde{+}\psi$, we have the existence of the Hukuhara difference of ζ and Θ, which we call the H-difference of ζ and Θ, and denoted by $\zeta\tilde{-}\Theta$. If H-difference exists, then

$$(\psi)^*(\beta)=(\zeta\tilde{-}\Theta)^*(\beta)=\zeta^*(\beta)-\Theta^*(\beta)$$

$$(\psi)_*(\beta)=(\zeta\tilde{-}\Theta)_*(\beta)=\zeta_*(\beta)-\Theta_*(\beta). \tag{7}$$

Theorem 2.4. [13, 26] *The space* $\mathbb{F}_0$ *dealing with a supremum metric i.e., for* $\psi,\Theta\in\mathbb{F}_0$

$$D(\psi,\Theta)=\sup_{0\leq\beta\leq1}H\left([\zeta]^\beta,[\Theta]^\beta\right), \tag{8}$$

it is a complete metric space, where H denote the well-known Hausdorff metric on space of intervals.

Definition 2.5. [10] A fuzzy-interval-valued map $\Psi:K\subset\mathbb{R}\to\mathbb{F}_0$ is called FIVF. For each $\beta\in(0,1]$, whose β-levels define the family of IVFs $\Psi_\beta:K\subset\mathbb{R}\to\mathcal{K}_C$ are given by $\Psi_\beta(z)=\left[\Psi_*(z,\beta),\Psi^*(z,\beta)\right]$ for all $z\in K$. Here, for each $\beta\in(0,1]$, the end point real functions $\Psi_*(.,\beta),\Psi^*(.,\beta):K\to\mathbb{R}$ are called lower and upper functions of Ψ.

The following conclusions can be drawn from the preceding literature review [10, 13, 22, 24]:

Definition 2.6. Let $\Psi:[u,v]\subset\mathbb{R}\to\mathbb{F}_0$ be a FIVF. Then fuzzy integral of Ψ over $[u,v]$, denoted by $(\mathrm{FR})\int_u^v\Psi(z)\mathrm{d}z$, it is given level-wise by

$$\left[(\mathrm{FR})\int_u^v\Psi(z)\mathrm{d}z\right]^\beta=(\mathrm{IR})\int_u^v\Psi_\beta(z)\mathrm{d}z$$

$$= \left\{ \int_u^v \Psi(z,\beta)\mathrm{d}z : \Psi(z,\beta) \in \mathcal{R}_{([u,v],\beta)} \right\}, \tag{9}$$

for all $\beta \in (0,1]$, where $\mathcal{R}_{([u,v],\beta)}$ denotes the collection of Riemannian integrable functions of IVFs. Ψ is *FR*-integrable over $[u,v]$ if $(\mathrm{FR})\int_u^v \Psi(z)\mathrm{d}z \in \mathbb{F}_0$. Note that, if both end point functions are Lebesgue-integrable, then Ψ is fuzzy Aumann-integrable function over $[u,v]$, see [13, 22, 24].

Theorem 2.7. *Let $\Psi : [u,v] \subset \mathbb{R} \to \mathbb{F}_0$ be a FIVF, whose β-levels define the family of IVFs $\Psi_\beta : [u,v] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\Psi_\beta(z) = \left[\Psi_*(z,\beta), \Psi^*(z,\beta)\right]$ for all $z \in [u,v]$ and for all $\beta \in (0,1]$. Then Ψ is FR-integrable over $[u,v]$ if and only if, $\Psi_*(z,\beta)$ and $\Psi^*(z,\beta)$ both are R-integrable over $[u,v]$. Moreover, if Ψ is FR-integrable over $[u,v]$, then.*

$$\left[(\mathrm{FR})\int_u^v \Psi(z)\mathrm{d}z\right]^\beta$$

$$= \left[(R)\int_u^v \Psi_*(z,\beta)\mathrm{d}z, (R)\int_u^v \Psi^*(z,\beta)\mathrm{d}z\right]$$

$$= (\mathrm{IR})\int_u^v \Psi_\beta(z)\mathrm{d}z \tag{10}$$

for all $\beta \in (0,1]$. For all $\beta \in (0,1]$, $\mathcal{FR}_{([\mu,v],\beta)}$ denotes the collection of all FR-integrable FIVFs over $[\mu,v]$.

Definition 2.8. Let K be convex set and $h : [0,1] \subseteq K \to \mathbb{R}$ such that $h \equiv 0$. Then FIVF $\Psi : K \to \mathbb{F}_C(\mathbb{R})$ is said to be:

- h-convex on K if

  $$\Psi(\xi z + (1-\xi)y) \preccurlyeq h(\xi)\Psi(z) \tilde{+} h(1-\xi)\Psi(y), \tag{11}$$

  for all $z, y \in K, \xi \in [0,1]$, where $\Psi(z) \succcurlyeq \tilde{0}$.
- h-concave on K if inequality (11) is reversed.
- affine h-convex on K if

  $$\Psi(\xi z + (1-\xi)y) = h(\xi)\Psi(z) \tilde{+} h(1-\xi)\Psi(y), \tag{12}$$

for all $z, y \in K, \xi \in [0,1]$, where $\Psi(z) \succcurlyeq \tilde{0}$.

Remark 2.9. The h-convex FIVFs have some very nice properties similar to convex FIVF,

If Ψ is h-convex FIVF, then $\Upsilon\Psi$ is also h-convex for $\Upsilon \geq 0$

.

If Ψ and Ψ both are h-convex FIVFs, then $\max(\Psi(z), \Psi(z))$ is also h-convex FIVF.

We now discuss some special cases of h-convex FIVFs:

(i) If $h(\xi) = \xi^s$, then h-convex FIVF becomes s-convex FIVF, that is

$$\Psi(\xi z + (1-\xi)y) \preccurlyeq \xi^s \Psi(z) \tilde{+} (1-\xi)^s \Psi(y),$$

for all $z, y \in K, \xi \in [0,1]$.

(ii) If $h(\xi) = \xi$, then h-convex FIVF becomes convex FIVF, see [25], that is

$$\Psi(\xi z + (1-\xi)y) \preccurlyeq \xi \Psi(z) \tilde{+} (1-\xi) \Psi(y),$$

for all $z, y \in K, \xi \in [0,1]$.

(iii) If $h(\xi) \equiv 1$, then h-convex FIVF becomes P-convex FIVF, that is

$$\Psi(\xi z + (1-\xi)y) \preccurlyeq \Psi(z) \tilde{+} \Psi(y),$$

for all $z, y \in K, \xi \in [0,1]$.

Note that, special cases (i) and (iii) are also new ones.

Theorem 2.10. *Let K be convex set, non-negative real valued function $h : [0,1] \subseteq K \to \mathbb{R}$ such that $h \equiv 0$ and let $\Psi : K \to \mathbb{F}_C(\mathbb{R})$ be a FIVF, whose β-levels define the family of interval valued functions $\Psi_\beta : K \subset \mathbb{R} \to \mathcal{K}_C^+ \subset \mathcal{K}_C$ are given by*

$$\Psi_\beta(z) = \left[\Psi_*(z,\beta), \Psi^*(z,\beta)\right], \tag{13}$$

for all $z \in K$ and for all $\beta \in [0,1]$. Then Ψ is h-convex on K, if and only if, for all $\beta \in [0,1]$, $\Psi_*(z,\beta)$ and $\Psi^*(z,\beta)$ are h-convex.

Proof. Assume that for each $\beta \in [0,1]$, $\Psi_*(z,\beta)$ and $\Psi^*(z,\beta)$ are h-convex on K. Then from (11), we have.

$$\Psi_*(\xi z + (1-\xi)y, \beta)$$

$$\leq h(\xi)\Psi_*(z,\beta) + h(1-\xi)\Psi_*(y,\beta)$$

for all $z, y \in K, \xi \in [0,1]$,and

$$\Psi^*(\xi z + (1-\xi)y, \beta)$$

$$\leq h(\xi)\Psi^*(z,\beta) + h(1-\xi)\Psi^*(y,\beta),$$

for all $z, y \in K, \xi \in [0,1]$.

Then by (13), (4) and (6), we obtain

$$\Psi_\beta(\xi z + (1-\xi)y)$$

$$= \left[\Psi_*(\xi z + (1-\xi)y, \beta), \Psi^*(\xi z + (1-\xi)y, \beta)\right],$$

$$\leq_I \left[h(\xi)\Psi_*(z,\beta), h(\xi)\Psi^*(z,\beta)\right]$$

$$+\left[h(1-\xi)\Psi_*(y,\beta), h(1-\xi)\Psi^*(y,\beta)\right],$$

that is

$$\Psi(\xi z+(1-\xi)y) \preccurlyeq h(\xi)\Psi(z)\tilde{+}h(1-\xi)\Psi(y), \forall z,y\in K, \xi\in[0,1].$$

Hence, Ψ is h-convex FIVF on K.

Conversely, let Ψ is h-convex FIVF on K. Then for all $z, y \in K$ and $\xi \in [0,1]$, we have

$$\Psi(\xi z+(1-\xi)y) \preccurlyeq h(\xi)\Psi(z)\tilde{+}h(1-\xi)\Psi(y).$$

Therefore, from (13), we have

$$\Psi_\beta(\xi z+(1-\xi)y)$$

$$=\left[\Psi_*(\xi z+(1-\xi)y,\beta), \Psi^*(\xi z+(1-\xi)y,\beta)\right].$$

Again, from (13), we obtain

$$h(\xi)\Psi_\beta(z)+h(1-\xi)\Psi_\beta(z)$$

$$=\left[h(\xi)\Psi_*(z,\beta), h(\xi)\Psi^*(z,\beta)\right]$$

$$+\left[h(1-\xi)\Psi_*(y,\beta), h(1-\xi)\Psi^*(y,\beta)\right], \tag{14}$$

for all $z, y \in K$ and $\xi \in [0,1]$. Then by h-convexity of Ψ, we have for all $z, y \in K$ and $\xi \in [0,1]$ such that

$$\Psi_*(\xi z+(1-\xi)y,\beta) \le h(\xi)\Psi_*(z,\beta)+h(1-\xi)\Psi_*(y,\beta),$$

and

$$\Psi^*(\xi z+(1-\xi)y,\beta)$$

$$\le h(\xi)\Psi^*(z,\beta)+h(1-\xi)\Psi^*(y,\beta),$$

for each $\beta \in [0,1]$. Hence, this concludes the proof.

Remark 2.11. If $\Psi_*(z,\beta)=\Psi^*(z,\beta)$ with $\beta=1$, then h-convex FIVF reduces to the classical h-convex function, see [29]. If $\Psi_*(z,\beta)=\Psi^*(z,\beta)$ with $\beta=1$ and $h(\xi)=\xi^s$ with $s\in(0,1)$ then h-convex FIVF becomes the classical s-convex function, see [6].

If $\Psi_*(z,\beta)=\Psi^*(z,\beta)$ with $\beta=1$ and $h(\xi)=\xi$, then h-convex FIVF reduces to the classical convex function.

If $\Psi_*(z,\beta)=\Psi^*(z,\beta)$ with $\beta=1$ and $h(\xi)=1$ then h-convex FIVF becomes the classical p-convex function, see [14].

Example 2.12. We consider $h(\xi)=\xi$, for $\xi\in[0,1]$ and the FIVF $\Psi:[0,1]\to\mathbb{F}_C(\mathbb{R})$ defined by

$$\Psi(z)(\sigma)=\begin{cases}\frac{\sigma}{2z^2}\sigma\in\left[0,2z^2\right]\\ \frac{4z^2-\sigma}{2z^2}\sigma\in\left(2z^2,4z^2\right]\\ 0\text{otherwise,}\end{cases}$$

Then, for each $\beta\in[0,1]$, we have $\Psi_\beta(z)=\left[2\beta z^2,(4-2\beta)z^2\right]$. Since end point functions $\Psi_*(z,\beta), \Psi^*(z,\beta)$ are h-convex functions for each $\beta\in[0,1]$. Hence $\Psi(z)$ is h-convex FIVF.

3 Main Results

In this section, we propose fuzzy-interval H–H inequalities for h-convex FIVFs. Furthermore, several examples are given to demonstrate the applicability of the theory produced in this research.

Theorem 3.1. *Let $\Psi:[\sqcap,\nu]\to\mathbb{F}_C(\mathbb{R})$ be a h-convex FIVF with non-negative real valued function $h:[0,1]\to\mathbb{R}$ and $h\left(\frac{1}{2}\right)\neq 0$, and for all $\beta\in[0,1]$, $\Psi_\beta: K\subset\mathbb{R}\to\mathcal{K}_C{}^+\subset\mathcal{K}_C$ represent the family of IVFs through β-levels. If $\Psi\in\mathcal{FR}_{([\sqcap,\nu],\beta)}$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi\left(\frac{\sqcap+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-\sqcap}(FR)\int_\sqcap^\nu\Psi(\ddagger)d\ddagger$$

$$\preccurlyeq\left[\Psi(\sqcap)\tilde{+}\Psi(\nu)\right]\int_0^1 h(\xi)d\xi. \tag{15}$$

Proof. Let $\Psi:[\sqcap,\nu]\to\mathbb{F}_C(\mathbb{R})$, h-convex FIVF. Then, by hypothesis, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi\left(\frac{\sqcap+\nu}{2}\right)\preccurlyeq\Psi(\xi\sqcap+(1-\xi)\nu)\tilde{+}\Psi((1-\xi)\sqcap+\xi\nu).$$

Therefore, for every $\beta\in[0,1]$, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi*\left(\frac{\sqcap+\nu}{2},\beta\right)$$
$$\le\Psi*(\xi\sqcap+(1-\xi)\nu,\beta)+\Psi*((1-\xi)\sqcap+\xi\nu,\beta),$$
$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi^*\left(\frac{\sqcap+\nu}{2},\beta\right)$$
$$\le\Psi^*(\xi\sqcap+(1-\xi)\nu,\beta)+\Psi^*((1-\xi)\sqcap+\xi\nu,\beta).$$

Then

$$\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1 \Psi_*\left(\frac{\mu+\nu}{2},\beta\right)\mathrm{d}\xi \le \int_0^1 \Psi_*(\xi\mu+(1-\xi)\nu,\beta)\mathrm{d}\xi + \int_0^1 \Psi_*((1-\xi)\nu+\xi\nu,\beta)\mathrm{d}\xi,$$

$$\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1 \Psi^*\left(\frac{\mu+\nu}{2},\beta\right)\mathrm{d}\xi \le \int_0^1 \Psi^*(\xi\mu+(1-\xi)\nu,\beta)\mathrm{d}\xi + \int_0^1 \Psi^*((1-\xi)\mu+\xi\nu,\beta)\mathrm{d}\xi.$$

It follows that

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi_*\left(\frac{\mu+\nu}{2},\beta\right) \le \frac{2}{\nu-\mu}\int_\mu^\nu \Psi_*(z,\beta)\mathrm{d}z,$$

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi^*\left(\frac{\mu+\nu}{2},\beta\right) \le \frac{2}{\nu-\mu}\int_\mu^\nu \Psi^*(z,\beta)\mathrm{d}z.$$

That is

$$\frac{1}{h\left(\frac{1}{2}\right)}\left[\Psi_*\left(\frac{\mu+\nu}{2},\beta\right),\Psi^*\left(\frac{\mu+\nu}{2},\beta\right)\right]$$

$$\le_I \frac{2}{\nu-\mu}\left[\int_\mu^\nu \Psi_*(\mu,\beta)\mathrm{d}z, \int_\mu^\nu \Psi^*(z,\beta)\mathrm{d}z\right].$$

Thus,

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi\left(\frac{\mu+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-\mu}(\mathrm{FR})\int_\mu^\nu \Psi(z)\mathrm{d}z. \tag{16}$$

In a similar way as above, we have

$$\frac{1}{\nu-\mu}(\mathrm{FR})\int_\mu^\nu \Psi(z)\mathrm{d}z\preccurlyeq\left[\Psi(\mu)\tilde{+}\Psi(\nu)\right]\int_0^1 h(\xi)\mathrm{d}\xi. \tag{17}$$

Combining (16) and (17), we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi\left(\frac{\mu+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-\mu}(\mathrm{FR})\int_\mu^\nu \Psi(z)\mathrm{d}z$$

$$\preccurlyeq\left[\Psi(\mu)\tilde{+}\Psi(\nu)\right]\int_0^1 h(\xi)\mathrm{d}\xi.$$

Hence, the required result.

Remark 3.1. If $h(\xi)=\xi^s$, then Theorem 3.1 reduces to the result for s-convex FIVF which is also new one:

$$2^{s-1}\Psi\left(\frac{\mu+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-\mu}(\mathrm{FR})\int_\mu^\nu \Psi(z)\mathrm{d}z$$

$$\preccurlyeq\frac{1}{s+1}\left[\Psi(\mu)\tilde{+}\Psi(\nu)\right].$$

If $h(\xi)=\xi$, then Theorem 3.1 reduces to the result for convex FIVF which is also new one:

$$\Psi\left(\frac{\mu+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-\mu}(\mathrm{FR})\int_\mu^\nu \Psi(z)\mathrm{d}z\preccurlyeq\frac{\Psi(\mu)\tilde{+}\Psi(\nu)}{2}.$$

If $h(\xi)\equiv 1$ then Theorem 3.1 reduces to the result for P FIVF which is also new one:

$$\frac{1}{2}\Psi\left(\frac{\mu+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-\mu}(FR)\int_\mu^\nu \Psi(z)dz\preccurlyeq\Psi(\mu)\tilde{+}\Psi(\nu).$$

If $\Psi_*(z,\beta)=\Psi^*(z,\beta)$ with $\beta=1$, then Theorem 3.1 reduces to the result for classical h-convex function, see [29]:

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi\left(\frac{\mu+\nu}{2}\right)\le\frac{1}{\nu-\mu}(R)\int_\mu^\nu \Psi(z)\mathrm{d}z$$

$$\le[\Psi(\mu)+\Psi(\nu)]\int_0^1 h(\xi)\mathrm{d}\xi.$$

If $\Psi_*(z,\beta)=\Psi^*(z,\beta)$ with $\beta=1$ and $h(\xi)=\xi^s$, then Theorem 3.1 reduces to the result for classical s-convex function, see [21]:

$$2^{s-1}\Psi\left(\frac{\mu+\nu}{2}\right)\le\frac{1}{\nu-\mu}(R)\int_\mu^\nu \Psi(z)\mathrm{d}z$$

$$\le\frac{1}{s+1}[\Psi(\mu)+\Psi(\nu)].$$

If $\Psi_*(z,\beta)=\Psi^*(z,\beta)$ with $\beta=1$ and $h(\xi)=\xi$, then Theorem 3.1 reduces to the result for classical convex function, see [16, 17]:

$$\Psi\left(\frac{\mu+\nu}{2}\right)\le\frac{1}{\nu-\mu}(R)\int_\mu^\nu \Psi(\mu)\mathrm{d}z\le\frac{\Psi(\mu)+\Psi(\nu)}{2}.$$

If $\Psi_*(z,\beta)=\Psi^*(z,\beta)$ with $\beta=1$ and $h(\xi)\equiv 1$ then Theorem 3.1 reduces to the result for classical P-convex function, see [14]:

$$\frac{1}{2}\Psi\left(\frac{\mu+\nu}{2}\right) \le \frac{1}{\nu-\mu}(R)\int_{\mu}^{\nu}\Psi(z)\mathrm{d}z \le \Psi(\mu)+\Psi(\nu).$$

Example 3.1 We consider $h(\xi)=\xi$, for $\xi\in[0,1]$, and the FIVF $\Psi : [\mu,\nu]=[0,2]\to\mathbb{F}_C(\mathbb{R})$ defined by,

$$\Psi(z)(\sigma)=\begin{cases}\frac{\sigma}{2z^2}\sigma\in\left[0,2z^2\right]\\ \frac{4z^2-\sigma}{2z^2}\sigma\in\left(2z^2,4z^2\right]\\ 0\text{otherwise,}\end{cases}$$

Then, for each $\beta\in[0,1]$, we have $\Psi_\beta(z)=\left[2\beta z^2,(4-2\beta)z^2\right]$. Since end point functions $\Psi_*(z,\beta)=2\beta z^2$, $\Psi^*(z,\beta)=(4-2\beta)z^2$ are h-convex functions for each $\beta\in[0,1]$. Hence $\Psi(z)$ is h-convex FIVF. We now computing the following

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi_*\left(\frac{\mu+\nu}{2},\beta\right) \le \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\Psi_*(z,\beta)\mathrm{d}z$$

$$\le \left[\Psi_*(\mu,\beta)+\Psi_*(\nu,\beta)\right]\int_0^1 h(\xi)\mathrm{d}\xi.$$

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi_*\left(\frac{\mu+\nu}{2},\beta\right)=\Psi_*(1,\beta)=2\beta,$$

$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\Psi_*(z,\beta)\mathrm{d}z=\frac{1}{2}\int_0^2 2\beta z^2\mathrm{d}z=\frac{8\beta}{3},$$

$$\left[\Psi_*(\mu,\beta)+\Psi_*(\nu,\beta)\right]\int_0^1 h(\xi)\mathrm{d}\xi=4\beta,$$

for all $\beta\in[0,1]$. That means

$$2\beta\le\frac{8\beta}{3}\le 4\beta.$$

Similarly, it can be easily show that

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi^*\left(\frac{\mu+\nu}{2},\beta\right) \le \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\Psi^*(z,\beta)\mathrm{d}z$$

$$\le \left[\Psi^*(\nu,\beta)+\Psi^*(\nu,\beta)\right]\int_0^1 h(\xi)\mathrm{d}\xi.$$

for all $\beta\in[0,1]$, such that

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi^*\left(\frac{\mu+\nu}{2},\beta\right)=\Psi_*(1,\beta)=(4-2\beta),$$

$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\Psi^*(z,\beta)\mathrm{d}z=\frac{1}{2}\int_0^2(4-2\beta)z^2\mathrm{d}z=\frac{4(4-2\beta)}{3},$$

$$\left[\Psi^*(\mu,\beta)+\Psi^*(\nu,\beta)\right]\int_0^1 h(\xi)\mathrm{d}\xi=2(4-2\beta).$$

From which, we have

$$(4-2\beta)\le\frac{4(4-2\beta)}{3}\le 2(4-2\beta),$$

that is

$$[2\beta,(4-2\beta)]\le_I\left[\frac{8\beta}{3},\frac{4(4-2\beta)}{3}\right]\le_I[4\beta,2(4-2\beta)],$$

for all $\beta\in[0,1]$.

Hence,

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Psi\left(\frac{\mu+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-\mu}(\mathrm{FR})\int_{\mu}^{\nu}\Psi(z)\mathrm{d}z$$

$$\preccurlyeq\left[\Psi(\mu)\tilde{+}\Psi(\nu)\right]\int_0^1 h(\xi)\mathrm{d}\xi.$$

Theorem 3.2. *Let $\Psi : [\mu,\nu]\to\mathbb{F}_C(\mathbb{R})$ be a h-convex FIVF with non-negative real valued function $h : [0,1]\to\mathbb{R}$ and $h\left(\frac{1}{2}\right)\neq 0$, and for all $\beta\in[0,1]$, $\Psi_\beta : K\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ represent the family of IVFs through β-levels. If $\Psi\in\mathcal{FR}_{([\mu,\nu],\beta)}$, then*

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\Psi\left(\frac{\mu+\nu}{2}\right)\preccurlyeq\rhd_2\preccurlyeq\frac{1}{\nu-\mu}(\mathrm{FR})\int_{\mu}^{\nu}\Psi(z)\mathrm{d}z\preccurlyeq\rhd_1$$

$$\preccurlyeq\left[\Psi(\mu)\tilde{+}\Psi(\nu)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi,$$

where

$$\rhd_1 = \left[\frac{\Psi(\mu)\tilde{+}\Psi(\nu)}{2}\tilde{+}\Psi\left(\frac{\mu+\nu}{2}\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi,$$

$$\rhd_2 = \frac{1}{4h\left(\frac{1}{2}\right)}\left[\Psi\left(\frac{3\mu+\nu}{4}\right)\tilde{+}\Psi\left(\frac{\mu+3\nu}{4}\right)\right],$$

and $\rhd_1 = [\rhd_{1*}, \rhd_1^*], \rhd_2 = [\rhd_{2*}, \rhd_2^*]$.

Proof. Take $\left[\mu, \frac{\mu+\nu}{2}\right]$, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi\left(\frac{\xi\mu+(1-\xi)\frac{\mu+\nu}{2}}{2}+\frac{(1-\xi)\mu+\xi\frac{\mu+\nu}{2}}{2}\right)$$

$$\preccurlyeq\Psi\left(\xi\mu+(1-\xi)\frac{\mu+\nu}{2}\right)\tilde{+}\Psi\left((1-\xi)\mu+\xi\frac{\mu+\nu}{2}\right).$$

Therefore, for every $\beta\in[0,1]$, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi_*\left(\frac{\xi\mu+(1-\xi)\frac{\mu+\nu}{2}}{2}+\frac{(1-\xi)\mu+\xi\frac{\mu+\nu}{2}}{2},\beta\right)$$

$$\leq \Psi_*\left(\xi\mu+(1-\xi)\frac{\mu+\nu}{2},\beta\right)+\Psi_*\left((1-\xi)\mu+\xi\frac{\mu+\nu}{2},\beta\right),$$

$$\frac{1}{h\left(\frac{1}{2}\right)}\Psi^*\left(\frac{\xi\mu+(1-\xi)\frac{\mu+\nu}{2}}{2}+\frac{(1-\xi)\mu+\xi\frac{\mu+\nu}{2}}{2},\beta\right)$$
$$\leq \Psi^*\left(\xi\mu+(1-\xi)\frac{\mu+\nu}{2},\beta\right)+\Psi^*\left((1-\xi)\mu+\xi\frac{\mu+\nu}{2},\beta\right).$$

In consequence, we obtain

$$\frac{1}{4h\left(\frac{1}{2}\right)}\Psi_*\left(\frac{3\mu+\nu}{4},\beta\right)\leq\frac{1}{\nu-\mu}\int_\mu^{\frac{\mu+\nu}{2}}\Psi_*(z,\beta)\mathrm{d}z,$$
$$\frac{1}{4h\left(\frac{1}{2}\right)}\Psi^*\left(\frac{3\mu+\nu}{4},\beta\right)\leq\frac{1}{\nu-\mu}\int_\mu^{\frac{\mu+\nu}{2}}\Psi^*(z,\beta)\mathrm{d}z.$$

That is

$$\frac{1}{4h\left(\frac{1}{2}\right)}\left[\Psi_*\left(\frac{3\mu+\nu}{4},\beta\right),\Psi^*\left(\frac{3\mu+\nu}{4},\beta\right)\right]$$

$$\leq_I \frac{1}{\nu-\mu}\left[\int_\mu^{\frac{\mu+\nu}{2}}\Psi_*(z,\beta)\mathrm{d}z,\int_\mu^{\frac{\mu+\nu}{2}}\Psi^*(z,\beta)\mathrm{d}z\right].$$

It follows that

$$\frac{1}{4h\left(\frac{1}{2}\right)}\Psi\left(\frac{3\mu+\nu}{4}\right)\preccurlyeq\frac{1}{\nu-\mu}\int_\mu^{\frac{\mu+\nu}{2}}\Psi(z)\mathrm{d}z. \tag{18}$$

In a similar way as above, we have

$$\frac{1}{4h\left(\frac{1}{2}\right)}\Psi\left(\frac{\mu+3\nu}{4}\right)\preccurlyeq\frac{1}{\nu-\mu}\int_{\frac{\mu+\nu}{2}}^{\nu}\Psi(z)\mathrm{d}z. \tag{19}$$

Combining (18) and (19), we have

$$\frac{1}{4h\left(\frac{1}{2}\right)}\left[\Psi\left(\frac{3\mu+\nu}{4}\right)\tilde{+}\Psi\left(\frac{\mu+3\nu}{4}\right)\right]\preccurlyeq\frac{1}{\nu-\mu}\int_\mu^\nu\Psi(z)\mathrm{d}z.$$

Using Theorem 3.1, we have

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\Psi\left(\frac{\mu+\nu}{2}\right)=\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\Psi\left(\frac{1}{2}.\frac{3\mu+\nu}{4}+\frac{1}{2}.\frac{\mu+3\nu}{4}\right).$$

Therefore, for every $\beta\in[0,1]$, we have

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\Psi_*\left(\frac{\mu+\nu}{2},\beta\right)=\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\Psi_*\left(\frac{1}{2}.\frac{3\mu+\nu}{4}+\frac{1}{2}.\frac{\mu+3\nu}{4},\beta\right),$$
$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\Psi^*\left(\frac{\mu+\nu}{2},\beta\right)=\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\Psi^*\left(\frac{1}{2}.\frac{3\mu+\nu}{4}+\frac{1}{2}.\frac{\mu+3\nu}{4},\beta\right),$$

$$\leq\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\left[h\left(\frac{1}{2}\right)\Psi_*\left(\frac{3\mu+\nu}{4},\beta\right)+h\left(\frac{1}{2}\right)\Psi_*\left(\frac{\mu+3\nu}{4},\beta\right)\right],$$
$$\leq\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\left[h\left(\frac{1}{2}\right)\Psi^*\left(\frac{3\mu+\nu}{4},\beta\right)+h\left(\frac{1}{2}\right)\Psi^*\left(\frac{\mu+3\nu}{4},\beta\right)\right],$$

$$=\rhd_{2*},$$
$$=\rhd_2^*,$$

$$\leq\frac{1}{\nu-\mu}\int_\mu^\nu\Psi_*(z,\beta)\mathrm{d}z,$$
$$\leq\frac{1}{\nu-\mu}\int_\mu^\nu\Psi^*(z,\beta)\mathrm{d}z,$$

$$\leq\left[\frac{\Psi_*(\mu,\beta)+\Psi_*(\nu,\beta)}{2}+\Psi_*\left(\frac{\mu+\nu}{2},\beta\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi,$$
$$\leq\left[\frac{\Psi^*(\mu,\beta)+\Psi^*(\nu,\beta)}{2}+\Psi^*\left(\frac{\mu+\nu}{2},\beta\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi,$$

$$=\rhd_{1*},$$
$$=\rhd_1^*,$$

$$\leq \left[\frac{\Psi_*(\mu,\beta)+\Psi_*(\nu,\beta)}{2} + h(\xi)\begin{pmatrix}\Psi_*(\mu,\beta)+ \\ \Psi_*(\nu,\beta)\end{pmatrix}\right]\int_0^1 h(\xi)\mathrm{d}\xi,$$
$$\leq \left[\frac{\Psi^*(\mu,\beta)+\Psi^*(\nu,\beta)}{2} + h(\xi)\begin{pmatrix}\Psi^*(\mu,\beta)+ \\ \Psi^*(\nu,\beta)\end{pmatrix}\right]\int_0^1 h(\xi)\mathrm{d}\xi,$$

$$= \left[\Psi_*(\mu,\beta)+\Psi_*(\nu,\beta)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi,$$
$$= \left[\Psi^*(\mu,\beta)+\Psi^*(\nu,\beta)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi,$$

that is

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\Psi\left(\frac{\mu+\nu}{2}\right) \preccurlyeq \rhd_2 \preccurlyeq \frac{1}{\nu-\mu}(\mathrm{FR})\int_\mu^\nu \Psi(z)\mathrm{d}z \preccurlyeq \rhd_1$$

$$\preccurlyeq \left[\Psi(\mu)\tilde{+}\Psi(\nu)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi.$$

Hence, the proof has been completed.

Example 3.2. We consider $h(\xi)=\xi$, for $\xi\in[0,1]$, and the FIVF $\Psi:[\mu,\nu]=[0,2]\to\mathbb{F}_C(\mathbb{R})$ defined by, $\Psi_\beta(z)=\left[2\beta z^2,(4-2\beta)z^2\right]$, as in Example 3.1, then $\Psi(z)$ is h-convex FIVF and satisfying (10). We have $\Psi_*(z,\beta)=\beta z$ and $\Psi^*(z,\beta)=(4-2\beta)z$. We now computing the following.

$$\left[\Psi_*(\mu,\beta)+\Psi_*(\nu,\beta)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi = 4\beta,$$
$$\left[\Psi^*(\mu,\beta)+\Psi^*(\nu,\beta)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi = 4(2-\beta),$$

$$\rhd_{1*} = \left[\frac{\Psi_*(\mu,\beta)+\Psi_*(\nu,\beta)}{2}+\Psi_*\left(\frac{\mu+\nu}{2},\beta\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi = 3\beta,$$
$$\rhd_1^* = \left[\frac{\Psi^*(\mu,\beta)+\Psi^*(\nu,\beta)}{2}+\Psi^*\left(\frac{\mu+\nu}{2},\beta\right)\right]\int_0^1 h(\xi)\mathrm{d}\xi = 3(2-\beta),$$

$$\rhd_{2*} = \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\left[h\left(\frac{1}{2}\right)\Psi_*\left(\frac{3\mu+\nu}{4},\beta\right)+h\left(\frac{1}{2}\right)\Psi_*\left(\frac{\mu+3\nu}{4},\beta\right)\right] = \frac{5}{2}\beta,$$
$$\rhd_2^* = \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\left[h\left(\frac{1}{2}\right)\Psi^*\left(\frac{3\mu+\nu}{4},\beta\right)+h\left(\frac{1}{2}\right)\Psi^*\left(\frac{\mu+3\nu}{4},\beta\right)\right] = \frac{5}{2}(2-\beta).$$

Then we obtain that

$$2\beta \leq \frac{5}{2}\beta \leq \frac{8\beta}{3} \leq 3\beta \leq 4\beta,$$
$$2(2-\beta) \leq \frac{5}{2}(2-\beta) \leq \frac{8(2-\beta)}{3} \leq 3(2-\beta) \leq 4(2-\beta).$$

Hence, Theorem 3.2 is verified.

Following results find the new versions of H–H inequalities for the product of two h-convex FIVFs.

Theorem 3.3. *Let $\Psi,\mathcal{J}:[\mu,\nu]\to\mathbb{F}_C(\mathbb{R})$ be two h-convex FIVFs with non-negative real valued functions $h_1,h_2:[0,1]\to\mathbb{R}$, and for all $\beta\in[0,1]$, $\Psi_\beta: K\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ represent the family of IVFs through β-levels. If $\Psi,\mathcal{J}$ and $\Psi\mathcal{J}\in\mathcal{FR}_{([\mu,\nu],\beta)}$, then*

$$\frac{1}{\nu-\mu}(\mathrm{FR})\int_\mu^\nu \Psi(\mu)\tilde{\times}\mathcal{J}(z)\mathrm{d}z$$

$$\preccurlyeq \mathcal{M}(\mu,\nu)\int_0^1 h_1(\xi)h_2(\xi)\mathrm{d}\xi$$

$$\tilde{+}\mathcal{N}(\mu,\nu)\int_0^1 h_1(\xi)h_2(1-\xi)\mathrm{d}\xi,$$

where $\mathcal{M}(\mu,\nu)=\Psi(\mu)\tilde{\times}\mathcal{J}(\mu)\tilde{+}\Psi(\nu)\tilde{\times}\mathcal{J}(\nu)$, $\mathcal{N}(\mu,\nu)=\Psi(\mu)\tilde{\times}\mathcal{J}(\nu)\tilde{+}\Psi(\nu)\tilde{\times}\mathcal{J}(\mu)$, and $\mathcal{M}_\beta(\mu,\nu)=\left[\mathcal{M}*((\mu,\nu),\beta),\mathcal{M}^*((\mu,\nu),\beta)\right]$ and $\mathcal{N}_\beta(\mu,\nu)=\left[\mathcal{N}*((\mu,\nu),\beta),\mathcal{N}^*((\mu,\nu),\beta)\right]$.

Example 3.3. We consider $h_1(\xi)=\xi, h_2(\xi)\equiv 1$, for $\xi\in[0,1]$, and the fuzzy interval-valued functions $\Psi,\mathcal{J}:[\mu,\nu]=[0,1]\to\mathbb{F}_C(\mathbb{R})$ defined by,

$$\Psi(z)(\sigma)=\begin{cases}\frac{\sigma}{2z^2}\sigma\in\left[0,2z^2\right]\\ \frac{4z^2-\sigma}{2z^2}\sigma\in\left(2z^2,4z^2\right]\\ 0\text{otherwise,}\end{cases}$$

$$\mathcal{J}(z)(\sigma)=\begin{cases}\frac{\sigma}{z}\sigma\in[0,z]\\ \frac{2z-\sigma}{z}\sigma\in(z,2z]\\ 0\text{otherwise,}\end{cases}$$

Then, for each $\beta\in[0,1]$, we have $\Psi_\beta(z)=\left[2\beta z^2,(4-2\beta)z^2\right]$ and $\mathcal{J}_\beta(z)=[\beta z,(2-\beta)z]$. Since end point functions $\Psi_*(z,\beta)=2\beta z^2$, $\Psi^*(z,\beta)=(4-2\beta)z^2$ and $\mathcal{J}*(z,\beta)=\beta z$, $\mathcal{J}^*(z,\beta)=(2-\beta)z$ h_1 and h_2-convex functions for each $\beta\in[0,1]$, respectively. Hence $\Psi,\mathcal{J}$ both are h-convex FIVFs**.** We now computing the following

$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu} \Psi_*(z,\beta)\times \mathcal{J}*(z,\beta)\mathrm{d}z = \frac{\beta^2}{2},$$
$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu} \Psi^*(z,\beta)\times \mathcal{J}^*(z,\beta)\mathrm{d}z = \frac{(2-\beta)^2}{2},$$

$$\mathcal{M}*((\mu,\nu),\beta)\int_0^1 h_1(\xi)h_2(\xi)\mathrm{d}\xi = \beta^2,$$
$$\mathcal{M}^*((\mu,\nu),\beta)\int_0^1 h_1(\xi)h_2(\xi)\mathrm{d}\xi = (2-\beta)^2,$$

$$\mathcal{N}*((\mu,\nu),\beta)\int_0^1 h_1(\xi)h_2(1-\xi)\mathrm{d}\xi = 0$$
$$\mathcal{N}^*((\mu,\nu),\beta)\int_0^1 h_1(\xi)h_2(1-\xi)\mathrm{d}\xi = 0,$$

for each $\beta\in[0,1]$, that means

$$\frac{\beta^2}{2} \le \beta^2+0=\beta^2,$$
$$\frac{(2-\beta)^2}{2} \le (2-\beta)^2+0=(2-\beta)^2,$$

Hence, Theorem 3.3 is demonstrated.

Theorem 3.4 *Let $\Psi,\mathcal{J}:[\sqcap,\nu]\to\mathbb{F}_C(\mathbb{R})$ be two h-convex FIVFs with non-negative real valued functionsh$_1,h_2:[0,1]\to\mathbb{R}$, respectively and $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\neq 0$, and for all$\beta\in[0,1]$, $\Psi_\beta: K\subset\mathbb{R}\to\mathcal{K}_C^{+}\subset\mathcal{K}_C$ represent the family of IVFs through β-levels. If $\Psi\mathcal{J}\in\mathcal{FR}_{([\sqcap,\nu],\beta)}$, then*

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Psi\left(\frac{\sqcap+\nu}{2}\right)\tilde{\times}\mathcal{J}\left(\frac{\sqcap+\nu}{2}\right)$$

$$\preccurlyeq \frac{1}{\nu-\sqcap}(FR)\int_{\sqcap}^{\nu}\Psi(\ddagger)\tilde{\times}\mathcal{J}(\ddagger)\mathrm{d}\ddagger$$

$$\tilde{+}\mathcal{M}(\sqcap,\nu)\int_0^1 h_1(\xi)h_2(1-\xi)\mathrm{d}\xi\ \tilde{+}\ \mathcal{N}(\sqcap,\nu)\int_0^1 h_1(\xi)h_2(\xi)\mathrm{d}\xi,$$

where $\mathcal{M}(\sqcap,\nu)=\Psi(\sqcap)\tilde{\times}\mathcal{J}(\sqcap)\tilde{+}\Psi(\nu)\tilde{\times}\mathcal{J}(\nu)$, $\mathcal{N}(\sqcap,\nu)=\Psi(\sqcap)\tilde{\times}\mathcal{J}(\nu)\tilde{+}\Psi(\nu)\tilde{\times}\mathcal{J}(\sqcap)$, *and* $\mathcal{M}_\beta(\sqcap,\nu)=\left[\mathcal{M}*((\sqcap,\nu),\beta),\mathcal{M}^*((\sqcap,\nu),\beta)\right]$ *and* $\mathcal{N}_\beta(\sqcap,\nu)=\left[\mathcal{N}*((\sqcap,\nu),\beta),\mathcal{N}^*((\sqcap,\nu),\beta)\right]$.

Proof. By hypothesis, for each $\beta\in[0,1]$, we have

$$\Psi*\left(\frac{\sqcap+\nu}{2},\beta\right)\times\mathcal{J}*\left(\frac{\sqcap+\nu}{2},\beta\right)$$
$$\Psi^*\left(\frac{\sqcap+\nu}{2},\beta\right)\times\mathcal{J}^*\left(\frac{\sqcap+\nu}{2},\beta\right)$$

$$\le h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\times\mathcal{J}*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\\+\Psi*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\times\mathcal{J}*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\end{bmatrix}$$

$$+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\times\mathcal{J}*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\\+\Psi*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\times\mathcal{J}*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\end{bmatrix},$$

$$\le h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi^*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\times\mathcal{J}^*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\\+\Psi^*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\times\mathcal{J}^*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\end{bmatrix}$$
$$+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi^*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\times\mathcal{J}^*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\\+\Psi^*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\times\mathcal{J}^*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\end{bmatrix},$$

$$\le h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\times\mathcal{J}*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\\+\Psi*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\times\mathcal{J}*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\end{bmatrix}$$
$$+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\begin{pmatrix}h_1(\xi)\Psi*(\sqcap,\beta)+\\h_1(1-\xi)\Psi*(\nu,\beta)\end{pmatrix}\\\times\begin{pmatrix}h_2(1-\xi)\mathcal{J}*(\sqcap,\beta)+\\h_2(\xi)\mathcal{J}*(\nu,\beta)\end{pmatrix}\\+\begin{pmatrix}h_1(1-\xi)\Psi*(\sqcap,\beta)+\\h_1(\xi)\Psi*(\nu,\beta)\end{pmatrix}\\\times\begin{pmatrix}h_2(\xi)\mathcal{J}*(\sqcap,\beta)+\\h_2(1-\xi)\mathcal{J}*(\nu,\beta)\end{pmatrix}\end{bmatrix},$$

$$\le h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi^*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\times\mathcal{J}^*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\\+\Psi^*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\times\mathcal{J}^*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\end{bmatrix}$$
$$+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\begin{pmatrix}h_1(\xi)\Psi^*(\sqcap,\beta)+\\h_1(1-\xi)\Psi^*(\nu,\beta)\end{pmatrix}\\\times\begin{pmatrix}h_2(1-\xi)\mathcal{J}^*(\sqcap,\beta)+\\h_2(\xi)\mathcal{J}^*(\nu,\beta)\end{pmatrix}\\+\begin{pmatrix}h_1(1-\xi)\Psi^*(\sqcap,\beta)+\\h_1(\xi)\Psi^*(\nu,\beta)\end{pmatrix}\\\times\begin{pmatrix}h_2(\xi)\mathcal{J}^*(\sqcap,\beta)+\\h_2(1-\xi)\mathcal{J}^*(\nu,\beta)\end{pmatrix}\end{bmatrix},$$

$$= h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\begin{bmatrix}\Psi*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\times\mathcal{J}*\begin{pmatrix}\xi u+\\(1-\xi)\nu,\beta\end{pmatrix}\\+\Psi*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\times\mathcal{J}*\begin{pmatrix}(1-\xi)u+\\\xi\nu,\beta\end{pmatrix}\end{bmatrix}$$

$$+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\left\{\begin{array}{c}h_1(\xi)h_2(\xi)+\\ h_1(1-\xi)h_2(1-\xi)\end{array}\right\}\mathcal{N}*((\sqcap,\nu),\beta)\\ +\left\{\begin{array}{c}h_1(\xi)h_2(1-\xi)+\\ h_1(1-\xi)h_2(\xi)\end{array}\right\}\mathcal{M}*((\sqcap,\nu),\beta)\end{array}\right],$$

$$=h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\Psi^*\left(\begin{array}{c}\xi u+\\ (1-\xi)\nu,\beta\end{array}\right)\times\mathcal{J}^*\left(\begin{array}{c}\xi u+\\ (1-\xi)\nu,\beta\end{array}\right)\\ +\Psi^*\left(\begin{array}{c}(1-\xi)u+\\ \xi\nu,\beta\end{array}\right)\times\mathcal{J}^*\left(\begin{array}{c}(1-\xi)u+\\ \xi\nu,\beta\end{array}\right)\end{array}\right]$$

$$+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\left\{\begin{array}{c}h_1(\xi)h_2(\xi)+\\ h_1(1-\xi)h_2(1-\xi)\end{array}\right\}\mathcal{N}^*((\sqcap,\nu),\beta)\\ +\left\{\begin{array}{c}h_1(\xi)h_2(1-\xi)+\\ h_1(1-\xi)h_2(\xi)\end{array}\right\}\mathcal{M}^*((\sqcap,\nu),\beta)\end{array}\right],$$

Integrating over [0, 1], we have

$$\begin{aligned}&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Psi*\left(\frac{\sqcap+\nu}{2},\beta\right)\times\mathcal{J}*\left(\frac{\sqcap+\nu}{2},\beta\right)\\ &\le\frac{1}{\nu-\sqcap}(R)\int_\sqcap^\nu\Psi*(\ddagger,\beta)\times\mathcal{J}*(\ddagger,\beta)\mathrm{d}\ddagger\\ &+\mathcal{M}*((\sqcap,\nu),\beta)\int_0^1h_1(\xi)h_2(1-\xi)\mathrm{d}\xi\\ &+\mathcal{N}*((\sqcap,\nu),\beta)\int_0^1h_1(\xi)h_2(\xi)\mathrm{d}\xi,\\ &\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Psi^*\left(\frac{\sqcap+\nu}{2},\beta\right)\times\mathcal{J}^*\left(\frac{\sqcap+\nu}{2},\beta\right)\\ &\le\frac{1}{\nu-\sqcap}(R)\int_\sqcap^\nu\Psi^*(\ddagger,\beta)\times\mathcal{J}^*(\ddagger,\beta)\mathrm{d}\ddagger\\ &+\mathcal{M}^*((\sqcap,\nu),\beta)\int_0^1h_1(\xi)h_2(1-\xi)\mathrm{d}\xi\\ &+\mathcal{N}^*((\sqcap,\nu),\beta)\int_0^1h_1(\xi)h_2(\xi)\mathrm{d}\xi,\end{aligned}$$

that is

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Psi\left(\frac{\sqcap+\nu}{2}\right)\tilde{\times}\mathcal{J}\left(\frac{\sqcap+\nu}{2}\right)$$

$$\preccurlyeq\frac{1}{\nu-\sqcap}(FR)\int_\sqcap^\nu\Psi(\ddagger)\tilde{\times}\mathcal{J}(\ddagger)\mathrm{d}\ddagger$$

$$\tilde{+}\mathcal{M}(\sqcap,\nu)\int_0^1h_1(\xi)h_2(1-\xi)\mathrm{d}\xi\tilde{+}\mathcal{N}(\sqcap,\nu)\int_0^1h_1(\xi)h_2(\xi)\mathrm{d}\xi,$$

Hence, the required result.

Example 3.4. We consider $h_1(\xi)=\xi, h_2(\xi)\equiv 1$, for $\xi\in[0,1]$, and the FIVFs $\Psi,\mathcal{J}:[\sqcap,\nu]=[0,1]\to\mathbb{F}_C(\mathbb{R})$, as in Example 3.3. Then, for each $\beta\in[0,1]$, we have $\Psi_\beta(\ddagger)=\left[2\beta\ddagger^2,(4-2\beta)\ddagger^2\right]$ and $\mathcal{J}_\beta(\ddagger)=[\beta\ddagger,(2-\beta)\ddagger]$ and, $\Psi(\ddagger)$, $\mathcal{J}(\ddagger)$ both are h_1-convex and h_2-convex FIVFs, respectively. We have $\Psi*(\ddagger,\beta)=2\beta\ddagger^2, \Psi^*(\ddagger,\beta)=(4-2\beta)\ddagger^2$ and $\mathcal{J}*(\ddagger,\beta)=\beta\ddagger, \mathcal{J}^*(\ddagger,\beta)=(2-\beta)\ddagger$. We now computing the following.

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Psi*\left(\frac{\sqcap+\nu}{2},\beta\right)\times\mathcal{J}*\left(\frac{\sqcap+\nu}{2},\beta\right)=\frac{\beta^2}{4},$$
$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Psi^*\left(\frac{\sqcap+\nu}{2},\beta\right)\times\mathcal{J}^*\left(\frac{\sqcap+\nu}{2},\beta\right)=\frac{(2-\beta)^2}{4},$$

$$\frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi*(\ddagger,\beta)\times\mathcal{J}*(\ddagger,\beta)\mathrm{d}\ddagger=\frac{\beta^2}{2},$$
$$\frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi^*(\ddagger,\beta)\times\mathcal{J}^*(\ddagger,\beta)\mathrm{d}\ddagger=\frac{(2-\beta)^2}{2},$$

$$\mathcal{M}*((\sqcap,\nu),\beta)\int_0^1h_1(\xi)h_2(1-\xi)\mathrm{d}\xi=\beta^2,$$
$$\mathcal{M}^*((\sqcap,\nu),\beta)\int_0^1h_1(\xi)h_2(1-\xi)\mathrm{d}\xi=(2-\beta)^2,$$

$$\mathcal{N}*((\sqcap,\nu),\beta)\int_0^1h_1(\xi)h_2(\xi)\mathrm{d}\xi=0,$$
$$\mathcal{N}^*((\sqcap,\nu),\beta)\int_0^1h_1(\xi)h_2(\xi)\mathrm{d}\xi=0,$$

for each $\beta\in[0,1]$, we conclude that

$$\frac{\beta^2}{4}\le\frac{\beta^2}{2}+\beta^2+0=\frac{3\beta^2}{2},$$
$$\frac{(2-\beta)^2}{4}\le\frac{(2-\beta)^2}{2}+(2-\beta)^2+0=\frac{3(2-\beta)^2}{2},$$

hence, Theorem 3.4 is demonstrated.

We now give *HH*-Fej´er inequalities for *h*-convex FIVFs. Firstly, we obtain the second *HH*-Fej´er inequality for *h*-convex FIVF.

Theorem 3.5. *Let* $\Psi:[\sqcap,\nu]\to\mathbb{F}_C(\mathbb{R})$ *be a h-convex FIVF with* $\sqcap<\nu$ *and* $h:[0,1]\to\mathbb{R}^+$, *and for all* $\beta\in[0,1]$, $\Psi_\beta:K\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ *represent the family of IVFs through β-levels. If* $\Psi\in\mathcal{FR}_{([\sqcap,\nu],\beta)}$ *and* $\nabla:[\sqcap,\nu]\to\mathbb{R}, \nabla(\ddagger)\ge 0$, *symmetric with respect to* $\frac{\sqcap+\nu}{2}$, *then*

$$\frac{1}{\nu-\sqcap}(\mathrm{FR})\int_\sqcap^\nu\Psi(\ddagger)\nabla(\ddagger)\mathrm{d}\ddagger$$

$$\preccurlyeq\left[\Psi(\sqcap)\tilde{+}\Psi(\nu)\right]\int_0^1h(\xi)\nabla((1-\xi)\sqcap+\xi\nu)\mathrm{d}\xi. \tag{20}$$

Proof. Let Ψ be a *h*-convex FIVF. Then, for each $\beta\in[0,1]$, we have

$$\begin{aligned}&\Psi*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla(\xi\sqcap+(1-\xi)\nu)\\ &\le(h(\xi)\Psi*(\sqcap,\beta)+h(1-\xi)\Psi*(\nu,\beta))\nabla\left(\begin{array}{c}\xi u+\\ (1-\xi)\nu\end{array}\right),\\ &\Psi^*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla(\xi\sqcap+(1-\xi)\nu)\\ &\le(h(\xi)\Psi^*(\sqcap,\beta)+h(1-\xi)\Psi^*(\nu,\beta))\nabla\left(\begin{array}{c}\xi u+\\ (1-\xi)\nu\end{array}\right).\end{aligned} \tag{21}$$

And

$$\begin{aligned}&\Psi_*((1-\xi)\sqcap+\xi\nu,\beta)\nabla((1-\xi)\sqcap+\xi\nu)\\&\leq(h(1-\xi)\Psi_*(\sqcap,\beta)+h(\xi)\Psi_*(\nu,\beta))\nabla\begin{pmatrix}(1-\xi)u\\+\xi\nu\end{pmatrix},\\&\Psi^*((1-\xi)\sqcap+\xi\nu,\beta)\nabla((1-\xi)\sqcap+\xi\nu)\\&\leq(h(1-\xi)\Psi^*(\sqcap,\beta)+h(\xi)\Psi^*(\nu,\beta))\nabla\begin{pmatrix}(1-\xi)u\\+\xi\nu\end{pmatrix}.\end{aligned}\tag{22}$$

After adding (21) and (22), and integrating over $[0,1]$, we get

$$\begin{aligned}&\int_0^1\Psi_*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla(\xi\sqcap+(1-\xi)\nu)d\xi\\&+\int_0^1\Psi_*((1-\xi)\sqcap+\xi\nu,\beta)\nabla((1-\xi)\sqcap+\xi\nu)d\xi\\&\leq\int_0^1\begin{bmatrix}\Psi_*(\sqcap,\beta)\begin{Bmatrix}h(\xi)\nabla(\xi\sqcap+(1-\xi)\nu)\\+h(1-\xi)\nabla((1-\xi)\sqcap+\xi\nu)\end{Bmatrix}\\+\Psi_*(\nu,\beta)\begin{Bmatrix}h(1-\xi)\nabla(\xi\sqcap+(1-\xi)\nu)\\+h(\xi)\nabla((1-\xi)\sqcap+\xi\nu)\end{Bmatrix}\end{bmatrix}d\xi,\\&\int_0^1\Psi^*((1-\xi)\sqcap+\xi\nu,\beta)\nabla((1-\xi)\sqcap+\xi\nu)d\xi\\&+\int_0^1\Psi^*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla(\xi\sqcap+(1-\xi)\nu)d\xi\\&\leq\int_0^1\begin{bmatrix}\Psi^*(\sqcap,\beta)\begin{Bmatrix}h(\xi)\nabla(\xi\sqcap+(1-\xi)\nu)\\+h(1-\xi)\nabla((1-\xi)\sqcap+\xi\nu)\end{Bmatrix}\\+\Psi^*(\nu,\beta)\begin{Bmatrix}h(1-\xi)\nabla(\xi\sqcap+(1-\xi)\nu)\\+h(\xi)\nabla((1-\xi)\sqcap+\xi\nu)\end{Bmatrix}\end{bmatrix}d\xi.\end{aligned}$$

$$\begin{aligned}&=2\Psi_*(\sqcap,\beta)\int_0^1h(\xi)\nabla(\xi\sqcap+(1-\xi)\nu)\,d\xi\\&+2\Psi_*(\nu,\beta)\int_0^1h(\xi)\nabla((1-\xi)\sqcap+\xi\nu)\,d\xi,\\&=2\Psi^*(\sqcap,\beta)\int_0^1h(\xi)\nabla(\xi\sqcap+(1-\xi)\nu)\,d\xi\\&+2\Psi^*(\nu,\beta)\int_0^1h(\xi)\nabla((1-\xi)\sqcap+\xi\nu)\,d\xi.\end{aligned}$$

Since ∇ is symmetric, then

$$\begin{aligned}&=2[\Psi_*(\sqcap,\beta)+\Psi_*(\nu,\beta)]\int_0^1h(\xi)\nabla\begin{pmatrix}(1-\xi)u\\+\xi\nu\end{pmatrix}d\xi,\\&=2[\Psi^*(\sqcap,\beta)+\Psi^*(\nu,\beta)]\int_0^1h(\xi)\nabla\begin{pmatrix}(1-\xi)u\\+\xi\nu\end{pmatrix}d\xi.\end{aligned}\tag{23}$$

Since

$$\begin{aligned}&\int_0^1\Psi_*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla(\xi\sqcap+(1-\xi)\nu)d\xi\\&=\int_0^1\Psi_*((1-\xi)\sqcap+\xi\nu,\beta)\nabla((1-\xi)\sqcap+\xi\nu)d\xi\\&=\frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi_*(\ddagger,\beta)\nabla(\ddagger)d\ddagger\\&\int_0^1\Psi^*((1-\xi)\sqcap+\xi\nu,\beta)\nabla((1-\xi)\sqcap+\xi\nu)d\xi\\&=\int_0^1\Psi^*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla(\xi\sqcap+(1-\xi)\nu)d\xi\\&=\frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi^*(\ddagger,\beta)\nabla(\ddagger)d\ddagger.\end{aligned}\tag{24}$$

From (23) and (24), we have

$$\begin{aligned}&\frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi_*(\ddagger,\beta)\nabla(\ddagger)d\ddagger\\&\leq[\Psi_*(\sqcap,\beta)+\Psi_*(\nu,\beta)]\int_0^1h(\xi)\nabla((1-\xi)\sqcap+\xi\nu)\,d\xi,\\&\frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi^*(\ddagger,\beta)\nabla(\ddagger)d\ddagger\\&\leq[\Psi^*(\sqcap,\beta)+\Psi^*(\nu,\beta)]\int_0^1h(\xi)\nabla((1-\xi)\sqcap+\xi\nu)\,d\xi,\end{aligned}$$

that is

$$\left[\frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi_*(\ddagger,\beta)\nabla(\ddagger)d\ddagger,\ \frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi^*(\ddagger,\beta)\nabla(\ddagger)d\ddagger\right]$$

$$\leq_I\left[\Psi_*(\sqcap,\beta)+\Psi_*(\nu,\beta),\ \Psi^*(\sqcap,\beta)+\Psi^*(\nu,\beta)\right]\int_0^1h(\xi)\nabla((1-\xi)\sqcap+\xi\nu)\,d\xi$$

hence

$$\frac{1}{\nu-\sqcap}(FR)\int_\sqcap^\nu\Psi(\ddagger)\nabla(\ddagger)d\ddagger$$

$$\preccurlyeq\left[\Psi(\sqcap)\,\tilde{+}\,\Psi(\nu)\right]\int_0^1h(\xi)\nabla((1-\xi)\sqcap+\xi\nu)d\xi,$$

this concludes the proof.

Next, we construct first *HH*-Fej´er inequality for *h*-convex FIVF, which generalizes first *HH*-Fej´er inequality for *h*-convex function, see [29].

Theorem 3.6 *Let $\Psi:[\sqcap,\nu]\to\mathbb{F}_C(\mathbb{R})$ be a h-convex FIVF with $\sqcap<\nu$ and $h:[0,1]\to\mathbb{R}^+$, and for all $\beta\in[0,1]$, $\Psi_\beta:K\subset\mathbb{R}\to\mathcal{K}_C{}^+\subset\mathcal{K}_C$ represent the family of IVFs through β-levels. If $\Psi\in\mathcal{FR}_{([\sqcap,\nu],\beta)}$ and $\nabla:[\sqcap,\nu]\to\mathbb{R}$, $\nabla(\ddagger)\geq0$, symmetric with respect to $\frac{\sqcap+\nu}{2}$, and $\int_\sqcap^\nu\nabla(\ddagger)d\ddagger>0$, then*

$$\Psi\left(\frac{\sqcap+\nu}{2}\right)\preccurlyeq\frac{2h(\frac12)}{\int_\sqcap^\nu\nabla(\ddagger)d\ddagger}(FR)\int_\sqcap^\nu\Psi(\ddagger)\nabla(\ddagger)d\ddagger.\tag{25}$$

Proof. Since Ψ is a *h*-convex, then for $\beta\in[0,1]$, we have

$$\begin{aligned}&\Psi_*\left(\tfrac{\sqcap+\nu}{2},\beta\right)\\&\leq h(\tfrac12)\begin{pmatrix}\Psi_*(\xi\sqcap+(1-\xi)\nu,\beta)\\+\Psi_*((1-\xi)\sqcap+\xi\nu,\beta)\end{pmatrix},\\&\Psi^*\left(\tfrac{\sqcap+\nu}{2},\beta\right)\\&\leq h\left(\tfrac12\right)\begin{pmatrix}\Psi^*(\xi\sqcap+(1-\xi)\nu,\beta)\\+\Psi^*((1-\xi)\sqcap+\xi\nu,\beta)\end{pmatrix},\end{aligned}\tag{26}$$

By multiplying (26) by $\nabla(\xi\sqcap+(1-\xi)\nu)=\nabla((1-\xi)\sqcap+\xi\nu)$ and integrate it by ξ over $[0,1]$, we obtain

$$\Psi*\left(\frac{\sqcap+\nu}{2},\beta\right)\int_0^1\nabla((1-\xi)\sqcap+\xi\nu)\mathrm{d}\xi$$
$$\leq h\left(\frac{1}{2}\right)\left(\begin{array}{c}\int_0^1\Psi*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla\left(\begin{array}{c}\xi u+\\(1-\xi)\nu\end{array}\right)\mathrm{d}\xi\\+\int_0^1\Psi*((1-\xi)\sqcap+\xi\nu,\beta)\nabla\left(\begin{array}{c}(1-\xi)u\\+\xi\nu\end{array}\right)\mathrm{d}\xi\end{array}\right)$$
$$\Psi^*\left(\frac{\sqcap+\nu}{2},\beta\right)\int_0^1\nabla((1-\xi)\sqcap+\xi\nu)\mathrm{d}\xi$$
$$\leq h\left(\frac{1}{2}\right)\left(\begin{array}{c}\int_0^1\Psi^*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla\left(\begin{array}{c}\xi u+\\(1-\xi)\nu\end{array}\right)\mathrm{d}\xi\\+\int_0^1\Psi^*((1-\xi)\sqcap+\xi\nu,\beta)\nabla\left(\begin{array}{c}(1-\xi)u\\+\xi\nu\end{array}\right)\mathrm{d}\xi\end{array}\right) \tag{27}$$

Since

$$\begin{array}{c}\int_0^1\Psi*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla(\xi\sqcap+(1-\xi)\nu)d\xi\\=\int_0^1\Psi*((1-\xi)\sqcap+\xi\nu,\beta)\nabla((1-\xi)\sqcap+\xi\nu)d\xi,\\=\frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi*(\ddagger,\beta)\nabla(\ddagger)\mathrm{d}\ddagger,\\\int_0^1\Psi^*((1-\xi)\sqcap+\xi\nu,\beta)\nabla((1-\xi)\sqcap+\xi\nu)d\xi\\=\int_0^1\Psi^*(\xi\sqcap+(1-\xi)\nu,\beta)\nabla(\xi\sqcap+(1-\xi)\nu)d\xi,\\=\frac{1}{\nu-\sqcap}\int_\sqcap^\nu\Psi^*(\ddagger,\beta)\nabla(\ddagger)\mathrm{d}\ddagger,\end{array} \tag{28}$$

From (27) and (28), we have

$$\Psi_*\left(\frac{\mu+\nu}{2},\beta\right)\leq\frac{2h\left(\frac{1}{2}\right)}{\int_\mu^\nu\nabla(z)\mathrm{d}z}\int_\mu^\nu\Psi_*(z,\beta)\nabla(z)\mathrm{d}z,$$
$$\Psi^*\left(\frac{\mu+\nu}{2},\beta\right)\leq\frac{2h\left(\frac{1}{2}\right)}{\int_\mu^\nu\nabla(z)\mathrm{d}z}\int_\mu^\nu\Psi^*(z,\beta)\nabla(z)\mathrm{d}z.$$

From which, we have

$$\left[\Psi_*\left(\frac{\mu+\nu}{2},\beta\right),\Psi^*\left(\frac{\mu+\nu}{2},\beta\right)\right]$$
$$\leq\frac{2h\left(\frac{1}{2}\right)}{l\int_\mu^\nu\nabla(z)\mathrm{d}z}\left[\int_\mu^\nu\Psi_*(z,\beta)\nabla(z)\mathrm{d}z,\int_\mu^\nu\Psi^*(z,\beta)\nabla(z)\mathrm{d}z\right],$$

that is

$$\Psi\left(\frac{\mu+\nu}{2}\right)\preccurlyeq\frac{2h\left(\frac{1}{2}\right)}{\int_\mu^\nu\nabla(z)\mathrm{d}z}(FR)\int_\mu^\nu\Psi(z)\nabla(z)\mathrm{d}z,$$

Then we complete the proof.

Remark 3.2. If $h(\xi)=\xi$, then inequalities in Theorem 10 and 11 reduces for convex FIVFs which are also new one.

If $\Psi_*(z,\beta)=\Psi^*(z,\beta)$ with $\beta=1$ and $h(\xi)=\xi$, then Theorem 10 and 11reduce to classical first and second *HH*-Fejér inequality for convex function, see [31].

Example 3.5. We consider $h(\xi)=\xi$ and the FIVF $\Psi:[1,4]\to\mathbb{F}_C(\mathbb{R})$ defined by,

$$\Psi(z)(\sigma)=\begin{cases}\frac{\sigma-e^z}{e^z},\sigma\in[e^z,2e^z],\\\frac{4e^z-\sigma}{2e^z},\sigma\in(2e^z,4e^z],\\0,\text{otherwise.}\end{cases}$$

Then, for each $\beta\in[0,1]$, we have $\Psi_\beta(z)=[(1+\beta)e^z,2(2-\beta)e^z]$. Since end point functions $\Psi_*(z,\beta),\Psi^*(z,\beta)$ are *h*-convex functions, for each $\beta\in[0,1]$, then $\Psi(z)$ is *h*-convex FIVF. If

$$\nabla(z)=\begin{cases}z-1,\sigma\in\left[1,\frac{5}{2}\right],\\4-z,\sigma\in\left(\frac{5}{2},4\right].\end{cases}$$

Then, we have

$$\frac{1}{3}\int_1^4\Psi_*(z,\beta)\nabla(z)\mathrm{d}z=\frac{1}{3}\int_1^4\Psi_*(z,\beta)\nabla(z)\mathrm{d}z$$
$$=\frac{1}{3}\int_1^{\frac{5}{2}}\Psi_*(z,\beta)\nabla(z)\mathrm{d}z+\frac{1}{3}\int_{\frac{5}{2}}^4\Psi_*(z,\beta)\nabla(z)\mathrm{d}z,$$
$$\frac{1}{3}\int_1^4\Psi^*(z,\beta)\nabla(z)\mathrm{d}z=\frac{1}{3}\int_1^4\Psi^*(z,\beta)\nabla(z)\mathrm{d}z$$
$$=\frac{1}{3}\int_1^{\frac{5}{2}}\Psi^*(z,\beta)\nabla(z)\mathrm{d}z+\frac{1}{3}\int_{\frac{5}{2}}^4\Psi^*(z,\beta)\nabla(z)\mathrm{d}z,$$

$$\begin{array}{c}=\frac{1}{3}(1+\beta)\int_1^{\frac{5}{2}}e^z(z-1)\mathrm{d}z+\frac{1}{3}(1+\beta)\int_{\frac{5}{2}}^4e^z(4-z)\mathrm{d}z\\\approx 11(1+\beta)\\=\frac{2}{3}(2-\beta)\int_1^{\frac{5}{2}}e^z(z-1)\mathrm{d}z+\frac{2}{3}(2-\beta)\int_{\frac{5}{2}}^4e^z(4-z)\mathrm{d}z\\\approx 22(2-\beta),\end{array} \tag{29}$$

and

$$\left[\Psi_*(\mu,\beta)+\Psi_*(\nu,\beta)\right]\int_0^1\xi\nabla((1-\xi)\mu+\xi\nu)\,d\xi$$

$$\left[\Psi^*(\mu,\beta)+\Psi^*(\nu,\beta)\right]\int_0^1\xi\nabla((1-\xi)\mu+\xi\nu)\,d\xi$$

$$\begin{array}{c}=(1+\beta)\left[e+e^4\right]\left[\int_0^{\frac{1}{2}}3\xi^2\mathrm{d}z+\int_{\frac{1}{2}}^1\xi(3-3\xi)\mathrm{d}\xi\right]\\\approx\frac{43}{2}(1+\beta)\\=2(2-\beta)\left[e+e^4\right]\left[\int_0^{\frac{1}{2}}3\xi^2\mathrm{d}z+\int_{\frac{1}{2}}^1\xi(3-3\xi)\mathrm{d}\xi\right]\\\approx 43(2-\beta).\end{array} \tag{30}$$

From (31) and (30), we have

$$[11(1+\beta), 22(2-\beta)] \leq_I \left[\frac{43}{2}(1+\beta), 43(2-\beta)\right],$$

for each $\beta \in [0,1]$. Hence, Theorem 3.5 is verified.

For Theorem 3.6, we have

$$\begin{aligned} \Psi_*\left(\tfrac{\mu+\nu}{2},\beta\right) &\approx \tfrac{61}{5}(1+\beta), \\ \Psi^*\left(\tfrac{\mu+\nu}{2},\beta\right) &\approx \tfrac{122}{5}(2-\beta), \end{aligned} \tag{31}$$

$$\int_\mu^\nu \nabla(z)dz = \int_1^{\frac{5}{2}} (z-1)\mathrm{d}z \int_{\frac{5}{2}}^4 (4-z)\mathrm{d}z = \frac{9}{4},$$

$$\begin{aligned} \tfrac{1}{\int_\mu^\nu \nabla(z)\mathrm{d}z} \int_1^4 \Psi_*(z,\beta)\nabla(z)\mathrm{d}z &\approx \tfrac{73}{5}(1+\beta) \\ \tfrac{1}{\int_\mu^\nu \nabla(z)dz} \int_1^4 \Psi^*(z,\beta)\nabla(z)\mathrm{d}z &\approx \tfrac{293}{10}(2-\beta) \end{aligned} \tag{32}$$

From (31) and (32), we have

$$\left[\frac{61}{5}(1+\beta), 24.4(2-\beta)\right] \leq_I \left[\frac{73}{5}(1+\beta), \frac{293}{10}(2-\beta)\right].$$

Hence, Theorem 11 is verified.

4 Conclusion

This study introduced the class of *h*-convex FIVFs and established some new H–H inequalities by means of fuzzy order relation on fuzzy-interval space. Moreover, we established strong relationship between H–H–Fej´er type inequality and *h*-convex FIVF. We provided relevant examples to demonstrate the application of the theory produced in this research. To construct fuzzy-interval inequalities of FIVFs, we plan to use a variety of convex FIVFs. We hope that this notion will assist other authors in remunerating their contributions in other sectors of knowledge. In future, we try to explore this concept using different fuzzy fractional integral operators.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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