



Multiple Attribute Decision-Making Method Based upon Intuitionistic Fuzzy Partitioned Dual Maclaurin Symmetric Mean Operators

HongJuan Wang¹ · Yi Liu² · Fang Liu^{1,2} · Jun Lin³

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Abstract

We propound the idea of the partitioned dual Maclaurin symmetric mean (PDMSM) operator stimulated by the partitioned Maclaurin symmetric mean, suppose that we can partition overall attributes into some portions and the attributes are inter-related in the same portion, but the attributes are not interrelated in different portions. We can deal with decision-making issues using PDMSM operator in the intuitionistic fuzzy environment. We also analysis features and peculiar instance of the PDMSM operator. And, we extend the PDMSM operator to introduce the intuitionistic fuzzy partitioned dual Maclaurin symmetric mean operator and the weighted intuitionistic fuzzy partitioned dual Maclaurin symmetric mean operator. Then, we analysis several characteristics and peculiar instances of the developed operators. A new multiple attribute decision-making (MADM) approach grounded on the established weighted intuitionistic fuzzy partitioned dual Maclaurin symmetric mean operator is propounded; the MADM method is to choose the optimal alternative from several alternatives. Finally, we demonstrate the designed method is more general and effective than existing methods through comparative analysis.

Keywords Fuzzy sets · Intuitionistic fuzzy numbers · Partitioned dual Maclaurin symmetric mean · Multiple attribute decision making

1 Introduction

Multiple attribute decision-making (MADM) method is to choose the best alternative from viable alternatives; it has attracted more and more attention of people and been used in many aspects, such as financial risk management, location selection and so forth. Owing to the intricacy of

MADM issues and multiple attribute group decision-making (MAGDM) issues, using precise values to depict attribute values of alternatives is difficult. On this case, Zadeh [1] developed the fuzzy sets (FSs), which utilize the membership degree to estimate alternatives and have been studied extensively. However, the FSs have only the membership degree and are usual hard to depict much more complex fuzzy information. For purpose of surmounting above shorting, Atanassov [2] presented the intuitionistic fuzzy set (IFS); it is an extensiveness of FSs increasing non-membership degree and hesitant degree. Further, Xu and Yager [3] established the intuitionistic fuzzy numbers (IFNs) and operational rules of IFNs. In recent years, the IFNs have been diffusely used to cope with MADM issues, Garg et al. [4] introduced improved possibility degree method to rank intuitionistic fuzzy numbers, Joshi [5] established a new bi-parametric exponential information measure based on IFS, IFS combined with linguistic sets [6–8] and MAGDM issues [9–12]. Molodtsov [13] introduced soft sets, LEE [14] introduced bipolar valued fuzzy sets, and Mahmood [15] established bipolar valued soft sets; they are also generalizations of fuzzy sets and used in decision making [16, 17]. In order to resolve decision-making problems (DMPs), the

✉ Jun Lin
junlin@hcnu.edu.cn

HongJuan Wang
whonjj@126.com

Yi Liu
liuyiy1@126.com

Fang Liu
lytb@163.com

¹ School of Mathematics and Information Sciences, Neijiang Normal University, Neijiang 641000, Sichuan, China

² Data Recovery Key Laboratory of Sichuan Province, Neijiang Normal University, Neijiang 641000, Sichuan, China

³ School of Big Data and Computer Science, Hechi University, Yizhou 546300, Guangxi, China

aggregation operators (AOs) are one of the most popular tools, which combine all the input individual arguments into a single argument, many studies have been made [18–21]. The weighted arithmetic operators were introduced by Xu [22] and Xu et al. [3, 23] proposed the weighted geometric operators based on the IFNs. In these AOs, the attributes are independent of each other. However, the relation of attribute cannot be ignored in DMPs. So, some operators consider that there is correlation between the aggregated arguments. In [24], Bonferroni developed the Bonferroni mean (BM); it presumes that correlation exists between any two arguments. Xu et al. [23] propounded the intuitionistic fuzzy BM (IFBM) operator and studied some desirable properties of IFBM operator. Xia et al. [25] put forward the geometric Bonferroni mean (GBM) by considering both the Bonferroni mean and the geometric mean for IFNs. Dutta et al. [26] introduced aggregation operators to fuse Extended Comparative Linguistic Expressions with Symbolic Translation (ELICIT) information, namely ELICITBM, ELICITEBM and ELICITPBM. The Heronian mean (HM) which can capture correlation between two arguments was proposed by Sykora [27]. Liu et al. [28] introduced partitioned HM operator in linguistic intuitionistic fuzzy environment. Mo et al. [29] introduced Archimedean geometric Heronian mean operators in dual hesitant fuzzy environment.

The Maclaurin symmetric mean (MSM) was originally proposed by Maclaurin [30], which is a popular operator with aggregation function, it can capture the relation among multiple input arguments and then was extended by Detemple and Robertson [31], while BM and HM capture the relationship between two input arguments. The MSM operator was used to solve DMPs [32–35] with linguistic fuzzy set, used in IFS environment [36], used in q -rung interval-valued orthopair fuzzy environment [37], and used in dual hesitant fuzzy soft set environment [38]. Further, Qin and Liu [19] established the intuitionistic fuzzy MSM (IFMSM) and the weighted IFMSM (WIFMSM). In [39], Qin and Liu introduced the dual Maclaurin symmetric mean (DMSM) operator, which considers the interrelationship among multiple input arguments; there are some generalizations of DMSM. Shi and Xiao [40] established the reducible weighted dual Maclaurin symmetric mean (RWDMSM) operator, Qin and Liu [39] purposed some operators (the uncertain linguistic Choquet dual Maclaurin symmetric mean (ULCDMSM), the uncertain linguistic dual Maclaurin symmetric mean (ULDMSM), the uncertain linguistic weighted dual Maclaurin symmetric mean (ULWDMMS)), Darko et al. [41] established operators (dual hesitant fuzzy dual MSM and weighted dual hesitant fuzzy dual MSM operators), and Wang et al. [42] extended DMSM to interval-valued

2-tuple linguistic Pythagorean fuzzy numbers (IV2TLP-FNs); these operators are able to reflect relationship among multiple attributes. However, in real decision-making situations, relation may not exist between attributes. For example, selecting a laptop from several laptops, we can consider following four attributes: basic requirements (Z_1), customer comments (Z_2), price comparison (Z_3), appearance characteristic (Z_4). These four attributes can be divided into two partitions: $P_1 = \{Z_1, Z_4\}$ and $P_2 = \{Z_2, Z_3\}$. We can know that the attribute Z_1 is interrelated to Z_4 , they are all in P_1 ; the attribute Z_2 is interrelated to Z_3 , they are all in P_2 . But, there does not exist relationship between the partitions P_1 and P_2 . Based on the condition, the partitioned Bonferroni mean (PBM) operator was presented by Dutta and Guha [43], Liu et al. [44] propounded the partitioned Maclaurin symmetric mean (PMSM) operator based on IFNs, these operators which suppose that total attributes are split into a few parts, and there exists relation between attributes in the same part and does not exist relation between attributes of diverse parts.

The DMSM operator considers that attributes are existing multiple relation. As previous study, it is easily found that generalizations of DMSM just only consider the case that attributes are relevant. But, in real DMPs, there may be no relationship between some attributes. Therefore, it is necessary that introduced the partitioned DMSM (PDMSM) to consider there is no relationship between some attributes. In this paper, our goals are: (1) to establish the PDMSM operator to surmount the drawback of the above discussion; (2) to put forward the intuitionistic fuzzy partitioned dual Maclaurin symmetric mean (IFPDMSM) operator and the weighted intuitionistic fuzzy partitioned dual Maclaurin symmetric mean (WIFPDMSM) operator; (3) to consider some properties, theorems and peculiar instances of the purposed operators; (4) to design a MADM methodology about the presented WIFPDMSM operator; and (5) to explain advantages of the established MADM approach.

The rest of the paper is described below. The Sect. 2 briefly gives notions of IFNs and DMSM. Besides, some properties of the DMSM are listed. In Sect. 3, we establish the PDMSM operator; we also give the features and particular case of the PDMSM operator. The Sect. 4 puts forward the IFPDMSM operator and the WIFPDMSM operator. The properties and peculiar instances of our introduced operators are too shown. The Sect. 5 designs a MADM methodology grounded on the put forward WIFPDMSM operator. Besides, giving an example and comparing the presented approach with other approaches to explain the presented MADM approach is more advantaged. The Sect. 6 is Conclusion and Future Studies.

2 Preliminaries

Firstly, we tersely reviewed some basal conceptions such as definitions, operational laws and ranking method of IFNs. Then, the Maclaurin symmetric mean and the dual Maclaurin symmetric mean are brushed up on in this section.

2.1 IFSs

Definition 1 [2] Suppose \tilde{Y} be a domain of discourse and $y \in \tilde{Y}$. Then an intuitionistic fuzzy set Γ on \tilde{Y} is indicated:

$$\Gamma = \{ \langle y, \rho_\Gamma(y), \tau_\Gamma(y) \rangle | y \in \tilde{Y} \} \tag{1}$$

where ρ_Γ and τ_Γ represent the membership degree (MD) and non-membership degree (NMD) to Γ , respectively, and $0 \leq \rho_\Gamma(y), \tau_\Gamma(y) \leq 1, 0 \leq \rho_\Gamma(y) + \tau_\Gamma(y) \leq 1$.

The pair $(\rho_\Gamma(y), \tau_\Gamma(y))$ is termed as intuitionistic fuzzy number (IFN) [22]. An IFN can be signified as $\theta = (\rho_\theta, \tau_\theta)$ for convenience, meeting $0 \leq \rho_\theta, \tau_\theta \leq 1$ and $0 \leq \rho_\theta + \tau_\theta \leq 1$.

Definition 2 [22] Let $\theta_1 = (\rho_1, \tau_1)$ and $\theta_2 = (\rho_2, \tau_2)$ be two IFNs. The operational laws of IFNs are represented as follows:

- (1) $\theta_1 \oplus \theta_2 = (1 - (1 - \rho_1)(1 - \rho_2), \tau_1 \tau_2)$,
- (2) $\theta_1 \otimes \theta_2 = (\rho_1 \rho_2, 1 - (1 - \tau_1)(1 - \tau_2))$,
- (3) $\lambda \theta_1 = (1 - (1 - \rho_1)^\lambda, \tau_1^\lambda)$, where λ is a real number and $\lambda > 0$,
- (4) $\theta_1^\lambda = (\rho_1^\lambda, 1 - (1 - \tau_1)^\lambda)$, where λ is a real number and $\lambda > 0$.

By Definition 2, some operational laws of IFNs can be obtained easily and listed as follows:

- (1) $\theta_1 \oplus \theta_2 = \theta_2 \oplus \theta_1$,
- (2) $\theta_1 \otimes \theta_2 = \theta_2 \otimes \theta_1$,
- (3) $\lambda(\theta_1 \oplus \theta_2) = \lambda \theta_1 \oplus \lambda \theta_2$, where λ is a real number and $\lambda > 0$,
- (4) $(\theta_1 \otimes \theta_2)^\lambda = \theta_1^\lambda \otimes \theta_2^\lambda$, where λ is a real number and $\lambda > 0$,
- (5) $\lambda_1 \theta_1 \oplus \lambda_2 \theta_1 = (\lambda_1 + \lambda_2) \theta_1$, where λ_1, λ_2 are real numbers and $\lambda_1, \lambda_2 > 0$,
- (6) $\theta_1^{\lambda_1} \otimes \theta_1^{\lambda_2} = \theta_1^{\lambda_1 + \lambda_2}$, where λ_1, λ_2 are real numbers and $\lambda_1, \lambda_2 > 0$.

In order to rank two IFNs, the score function and accuracy function are introduced.

Definition 3 [45] If $\theta = (\rho_\theta, \tau_\theta)$ be an IFN, then score function S of the IFN θ is described:

$$S(\theta) = \rho_\theta - \tau_\theta \tag{2}$$

where $S(\theta) \in [0, 1]$.

Definition 4 [46] If $\theta = (\rho_\theta, \tau_\theta)$ be an IFN, then accuracy function H of the IFN θ is shown:

$$H(\theta) = \rho_\theta + \tau_\theta \tag{3}$$

where $H(\theta) \in [0, 1]$.

Definition 5 [3] Suppose $\theta_1 = (\rho_1, \tau_1)$ and $\theta_2 = (\rho_2, \tau_2)$ be two IFNs, then

- (1) if $S(\theta_1) > S(\theta_2)$, then $\theta_1 > \theta_2$,
- (2) if $S(\theta_1) = S(\theta_2)$, then if $H(\theta_1) > H(\theta_2)$, then $\theta_1 > \theta_2$, if $H(\theta_1) = H(\theta_2)$, then $\theta_1 = \theta_2$.

2.2 The MSM Operator and DMSM Operator

Definition 6 [30] For given nonnegative real numbers $z_l (l = 1, 2, \dots, f)$, and $b = 1, 2, \dots, f$. The MSM operator is given by:

$$\begin{aligned} & \text{MSM}^{(b)}(z_1, z_2, \dots, z_f) \\ &= \left(\frac{\sum_{1 \leq l_1 < \dots < l_b \leq f} \left(\prod_{q=1}^b z_{l_q} \right)}{C_f^b} \right)^{\frac{1}{b}} \end{aligned} \tag{4}$$

where (l_1, l_2, \dots, l_b) contains all the b-tuple combination of $(1, 2, \dots, f)$ and $C_f^b = \frac{f!}{b!(f-b)!}$.

Definition 7 [39] For given nonnegative real numbers $z_l (l = 1, 2, \dots, f)$, and $b = 1, 2, \dots, f$. The DMSM operator is described by:

$$\begin{aligned} & \text{DMSM}^{(b)}(z_1, z_2, \dots, z_f) \\ &= \frac{1}{b} \left(\prod_{1 \leq l_1 < \dots < l_b \leq f} \left(\sum_{q=1}^b z_{l_q} \right)^{\frac{1}{C_f^b}} \right) \end{aligned} \tag{5}$$

where (l_1, l_2, \dots, l_b) contains all the b-tuple combination of $(1, 2, \dots, f)$ and $C_f^b = \frac{f!}{b!(f-b)!}$.

The DMSM operator has the following properties [39]:

- (1) if $z_l = z (l = 1, 2, \dots, f)$, then $\text{DMSM}^{(b)}(z_1, z_2, \dots, z_f) = z$,
- (2) if $z_l \leq a_l (l = 1, 2, \dots, f)$, then

$$\text{DMSM}^{(b)}(z_1, z_2, \dots, z_f) \leq \text{DMSM}^{(b)}(a_1, a_2, \dots, a_f),$$

- (3) $\min_l \{z_l\} \leq \text{DMSM}^{(b)}(z_1, z_2, \dots, z_f) \leq \max_l \{z_l\}$.

3 The PDMSM Operator

The DMSM operator in the internal structure presents that input arguments are correlated to other input arguments. But in many real problems, input arguments may be partitioned into some parts and input argument be interrelated to other input arguments in the same part, interrelation does not exist between input arguments in different parts.

Let $Z_l(l = 1, 2, \dots, f)$ be input arguments, and $z_l(l = 1, 2, \dots, f)$ be values, where z_l is the evaluating value of argument $Z_l, z_l \geq 0, 1 \leq l \leq f$. Suppose that the collection Z is divided into c different partitions P_1, P_2, \dots, P_c , meeting $P_l \cap P_q = \emptyset, l, q \in \{1, 2, \dots, c\}, l \neq q$ and $\bigcup_{t=1}^c P_t = Z$. We assume that the arguments are relevant in the identical partition and the arguments are irrelevant in diverse partitions.

In the following, we introduce the partitioned dual Maclaurin symmetric mean operator.

Definition 8 For given nonnegative real numbers $z_l(l = 1, 2, \dots, f), c$ indicates the number of partitions, $b = 1, 2, \dots, o_t, o_t$ is the amount of attributes in P_t . The PDMSM operator is defined:

$$PDMSM^{(b)}(z_1, z_2, \dots, z_f) = \frac{1}{c} \sum_{t=1}^c \left(\frac{\prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\sum_{q=1}^b z_{l_q} \right)^{\frac{1}{c_{o_t}^b}}}{b} \right) \tag{6}$$

where (l_1, l_2, \dots, l_b) contains all the b -tuple combination of $(1, 2, \dots, o_t)$ and $C_{o_t}^b = \frac{o_t!}{b!(o_t-b)!}$.

Next, we will verify the characters of the introduced PDMSM operator.

Proposition 1 (Idempotency) Suppose $z_1 = z_2 = \dots = z_f = z$, then

$$PDMSM^{(b)}(z_1, z_2, \dots, z_f) = z.$$

Proof By Definition 8 and $z_1 = z_2 = \dots = z_f = z$, we know

$$PDMSM^{(b)}(z_1, z_2, \dots, z_f) = \frac{1}{c} \sum_{t=1}^c \left(\frac{\prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\sum_{q=1}^b z_{l_q} \right)^{\frac{1}{c_{o_t}^b}}}{b} \right) = \frac{1}{c} \sum_{t=1}^c \left(\frac{\prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\sum_{q=1}^b z \right)^{\frac{1}{c_{o_t}^b}}}{b} \right) = \frac{1}{c} \sum_{t=1}^c \left(\frac{z^{\frac{b}{c_{o_t}^b}}}{b} \right) = z,$$

which completes the proof. □

Proposition 2 (Monotonicity) Let $y_l, z_l(l = 1, 2, \dots, f)$ be two groups of nonnegative real numbers and $y_l \leq z_l, l = 1, 2, \dots, f$, then

$$PDMSM^{(b)}(y_1, y_2, \dots, y_f) \leq PDMSM^{(b)}(z_1, z_2, \dots, z_f)$$

when the parameter b is fixed.

Proof By the definition of the PDMSM operator and $y_l \leq z_l$ for all l , we get $\sum_{q=1}^b y_{l_q} \leq \sum_{q=1}^b z_{l_q}$ and

$$\frac{1}{b} \left(\prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\sum_{q=1}^b y_{l_q} \right)^{\frac{1}{c_{o_t}^b}} \right) \leq \frac{1}{b} \left(\prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\sum_{q=1}^b z_{l_q} \right)^{\frac{1}{c_{o_t}^b}} \right).$$

Further, we obtain

$$\frac{1}{c} \sum_{t=1}^c \left(\frac{\prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\sum_{q=1}^b y_{l_q} \right)^{\frac{1}{c_{o_t}^b}}}{b} \right) \leq \frac{1}{c} \sum_{t=1}^c \left(\frac{\prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\sum_{q=1}^b z_{l_q} \right)^{\frac{1}{c_{o_t}^b}}}{b} \right).$$

which completes the proof. □

Proposition 3 (Boundedness) *Let $z_l(l = 1, 2, \dots, f)$ be a collection of nonnegative real numbers, $z^- = \min_l\{z_l\}$ and $z^+ = \max_l\{z_l\}$, then*

$$z^- \leq PDMSM^{(b)}(z_1, z_2, \dots, z_f) \leq z^+.$$

Proof Because $z^- = \min_l\{z_l\} \leq z_l$, and owing to the Propositions 1 and 2, we can have

$$\begin{aligned} z^- &= PDMSM^{(b)}(z^-, z^-, \dots, z^-) \\ &\leq PDMSM^{(b)}(z_1, z_2, \dots, z_f). \end{aligned}$$

In the same way, we can have

$$\begin{aligned} PDMSM^{(b)}(z_1, z_2, \dots, z_f) \\ \leq PDMSM^{(b)}(z^+, z^+, \dots, z^+) = z^+. \end{aligned}$$

Therefore, $z^- \leq PDMSM^{(b)}(z_1, z_2, \dots, z_f) \leq z^+$. □

Moreover, the effect of b towards the established PDMSM operator can be studied.

Theorem 1 *For given nonnegative real numbers $z_l(l = 1, 2, \dots, f)$, then PDMSM operator monotonically increases with respect to the parameter b .*

Proof By [39], we can have the DMSM operator inequality

$$\begin{aligned} DMSM^{(1)}(z_1, z_2, \dots, z_f) &\leq DMSM^{(2)}(z_1, z_2, \dots, z_f) \\ &\leq \dots \leq DMSM^{(f)}(z_1, z_2, \dots, z_f). \end{aligned}$$

Therefore, based on Eq. (6), we can obtain

$$\begin{aligned} PDMSM^{(1)}(z_1, z_2, \dots, z_f) \\ \leq PDMSM^{(2)}(z_1, z_2, \dots, z_f) \leq \dots \\ \leq PDMSM^{(\min\{o_i\})}(z_1, z_2, \dots, z_f) \end{aligned}$$

where $b = 1, 2, \dots, \min_l\{o_l\}$. □

Theorem 2 *For given nonnegative real numbers $z_l(l = 1, 2, \dots, f)$, then*

$$\begin{aligned} \min\{PDMSM^{(b)}(z_1, z_2, \dots, z_f)\} \\ = PDMSM^{(1)}(z_1, z_2, \dots, z_f) = \frac{1}{c} \sum_{t=1}^c \left(\prod_{l_q=1}^{o_t} z_{l_q}^{\frac{1}{c^b}} \right) \end{aligned}$$

and

$$\begin{aligned} \max\{PDMSM^{(b)}(z_1, z_2, \dots, z_f)\} \\ = PDMSM^{(\min\{o_i\})}(z_1, z_2, \dots, z_f). \end{aligned}$$

when $c = 1$, the established PDMSM operator reduces to the DMSM operator [39], listed as:

$$\begin{aligned} PDMSM^{(b)}(z_1, z_2, \dots, z_f) \\ = \sum_{t=1}^1 \left(\frac{\prod_{1 \leq l_1 < \dots < l_b \leq o_1} \left(\sum_{q=1}^b z_{l_q} \right)^{\frac{1}{c^b}}}{b} \right) \\ = \frac{1}{b} \left(\prod_{1 \leq l_1 < \dots < l_b \leq f} \left(\sum_{q=1}^b z_{l_q} \right)^{\frac{1}{c^b}} \right). \end{aligned}$$

4 The PDMSM Operator Based on the IFNs

In the following, we will extend the PDMSM operator to propound the partitioned dual Maclaurin symmetric mean operator for IFNs and the weighted partitioned dual Maclaurin symmetric mean operator for IFNs. The detailed description is shown as follows.

4.1 The IFPDMSM Operator

Definition 9 For given IFNs $\theta_l = (\rho_l, \tau_l)(l = 1, 2, \dots, f)$ partitioned into c distinct partitions P_1, P_2, \dots, P_c , $b = 1, 2, \dots, o_t$, o_t indicates the amount of attributes in P_t . The IFPDMSM operator is described:

$$\begin{aligned} IFPDMSM^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\ = \frac{1}{c} \left(\bigoplus_{t=1}^c \left(\frac{\bigotimes_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\bigoplus_{q=1}^b \theta_{l_q} \right)^{\frac{1}{c^b}}}{b} \right) \right) \end{aligned} \tag{7}$$

where (l_1, l_2, \dots, l_b) contains overall the b -tuple combination of $(1, 2, \dots, o_t)$ and $C_{o_t}^b = \frac{o_t!}{b!(o_t-b)!}$.

Theorem 3 *For given IFNs $\theta_l = (\rho_l, \tau_l)(l = 1, 2, \dots, f)$. The aggregated result of formula (7) is still an IFN, described as follows:*

$$\text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) = \left(1 - \left(\prod_{i=1}^c \left(1 - \prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right) \right)^{\frac{1}{c^b o_t}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \left(\prod_{i=1}^c \left(1 - \prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(1 - \prod_{q=1}^b \tau_{l_q} \right) \right)^{\frac{1}{c^b o_t}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \tag{8}$$

Proof By the operational laws of the IFNs, we can get

$$\bigoplus_{q=1}^b \theta_{l_q} = \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}), \prod_{q=1}^b \tau_{l_q} \right)$$

and

$$\left(\bigoplus_{q=1}^b \theta_{l_q} \right)^{\frac{1}{c^b o_t}} = \left(\left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{c^b o_t}}, 1 - \left(1 - \prod_{q=1}^b \tau_{l_q} \right)^{\frac{1}{c^b o_t}} \right)$$

Then, we obtain

$$\begin{aligned} & \bigotimes_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\bigoplus_{q=1}^b \theta_{l_q} \right)^{\frac{1}{c^b o_t}} \\ &= \left(\prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{c^b o_t}}, \right. \\ & \quad \left. 1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q} \right)^{\frac{1}{c^b o_t}} \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{b} \left(\bigotimes_{1 \leq l_1 < \dots < l_b \leq o_t} \left(\bigoplus_{q=1}^b \theta_{l_q} \right)^{\frac{1}{c^b o_t}} \right) \\ &= \left(1 - \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right) \right)^{\frac{1}{c^b o_t}}, \right. \\ & \quad \left. \left(1 - \prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(1 - \prod_{q=1}^b \tau_{l_q} \right)^{\frac{1}{c^b o_t}} \right)^{\frac{1}{b}} \right) \end{aligned}$$

Therefore, we can get

$$\text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) = \left(1 - \left(\prod_{t=1}^c \left(1 - \prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right) \right)^{\frac{1}{c o_t^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}},$$

$$\left(\prod_{t=1}^c \left(1 - \prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(1 - \prod_{q=1}^b \tau_{l_q} \right) \right)^{\frac{1}{c o_t^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}.$$

Next,

$$0 \leq 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right) \right)^{\frac{1}{c o_t^b}} \right)^{\frac{1}{b}} \leq 1$$

$$0 \leq 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right) \right)^{\frac{1}{c o_t^b}} \right)^{\frac{1}{b}} + \left(\prod_{t=1}^c \left(1 - \prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(1 - \prod_{q=1}^b \tau_{l_q} \right) \right)^{\frac{1}{c o_t^b}} \right)^{\frac{1}{b}} \leq 1$$

and

$$0 \leq \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q} \right) \right)^{\frac{1}{c o_t^b}} \right)^{\frac{1}{b}} \leq 1$$

$$\leq 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q} \right) \right)^{\frac{1}{c o_t^b}} \right)^{\frac{1}{b}} + \left(\prod_{t=1}^c \left(1 - \prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(1 - \prod_{q=1}^b \tau_{l_q} \right) \right)^{\frac{1}{c o_t^b}} \right)^{\frac{1}{b}} = 1.$$

Because $0 \leq \rho_{l_q} \leq 1$, $0 \leq \tau_{l_q} \leq 1$ and $0 \leq \rho_{l_q} + \tau_{l_q} \leq 1$, we can have $0 \leq \tau_{l_q} \leq 1 - \rho_{l_q} \leq 1$. Thus,

Thus, we complete the proof. □

Also, we can get the IFPDMSM operator has following desirable properties.

Proposition 4 (Idempotency) *If $\theta_l (l = 1, 2, \dots, f)$ are equal, i.e., $\theta_l = (\rho_l, \tau_l) = \theta = (\rho, \tau) (l = 1, 2, \dots, f)$. Then,*

$$\text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) = \theta.$$

Proof On the ground of Eq. (8), we can have

$$\text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f)$$

$$\begin{aligned} &= \left(1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q} \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \right) \\ &= \left(1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho) \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \right) \\ &= \left(1 - \left(\prod_{t=1}^c \left(1 - \left((1 - (1 - \rho)^b \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \left(\prod_{t=1}^c \left(1 - \left((1 - \tau^b)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \right) \right) \\ &= \left(1 - \left(\prod_{t=1}^c \left((1 - \rho)^b \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \left(\prod_{t=1}^c \left(\tau^b \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \right) = (\rho, \tau) = \theta. \end{aligned}$$

□

Proposition 5 (Commutativity) *For given IFNs $\theta_l = (\rho_l, \tau_l)$ and $\check{\theta}_l = (\check{\rho}_l, \check{\tau}_l) (l = 1, 2, \dots, f)$. Suppose $\check{\theta}_l = (\check{\rho}_l, \check{\tau}_l)$ is any permutation of $\theta_l = (\rho_l, \tau_l)$ for all l , then*

$$\begin{aligned} & \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\ &= \text{IFPDMSM}^{(b)}(\check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_f). \end{aligned}$$

Proof According to Eq. (8), we can get

$$\begin{aligned} & \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\ &= \left(1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right) \right)^{\frac{1}{c^{b/o_t}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \\ & \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q} \right)^{\frac{1}{c^{b/o_t}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}. \end{aligned}$$

$$\begin{aligned} & \text{IFPDMSM}^{(b)}(\check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_f) \\ &= \left(1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \check{\rho}_{l_q}) \right) \right)^{\frac{1}{c^{b/o_t}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \\ & \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \check{\tau}_{l_q} \right)^{\frac{1}{c^{b/o_t}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}. \end{aligned}$$

Because $\check{\theta}_l = (\check{\rho}_l, \check{\tau}_l)$ is any permutation of $\theta_l = (\rho_l, \tau_l)$ for all l , we have

$$\begin{aligned} & \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\ &= \text{IFPDMSM}^{(b)}(\check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_f). \end{aligned}$$

□

Proposition 6 (Monotonicity) Let $\theta_l = (\rho_l, \tau_l)$ and $\check{\theta}_l = (\check{\rho}_l, \check{\tau}_l) (l = 1, 2, \dots, f)$ be arbitrary two collections of IFNs, satisfying $\rho_l \geq \check{\rho}_l, \tau_l \leq \check{\tau}_l$ for all l , then

$$\begin{aligned} & \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\ & \geq \text{IFPDMSM}^{(b)}(\check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_f) \end{aligned}$$

when the parameter b is fixed.

Proof Suppose that $\text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) = \theta = (\rho, \tau)$ and $\text{IFPDMSM}^{(b)}(\check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_f) = \check{\theta} = (\check{\rho}, \check{\tau})$, then

$$\begin{aligned} \rho &= 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right) \right)^{\frac{1}{c^{b/o_t}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \\ \check{\rho} &= 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \check{\rho}_{l_q}) \right) \right)^{\frac{1}{c^{b/o_t}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \end{aligned}$$

$$\begin{aligned} \tau &= \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q} \right)^{\frac{1}{c^{b/o_t}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \\ \check{\tau} &= \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \check{\tau}_{l_q} \right)^{\frac{1}{c^{b/o_t}}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}. \end{aligned}$$

Because $\rho_l \geq \check{\rho}_l, \tau_l \leq \check{\tau}_l$, then

$$\begin{aligned}
 & 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \\
 & \geq 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \check{\rho}_{l_q}) \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \\
 & \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q} \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \\
 & \leq \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \check{\tau}_{l_q} \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}.
 \end{aligned}$$

That is $\rho \geq \check{\rho}$ and $\tau \leq \check{\tau}$, then $\theta \geq \check{\theta}$. Therefore,

$$\begin{aligned}
 & \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\
 & \geq \text{IFPDMSM}^{(b)}(\check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_f).
 \end{aligned}$$

Proposition 7 (Boundedness) *For given IFNs $\theta_l = (\rho_l, \tau_l)(l = 1, 2, \dots, f)$, $\theta^- = \min_l \{\theta_l\}$ and $\theta^+ = \max_l \{\theta_l\}, l = 1, 2, \dots, f$, then*

$$\theta^- \leq \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \leq \theta^+.$$

Proof Owing to $\theta^- = \min_l \{\theta_l\} \leq \theta_l$, and Propositions 4 and 6, we have

$$\begin{aligned}
 & \theta^- = \text{IFPDMSM}^{(b)}(\theta^-, \theta^-, \dots, \theta^-) \\
 & \leq \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f)
 \end{aligned}$$

and

$$\begin{aligned}
 & \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\
 & \leq \text{IFPDMSM}^{(b)}(\theta^+, \theta^+, \dots, \theta^+) = \theta^+.
 \end{aligned}$$

Thus, we have $\theta^- \leq \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \leq \theta^+$.

Furthermore, we study the impact of b for the IFPDMSM operator and an example.

Lemma 1 [47] *Suppose $y_l > 0, z_l > 0(l = 1, 2, \dots, f)$ meeting $\sum_{l=1}^f z_l = 1$, then $\prod_{l=1}^f y_l^{z_l} \leq \sum_{l=1}^f y_l z_l$.*

Theorem 4 *Let $\theta_l = (\rho_l, \tau_l)(l = 1, 2, \dots, f)$ be a group of IFNs, and $b = 1, 2, \dots, \min_l \{o_t\}$. The IFPDMSM operator monotonically increases with respect to the parameter b .*

Proof Based on Theorem 3, we have

$$\begin{aligned}
 & \text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\
 & = \left(1 - \prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \\
 & \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q} \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}.
 \end{aligned}$$

Assume that

$$\begin{aligned}
 & G(b) = 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \\
 & H(b) = \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q} \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}.
 \end{aligned}$$

Next, we prove function $G(b)$ monotonically increases as b increases. Based on the dual Maclaurin inequality and Lemma 1, we have

$$\begin{aligned}
 & \prod_{1 \leq l_1 < \dots < l_b \leq o_t} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{C_{o_t}^b}} \\
 & \leq \sum_{1 \leq l_1 < \dots < l_b \leq o_t} \frac{1 - \prod_{q=1}^b (1 - \rho_{l_q})}{C_{o_t}^b} \\
 & \Rightarrow 1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{C_{o_t}^b}} \\
 & \geq 1 - \sum_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \frac{1 - \prod_{q=1}^b (1 - \rho_{l_q})}{C_{o_t}^b} = \sum_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \frac{\prod_{q=1}^b (1 - \rho_{l_q})}{C_{o_t}^b} \\
 & \Rightarrow \frac{1}{b} \ln \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{C_{o_t}^b}} \right) \\
 & \geq \frac{1}{b} \ln \left(\sum_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \frac{\prod_{q=1}^b (1 - \rho_{l_q})}{C_{o_t}^b} \right) \\
 & \Rightarrow \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{C_{o_t}^b}} \right)^{\frac{1}{b}} \\
 & \geq \left(\sum_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \frac{\prod_{q=1}^b (1 - \rho_{l_q})}{C_{o_t}^b} \right)^{\frac{1}{b}} \\
 & \Rightarrow \prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{C_{o_t}^b}} \right)^{\frac{1}{b}} \\
 & \geq \prod_{t=1}^c \left(\sum_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \frac{\prod_{q=1}^b (1 - \rho_{l_q})}{C_{o_t}^b} \right)^{\frac{1}{b}}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 G(b) &= 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q}) \right)^{\frac{1}{C_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \\
 &\leq 1 - \left(\prod_{t=1}^c \left(\sum_{1 \leq l_1 < \dots < l_b \leq o_t} \frac{\prod_{q=1}^b (1 - \rho_{l_q})}{C_{o_t}^b} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}.
 \end{aligned}$$

In the following, we take the proof by contradiction methodology. Assume $G(b)$ is monotonically decreasing as b increases, then $G(\min_t \{o_t\}) < \dots < G(2) < G(1)$ and we can have

$$\begin{aligned}
 G(1) &\leq 1 - \left(\prod_{t=1}^c \left(\sum_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o_t}} \frac{\prod_{q=1}^1 (1 - \rho_{l_q})}{C_{o_t}^1} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \\
 &= 1 - \left(\prod_{t=1}^c \frac{\sum_{l_q=1}^{o_t} (1 - \rho_{l_q})}{o_t} \right)^{\frac{1}{c}}.
 \end{aligned}$$

In addition, we assume that $o_t = o$, then we can have

$$G(\min_t \{o_t\}) = G(o) = 1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots \\ < l_b \leq o}} \left(1 - \prod_{q=1}^o (1 - \rho_{l_q}) \right)^{\frac{1}{o}} \right)^{\frac{1}{c}} \right)$$

$$= 1 - \left(\prod_{t=1}^c \left(\prod_{q=1}^o (1 - \rho_{l_q}) \right)^{\frac{1}{o}} \right)^{\frac{1}{c}}.$$

According to the assumption $G(o) < G(1)$, we can get

$$G(o) = 1 - \left(\prod_{t=1}^c \left(\prod_{l_q=1}^o (1 - \rho_{l_q}) \right)^{\frac{1}{o}} \right)^{\frac{1}{c}} < G(1)$$

$$\leq 1 - \left(\prod_{t=1}^c \frac{\sum_{l_q=1}^{o_t} (1 - \rho_{l_q})}{o_t} \right)^{\frac{1}{c}}$$

$$= 1 - \left(\prod_{t=1}^c \frac{\sum_{l_q=1}^o (1 - \rho_{l_q})}{o} \right)^{\frac{1}{c}}$$

$$\Rightarrow \left(\prod_{l_q=1}^o (1 - \rho_{l_q}) \right)^{\frac{1}{o}} > \frac{\sum_{l_q=1}^o (1 - \rho_{l_q})}{o}.$$

Obviously, $\left(\prod_{l_q=1}^o (1 - \rho_{l_q}) \right)^{\frac{1}{o}} > \frac{\sum_{l_q=1}^o (1 - \rho_{l_q})}{o}$ is contradiction to the Lemma 1. Thus, as the b increases, function $G(b)$ is monotonically increasing. In the same way, we can prove function $H(b)$ is monotonically decreasing as the b increases.

In accordance with the previous analysis, we can have

$$\text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f)$$

$$< \text{IFPDMSM}^{(b+1)}(\theta_1, \theta_2, \dots, \theta_f).$$

Therefore, the IFPDMSM operator monotonically increases with respect to the parameter b .

Theorem 5 Let $\theta_l = (\rho_l, \tau_l) (l = 1, 2, \dots, f)$ be a group of IFNs, and $b = 1, 2, \dots, \min_t \{o_t\}$. Then

$$\min\{\text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f)\}$$

$$= \text{IFPDMSM}^{(1)}(\theta_1, \theta_2, \dots, \theta_f)$$

$$= \left(1 - \left(\prod_{t=1}^c \left(1 - \prod_{l_q=1}^{o_t} (\rho_{l_q})^{\frac{1}{o_t}} \right) \right)^{\frac{1}{c}}, \right.$$

$$\left. \left(\prod_{t=1}^c \left(1 - \prod_{l_q=1}^{o_t} (1 - \tau_{l_q})^{\frac{1}{o_t}} \right) \right)^{\frac{1}{c}} \right)$$

and

$$\max\{\text{IFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f)\}$$

$$= \text{IFPDMSM}^{(\min_t \{o_t\})}(\theta_1, \theta_2, \dots, \theta_f).$$

Example 1 Let $\theta_1 = (0.5, 0.3), \theta_2 = (0.4, 0.4), \theta_3 = (0.6, 0.2), \theta_4 = (0.7, 0.2)$ and $\theta_5 = (0.3, 0.4)$ be five IFNs. Suppose these five IFNs are partitioned into two groups $P_1 = \{\theta_1, \theta_3, \theta_5\}$ and $P_2 = \{\theta_2, \theta_4\}$. Here we use the IFPDMSM to aggregate these five IFNs. In general, we let $b = 2$, then

$$\left(\prod_{t=1}^2 \left(1 - \prod_{1 \leq l_1 < l_2 \leq o_t} \left(1 - \prod_{q=1}^2 \tau_{l_q} \right)^{\frac{1}{c_{o_t}^2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$= \left(\left((1 - (1 - 0.3 \times 0.2))^{\frac{1}{3}} \times (1 - 0.3 \times 0.4)^{\frac{1}{3}} \right. \right.$$

$$\left. \times (1 - 0.2 \times 0.4)^{\frac{1}{3}} \right)^{\frac{1}{2}} \times \left((1 - (1 - 0.4 \times 0.2))^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$= 0.2888.$$

Similarly,

$$1 - \left(\prod_{t=1}^2 \left(1 - \prod_{\substack{1 \leq l_1 < \dots < l_b \leq o_t}} \left(1 - \prod_{q=1}^2 (1 - \rho_{l_q}) \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} = 0.5265.$$

Therefore, $IFPDMSM^{(2)}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (0.5265, 0.2888).$

4.2 The WIFPDMSM Operator

In Definition 9, the IFPDMSM operator does not view the importance of each input element. But, input elements always have diverse significance in practical DMPs, so we should consider that the weight of elements may be different. Following, We introduce the WIFPDMSM operator for IFNs which considers the importance of elements.

Definition 10 For given IFNs $\theta_l = (\rho_l, \tau_l)(l = 1, 2, \dots, f)$ partitioned into c distinct partitions P_1, P_2, \dots, P_c and the

weight of IFN $\theta_l(l = 1, 2, \dots, f)$ is ω_l , meeting $\omega_l \in [0, 1]$ and $\sum_{l=1}^f \omega_l = 1, b = 1, 2, \dots, o_t, o_t$ signifies the number of attributes in P_t . Then the WIFPDMSM operator of the $\theta_l(l = 1, 2, \dots, f)$ is defined as:

$$WIFPDMSM^{(b)}(\theta_1, \theta_2, \dots, \theta_f) = \frac{1}{c} \bigoplus_{t=1}^c \left(\frac{\bigotimes_{\substack{1 \leq l_1 < \dots < l_b \leq o_t}} \left(\bigoplus_{q=1}^b (\omega_{l_q} \theta_{l_q}) \right)^{\frac{1}{c_{o_t}^b}}}{b} \right) \quad (9)$$

where (l_1, l_2, \dots, l_b) contains all the b -tuple combination of $(1, 2, \dots, o_t)$ and $C_{o_t}^b = \frac{o_t!}{b!(o_t-b)!}$.

Theorem 6 For given IFNs $\theta_l = (\rho_l, \tau_l)(l = 1, 2, \dots, f)$, the weight of IFN $\theta_l(l = 1, 2, \dots, f)$ is ω_l , meeting $\omega_l \in [0, 1]$ and $\sum_{l=1}^f \omega_l = 1$. The aggregated result of formula(9) is too an IFN, listed:

$$WIFPDMSM^{(b)}(\theta_1, \theta_2, \dots, \theta_f) = \left(1 - \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots < l_b \leq o_t}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q})^{\omega_{l_q}} \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}}, \left(\prod_{t=1}^c \left(1 - \prod_{\substack{1 \leq l_1 < \dots < l_b \leq o_t}} \left(1 - \prod_{q=1}^b \tau_{l_q}^{\omega_{l_q}} \right)^{\frac{1}{c_{o_t}^b}} \right)^{\frac{1}{b}} \right)^{\frac{1}{c}} \right). \quad (10)$$

Table 1 Decision matrix Λ given by the decision maker

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Y_1	(0.5,0.3)	(0.6,0.3)	(0.6,0.2)	(0.4,0.4)	(0.6,0.3)	(0.4,0.3)
Y_2	(0.6,0.3)	(0.3,0.4)	(0.7,0.2)	(0.6,0.3)	(0.5,0.2)	(0.6,0.4)
Y_3	(0.6,0.3)	(0.5,0.2)	(0.4,0.4)	(0.5,0.3)	(0.4,0.2)	(0.3,0.3)
Y_4	(0.6,0.2)	(0.7,0.3)	(0.6,0.2)	(0.6,0.3)	(0.8,0.2)	(0.7,0.2)

Table 2 Ranking results by our developed method for parameter

Parameter b, c	Score value $S(\check{\theta}_l)(l = 1, 2, 3, 4)$	Sorting
$b = 1, c = 2$	$S(\check{\theta}_1) = -0.706, S(\check{\theta}_2) = -0.707, S(\check{\theta}_3) = -0.727, S(\check{\theta}_4) = -0.625$	$Y_4 > Y_1 > Y_2 > Y_3$
$b = 2, c = 2$	$S(\check{\theta}_1) = -0.697, S(\check{\theta}_2) = -0.692, S(\check{\theta}_3) = -0.717, S(\check{\theta}_4) = -0.616$	$Y_4 > Y_2 > Y_1 > Y_3$
$b = 3, c = 2$	$S(\check{\theta}_1) = -0.694, S(\check{\theta}_2) = -0.687, S(\check{\theta}_3) = -0.712, S(\check{\theta}_4) = -0.613$	$Y_4 > Y_2 > Y_1 > Y_3$
$b = 1, c = 1$	$S(\check{\theta}_1) = -0.720, S(\check{\theta}_2) = -0.710, S(\check{\theta}_3) = -0.730, S(\check{\theta}_4) = -0.635$	$Y_4 > Y_2 > Y_1 > Y_3$
$b = 2, c = 1$	$S(\check{\theta}_1) = -0.704, S(\check{\theta}_2) = -0.697, S(\check{\theta}_3) = -0.720, S(\check{\theta}_4) = -0.622$	$Y_4 > Y_2 > Y_1 > Y_3$
$b = 3, c = 1$	$S(\check{\theta}_1) = -0.070, S(\check{\theta}_2) = -0.053, S(\check{\theta}_3) = -0.093, S(\check{\theta}_4) = -0.050$	$Y_4 > Y_2 > Y_1 > Y_3$

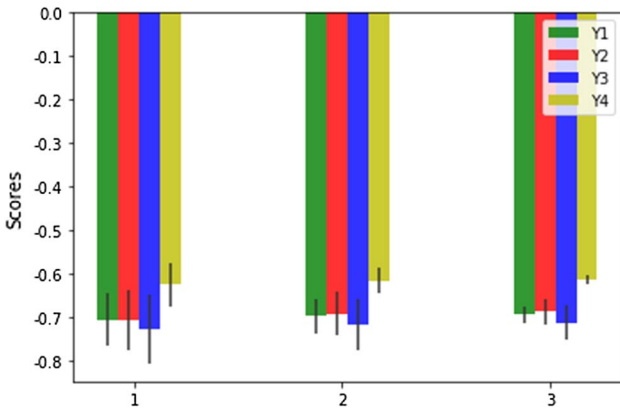


Fig. 1 Ranking results by our developed method for $c = 2, b = 1, 2, 3$

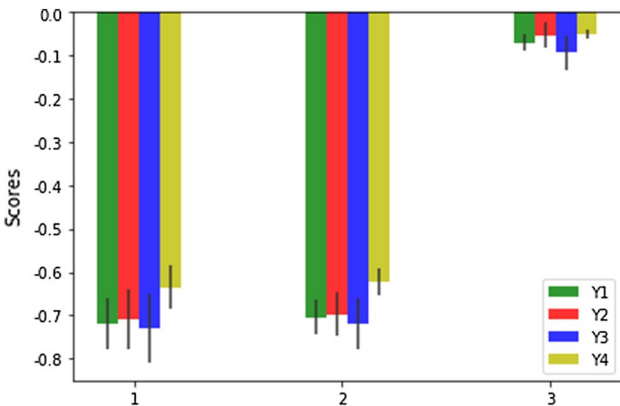


Fig. 2 Ranking results by our developed method for $c = 1, b = 1, 2, 3$

Likewise, we can analyze the WIFPDMSM operator has some desirable properties.

Proposition 8 (Commutativity) For given IFNs $\theta_l = (\rho_l, \tau_l)$ and $\check{\theta}_l = (\check{\rho}_l, \check{\tau}_l) (l = 1, 2, \dots, f)$. Suppose $\check{\theta}_l = (\check{\rho}_l, \check{\tau}_l)$ is any permutation of $\theta_l = (\rho_l, \tau_l)$ for all l , then

$$\begin{aligned} & \text{WIFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\ &= \text{WIFPDMSM}^{(b)}(\check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_f). \end{aligned}$$

Proposition 9 (Monotonicity) For given IFNs $\theta_l = (\rho_l, \tau_l)$ and $\check{\theta}_l = (\check{\rho}_l, \check{\tau}_l) (l = 1, 2, \dots, f)$, satisfying the condition $\rho_l \geq \check{\rho}_l, \tau_l \leq \check{\tau}_l$, then

$$\begin{aligned} & \text{WIFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \\ & \geq \text{WIFPDMSM}^{(b)}(\check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_f). \end{aligned}$$

Furthermore, we discuss the effect of parameter b about the WIFPDMSM operator and particular case.

Theorem 7 Let $\theta_l = (\rho_l, \tau_l) (l = 1, 2, \dots, f)$ be a collection of IFNs, $b = 1, 2, \dots, \min_l \{o_l\}$. The WIFPDMSM operator monotonically increases with respect to the b .

Theorem 8 Let $\theta_l = (\rho_l, \tau_l) (l = 1, 2, \dots, f)$ be a set of IFNs, $b = 1, 2, \dots, \min_l \{o_l\}$. Then

$$\begin{aligned} & \min \{ \text{WIFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \} \\ &= \text{WIFPDMSM}^{(1)}(\theta_1, \theta_2, \dots, \theta_f) \\ &= \left(1 - \left(\prod_{i=1}^c \left(1 - \prod_{l_q=1}^{o_i} \left(1 - (1 - \rho_{l_q})^{\omega_{l_q}} \right)^{\frac{1}{o_i}} \right) \right)^{\frac{1}{c}} \right) \\ & \quad \left(\prod_{i=1}^c \left(1 - \prod_{l_q=1}^{o_i} \left(1 - \tau_{l_q}^{\omega_{l_q}} \right)^{\frac{1}{o_i}} \right) \right)^{\frac{1}{c}} \end{aligned}$$

and

$$\begin{aligned} & \max \{ \text{WIFPDMSM}^{(b)}(\theta_1, \theta_2, \dots, \theta_f) \} \\ &= \text{WIFPDMSM}^{(\min_l \{o_l\})}(\theta_1, \theta_2, \dots, \theta_f). \end{aligned}$$

when $c = 1$, we can get

Table 3 New decision matrix Λ' got by the decision maker

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Y_1	(0.5, 0.3)	(0.6, 0.3)	(0.6, 0.2)	(0.4, 0.4)	(0.6, 0.3)	(0.4, 0.3)
Y_2	(0.6, 0.3)	(0.3, 0.4)	(0.7, 0.2)	(0.6, 0.3)	(0.5, 0.2)	(0.6, 0.4)
Y_3	(0.5, 0.4)	(0.3, 0.3)	(0.3, 0.5)	(0.4, 0.4)	(0.3, 0.3)	(0.2, 0.5)
Y_4	(0.6, 0.2)	(0.7, 0.3)	(0.6, 0.2)	(0.6, 0.3)	(0.8, 0.2)	(0.7, 0.2)

Table 4 Sorting of diverse methodologies of Example 2

Methodologies	Score values	Sorting
Xu and Yager [3]	$S(\tilde{\theta}_1) = 0.220, S(\tilde{\theta}_2) = 0.214,$ $S(\tilde{\theta}_3) = 0.144, S(\tilde{\theta}_4) = 0.425$	$Y_4 > Y_1 > Y_2 > Y_3$
Qin and Liu ($b = 2$) [19]	$S(\tilde{\theta}_1) = -0.701, S(\tilde{\theta}_2) = -0.694,$ $S(\tilde{\theta}_3) = -0.718, S(\tilde{\theta}_4) = -0.619$	$Y_4 > Y_2 > Y_1 > Y_3$
Xia et al. ($q = p = 1$) [25]	$S(\tilde{\theta}_1) = 0.887, S(\tilde{\theta}_2) = 0.890,$ $S(\tilde{\theta}_3) = 0.873, S(\tilde{\theta}_4) = 0.923$	$Y_4 > Y_2 > Y_1 > Y_3$
Shi and Xiao* ($b = 2$) [40]	$S(\tilde{\theta}_1) = 0.222, S(\tilde{\theta}_2) = 0.248,$ $S(\tilde{\theta}_3) = 0.164, S(\tilde{\theta}_4) = 0.433$	$Y_4 > Y_2 > Y_1 > Y_3$
Shi and Xiao** ($b = 2$) [40]	$S(\tilde{\theta}_1) = 0.219, S(\tilde{\theta}_2) = 0.246,$ $S(\tilde{\theta}_3) = 0.163, S(\tilde{\theta}_4) = 0.434$	$Y_4 > Y_2 > Y_1 > Y_3$
The developed approach ($b = 2, c = 2$)	$S(\tilde{\theta}_1) = -0.697, S(\tilde{\theta}_2) = -0.692,$ $S(\tilde{\theta}_3) = -0.717, S(\tilde{\theta}_4) = -0.616$	$Y_4 > Y_2 > Y_1 > Y_3$

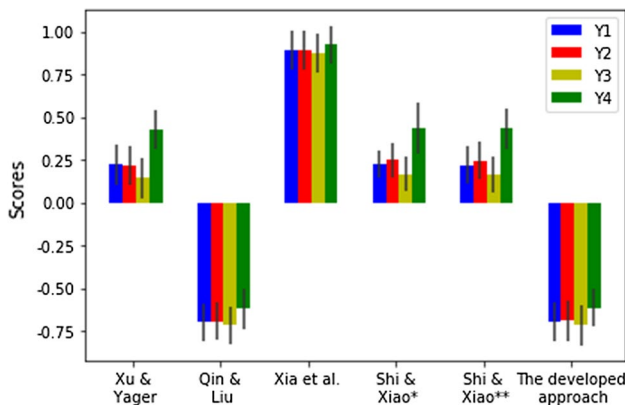


Fig. 3 Sorting of diverse methodologies of Example 2

$$WIFPDMSM^{(b)}(\theta_1, \theta_2, \dots, \theta_f)$$

$$= \left[1 - \left[1 - \prod_{\substack{1 \leq l_1 < \dots < l_b \leq f}} \left(1 - \prod_{q=1}^b (1 - \rho_{l_q})^{\omega_{l_q}} \right)^{\frac{1}{c^b}} \right]^{\frac{1}{b}} \right. \\ \left. \left(1 - \prod_{1 \leq l_1 < \dots < l_b \leq f} \left(1 - \prod_{q=1}^b \tau_{l_q}^{\omega_{l_q}} \right)^{\frac{1}{c^b}} \right)^{\frac{1}{b}} \right]$$

5 MADM Approach Based on

Following, we will apply an established MADM methodology to solve real decision-making problem. Suppose $Y = \{Y_1, Y_2, \dots, Y_n\}$ be a group of alternatives, $Z = \{Z_1, Z_2, \dots, Z_f\}$ be a set of attributes, let $\omega = (\omega_1, \omega_2, \dots, \omega_f)$ is weight vector of attributes and $\sum_{l=1}^f \omega_l = 1, 0 \leq \omega_l \leq 1 (l = 1, 2, \dots, f)$. Decision maker can use the IFNs to evaluate the alternative Y_l with respect to attribute Z_q and represent it as $\theta_{l_q} = (\rho_{l_q}, \tau_{l_q})$, where $\rho_{l_q}, \tau_{l_q} \in [0, 1]$ and $0 \leq \rho_{l_q} + \tau_{l_q} \leq 1$, we can get a decision matrix $\Lambda = [\theta_{l_q}]_{n \times f}, 1 \leq l \leq n, 1 \leq q \leq f$. Suppose that $Z = \{Z_1, Z_2, \dots, Z_f\}$ is partitioned into c distinct groups P_1, P_2, \dots, P_c , there has relationship between attributes of the identical group and does not have relationship between attributes of disparate groups. The steps of the propounded method are listed:

Step 1 In DMPs, attributes can be cut into two types, one is benefit attribute, and the other is cost attribute. In order to eliminate the effect of different attribute types, we can transform the decision matrix $\Lambda = [\theta_{l_q}]_{n \times f}$ into the normalization matrix $\tilde{\Lambda} = [\tilde{\theta}_{l_q}]_{n \times f}, 1 \leq l \leq n, 1 \leq q \leq f$. We use the following formula to obtain the normalization matrix $\tilde{\Lambda} = [\tilde{\theta}_{l_q}]_{n \times f}$

$$\tilde{\theta}_{l_q} = (\tilde{\rho}_{l_q}, \tilde{\tau}_{l_q}) = \begin{cases} (\rho_{l_q}, \tau_{l_q}), & \text{for the benefit attributes} \\ (\tau_{l_q}, \rho_{l_q}), & \text{for the cost attributes} \end{cases}$$

where $1 \leq l \leq n, 1 \leq q \leq f$.

Step 2 Use the proposed WIFPDMSM operator to get

$$\check{\theta}_l = (\check{\rho}_l, \check{\tau}_l) = \text{WIFPDMSM}^{(b)}(\check{\theta}_{l_1}, \check{\theta}_{l_2}, \dots, \check{\theta}_{l_f}) \quad (11)$$

where $\check{\theta}_l$ is the preference value of alternative Y_l , where $l = 1, 2, \dots, n$.

Step 3 Compute values $S(\check{\theta}_l)$ and $H(\check{\theta}_l)$ of the aggregated result $\check{\theta}_l (l = 1, 2, \dots, n)$.

Step 4 According to the comparative way, making a comparing between the computed values $S(\check{\theta}_1), S(\check{\theta}_2), \dots, S(\check{\theta}_n)$ and accuracy values $H(\check{\theta}_1), H(\check{\theta}_2), \dots, H(\check{\theta}_n)$, then ranking all alternatives to choose the greatest one(s).

5.1 Problem Description

Example 2 In the north, there are often droughts in summer, which has some effect on the crops and causes the loss of many farmers. In order to cut the loss of farmers as soon as possible, the local government asked a company to build a dam. After the company goes through many surveys, there are four possible locations Y_1, Y_2, Y_3, Y_4 and decision maker assesses the four possible locations from six attributes, i.e., the cost of location (Z_1), the size of location (Z_2), the surrounding environment (Z_3), the labor cost (Z_4), the material cost (Z_5) and the transportation condition (Z_6). Let the weights of attributes Z_1, Z_2, Z_3, Z_4, Z_5 and Z_6 be 0.2, 0.2, 0.2, 0.1, 0.1 and 0.2, respectively, the weight vector is $(0.2, 0.2, 0.2, 0.1, 0.1, 0.2)^T$. Furthermore, suppose that the attributes Z_1, Z_2, Z_3, Z_4, Z_5 and Z_6 are cut into two groups P_1 and P_2 , where $P_1 = \{Z_1, Z_4, Z_5\}$ and $P_2 = \{Z_2, Z_3, Z_6\}$. The decision maker evaluates the location $Y_l (l = 1, 2, 3, 4)$ with respect to the attribute $Z_q (q = 1, 2, 3, 4, 5, 6)$ by the IFNs. Therefore, the decision matrix $\Lambda = [\theta_{lq}]_{4 \times 6}$ can be obtained, where $\theta_{lq} = (\rho_{lq}, \tau_{lq}), 1 \leq l \leq 4, 1 \leq q \leq 6$, which is shown as follows (Table 1).

5.2 Decision-Making Process About the Designed Method

Step 1 Standardize the matrix Λ . Because total attributes are beneficial, standardization does not need.

Step 2 Aggregate all performance value $\check{\theta}_{lq} (1 \leq l \leq 4, 1 \leq q \leq 6)$. And founded on Eq. (11), we get the aggregated result $\check{\theta}_l$ for each alternative $Y_l (l = 1, 2, 3, 4)$. (Without loss of generality, here we take $b = 2, c = 2$)

$$\check{\theta}_1 = (0.116, 0.813),$$

$$\check{\theta}_2 = (0.127, 0.820),$$

$$\check{\theta}_3 = (0.095, 0.811),$$

$$\check{\theta}_4 = (0.167, 0.783).$$

Step 3 Compute value $S(\check{\theta}_l)$ and $H(\check{\theta}_l)$ of the overall preference value $\check{\theta}_l (l = 1, 2, 3, 4)$, presented below:

$$S(\check{\theta}_1) = -0.697, S(\check{\theta}_2) = -0.692,$$

$$S(\check{\theta}_3) = -0.717, S(\check{\theta}_4) = -0.616,$$

$$H(\check{\theta}_1) = 0.929, H(\check{\theta}_2) = 0.947,$$

$$H(\check{\theta}_3) = 0.906, H(\check{\theta}_4) = 0.950.$$

Step 4 In accordance with the Step 3, the sorting result is $Y_4 > Y_2 > Y_1 > Y_3$. Therefore, the greatest alternative is Y_4 .

Furthermore, we can use the parameter b to describe the preference of decision maker in real situations. So, decision maker need to choose the suitable value based on preference. The decision maker is either optimistic or pessimistic when values of the parameter b are different [44]. We discuss the effect of the different value of the parameter b about the developed method. In Table 2 and Figs. 1 and 2, we can obtain the results by the propounded method.

From Table 2 and Figs. 1 and 2., we can get: with the parameter b changing based on the subjective preference of decision maker, the sorting results are slightly different,

Table 5 The characteristics of the distinct approaches

Approaches	Depict fuzzy information using NMD	Capture relationship for each two attributes of the identical group	Capture relationship for attributes of the identical group	Consider the parts for the attributes
Xu and Yager [3]	Yes	No	No	No
Qin and Liu [19]	Yes	Yes	Yes	No
Xia et al. [25]	Yes	Yes	No	No
Shi and Xiao* [40]	Yes	Yes	Yes	No
Shi and Xiao** [40]	Yes	Yes	Yes	No
The developed approach	Yes	Yes	Yes	Yes

which denotes that the WIFPDMSM operator can reflect the preference of decision maker, but the optimal alternative is all Y_4 . And we can get score value increases based on the developed operator when the value of b increases and the value of c is fixed.

5.3 Validation Test

We examine the effectiveness of the developed approach through Wang and Triantaphylou introduced three test criterion, the specific content of three test criterion is in [48].

Firstly, to verify the effectiveness of the developed methodology by Test Criterion 1 [48]. The new decision matrix is got by decision maker through replacing alternative Y_3 by a worse alternative Y'_3 , shown in Table 3.

We use the established methodology to resolve the new decision-making problem. The outcome of each alternative is calculated as

$$\begin{aligned}\check{\theta}_1 &= (0.116, 0.813), \check{\theta}_2 = (0.127, 0.820), \\ \check{\theta}_3 &= (0.064, 0.860), \check{\theta}_4 = (0.167, 0.783).\end{aligned}$$

So, the score value of each alternative is computed as

$$\begin{aligned}S(\check{\theta}_1) &= -0.697, S(\check{\theta}_2) = -0.692, \\ S(\check{\theta}_3) &= -0.796, S(\check{\theta}_4) = -0.616.\end{aligned}$$

Founded on the above discussion, the sorting is $Y_4 > Y_2 > Y_1 > Y_3$, the optimal alternative is still alternative Y_4 in the new decision-making problem, which is similar with the result obtained by the original problem. So the optimal alternative is not changed when the non-optimal alternative is replaced with another worse alternative.

Next, we examine the effectiveness of our presented methodology utilizing Test Criterion 2,3 [48]. Firstly, the initial decision issue decomposes as the three smaller problems $\{Y_1, Y_2, Y_4\}$, $\{Y_1, Y_3, Y_4\}$, $\{Y_2, Y_3, Y_4\}$. Then, we use the developed method in this paper to solve the three smaller problems, respectively, the corresponding sorting results are obtained as $Y_4 > Y_2 > Y_1$, $Y_4 > Y_1 > Y_3$, $Y_4 > Y_2 > Y_3$. Accordingly, the combination ranking is $Y_4 > Y_2 > Y_1 > Y_3$, which is identical with sorting of the initial decision issue and meets the transitivity property. Thus, the proposed approach satisfies the Test criterion 2 and Test criterion 3.

5.4 Comparison Analysis

In the following, in order to confirm the validity of the propounded methodology, we compare the presented methodology with other methods, such as Xu and Yager's MADM approach [3] using the IFWG operator, Qin and Liu's MADM methodology [19] utilizing the WIFMSM operator, Xia et al.'s MADM approach [25] using the WIFGBM

operator, Shi and Xiao's MADM method [40] utilizing the IFRWDMSM and the IFRWMSM operator based on Example 2, and the result of each method is shown in Table 4 and Fig. 3 (* denotes the method [40] grounded on the IFRWDMSM operator, ** denotes the method [40] based on the IFRWMSM operator). In Table 4 and Fig. 3., we can get that the decision outcomes through applying different approaches have subtle difference. But, the most satisfied option is all Y_4 . Furthermore, we analyze why the developed method outperforms other methods, listed as follows:

- (1) MADM methodology utilizes the WIFMSM operator propounded by Qin and Liu [19], MADM method utilizes the WIFGBM operator established by Xia et al. [25], MADM approach utilizes the IFRWDMSM and the IFRWMSM operator introduced by Shi and Xiao [40], but our developed method uses the propounded WIFPDMSM operator. From Table 4, we can see that the sortings of Qin and Liu's MADM methodology [19], Xia et al.'s MADM method [25] and Shi and Xiao's MADM approach [40] are same to the designed MADM method based on the WIFPDMSM operator, namely $Y_4 > Y_2 > Y_1 > Y_3$. The proposed method considers the interrelationship betwixt attributes in the identical group; there are other methods capture correlation between attributes except the Xia et al.'s MADM method [25] which captures the correlation for any two attributes. In Example 2, attributes Z_1, Z_4 and Z_5 are related to each other and attributes Z_2, Z_3 and Z_6 are so too, but there is no interrelationship between the groups $\{Z_1, Z_4, Z_5\}$ and $\{Z_2, Z_3, Z_6\}$. Obviously, our method based on the introduced WIFPDMSM operator effectively settles Example 2, which views the relationship betwixt any two members of the identical set, it averts the influence of irrelevant members. Qin and Liu's MADM approach [19], Xia et al.'s MADM approach [25] and Shi and Xiao's MADM approach [40] consider correlation between attributes, but these attributes are independent each other, for example among the attributes Z_1 and Z_2 , among the attributes Z_1 and Z_3 , among the attributes Z_1 and Z_6 , among the attributes Z_4 and Z_2 , among the attributes Z_4 and Z_3 , among the attributes Z_4 and Z_6 , among the attributes Z_5 and Z_2 , among the attributes Z_5 and Z_3 , among the attributes Z_5 and Z_6 . Thus, the developed methodology grounded on the WIFPDMSM operator contrasting with Qin and Liu's MADM methodology [19], Xia et al.'s MADM method [25] and Shi and Xiao's MADM method [40] is more general.
- (2) The sorting of MADM approach [3] grounded on the IFWG operator introduced by Xu and Yager is slightly different from the sorting of the established method,

because Xu and Yager's MADM approach uses ordinary weighted geometric operator, which does not view relationship for the attributes. In Example 2, attributes Z_1, Z_4 and Z_5 are relevant and attributes Z_2, Z_3 and Z_6 are relevant. Therefore, the developed method successfully settles Example 2.

- (3) In real DMPs, some attributes may be independent each other. Thus, the developed method is more suitably used in practical MADM issues by contrasting with the methods [19, 25, 40]. And in real DMPs, relationship of attribute may be not ignored. Therefore, the designed method is more generally used in practical MADM issues by contrasting with the approach [3].

By previous analysis, the characteristics of the put forward methodology contrast with other methodologies, results are listed in Table 5 (* denotes the method [40] grounded on the IFRWDMSM operator, ** denotes the method [40] based on the IFRWMSM operator). From it, we can obtain: (1) Xu and Yager's MADM approach [3] does not consider correlation between attributes, Xia et al.'s MADM approach [25] capture relationship between any two attributes, Qin and Liu's MADM methodology [19], Shi and Xiao's MADM approach [40] and the propounded method consider interrelationship for the attributes; (2) the developed MADM method partitions overall attributes into some parts and the attributes are interrelated in the same part, but the attributes are not interrelated in different parts, while Qin and Liu's MADM methodology [19], Xia et al.'s MADM methodology [25] and Shi and Xiao's MADM approach [40] don't split attributes into several portions; (3) the proposed approach and Xia et al.'s MADM approach [25] consider the interrelationship between attributes. Our proposed MADM method views interrelationship betwixt members of the identical parts, but Xia et al.'s MADM methodology [25] just finds relation betwixt any two members; (4) our introduced approach can surmount the disadvantage of Qin and Liu's MADM methodology [19], Xia et al.'s MADM approach [25] and Shi and Xiao's MADM approach [40], where the four approaches do not cut the attributes into some partitions, and Xu and Yager's MADM approach [3] does not capture the interrelationship between attributes. In conclusion, the propounded methodology is more ordinary and appropriate than other methods [3, 19, 25, 40].

6 Conclusion and Future Studies

The DMSM operator considers that attributes are existing multiple relation. We easily find that the generalizations of the DMSM just only consider the case that attributes are

relevant. But, in real DMPs, there may be not relationship between attributes So this paper has put forward the PDMSM operator. The purposed PDMSM operator has been expanded to the IFPDMSM and the WIFPDMSM operator for intuitionistic fuzzy environment. A few features and particular instances of the established operators have been analyzed. In accordance with the propounded WIFPDMSM operator, a new MADM approach has been designed for intuitionistic fuzzy environment. The established method breaks attributes into some portions, considering attributes of the identical portion are interrelated, while attributes of diverse portions are independent each other. We have applied an example to contrast our established MADM approach with the ones of [3, 19, 25, 40], comparative results show our introduced MADM methodology is more effective than Qin and Liu's MADM methodology [19], Xia et al.'s MADM approach [25], Shi and Xiao's MADM approach [40], and Xu and Yager's MADM approach [3] in which does not capture the interrelationship between attributes to deal with MADM problems with intuitionistic fuzzy environment. In our future study, the presented operators will be utilized to fuzzy linguistic, which is admmissive to utilize multiple linguistic fuzzy sets in fuzzy linguistic, it has been wide utilized in DMPs because which allows decision makers to use linguistic fuzzy set to voice their preferences. And the introduced operators can be used in proportional hesitant fuzzy 2-tuple linguistic term set [49] to deal with DMPs. The study is in intuitionistic fuzzy environment, spherical fuzzy sets, Pythagorean fuzzy sets and q-rung orthopair fuzzy sets are extension of IFS, and the introduced operator PDMSM can be used for spherical fuzzy sets [50, 51] which introduced operations of spherical fuzzy sets and Pythagorean fuzzy sets [52] which introduced operators and q-rung orthopair fuzzy sets [53, 54] which introduced a new type and ranking of q-rung orthopair fuzzy sets to solve DMPS. The established operator PDMSM can combine with TOPSIS method [55, 56] which established similarity and distance of Complex Interval-Valued q-Rung Orthopair Fuzzy Sets and consider the unknown weights of the attributes to solve DMPS. These are future works.

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Declarations

Conflict of interest There is no any conflicts of interest among all authors.

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