




Non-linear characterization of Jordan $*$ -isomorphisms via maps on positive cones of C^* -algebras

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Abstract. We study maps between positive definite or positive semi-definite cones of unital C^* -algebras. We describe surjective maps that preserve

- (1) the norm of the quotient or product of elements;
- (2) the spectrum of the quotient or product of elements;
- (3) the spectral seminorm of the quotient or product of elements.

These maps relate to the Jordan $*$ -isomorphisms between the specified C^* -algebras. While a surjection between positive definite cones that preserves the norm of the quotient of elements may not be extended to a linear map between the underlying C^* -algebras, the other types of surjections can be extended to a Jordan $*$ -isomorphism or a Jordan $*$ -isomorphism followed by 2-sided multiplication by a positive invertible element. We also study conditions for the centrality of positive invertible elements. We generalize “the corollary” regarding surjections between positive semidefinite cones of unital C^* -algebras. Applying it, we provide positive solutions to the problem posed by Molnár for general unital C^* -algebras.

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1. Introduction

A Jordan $*$ -isomorphism plays a crucial role in the study of mappings between C^* -algebras. It preserves various quantities, sets, and structures of C^* -algebras. Conversely, we can characterize a Jordan $*$ -isomorphism by its preservation of specific quantities, sets, and structures. Kadison’s theorem [14] is a celebrated result that states that a surjective complex linear unital isometry between unital C^* -algebras is a Jordan $*$ -isomorphism. Kadison

also showed that a unital order isomorphism is a Jordan $*$ -isomorphism [15]. Therefore, if a unital complex linear bijection preserves positive elements, then it is a Jordan $*$ -isomorphism. If a surjective complex linear map preserves the unitary group, it is a Jordan $*$ -isomorphism multiplied by a unitary [29].

On the other hand, Molnár [24] initiated the study of the so-called multiplicatively spectrum-preserving maps. A map $T: B_1 \rightarrow B_2$ between Banach algebras B_1 and B_2 is called multiplicatively spectrum-preserving if $\sigma(T(a)T(b)) = \sigma(ab)$ for $a, b \in B_1$. He presented non-linear characterizations of automorphisms on algebras of functions and operators. Several further investigations were obtained in this way. We mention only a few of the corresponding works [7–9, 11, 16, 27, 28, 30] on commutative Banach algebras and non-commutative Banach algebras [1–3, 5, 12, 13, 17, 20, 31].

In this paper, we study non-linear characterizations of Jordan $*$ -isomorphisms on positive (definite or semidefinite) cones of C^* -algebras. Here, “non-linear” means that we do not assume that the maps we are considering are linear in any sense. We investigate surjective maps that preserve the norm, the spectral seminorm, and the spectrum of the product as well as the quotient of elements between positive definite cones of C^* -algebras. The positive (definite or semidefinite) cone has a vibrant structure from algebraic and geometrical points of view. It has wide-ranging applications in various areas of mathematics and mathematical physics.

Molnár’s fascinating lectures [18] inspired our research on preserver problems related to algebras of operators and functions. During these lectures, he presented his and his colleague’s findings about preservers of the norm of the arithmetic mean and the geometric mean of positive invertible elements in a unital C^* -algebra (cf. [4, 6, 21]). He presented non-linear characterizations of Jordan $*$ -isomorphisms. He also discussed these maps during the lectures. All these findings inspired us to study surjective maps between positive (definite or semidefinite) cones of C^* -algebras.

We will start by clarifying the notation and introducing the necessary definitions and properties that we will use throughout the paper. In this paper, we use A , A_1 , and A_2 to denote unital C^* -algebras. We always write the unit as e . We define

$$A_{SA} = \{a \in A : a = a^*\},$$

the Jordan algebra of all self-adjoint elements in A . The positive semidefinite cone is

$$A_+ = \{a \in A_{SA} : a \geq 0\},$$

and the positive definite cone is

$$A_+^{-1} = \{a \in A_+ : a \text{ is invertible in } A\}.$$

The spectrum of $a \in A$ is denoted by $\sigma(a)$, and the spectral seminorm on A is

$$\|a\|_S = \sup\{|t| : t \in \sigma(a)\}.$$

It is worth noting that $\|a\| = \|a\|_S$ for a self-adjoint element a , in particular, $a \in A_+^{-1}$. A Jordan $*$ -isomorphism is a complex linear bijective map between two unital C^* -algebras that preserves the Jordan product $((ab+ba)/2$ for a, b) and the involution. It preserves squaring, the unit, and the invertibility. It also preserves commutativity. The same applies for the spectrum. Although a Jordan $*$ -isomorphism may not preserve multiplication, it is known to preserve the triple product: that is, $J(aba) = J(a)J(b)J(a)$, for a, b in the domain. It is worth noting that a Jordan $*$ -isomorphism preserves the norm. Kadison's theorem is a celebrated result stating that if a complex linear unital bijection preserves the norm, it is a Jordan $*$ -isomorphism [14, Theorem 7]. A complex linear unital bijection is a Jordan $*$ -isomorphism if it preserves the order in both directions, as stated in [15, Corollary 5].

Molnár presented the following striking result in [21]. Let $\phi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ be a map. We say that ϕ is positive homogeneous if the condition $\phi(ta) = t\phi(a)$ holds for all $a \in A_{1+}^{-1}$ and real numbers $t > 0$. We say that ϕ is order-preserving in both directions provided that $a \leq b$ if and only if $\phi(a) \leq \phi(b)$ for any $a, b \in A_{1+}^{-1}$. We say that ϕ is an order isomorphism if ϕ is a bijection and order-preserving in both directions. It is worth noting that if ϕ is a surjection and order-preserving in both directions, then ϕ is also an injection. (In fact, if $\phi(x) = \phi(x')$, then $\phi(x) \leq \phi(x')$ and $\phi(x') \leq \phi(x)$. Hence we have $x \leq x'$ and $x' \leq x$ simultaneously. Thus, $x = x'$ which means ϕ is an injection.) Thus, ϕ is a bijection, hence an order isomorphism.

Theorem 1.1 (Proposition 13 in [21]). *Suppose that $\phi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ is a positive homogeneous order isomorphism. Then there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\phi = \phi(e)^{\frac{1}{2}}J\phi(e)^{\frac{1}{2}}$ on A_{1+}^{-1} .*

After examining the definition of the Thompson metric, we can conclude that if ϕ is a positive homogeneous order isomorphism, it is also a Thompson isometry. By utilizing [10, Theorem 9] and carefully considering the matter at hand, we can prove Theorem 1.1. Theorem 1.1 is highly applicable in many situations, including in this paper. Furthermore, it should be mentioned that [10, Theorem 9] is proven by utilizing a profound result of Kadison [14, Theorem 7] mentioned above. It is worth pointing out that there is a typo in [21, Theorem 13], where $\phi: \mathcal{A} \rightarrow \mathcal{B}$ should read $\phi: \mathcal{A}_+^{-1} \rightarrow \mathcal{B}_+^{-1}$. Molnár refers to Theorem 1.1 as “the corollary” in his lectures [18] because several interesting results follow from it. We also apply “the corollary” several times in this paper.

In Sect. 2, we present characterizations of bijections between positive definite cones that preserve the norm of the quotients of positive elements (Theorem 2.3). These bijections are related to Jordan $*$ -isomorphisms as usual. However, they may not be extended to linear maps. We also provide a condition pertaining to centrality that allows the underlying bijection to be extended to a Jordan $*$ -isomorphism. Theorem 2.4 is about a bijection that preserves the spectrum or spectral seminorm of the quotient of positive invertible elements. This bijection is extended to a Jordan $*$ -isomorphism followed by 2-sided multiplication by a positive invertible element.

In Sect. 3, we are concerned with multiplicatively norm or spectrum-preserving bijections. As described in the second paragraph of this section, Molnár [24] initiated the study of multiplicatively spectrum-preserving maps between algebras of operators or functions. We study multiplicatively norm-preserving maps between positive definite cones of C^* -algebras (Theorem 3.6). We are also concerned with maps that preserve the norm of the triple product.

In Sect. 4, we investigate the conditions of centrality for positive invertible elements. Ogasawara [26] proved that a C^* -algebra is commutative if and only if squaring preserves the order for all positive semidefinite elements. Molnár introduced local monotonicity [23]. A function f is said to be locally monotone at a self-adjoint element a if $a \leq x$ implies $f(a) \leq f(x)$ for every self-adjoint element x . He proved that a self-adjoint element a is central if and only if the exponential function is locally monotone at a . Finally, Nagy [25] succeeded in proving that every strictly convex increasing function defined on an open interval unbounded from above is locally monotone at a if and only if it is central. Prior to that, Virosztek [32] had proved the corresponding result for a large class of functions. The proofs by Nagy and Virosztek are universally applicable to a broad category of functions, encompassing the squaring function. However, it requires a rather extensive computation. It is noteworthy to present a proof specifically for the squaring function only if it is sufficiently straightforward. We present a concise and straightforward direct proof establishing that the squaring function is locally monotone at a positive invertible element if and only if it is central (Proposition 4.3). Through its application, we illustrate specific norm conditions that are sufficient for the centrality of positive invertible elements.

In Sect. 5, we study maps between positive semidefinite cones. We prove “the corollary” for maps between semidefinite cones (Theorem 5.1). Applying Theorem 5.1, we exhibit results similar to those in Sect. 4 for the case of positive semidefinite cones. In particular, we show a generalization of Theorem 2.6 and Corollary 2.9 in [20] for the case of general unital C^* -algebras. Our results provide positive solutions to the problem posed by Molnár [20, p.194].

2. Maps that preserve the spectrum or norm of the quotients of elements in positive definite cones

We aim to prove that the map $x \mapsto (ax^2a)^{\frac{1}{2}}a^{-1}$ on A_+^{-1} preserves the norm in Proposition 2.1, for any $a \in A_+^{-1}$. We can extend this map to a linear map on A if and only if a is central in A (Proposition 2.2). Using this approach, we prove the main result of this section, Theorem 2.3. We observe that a bijection between positive definite cones, which preserves the norm of the quotient of any two elements, is related to a Jordan $*$ -isomorphism between the whole algebras. Nevertheless, this bijection generally cannot be extended to a linear map between entire algebras.

In the first draft of the paper, we apply the Gelfand-Naimark theorem to prove the following proposition. However, Lajos Molnár pointed out that

direct proof is possible. This adjustment not only streamlines the proof but also enhances the clarity and accessibility of our paper.

Proposition 2.1. *For any trio $a, x, y \in A_+^{-1}$ we have*

$$\|(ax^2a)^{\frac{1}{2}}a^{-1}\| = \|x\|$$

and

$$\|(ax^2a)^{\frac{1}{2}}(ay^2a)^{-\frac{1}{2}}\| = \|xy^{-1}\|.$$

Proof. We have

$$\begin{aligned} \|(ax^2a)^{\frac{1}{2}}a^{-1}\|^2 &= \|((ax^2a)^{\frac{1}{2}}a^{-1})^*((ax^2a)^{\frac{1}{2}}a^{-1})\| \\ &= \|a^{-1}(ax^2a)a^{-1}\| = \|x^2\| = \|x\|^2. \end{aligned}$$

Thus the first equation holds. For the second equation, we have

$$\begin{aligned} \|(ax^2a)^{\frac{1}{2}}(ay^2a)^{-\frac{1}{2}}\|^2 &= \|((ax^2a)^{\frac{1}{2}}(ay^2a)^{-\frac{1}{2}})^*((ax^2a)^{\frac{1}{2}}(ay^2a)^{-\frac{1}{2}})\| \\ &= \|(ay^2a)^{-\frac{1}{2}}ax^2a(ay^2a)^{-\frac{1}{2}}\| = \|xa(ay^2a)^{-\frac{1}{2}}\|^2 \\ &= \|(xa(ay^2a)^{-\frac{1}{2}})(xa(ay^2a)^{-\frac{1}{2}})^*\| \\ &= \|xa(ay^2a)^{-1}ax\| = \|xy^{-2}x\| = \|xy^{-1}\|^2. \end{aligned}$$

□

To prove the main result of this section, Theorem 2.3, we will utilize the following proposition.

Proposition 2.2. *Let $a \in A_+^{-1}$. Suppose that $\phi: A_+^{-1} \rightarrow A_+^{-1}$ is defined as $\phi(x) = (ax^2a)^{\frac{1}{2}}$, $x \in A_+^{-1}$. Then ϕ is additive if and only if a is a central element in A . In this case, $\phi(x) = ax$, $x \in A_+^{-1}$.*

Proof. If a is a central element, then by an elementary calculation, we have $\phi(x) = ax$ for every $x \in A_+^{-1}$, so ϕ is additive.

Suppose that $\phi(x) = (ax^2a)^{\frac{1}{2}}$, $x \in A_+^{-1}$ is an additive map. Then

$$(a(x^2 + xy + yx + y^2)a)^{\frac{1}{2}} = (a(x + y)^2a)^{\frac{1}{2}} = (ax^2a)^{\frac{1}{2}} + (ay^2a)^{\frac{1}{2}}$$

for every $x, y \in A_+^{-1}$. To simplify the equation, we can square both sides, resulting in:

$$axy a + ayx a = (ax^2a)^{\frac{1}{2}}(ay^2a)^{\frac{1}{2}} + (ay^2a)^{\frac{1}{2}}(ax^2a)^{\frac{1}{2}}.$$

Substituting $x = a^{-1}$ we obtain

$$ya + ay = 2(ay^2a)^{\frac{1}{2}}, \quad y \in A_+^{-1}. \tag{1}$$

Since ϕ is additive, the inverse map $\phi^{-1}(x) = (a^{-1}x^2a^{-1})^{\frac{1}{2}}$, $x \in A_+^{-1}$ is also additive. In a similar way as (1) we obtain

$$ya^{-1} + a^{-1}y = 2(a^{-1}y^2a^{-1})^{\frac{1}{2}}, \quad y \in A_+^{-1}. \tag{2}$$

Replacing y by aya in (2), we have

$$ay + ya = 2(ya^2y)^{\frac{1}{2}}, \quad y \in A_+^{-1}. \tag{3}$$

By comparing (1) and (3) we have

$$(ay)(ay)^* = ay^2a = ya^2y = (ay)^*(ay), \quad y \in A_+^{-1}.$$

Thus ay is normal for every $y \in A_+^{-1}$. On the other hand, the equation

$$\sigma(ay) = \sigma(y^{\frac{1}{2}}ay^{\frac{1}{2}}) \subset (0, \infty)$$

implies that the spectrum of ay consists of positive real numbers. It may be well known that a normal element, whose spectrum is confined to real numbers, is self-adjoint. This can be demonstrated by examining the closed subalgebra generated by the corresponding normal element, constituting a commutative C^* -algebra. As the spectra in a C^* -algebra and in a closed $*$ -subalgebra are identical, and in a commutative C^* -algebra, an element whose spectrum is confined to the real numbers is self-adjoint, it follows that the element ay is self-adjoint. This implies that a and y commute for every $y \in A_+^{-1}$, which further implies that a is a central element in A . Finally, a simple calculation shows that $\phi(x) = ax$ for $x \in A_+^{-1}$. \square

Theorem 2.3. *Let $\phi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ be a surjection. Then, the following are equivalent:*

- (i) $\|\phi(x)\phi(y)^{-1}\| = \|xy^{-1}\|, \quad x, y \in A_{1+}^{-1},$
- (ii) *there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that*

$$\phi(x) = (\phi(e)J(x)^2\phi(e))^{\frac{1}{2}}, \quad x \in A_{1+}^{-1}.$$

When ϕ satisfies the condition (i), or equivalently (ii), the map ϕ is additive if and only if $\phi(e)$ is a central element in A_2 . In this case,

$$\phi(x) = \phi(e)J(x), \quad x \in A_{1+}^{-1}.$$

Proof. We prove that (i) implies (ii). Define $\psi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ by $\psi(x) = \phi(x^{\frac{1}{2}})^2, x \in A_{1+}^{-1}$. Then ψ is a surjection. By calculation, we have

$$\begin{aligned} \|y^{-\frac{1}{2}}x^{\frac{1}{2}}\| &= \|(x^{\frac{1}{2}}y^{-\frac{1}{2}})^*\| = \|x^{\frac{1}{2}}y^{-\frac{1}{2}}\| = \|\phi(x^{\frac{1}{2}})\phi(y^{\frac{1}{2}})^{-1}\| \\ &= \|\psi(x)^{\frac{1}{2}}\psi(y)^{-\frac{1}{2}}\| = \|\psi(y)^{-\frac{1}{2}}\psi(x)^{\frac{1}{2}}\|. \end{aligned}$$

Then we have

$$\begin{aligned} \|y^{-\frac{1}{2}}xy^{-\frac{1}{2}}\| &= \|(y^{-\frac{1}{2}}x^{\frac{1}{2}})(y^{-\frac{1}{2}}x^{\frac{1}{2}})^*\| = \|y^{-\frac{1}{2}}x^{\frac{1}{2}}\|^2 \\ &= \|\psi(y)^{-\frac{1}{2}}\psi(x)^{\frac{1}{2}}\|^2 = \|\psi(y)^{-\frac{1}{2}}\psi(x)\psi(y)^{-\frac{1}{2}}\|. \end{aligned} \tag{4}$$

Similarly, we have

$$\|x^{-\frac{1}{2}}yx^{-\frac{1}{2}}\| = \|\psi(x)^{-\frac{1}{2}}\psi(y)\psi(x)^{-\frac{1}{2}}\|. \tag{5}$$

Since $y^{-\frac{1}{2}}xy^{-\frac{1}{2}}$ is positive, we have

$$\begin{aligned} \|y^{-\frac{1}{2}}xy^{-\frac{1}{2}}\| &= \sup\{t: t \in \sigma(y^{-\frac{1}{2}}xy^{-\frac{1}{2}})\} \\ &= \inf\{t: y^{-\frac{1}{2}}xy^{-\frac{1}{2}} \leq te\} = \inf\{t: x \leq ty\}. \end{aligned} \tag{6}$$

In the same way, we have

$$\|x^{-\frac{1}{2}}yx^{-\frac{1}{2}}\| = \inf\{s: y \leq sx\}. \tag{7}$$

Recall that the Thompson metric $d_T(\cdot, \cdot)$ on A_+^{-1} is

$$d_T(x, y) = \log \max\{\inf\{t: x \leq ty\}, \inf\{s: y \leq sx\}\}, \quad x, y \in A_+^{-1}.$$

It follows by (4), (5), (6) and (7) that for every $x, y \in A_{1+}^{-1}$

$$\begin{aligned} d_T(x, y) &= \log \max\{\|x^{-\frac{1}{2}}yx^{-\frac{1}{2}}\|, \|y^{-\frac{1}{2}}xy^{-\frac{1}{2}}\|\} \\ &= \log \max\{\|\psi(x)^{-\frac{1}{2}}\psi(y)\psi(x)^{-\frac{1}{2}}\|, \|\psi(y)^{-\frac{1}{2}}\psi(x)\psi(y)^{-\frac{1}{2}}\|\} \\ &= d_T(\psi(x), \psi(y)). \end{aligned}$$

We conclude that $\psi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ is a surjective Thompson isometry. By [10, Theorem 9] there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ and a central projection $P \in A_2$ such that

$$\psi(x) = \psi(e)^{\frac{1}{2}}(PJ(x) + (e - P)J(x^{-1}))\psi(e)^{\frac{1}{2}}, \quad x \in A_{1+}^{-1}. \tag{8}$$

Substituting $x = \frac{1}{2}e$ and $y = e$ in (4), we obtain by (8) that

$$\begin{aligned} \frac{1}{2} &= \left\| e^{-\frac{1}{2}}\frac{1}{2}ee^{-\frac{1}{2}} \right\| = \left\| \psi(e)^{-\frac{1}{2}}\psi\left(\frac{1}{2}e\right)\psi(e)^{-\frac{1}{2}} \right\| \\ &= \left\| PJ\left(\frac{1}{2}e\right) + (e - P)J\left(\left(\frac{1}{2}e\right)^{-1}\right) \right\| = \left\| \frac{1}{2}P + 2(e - P) \right\|, \end{aligned}$$

as P is a central projection, we have

$$= \max\left\{ \left\| \frac{1}{2}P \right\|, \|2(e - P)\| \right\}.$$

It follows that $e - P = 0$, so $\psi = \psi(e)^{\frac{1}{2}}J\psi(e)^{\frac{1}{2}}$. We conclude that

$$\phi(x) = \psi(x^2)^{\frac{1}{2}} = (\phi(e)J(x)^2\phi(e))^{\frac{1}{2}}, \quad x \in A_{1+}^{-1}.$$

Suppose that (ii) holds. We prove (i). Applying Proposition 2.1 we have

$$\begin{aligned} \|\phi(x)\phi(y)^{-1}\| &= \|(\phi(e)J(x)^2\phi(e))^{\frac{1}{2}}(\phi(e)J(y)^2\phi(e))^{-\frac{1}{2}}\| \\ &= \|J(x)J(y)^{-1}\|, \quad x, y \in A_{1+}^{-1}. \end{aligned} \tag{9}$$

On the other hand, as a Jordan $*$ -isomorphism preserves squaring, the triple product, the inverse, and the norm, we have

$$\begin{aligned} \|J(x)J(y)^{-1}\|^2 &= \|(J(x)J(y)^{-1})^*J(x)J(y)^{-1}\| \\ &= \|J(y)^{-1}J(x^2)J(y)^{-1}\| = \|J(y^{-1}x^2y^{-1})\| \\ &= \|y^{-1}x^2y^{-1}\| = \|xy^{-1}\|^2, \quad x, y \in A_{1+}^{-1}. \end{aligned}$$

Then by (9) we conclude

$$\|\phi(x)\phi(y)^{-1}\| = \|xy^{-1}\|$$

holds for every $x, y \in A_{1+}^{-1}$.

Suppose that $\phi(x) = (\phi(e)J(x)^2\phi(e))^{\frac{1}{2}}$, $x \in A_{1+}^{-1}$ and ϕ is additive. This implies that the mapping $y \mapsto (\phi(e)y^2\phi(e))^{\frac{1}{2}}$ is additive. Therefore, by Proposition 2.2, we conclude that $\phi(e)$ is a central element in A_2 . We infer by a simple calculation that $\phi(x) = \phi(e)J(x)$ for every $x \in A_{1+}^{-1}$. \square

Note that if $\|\phi(x)\phi(y)^{-1}\| = \|xy^{-1}\|$ holds for every $x, y \in A_{1+}^{-1}$ and $\phi(e)$ is not a central element in A_2 , then ϕ can not be extended to even an additive map from A_1 into A_2 .

The structure of a surjection that preserves the spectrum or the spectral seminorm of the quotient is relatively simple. It can be extended to a Jordan $*$ -isomorphism followed by 2-sided multiplication by an element in A_{2+}^{-1} . To elaborate, the following holds true.

Theorem 2.4. *Let $\phi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ be a surjection. Then, the following are equivalent:*

(i) *There exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that*

$$\phi(x) = \phi(e)^{\frac{1}{2}}J(x)\phi(e)^{\frac{1}{2}}, \quad x \in A_{1+}^{-1},$$

(ii) $\|\phi(x)\phi(y)^{-1}\|_S = \|xy^{-1}\|_S, \quad x, y \in A_{1+}^{-1},$

(iii) $\sigma(\phi(x)\phi(y)^{-1}) = \sigma(xy^{-1}), \quad x, y \in A_{1+}^{-1}.$

Proof. By the definition of the spectrum norm, it is evident that (iii) implies (ii).

We prove that (ii) implies (i). For every $x, y \in A_{1+}^{-1}$,

$$\begin{aligned} \|xy^{-1}\|_S &= \sup\{t: t \in \sigma(xy^{-1})\} = \sup\{t: t \in \sigma(y^{-\frac{1}{2}}xy^{-\frac{1}{2}})\} \\ &= \inf\{t: y^{-\frac{1}{2}}xy^{-\frac{1}{2}} \leq te\} = \inf\{t: x \leq ty\}. \end{aligned}$$

In the same way, $\|yx^{-1}\|_S = \inf\{t: y \leq tx\}$, $\|\phi(x)\phi(y)^{-1}\|_S = \inf\{t: \phi(x) \leq t\phi(y)\}$, and $\|\phi(y)\phi(x)^{-1}\|_S = \inf\{t: \phi(y) \leq t\phi(x)\}$. Due to the definition of the Thompson metric, we obtain

$$\begin{aligned} d_T(x, y) &= \log \max\{\|xy^{-1}\|_S, \|yx^{-1}\|_S\} \\ &= \log \max\{\|\phi(x)\phi(y)^{-1}\|_S, \|\phi(y)\phi(x)^{-1}\|_S\} = d_T(\phi(x), \phi(y)) \end{aligned}$$

for every $x, y \in A_{1+}^{-1}$; ϕ is a (surjective) Thompson isometry. Hence, according to [10, Theorem 9] there exist a Jordan $*$ -isomorphism J and a central projection $P \in A_2$ such that

$$\phi(x) = \phi(e)^{\frac{1}{2}}(PJ(x) + (e - P)J(x^{-1}))\phi(e)^{\frac{1}{2}}, \quad x \in A_{1+}^{-1}.$$

Letting $x = \frac{1}{2}e$, we have

$$\phi\left(\frac{1}{2}e\right) = \phi(e)^{\frac{1}{2}}\left(\frac{1}{2}P + 2(e - P)\right)\phi(e)^{\frac{1}{2}}.$$

So

$$\begin{aligned} \frac{1}{2} &= \left\| \frac{1}{2}ee^{-1} \right\|_S = \left\| \phi(e)^{\frac{1}{2}}\left(\frac{1}{2}P + 2(e - P)\right)\phi(e)^{-\frac{1}{2}} \right\|_S \\ &= \left\| \frac{1}{2}P + 2(e - P) \right\|_S. \end{aligned}$$

It follows that $e - P = 0$. We conclude that $\phi = \phi(e)^{\frac{1}{2}}J\phi(e)^{\frac{1}{2}}$.

Suppose that (i) holds. Then for every $x, y \in A_{1+}^{-1}$, we infer $\phi(x)\phi(y)^{-1} = \phi(e)^{\frac{1}{2}}J(x)J(y^{-1})\phi(e)^{-\frac{1}{2}}$, so

$$\begin{aligned} \sigma(\phi(x)\phi(y)^{-1}) &= \sigma(\phi(e)^{\frac{1}{2}}J(x)J(y^{-1})\phi(e)^{-\frac{1}{2}}) \\ &= \sigma(J(x)J(y^{-1})) = \sigma(J(x^{\frac{1}{2}})J(y^{-1})J(x^{\frac{1}{2}})) \\ &= \sigma(J(x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}})) = \sigma(x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}}) = \sigma(xy^{-1}). \end{aligned}$$

Thus, (iii) holds. □

3. Maps that preserve the spectrum or the norm of the products of elements in positive definite cones

In this section, we study not only multiplicatively spectrum-preserving maps but also multiplicatively norm-preserving maps and multiplicatively spectral seminorm-preserving maps on a positive definite cone. We prove the following as an application of Lemma 13 in [4].

Lemma 3.1. *Let $a, a' \in A_+^{-1}$. Then, the following are equivalent:*

- (i) $a = a'$,
- (ii) $\|ax\| = \|a'x\|$ for every $x \in A_+^{-1}$,
- (iii) $\|xax\| = \|xa'x\|$ for every $x \in A_+^{-1}$.

Proof. If $a = a'$, then (ii) and (iii) are apparent. Suppose that (iii) holds. Then, $a \leq a'$ follows from $\|xax\| \leq \|xa'x\|$, $x \in A_+^{-1}$ by [4, Lemma 13]. In the same manner, we infer that $a' \leq a$. Thus, we obtain (i). Suppose that (ii) holds. Since $(ax)^*(ax) = xa^2x$ for every $x \in A_+^{-1}$ as x and a are self-adjoint, so

$$\|xa^2x\| = \|ax\|^2 = \|a'x\|^2 = \|xa'^2x\|, \quad x \in A_+^{-1}.$$

As (iii) implies (i), we have $a^2 = a'^2$. Hence, $a = a'$. □

Lemma 3.2. *Let $a, a' \in A_+^{-1}$. Then, $a \leq a'$ if and only if $\|ay\|_S \leq \|a'y\|_S$ for every $y \in A_+^{-1}$.*

Proof. Suppose that $a \leq a'$. Then $y^{\frac{1}{2}}ay^{\frac{1}{2}} \leq y^{\frac{1}{2}}a'y^{\frac{1}{2}}$ for every $y \in A_+^{-1}$. Since $\sigma(ay) = \sigma(y^{\frac{1}{2}}ay^{\frac{1}{2}})$ (resp. $\sigma(a'y) = \sigma(y^{\frac{1}{2}}a'y^{\frac{1}{2}})$) for every $y \in A_+^{-1}$, which is a subset of the set of positive real numbers, we obtain

$$\begin{aligned} \|ay\|_S &= \sup\{t : t \in \sigma(ay)\} = \sup\{t : t \in \sigma(y^{\frac{1}{2}}ay^{\frac{1}{2}})\} \\ &\leq \sup\{t : t \in \sigma(y^{\frac{1}{2}}a'y^{\frac{1}{2}})\} = \sup\{t : t \in \sigma(a'y)\} = \|a'y\|_S \end{aligned}$$

for every $y \in A_+^{-1}$.

Conversely, suppose that $\|ay\|_S \leq \|a'y\|_S$ for every $y \in A_+^{-1}$. Since $\sigma(ay^2) = \sigma(yay)$ (resp. $\sigma(a'y^2) = \sigma(ya'y)$) and $\|yay\| = \|yay\|_S$ (resp. $\|ya'y\| = \|ya'y\|_S$) for every $y \in A_+^{-1}$, we have

$$\|yay\| = \|ay^2\|_S \leq \|a'y^2\|_S = \|ya'y\|$$

for every $y \in A_+^{-1}$. By [4, Lemma 13], we have $a \leq a'$. □

Corollary 3.3. *Let $a, a' \in A_+^{-1}$. Then $a = a'$ if and only if $\|ay\|_S = \|a'y\|_S$ holds for every $y \in A_+^{-1}$.*

A proof is apparent from Lemma 3.2 and is omitted.

Lemma 3.4. *Suppose that ϕ and ψ are surjections from A_{1+}^{-1} onto A_{2+}^{-1} such that*

$$\|\phi(y)\psi(x)\phi(y)\| = \|yxy\|, \quad x, y \in A_{1+}^{-1}$$

and $\psi(e) = e$. Then, there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\psi = J$ on A_{1+}^{-1} .

Proof. We show that ψ is order preserving in both directions. By Lemma 13 in [4] we have $x \leq x'$ if and only if $\|yxy\| \leq \|yx'y\|$ if and only if $\|\phi(y)\psi(x)\phi(y)\| \leq \|\phi(y)\psi(x')\phi(y)\|$ if and only if $\psi(x) \leq \psi(x')$ as $\phi(A_{1+}^{-1}) = A_{2+}^{-1}$ for every $x, x' \in A_{1+}^{-1}$. This means that ψ is order preserving in both directions. Hence, we see that ψ is an injection, so a bijection. We prove that ψ is positive homogeneous. Let $t > 0$ and $x \in A_{1+}^{-1}$ be arbitrary. We have

$$\begin{aligned} \|\phi(y)\psi(tx)\phi(y)\| &= \|ytxy\| \\ &= t\|yxy\| = t\|\phi(y)\psi(x)\phi(y)\| = \|\phi(y)t\psi(x)\phi(y)\| \end{aligned}$$

for every $y \in A_{1+}^{-1}$. Since $\phi(A_{1+}^{-1}) = A_{2+}^{-1}$, we have $\psi(tx) = t\psi(x)$ by Lemma 3.1. As t and x are arbitrary, we see that ψ is positive homogeneous. By Theorem 1.1 there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\psi = J$ on A_{1+}^{-1} since $\psi(e) = e$. □

As a result of Lemma 3.4, we characterize a surjective map that preserves the norm of the triple product.

Corollary 3.5. *Let $\phi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ be a surjection. Suppose that*

$$\|\phi(y)\phi(x)\phi(y)\| = \|yxy\|, \quad x, y \in A_{1+}^{-1}. \tag{10}$$

Then there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\phi = J$ on A_{1+}^{-1} . Conversely, a Jordan $*$ -isomorphism J satisfies

$$\|J(y)J(x)J(y)\| = \|yxy\|, \quad x, y \in A_{1+}^{-1}.$$

Proof. Letting $x = y$ in (10), we have $\|\phi(y)^3\| = \|y^3\|$. As y and $\phi(y)$ are positive, we have $\|\phi(y)\| = \|y\|$. Hence $\|\phi(y)^2\| = \|y^2\|$. Thus

$$\|\phi(y)\phi(e)\phi(y)\| = \|yey\| = \|\phi(y)^2\| = \|\phi(y)e\phi(y)\|$$

for every $y \in A_{1+}^{-1}$. As $\phi(A_{1+}^{-1}) = A_{2+}^{-1}$, applying Lemma 3.1 we have $\phi(e) = e$. Then by Lemma 3.4 we observe that there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\phi = J$ on A_{1+}^{-1} .

Conversely, as a Jordan $*$ -isomorphism preserves the triple product and the norm, we see that

$$\|J(y)J(x)J(y)\| = \|J(yxy)\| = \|yxy\|$$

for every $x, y \in A_{1+}^{-1}$. □

Theorem 3.6. *Suppose that $\phi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ is a surjection. The following are equivalent:*

- (i) *There exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\phi = J$ on A_{1+}^{-1} ,*
- (ii) $\|\phi(x)\phi(y)\| = \|xy\|, x, y \in A_{1+}^{-1}$,
- (iii) $\|\phi(x)\phi(y)\|_S = \|xy\|_S, x, y \in A_{1+}^{-1}$,
- (iv) $\sigma(\phi(x)\phi(y)) = \sigma(xy), x, y \in A_{1+}^{-1}$,
- (v) *for a positive real number p , $\|(\phi(x)^{\frac{p}{2}}\phi(y)^p\phi(x)^{\frac{p}{2}})^{\frac{1}{p}}\| = \|(x^{\frac{p}{2}}y^p x^{\frac{p}{2}})^{\frac{1}{p}}\|,$
 $x, y \in A_{1+}^{-1}$.*

Note that the equivalence between (i) and (ii) is already pointed out just after Theorem 25 ([19, p.36]). For clarity, we include proof.

Proof. We prove that (i) implies (ii). Suppose that there exists a Jordan $*$ -isomorphism J such that $\phi = J$ on A_{1+}^{-1} . Since J preserves squaring, the involution, the triple product, and the norm, we obtain

$$\begin{aligned} \|\phi(x)\phi(y)\|^2 &= \|J(x)J(y)\|^2 = \|(J(x)J(y))^*J(x)J(y)\| \\ &= \|J(y)J(x^2)J(y)\| = \|J(yx^2y)\| = \|yx^2y\| = \|xy\|^2, \end{aligned}$$

so (ii) holds.

We prove that (ii) implies (i). Suppose that $\|\phi(x)\phi(y)\| = \|xy\|$ holds for every $x, y \in A_{1+}^{-1}$. Letting $x = y$ we obtain $\|y^2\| = \|\phi(y)^2\|$, hence $\|y\| = \|\phi(y)\|$ since y and $\phi(y)$ are self-adjoint. Then, we have

$$\|\phi(e)\phi(y)\| = \|ey\| = \|y\| = \|\phi(y)\| = \|e\phi(y)\|$$

for every $y \in A_{1+}^{-1}$. As ϕ is a surjection we infer by Lemma 3.1 that $\phi(e) = e$. Put $\psi(x) = \phi(x^{\frac{1}{2}})^2$ for $x \in A_{1+}^{-1}$. It is apparent that ψ is a surjection. We infer that $\psi(e) = \phi(e^{\frac{1}{2}})^2 = \phi(e)^2 = e^2 = e$. Let x, y be arbitrary in A_{1+}^{-1} . Since $yxxy = (x^{\frac{1}{2}}y)^*x^{\frac{1}{2}}y$ and

$$(\phi(x^{\frac{1}{2}})\phi(y))^*\phi(x^{\frac{1}{2}})\phi(y) = \phi(y)\phi(x^{\frac{1}{2}})^2\phi(y) = \phi(y)\psi(x)\phi(y)$$

hold, we have

$$\|yxxy\| = \|x^{\frac{1}{2}}y\|^2 = \|\phi(x^{\frac{1}{2}})\phi(y)\|^2 = \|\phi(y)\psi(x)\phi(y)\|. \tag{11}$$

Then by Lemma 3.4 there exists a Jordan $*$ -isomorphism J such that $J = \psi$ on A_{1+}^{-1} . We infer by the definition of ψ that $\phi = J$ on A_{1+}^{-1} .

It is apparent that (iv) implies (iii).

Suppose that (iii) holds. We prove (i). We show that $\phi(e) = e$. Letting $x = y$ in (iii) we have $\|\phi(y)^2\|_S = \|y^2\|_S$. By the spectral mapping theorem, we have $\sigma(z^2) = \{t^2: t \in \sigma(z)\}$, so that $\|y^2\|_S = \|y\|_S^2$ and $\|\phi(y)^2\|_S = \|\phi(y)\|_S^2$. Thus we have $\|y\|_S = \|\phi(y)\|_S$. Thus we have

$$\|e\phi(y)\|_S = \|\phi(y)\|_S = \|y\|_S = \|ey\|_S = \|\phi(e)\phi(y)\|_S$$

for every $y \in A_{1+}^{-1}$. By applying Corollary 3.3 we have $\phi(e) = e$. Since $\sigma(xy) = \sigma(y^{\frac{1}{2}}xy^{\frac{1}{2}})$, we have

$$\|xy\|_S = \|y^{\frac{1}{2}}xy^{\frac{1}{2}}\|_S = \|y^{\frac{1}{2}}xy^{\frac{1}{2}}\|. \tag{12}$$

Similarly, we obtain

$$\|\phi(x)\phi(y)\|_S = \|\phi(y)^{\frac{1}{2}}\phi(x)\phi(y)^{\frac{1}{2}}\|. \tag{13}$$

Letting $\varphi(y) = \phi(y^2)^{\frac{1}{2}}$ for each $y \in A_{1+}^{-1}$, $\varphi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ is a surjection. By (12) and (13) we observe that

$$\|yxy\| = \|\varphi(y)\phi(x)\varphi(y)\|, \quad x, y \in A_{1+}^{-1}.$$

Then by Lemma 3.4 there exists a Jordan $*$ -isomorphism J such that $J = \phi$ on A_{1+}^{-1} .

A proof that (i) implies (iv) is as follows. As a Jordan $*$ -isomorphism J preserves squaring, the triple product, and the spectrum we have

$$\begin{aligned} \sigma(J(x)J(y)) &= \sigma(J(x)J(y^{\frac{1}{2}})^2) = \sigma(J(y^{\frac{1}{2}})J(x)J(y^{\frac{1}{2}})) \\ &= \sigma(J(y^{\frac{1}{2}}xy^{\frac{1}{2}})) = \sigma(y^{\frac{1}{2}}xy^{\frac{1}{2}}) = \sigma(xy). \end{aligned}$$

We prove that (v) implies (i). Let $p = 1$ first. As $x^{\frac{1}{2}}yx^{\frac{1}{2}} \in A_{1+}^{-1}$ for every $x, y \in A_{1+}^{-1}$, we infer $\|x^{\frac{1}{2}}yx^{\frac{1}{2}}\| = \|x^{\frac{1}{2}}yx^{\frac{1}{2}}\|_S$. We also have $\sigma(x^{\frac{1}{2}}yx^{\frac{1}{2}}) = \sigma(xy)$ for every $x, y \in A_{1+}^{-1}$. Hence, we have $\|x^{\frac{1}{2}}yx^{\frac{1}{2}}\| = \|xy\|_S$. Similarly, we have $\|\phi(x)^{\frac{1}{2}}\phi(y)\phi(x)^{\frac{1}{2}}\| = \|\phi(x)\phi(y)\|_S$. Hence, (v) is equivalent to (iii) if $p = 1$. We consider a general positive real number p . Put $\psi(x) = \phi(x^{\frac{1}{p}})^p$, $x \in A_{1+}^{-1}$. By a simple calculation, substituting x by $x^{\frac{1}{p}}$ and y by $y^{\frac{1}{p}}$ in (v), we obtain

$$\begin{aligned} \|(\psi(x)^{\frac{1}{2}}\psi(y)\psi(x)^{\frac{1}{2}})^{\frac{1}{p}}\| &= \|(\phi(x^{\frac{1}{p}})^{\frac{p}{2}}\phi(y^{\frac{1}{p}})^p\phi(x^{\frac{1}{p}})^{\frac{p}{2}})^{\frac{1}{p}}\| \\ &= \|(x^{\frac{1}{2}}yx^{\frac{1}{2}})^{\frac{1}{p}}\|, \quad x, y \in A_{1+}^{-1}. \end{aligned} \tag{14}$$

By the spectral mapping theorem we have $\sigma(a^{\frac{1}{p}}) = \sigma(a)^{\frac{1}{p}}$ for every positive invertible element a , we have

$$\|a^{\frac{1}{p}}\| = \|a^{\frac{1}{p}}\|_S = \|a\|_S^{\frac{1}{p}} = \|a\|^{\frac{1}{p}}$$

for every positive invertible element a . By (14) we have

$$\|\psi(x)^{\frac{1}{2}}\psi(y)\psi(x)^{\frac{1}{2}}\| = \|x^{\frac{1}{2}}yx^{\frac{1}{2}}\|, \quad x, y \in A_{1+}^{-1}.$$

It follows from the first part that

$$\|\psi(x)\psi(y)\|_S = \|xy\|_S, \quad x, y \in A_{1+}^{-1},$$

(iii) for ψ is observed. As we have already proven that (iii) and (i) are equivalent, a Jordan $*$ -isomorphism J exists, such as $\psi = J$. Then $\phi(x) = \psi(x^p)^{\frac{1}{p}} = J(x^p)^{\frac{1}{p}} = J(x)$, $x \in A_{1+}^{-1}$; (i) holds for ϕ . Suppose that $\phi = J$ on A_{1+}^{-1} . Since a Jordan $*$ -isomorphism J preserves the power and the triple product, we see that

$$(\phi(x)^{\frac{p}{2}}\phi(y)^p\phi(x)^{\frac{p}{2}})^{\frac{1}{p}} = (J(x^{\frac{p}{2}}y^p x^{\frac{p}{2}}))^{\frac{1}{p}},$$

and J preserves the norm. Thus, we have (v). □

Theorem 3.7. *Let $\phi_j: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ be a bijection for $j = 1, 2$. Suppose that*

$$\|\phi_1(x)\phi_2(y)\| = \|xy\|, \quad x, y \in A_{1+}^{-1}.$$

Suppose also that $\phi_1(e) = e$. Then, there exists a Jordan $$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\phi_1 = \phi_2 = J$ on A_{1+}^{-1} .*

Proof. We have

$$\begin{aligned} \|\phi_2(y)\phi_1(x^{\frac{1}{2}})^2\phi_2(y)\| &= \|\phi_1(x^{\frac{1}{2}})\phi_2(y)\|^2 \\ &= \|x^{\frac{1}{2}}y\|^2 = \|yxy\|, \quad x, y \in A_{1+}^{-1}. \end{aligned} \tag{15}$$

Put $\psi(x) = \phi_1(x^{\frac{1}{2}})^2$, $x \in A_{1+}^{-1}$. Then $\psi: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ is a bijection. Apparently, $\psi(e) = e$. Applying Lemma 3.4 there is a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\phi_1 = J$ on A_{1+}^{-1} . Then we have

$$\begin{aligned} \|\phi_2(e)J(x)\| &= \|(\phi_2(e)J(x))^*\| \\ &= \|J(x)\phi_2(e)\| = \|xe\| = \|x\| = \|J(x)\| = \|eJ(x)\| \end{aligned}$$

for every $x \in A_{1+}^{-1}$ we have $e = \phi_2(e)$ as J is surjective. Similarly as ϕ_1 , there is a Jordan $*$ -isomorphism $J': A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ such that $\phi_2 = J'$ on A_{1+}^{-1} . Let $y \in A_{1+}^{-1}$ be fixed. Then for every $x \in A_{1+}^{-1}$ we have

$$\begin{aligned} \|J(x)J'(y)\|^2 &= \|xy\|^2 = \|xy^2x\| = \|J(xy^2x)\| \\ &= \|J(x)J(y)^2J(x)\| = \|J(x)J(y)\|^2. \end{aligned}$$

Then $J'(y) = J(y)$ by Lemma 3.2. As y can be arbitrary we conclude that $J'(y) = J(y)$ for every $y \in A_{1+}^{-1}$. Thus $\phi_1 = \phi_2 = J$ on A_{1+}^{-1} . \square

Note that the assumption $\phi_1(e) = e$ is essential in Theorem 3.7 as the following example shows.

Example 3.8. Suppose that $a \in A_{1+}^{-1}$ is not a central element and $J: A_1 \rightarrow A_2$ is a Jordan $*$ -isomorphism. Let $\phi_1: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ be defined as $\phi_1(x) = (aJ(x)^2a)^{\frac{1}{2}}$, $x \in A_{1+}^{-1}$ and $\phi_2(x) = (a^{-1}J(x)^2a^{-1})^{\frac{1}{2}}$, $x \in A_{1+}^{-1}$. We infer that $\phi_1(e) = a$ and $\phi_1(x) = (\phi_1(e)J(x)^2\phi_1(e))^{\frac{1}{2}}$. We also have $\phi_2(y) = \phi_1(y^{-1})^{-1}$, $y \in A_{1+}^{-1}$. Hence, we have by Theorem 2.3 that

$$\|\phi_1(x)\phi_2(y)\| = \|\phi_1(x)\phi_1(y^{-1})^{-1}\| = \|x(y^{-1})^{-1}\| = \|xy\|$$

for every $x, y \in A_{1+}^{-1}$. As a is not a central element, neither ϕ_1 nor ϕ_2 are additive on A_{1+}^{-1} according to Theorem 2.3. Hence, neither ϕ_1 nor ϕ_2 are Jordan $*$ -isomorphisms.

4. Conditions for centrality of elements in the positive definite cones

In this section, we study certain conditions for the centrality of elements in the positive definite cones. The following is a version of Corollary 5 in [15]. We can prove this by applying Corollary 5 in [15]. We can also prove this using Theorem 1.1. Here we provide a proof as a corollary of Proposition 1 in [22].

Proposition 4.1. *Let $T: A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ be a bijection. Suppose that T is additive; i.e., $T(a + b) = T(a) + T(b)$ for all $a, b \in A_{1+}^{-1}$. Then there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $T(a) = T(e)^{\frac{1}{2}}J(a)T(e)^{\frac{1}{2}}$ for every $a \in A_{1+}^{-1}$.*

Proof. We prove that T preserves the arithmetic mean; $T\left(\frac{a+b}{2}\right) = \frac{T(a)+T(b)}{2}$ for every $a, b \in A_{1+}^{-1}$. First, for any $a \in A_{1+}^{-1}$ we have $T(a) = T\left(\frac{a}{2} + \frac{a}{2}\right) = 2T\left(\frac{a}{2}\right)$. Hence $\frac{T(a)}{2} = T\left(\frac{a}{2}\right)$ for every $a \in A_{1+}^{-1}$. It follows that

$$T\left(\frac{a+b}{2}\right) = \frac{1}{2}(T(a+b)) = \frac{T(a) + T(b)}{2}$$

for every $a, b \in A_{1+}^{-1}$. Thus T preserves the arithmetic mean. Then by [22, Proposition 1] that T is of the desired form. \square

Recall that the geometric mean $x\#y$ of $x, y \in A_+^{-1}$ is

$$x\#y = x^{\frac{1}{2}}(x^{-\frac{1}{2}}yx^{-\frac{1}{2}})^{\frac{1}{2}}x^{\frac{1}{2}}.$$

Let $a \in A_+^{-1}$ and $\phi(x) = a^2\#x^2$, $x \in A_+^{-1}$. Then ϕ is a bijection from A_+^{-1} onto itself. Note that if A is commutative, then $a^2\#x^2 = ax$. It means that ϕ is additive if A is commutative. We have the converse. Suppose that ϕ is additive. Then $a^{-1}\phi(x)a^{-1} = (a^{-1}x^2a^{-1})^{\frac{1}{2}}$ defines an additive bijection from A_+^{-1} onto itself. Then by Proposition 2.2, a^{-1} is central, hence so is a . Similarly, it occurs for $(b\#x)^2$. If A is commutative, then $(b\#x)^2 = bx$ for every $b, x \in A_+^{-1}$. On the other hand, we have the following.

Corollary 4.2. *Let $b \in A_+^{-1}$. Suppose that a map $\phi: A_+^{-1} \rightarrow A_+^{-1}$ which is defined as $\phi(x) = (b\#x)^2$, $x \in A_+^{-1}$ is additive. Then b is a central element in A .*

Proof. We infer that ϕ is a bijection by a simple calculation. We have $\phi(e) = (b\#e)^2 = b$. Hence, by Proposition 4.1 there exists a Jordan $*$ -isomorphism J such that $\phi = b^{\frac{1}{2}}Jb^{\frac{1}{2}}$. As $b\#x = b^{\frac{1}{2}}(b^{-\frac{1}{2}}xb^{-\frac{1}{2}})^{\frac{1}{2}}b^{\frac{1}{2}}$ we have

$$J(x) = (b^{-\frac{1}{2}}xb^{-\frac{1}{2}})^{\frac{1}{2}}b(b^{-\frac{1}{2}}xb^{-\frac{1}{2}})^{\frac{1}{2}}, \quad x \in A_+^{-1}.$$

As a Jordan $*$ -isomorphism preserves the triple product, we have

$$J(b^{\frac{1}{2}})J(x)J(b^{\frac{1}{2}}) = J(b^{\frac{1}{2}}xb^{\frac{1}{2}}) = x^{\frac{1}{2}}bx^{\frac{1}{2}}, \quad x \in A_+^{-1}. \tag{16}$$

On the other hand, as we have

$$b^{\frac{1}{2}}J(b)b^{\frac{1}{2}} = \phi(b) = (b\#b)^2 = b^2,$$

we obtain $J(b) = b$, hence $J(b^{\frac{1}{2}}) = b^{\frac{1}{2}}$. By (16) we obtain

$$b^{\frac{1}{2}}J(x)b^{\frac{1}{2}} = x^{\frac{1}{2}}bx^{\frac{1}{2}} \tag{17}$$

for every $x \in A_+^{-1}$. Substituting x^{-1} instead of x we have

$$b^{\frac{1}{2}}J(x)^{-1}b^{\frac{1}{2}} = b^{\frac{1}{2}}J(x^{-1})b^{\frac{1}{2}} = x^{-\frac{1}{2}}bx^{-\frac{1}{2}}. \tag{18}$$

Hence

$$b^{-\frac{1}{2}}J(x)b^{-\frac{1}{2}} = x^{\frac{1}{2}}b^{-1}x^{\frac{1}{2}}. \tag{19}$$

By (17) and (19) we have

$$b^{-\frac{1}{2}}x^{\frac{1}{2}}bx^{\frac{1}{2}}b^{-\frac{1}{2}} = b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-1}x^{\frac{1}{2}}b^{\frac{1}{2}},$$

so

$$(b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-\frac{1}{2}})^*(b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-\frac{1}{2}}) = (b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-\frac{1}{2}})(b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-\frac{1}{2}})^*.$$

This means that $b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-\frac{1}{2}}$ is normal. Alongside the equation

$$\sigma(b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-\frac{1}{2}}) = \sigma(x^{\frac{1}{2}}) \subset (0, \infty),$$

we observe that $b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-\frac{1}{2}}$ is self-adjoint, similar to the manner highlighted in the proof of Proposition 2.2. Hence, we have

$$b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-\frac{1}{2}} = (b^{\frac{1}{2}}x^{\frac{1}{2}}b^{-\frac{1}{2}})^* = b^{-\frac{1}{2}}x^{\frac{1}{2}}b^{\frac{1}{2}}.$$

It follows that $bx^{\frac{1}{2}} = x^{\frac{1}{2}}b$ holds for every x . Then b is a central element in A . □

We present a proof of a localized version of Ogasawara’s theorem [26] concerning commutativity. This result is a specific instance of a theorem of Nagy [25, Theorem 1] and Virosztek [32, Theorem 1] applied to the squaring functions, addressing the local monotonicity of a strictly convex increasing function. Our presentation provides a concise and straightforward direct proof, despite the application of two profound results: the Russo and Dye theorem [29, Corollary 1] and Kadison’s generalized Schwarz lemma [15, Theorem 1].

Proposition 4.3. *Let $a \in A_+^{-1}$. Then a is central if and only if $a^2 \leq x^2$ holds for every $x \in A_+^{-1}$ with $a \leq x$.*

Proof. If a is central, it is apparent that $a^2 \leq x^2$ holds for every $x \in A_+^{-1}$ with $a \leq x$.

We prove the converse. Suppose that $a \in A_+^{-1}$ which satisfies that $a^2 \leq x^2$ holds for every $x \in A_+^{-1}$ with $a \leq x$. We first prove $0 \leq ab + ba$ for every $b \in A$ with $0 \leq b$. For any $0 < t \leq 1$ we have $a \leq (1 - t)a + t(a + b)$. By the hypothesis, we have $a^2 \leq ((1 - t)a + t(a + b))^2$. By calculation, we obtain that

$$(2 - t)a^2 \leq (1 - t)(2a^2 + ab + ba) + t(a + b)^2.$$

Letting $t \rightarrow 0$, we have

$$0 \leq ab + ba.$$

Define a map $T: A \rightarrow A$ as $T(y) = \frac{1}{2}(a^{\frac{1}{2}}ya^{-\frac{1}{2}} + a^{-\frac{1}{2}}ya^{\frac{1}{2}})$, $y \in A$. Then T is positive since $T(b) = \frac{1}{2}a^{-\frac{1}{2}}(ab + ba)a^{-\frac{1}{2}} \geq 0$ for any $b \geq 0$. We also have $T(e) = e$. By the Russo and Dye theorem [29, Corollary 1], we see that $\|T\| = 1$. Kadison’s generalized Schwarz lemma [15, Theorem 1] asserts that $T(x^2) \geq T(x)^2$ for each self-adjoint element x . After calculating $T(x^2)$ and $T(x)^2$, we can observe that

$$a^{\frac{1}{2}}x^2a^{-\frac{1}{2}} + a^{-\frac{1}{2}}x^2a^{\frac{1}{2}} \geq a^{\frac{1}{2}}xa^{-1}xa^{\frac{1}{2}} + a^{-\frac{1}{2}}xaxa^{-\frac{1}{2}}.$$

By multiplying $a^{\frac{1}{2}}$ from the left and right-hand sides of each term of both sides of the inequality, we infer that

$$ax^2 + x^2a \geq axa^{-1}xa + xax.$$

Then

$$ax^2 - xax \geq axa^{-1}xa - x^2a = (ax - xa)a^{-1}xa.$$

Thus we obtain

$$(ax - xa)a^{-1}(ax - xa) \geq 0. \tag{20}$$

On the other hand, as $(ax - xa)^* = -(ax - xa)$, we obtain

$$0 \leq ((ax - xa)a^{-\frac{1}{2}})((ax - xa)a^{-\frac{1}{2}})^* = -(ax - xa)a^{-1}(ax - xa). \tag{21}$$

Combining (20) and (21), we infer that

$$\|(ax - xa)a^{-\frac{1}{2}}\|^2 = \|(ax - xa)a^{-1}(ax - xa)\| = 0.$$

It follows that $ax = xa$. As x is an arbitrary self-adjoint element, we conclude that a is a central element. □

Proposition 4.4. *Let $a \in A_+^{-1}$. The following are equivalent:*

- (i) a is a central element in A ,
- (ii) $\|axa^{-1}\| = \|x\|$ for every element $x \in A_+^{-1}$,
- (iii) $\|a^2x\| = \|axa\|$ for every element $x \in A_+^{-1}$,
- (iv) $\|ax\| = \|ax\|_S$ for every element $x \in A_+^{-1}$,
- (v) $\|a^2x\| = \|a^2x\|_S$ for every element $x \in A_+^{-1}$.

Proof. First, we prove that (i) and (iv) are equivalent. Suppose that (i) holds; a is a central element. Then, so is $a^{\frac{1}{2}}$ since $a^{\frac{1}{2}}$ is approximated by polynomials of a . Hence $ax = a^{\frac{1}{2}}xa^{\frac{1}{2}}$, so $\|ax\| = \|a^{\frac{1}{2}}xa^{\frac{1}{2}}\|$. As $a^{\frac{1}{2}}xa^{\frac{1}{2}}$ is positive, so self-adjoint, we have $\|a^{\frac{1}{2}}xa^{\frac{1}{2}}\| = \|a^{\frac{1}{2}}xa^{\frac{1}{2}}\|_S$. As $\sigma(a^{\frac{1}{2}}xa^{\frac{1}{2}}) = \sigma(ax)$, we have $\|a^{\frac{1}{2}}xa^{\frac{1}{2}}\|_S = \|ax\|_S$. Thus, (iv) holds.

Suppose that a is not a central element. We prove (iv) does not hold. As the above, we see that a^2 is not a central element. Applying Proposition 4.3, there exists $x_0 \in A_+^{-1}$ such that $a \leq x_0^{-1}$ and $a^2 \not\leq x_0^{-2}$. Put

$$t_0 = \inf\{t: a^2 \leq tx_0^{-2}\},$$

and

$$s_0 = \inf\{s: a \leq sx_0^{-1}\}.$$

Then $t_0 > 1$ as $a^2 \not\leq x_0^{-2}$. Since $a \leq x_0^{-1}$ we have $s_0 \leq 1$. We compute

$$t_0 = \inf\{t: x_0a^2x_0 \leq te\} = \sup\{t: t \in \sigma(x_0a^2x_0)\} = \|x_0a^2x_0\|,$$

where the last equality holds since $x_0a^2x_0$ is positive, hence self-adjoint. Since $x_0a^2x_0 = (ax_0)^*(ax_0)$ we have

$$\|x_0a^2x_0\| = \|ax_0\|^2.$$

On the other hand, we compute

$$\begin{aligned} s_0 &= \inf\{s : a \leq sx_0^{-1}\} = \inf\{s : x_0^{\frac{1}{2}}ax_0^{\frac{1}{2}} \leq se\} \\ &= \sup\{s : s \in \sigma(x_0^{\frac{1}{2}}ax_0^{\frac{1}{2}})\} = \sup\{s : s \in \sigma(ax_0)\} = \|ax_0\|_S. \end{aligned}$$

Since $t_0 > 1 \geq s_0^2$, we can conclude that $\|axa\| > \|ax_0\|_S$, so (iv) does not hold.

In a similar way as above, we see that (i) and (v) are equivalent.

By a calculation, we have

$$\|axa\| = \sup\{t : t \in \sigma(axa)\} = \sup\{t : t \in \sigma(a^2x)\} = \|a^2x\|_S.$$

Hence (iii) and (v) are equivalent.

Suppose that (ii) holds. By substituting axa in place of x in equation (ii), we have $\|a^2x\| = \|axa\|$. Thus (iii) holds.

Suppose that (iii) holds. Substituting $a^{-1}xa^{-1}$ in place of x in equation (iii), we have $\|axa^{-1}\| = \|x\|$. Thus (ii) holds. \square

5. Maps between positive semidefnite cones

In this section, we exhibit results on maps between positive semidefnite cones of unital C^* -algebras. Theorem 1.1, “the corollary” of Molnár, can be slightly generalized for the case of positive semidefnite cones of unital C^* -algebras as Theorem 5.1. It is in the same vein as “the corollary”. Notably, the Thompson metric is not well defined on the positive semidefnite cones. However, a representation theorem is possible for positively homogeneous order isomorphisms between positive semidefnite cones. As an application of Theorem 5.1, we provide a solution (Theorem 5.5) to the problem posed by Molnár [20, p.194].

Theorem 5.1. *Suppose that $\phi : A_{1+} \rightarrow A_{2+}$ is a positively homogeneous order isomorphism, that is,*

- (1) ϕ is a bijection,
- (2) $\phi(tx) = t\phi(x), \quad x \in A_{1+}, t \geq 0,$
- (3) $x \leq x'$ if and only if $\phi(x) \leq \phi(x')$.

Then there exists a Jordan $$ -isomorphism $J : A_1 \rightarrow A_2$ such that $\phi = \phi(e)^{\frac{1}{2}}J\phi(e)^{\frac{1}{2}}$ on A_{1+} .*

Proof. We first prove that $\phi(A_{1+}^{-1}) = A_{2+}^{-1}$. Suppose that $a \in A_{1+}^{-1}$. Let $b = \phi^{-1}(e)$, where e is the unit in A_2 . As a is invertible, $a^{-\frac{1}{2}}ba^{-\frac{1}{2}} \geq 0$, so there exists a positive real number $t_b > 0$ such that $b \leq t_b a$. As ϕ preserves the order and is positively homogeneous, we infer that

$$e = \phi(b) \leq \phi(t_b a) = t_b \phi(a).$$

Hence $\phi(a)$ is invertible, so $\phi(a) \in A_{2+}^{-1}$. Conversely, we infer in a similar way that if $c \in A_{2+}^{-1}$, then $\phi^{-1}(c) \in A_{1+}^{-1}$. Thus we have $\phi(A_{1+}^{-1}) = A_{2+}^{-1}$.

According to “the corollary”, there exists a Jordan $*$ -isomorphism $J : A_1 \rightarrow A_2$ such that $\phi = \phi(e)^{\frac{1}{2}}J\phi(e)^{\frac{1}{2}}$ on A_{1+}^{-1} . We prove that the equality

also holds on A_{1+} . Let $a \in A_{1+}$ be arbitrary. For any $t > 0$, $a + te \in A_{1+}^{-1}$. Then

$$\phi(a) \leq \phi(a + te) = \phi(e)^{\frac{1}{2}} J(a + te) \phi(e)^{\frac{1}{2}} = \phi(e)^{\frac{1}{2}} J(a) \phi(e)^{\frac{1}{2}} + t\phi(e)$$

as ϕ preserves the order. Letting $t \rightarrow 0$ we have

$$\phi(a) \leq \phi(e)^{\frac{1}{2}} J(a) \phi(e)^{\frac{1}{2}}.$$

As ϕ is a surjection, there exists $a' \in A_{1+}$ such that $\phi(a') = \phi(e)^{\frac{1}{2}} J(a) \phi(e)^{\frac{1}{2}}$. Hence we have $\phi(a) \leq \phi(a')$, so $a \leq a'$ as ϕ is an order isomorphism. Next, for any $t > 0$ we have

$$\phi(a') = \phi(e)^{\frac{1}{2}} J(a) \phi(e)^{\frac{1}{2}} \leq \phi(e)^{\frac{1}{2}} J(a + te) \phi(e)^{\frac{1}{2}} = \phi(a + te),$$

as J preserves the order and $a + te \in A_{1+}^{-1}$. Thus, we have $a' \leq a + te$ for every $t > 0$. Letting $t \rightarrow 0$, we have $a' \leq a$. It follows that $a = a'$ and $\phi(a) = \phi(a') = \phi(e)^{\frac{1}{2}} J(a) \phi(e)^{\frac{1}{2}}$. As $a \in A_{1+}$ is arbitrary, we conclude that $\phi = \phi(e)^{\frac{1}{2}} J \phi(e)^{\frac{1}{2}}$ on A_{1+} . □

The proposition below is about a characterization of the inequality $a \leq b$ for a, b in a positive semidefinite cone of a unital C^* -algebra. It is a generalization of a part of [4, Lemma 13] which applies to positive definite cones, as well as a part of [20, Lemma 2.8], which applies to positive semidefinite cones of von Neumann algebras. To prove it, we apply [4, Lemma 13].

Proposition 5.2. *Let $a, b \in A_+$. Suppose that $\|xax\| \leq \|xbx\|$ holds for every $x \in A_+^{-1}$. Then $a \leq b$. Suppose conversely that $a \leq b$. Then $\|xax\| \leq \|xbx\|$ holds for all $x \in A_+$. In particular, $a \leq b$ if and only if $\|xax\| \leq \|xbx\|$ for every $x \in A_+$.*

Proof. Suppose that $\|xax\| \leq \|xbx\|$ for every $x \in A_+^{-1}$. Let $x \in A_+^{-1}$ and $0 < t < 1$ be arbitrary. We have

$$\begin{aligned} \|x(a + te)x\| &\leq \|xax\| + \|x(te)x\| \leq \|xbx\| + \|x(te)x\| \\ &\leq \|x(b + te)x\| + 2\|x(te)x\|. \end{aligned} \tag{22}$$

As $0 < te \leq \sqrt{t}b + te$, we have by Lemma 13 in [4] that

$$\|x(te)x\| \leq \|x(\sqrt{t}b + te)x\| = \sqrt{t}\|x(b + \sqrt{t}e)x\|.$$

As $0 < t < 1$ we have $0 < b + te \leq b + \sqrt{t}e$, so by [4, Lemma 13],

$$\|x(b + te)x\| \leq \|x(b + \sqrt{t}e)x\|.$$

We obtain that

$$\|x(b + te)x\| + 2\|x(te)x\| \leq (1 + 2\sqrt{t})\|x(b + \sqrt{t}e)x\|.$$

Then by (22) we have

$$\|x(a + te)x\| \leq (1 + 2\sqrt{t})\|x(b + \sqrt{t}e)x\| = \|x(1 + 2\sqrt{t})(b + \sqrt{t}e)x\|.$$

As $x \in A_+^{-1}$ is arbitrary, by [4, Lemma 13] we have

$$a + te \leq (1 + 2\sqrt{t})(b + \sqrt{t}e).$$

Letting $t \rightarrow 0$, we obtain $a \leq b$.

Conversely, suppose that $a \leq b$. Then for every $x \in A_+$, $0 \leq xax \leq xbx$. Hence

$$\|xax\|_S = \sup\{t : t \in \sigma(xax)\} \leq \sup\{t : t \in \sigma(xbx)\} = \|xbx\|_S.$$

As xax and xbx are self-adjoint, the norm and the spectral seminorm coincide with each other, hence we obtain $\|xax\| \leq \|xbx\|$ for all $x \in A_+$. □

Corollary 5.3. *Let $a, b \in A_+$. Then, $a = b$ if and only if $\|xax\| = \|xbx\|$ for every $x \in A_+$ with $x \leq e$.*

Proof. Suppose that $\|xax\| = \|xbx\|$ for every $x \in A_+$ with $x \leq e$. Then for every positive real number t , we have $\|(tx)a(tx)\| = \|(tx)b(tx)\|$. Thus we have $\|yay\| = \|yby\|$ for every $y \in A_+$. By Proposition 5.2 we infer that $a = b$.

The converse statement is trivial. □

The following proposition is a version of Lemma 3.4.

Proposition 5.4. *Suppose that ϕ and ψ are surjections from A_{1+} onto A_{2+} such that*

$$\|yxy\| = \|\phi(y)\psi(x)\phi(y)\|, \quad x, y \in A_{1+} \tag{23}$$

and $\psi(e) = e$. Then, there exists a Jordan $$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\psi = J$ on A_{1+} .*

Proof. We prove that ψ is an order isomorphism from A_{1+} onto A_{2+} . Let $x, x' \in A_{1+}$. As ψ is a surjection, it is enough to prove that $x \leq x'$ if and only if $\psi(x) \leq \psi(x')$ for every $x, x' \in A_{1+}$. By (23) and Proposition 5.2, we have $x \leq x'$ if and only if $\|yxy\| \leq \|yx'y\|$ for every $y \in A_{1+}$ if and only if $\|\phi(y)\psi(x)\phi(y)\| \leq \|\phi(y)\psi(x')\phi(y)\|$ for every $y \in A_{1+}$. As ϕ is a surjection, it is equivalent to $\psi(x) \leq \psi(x')$ by Proposition 5.2. We conclude that $x \leq x'$ if and only if $\psi(x) \leq \psi(x')$ for every $x, x' \in A_{1+}$. We infer that ψ is an injection, so it is a bijection. As x and x' are arbitrary, we conclude that ψ is an order isomorphism. We prove that ψ is positively homogeneous. Let $x \in A_{1+}$ and $t \geq 0$ arbitrary. Then we have by (23) that

$$\begin{aligned} \|\phi(y)\psi(tx)\phi(y)\| &= \|y(tx)y\| \\ &= t\|yxy\| = t\|\phi(y)\psi(x)\phi(y)\| = \|\phi(y)(t\psi(x))\phi(y)\|, \quad y \in A_{1+}. \end{aligned}$$

As ϕ is a surjection, we have $\psi(tx) = t\psi(x)$ by Corollary 5.3. Thus, ψ is positively homogeneous. Due to Theorem 5.1 and the hypothesis that $\psi(e) = e$, we conclude that there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\psi = J$. □

The following theorem provides a solution to the problem posed by Molnár [20, p.194].

Theorem 5.5. *Suppose that $\phi: A_{1+} \rightarrow A_{2+}$ is a surjection. The following are equivalent:*

- (i) *There exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\phi = J$ on A_{1+} ,*

- (ii) $\|\phi(x)\phi(y)\| = \|xy\|, \quad x, y \in A_{1+},$
- (iii) $\|\phi(x)\phi(y)\|_S = \|xy\|_S, \quad x, y \in A_{1+},$
- (iv) $\sigma(\phi(x)\phi(y)) = \sigma(xy), \quad x, y \in A_{1+},$
- (v) for a positive real number $p,$

$$\|(\phi(x)^{\frac{p}{2}}\phi(y)^p\phi(x)^{\frac{p}{2}})^{\frac{1}{p}}\| = \|(x^{\frac{p}{2}}y^px^{\frac{p}{2}})^{\frac{1}{p}}\|, \quad x, y \in A_{1+}.$$

Proof. Suppose that (ii) holds. We prove (i). As $(ab)^*ab = ba^2b$ for every $a, b \in A_{1+}$, we infer by the equation (ii) that

$$\|yxy\| = \|x^{\frac{1}{2}}y\|^2 = \|\phi(x^{\frac{1}{2}})\phi(y)\|^2 = \|\phi(y)\phi(x^{\frac{1}{2}})^2\phi(y)\|, \quad x, y \in A_{1+} \quad (24)$$

Defining $\psi: A_{1+} \rightarrow A_{2+}$ by $\psi(x) = \phi(x^{\frac{1}{2}})^2, x \in A_{1+}$, ψ is a surjection, and by (24) we obtain

$$\|yxy\| = \|\phi(y)\psi(x)\phi(y)\|, \quad x, y \in A_{1+}. \quad (25)$$

We prove $\psi(e) = e$. Letting $x = y$ in the equation of (ii), we have $\|y^2\| = \|\phi(y)^2\|$. Thus we have

$$\begin{aligned} \|\phi(y)\phi(e)^2\phi(y)\| &= \|\phi(e)\phi(y)\|^2 \\ &= \|ey\|^2 = \|y^2\| = \|\phi(y)^2\| = \|\phi(y)e\phi(y)\|. \end{aligned}$$

By Corollary 5.3, we have $\phi(e)^2 = e$, so $\phi(e) = e$, hence $\psi(e) = e$. Applying Proposition 5.4, there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\psi = J$ on A_{1+} . For any $x \in A_{1+}$, $\phi(x) = \psi(x^2)^{\frac{1}{2}} = J(x^2)^{\frac{1}{2}} = J(x)$. Thus $\phi = J$ on A_{1+} .

We prove (v) implies (i). Suppose first we consider $p = 1$. In this case, the equation of (iii) is

$$\|\phi(x)^{\frac{1}{2}}\phi(y)\phi(x)^{\frac{1}{2}}\| = \|x^{\frac{1}{2}}yx^{\frac{1}{2}}\|, \quad x, y \in A_{1+}. \quad (26)$$

We prove $\phi(e) = e$. Inserting $x = y$ in (26), we obtain $\|\phi(x)^2\| = \|x^2\|$. Hence $\|\phi(x)\| = \|x\|$ for every $x \in A_{1+}$. Then by (26) we have

$$\begin{aligned} \|\phi(x)^{\frac{1}{2}}\phi(e)\phi(x)^{\frac{1}{2}}\| &= \|x^{\frac{1}{2}}ex^{\frac{1}{2}}\| \\ &= \|x\| = \|\phi(x)\| = \|\phi(x)^{\frac{1}{2}}e\phi(x)^{\frac{1}{2}}\|, \quad x \in A_{1+}. \end{aligned}$$

As ϕ is a surjection, we see that $\{\phi(x)^{\frac{1}{2}}: x \in A_{1+}\} = A_{2+}$. Then Corollary 5.3 ensures that $\phi(e) = e$. Define $\varphi: A_{1+} \rightarrow A_{2+}$ by $\varphi(x) = \phi(x^2)^{\frac{1}{2}}, x \in A_{1+}$. Then by (26) we obtain

$$\|\varphi(x)\phi(y)\varphi(x)\| = \|xyx\|, \quad x, y \in A_{1+}.$$

Then by Proposition 5.4 we observe that (i) holds.

We consider a general positive p . Define $\phi': A_{1+} \rightarrow A_{2+}$ by $\phi'(x) = \phi(x^{\frac{1}{p}})^p, x \in A_{1+}$. By a simple calculation, we infer that ϕ' is a surjection and

$$\|x^{\frac{1}{2}}yx^{\frac{1}{2}}\| = \|\phi'(x)^{\frac{1}{2}}\phi'(y)\phi'(x)^{\frac{1}{2}}\|, \quad x, y \in A_{1+}.$$

By the first part, a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ exists such that $\phi' = J$. Thus, we also see that $\phi = J$ on A_{1+} .

It is apparent that (v) implies (iv).

Suppose that (iii) holds. Let $x, y \in A_{1+}$ arbitrary. By the so-called Jacobson’s lemma, we have $\sigma(xy) \setminus \{0\} = \sigma(x^{\frac{1}{2}}yx^{\frac{1}{2}}) \setminus \{0\}$. Hence we have $\|xy\|_S = \|x^{\frac{1}{2}}yx^{\frac{1}{2}}\|_S$. As $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is self-adjoint, we have $\|x^{\frac{1}{2}}yx^{\frac{1}{2}}\|_S = \|x^{\frac{1}{2}}yx^{\frac{1}{2}}\|$. Therefore we have $\|xy\|_S = \|x^{\frac{1}{2}}yx^{\frac{1}{2}}\|$. In the same way we have $\|\phi(x)\phi(y)\|_S = \|\phi(x)^{\frac{1}{2}}\phi(y)\phi(x)^{\frac{1}{2}}\|$. Thus we have

$$\|\phi(x)^{\frac{1}{2}}\phi(y)\phi(x)^{\frac{1}{2}}\| = \|x^{\frac{1}{2}}yx^{\frac{1}{2}}\|, \quad x, y \in A_{1+}.$$

In other words, we have (v) for $p = 1$ holds for ϕ . In this case, we have already proven that (i) holds.

Suppose that (i); $\phi = J$ on A_{1+} for a Jordan $*$ -isomorphism J . We prove (iv). Let $x, y \in A_{1+}$ arbitrary. By the Jacobson lemma, $\sigma(xy) \setminus \{0\} = \sigma(x^{\frac{1}{2}}yx^{\frac{1}{2}}) \setminus \{0\}$. By a simple calculation we have $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is invertible if and only if xy is invertible. (Suppose that $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is invertible; $(x^{\frac{1}{2}}yx^{\frac{1}{2}})a = a(x^{\frac{1}{2}}yx^{\frac{1}{2}}) = e$ for some $a \in A_1$. It means that $x^{\frac{1}{2}}$ is also invertible. Hence $yx = x^{-\frac{1}{2}}(x^{\frac{1}{2}}yx^{\frac{1}{2}})x^{\frac{1}{2}}$ is invertible. Thus $xy = (yx)^*$ is invertible. Conversely, suppose that xy is invertible. As $(xy)^* = yx$ is also invertible, we have $xyyx = xy^2x$ is invertible. Hence, there exists $b \in A_1$ such that $bxy^2x = xy^2xb = e$. Hence, x is also invertible. Thus, the spectrum of x consists of positive real numbers. By the spectral mapping theorem, so is the spectrum of $x^{\frac{1}{2}}$. Thus, $x^{\frac{1}{2}}$ is invertible. Hence $x^{\frac{1}{2}}(yx)x^{-\frac{1}{2}} = x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is invertible.) It follows that $\sigma(x^{\frac{1}{2}}yx^{\frac{1}{2}}) = \sigma(xy)$. Similarly, we have $\sigma(J(x)^{\frac{1}{2}}J(y)J(x)^{\frac{1}{2}}) = \sigma(J(x)J(y))$. As a Jordan $*$ -isomorphism preserves the spectrum, the triple product, and the square root, we see that

$$\sigma(x^{\frac{1}{2}}yx^{\frac{1}{2}}) = \sigma(J(x)^{\frac{1}{2}}J(y)J(x)^{\frac{1}{2}}).$$

We conclude that $\sigma(J(x)J(y)) = \sigma(xy)$.

We prove that (i) implies (ii). Suppose that there exists a Jordan $*$ -isomorphism J such that $\phi = J$ on A_{1+} . Since J preserves squaring, the involution, the triple product, and the norm, we obtain for arbitrary $x, y \in A_{1+}$ that

$$\begin{aligned} \|\phi(x)\phi(y)\|^2 &= \|J(x)J(y)\|^2 = \|(J(x)J(y))^*J(x)J(y)\| \\ &= \|J(y)J(x^2)J(y)\| = \|J(yx^2y)\| = \|yx^2y\| = \|xy\|^2, \end{aligned}$$

so (ii) holds.

As a Jordan $*$ -isomorphism preserves the triple product and the power, it is easy to prove that (i) implies (v). □

The set $E(A) = \{a \in A_+ : 0 \leq a \leq e\}$ is known as the effect algebra. This structure holds significant importance in the quantum theory of measurements, as it establishes the foundation for unsharp measurements within quantum mechanics. The following result generalizes Corollary 2.9 in [20], which was stated for von Neumann algebras. Let $p > 0$. We define $x \diamond_p y = (x^{\frac{p}{2}}y^px^{\frac{p}{2}})^{\frac{1}{p}}$ for $x, y \in E(A)$. The involved operation for $p = 1$ is the usual sequential product.

Corollary 5.6. *Let $\phi : E(A_1) \rightarrow E(A_2)$ be a surjection and $p > 0$. It satisfies*

$$\|\phi(x) \diamond_p \phi(y)\| = \|x \diamond_p y\|, \quad x, y \in A_{1+}$$

if and only if there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\phi = J$ on $E(A_1)$.

Proof. We first prove that $\phi(tx) = t\phi(x)$ for every $x \in E(A_1)$ and $0 < t \leq 1$. Suppose that $x, y \in E(A_1)$ and $0 < t \leq 1$. Then we have $ty \in E(A_1)$ and

$$\begin{aligned} \|(\phi(x)^{\frac{p}{2}}\phi(ty)^p\phi(x)^{\frac{p}{2}})^{\frac{1}{p}}\| &= \|(x^{\frac{p}{2}}(ty)^p x^{\frac{p}{2}})^{\frac{1}{p}}\| = t\|(x^{\frac{p}{2}}y^p x^{\frac{p}{2}})^{\frac{1}{p}}\| \\ &= t\|(\phi(x)^{\frac{p}{2}}\phi(y)^p\phi(x)^{\frac{p}{2}})^{\frac{1}{p}}\| = \|(\phi(x)^{\frac{p}{2}}(t\phi(y))^p\phi(x)^{\frac{p}{2}})^{\frac{1}{p}}\|. \end{aligned}$$

As $\{\phi(x)^{\frac{p}{2}}: x \in E(A_1)\} = E(A_2)$, we have $\phi(ty) = t\phi(y)$ by Corollary 5.3.

Define $\tilde{\phi}: A_{1+} \rightarrow A_{2+}$ as

$$\tilde{\phi}(a) = \begin{cases} \|a\|\phi\left(\frac{a}{\|a\|}\right), & a \neq 0 \\ 0, & a = 0. \end{cases} \tag{27}$$

It is easy to see that $\tilde{\phi}$ is a surjection. By the first part, we infer that $\phi = \tilde{\phi}$ on $E(A_1)$. We also see by a simple calculation that

$$\|(\tilde{\phi}(x)^{\frac{p}{2}}\tilde{\phi}(y)^p\tilde{\phi}(x)^{\frac{p}{2}})^{\frac{1}{p}}\| = \|(x^{\frac{p}{2}}y^p x^{\frac{p}{2}})^{\frac{1}{p}}\|, \quad x, y \in A_{1+}.$$

It follows by Theorem 5.5 that there exists a Jordan $*$ -isomorphism $J: A_1 \rightarrow A_2$ such that $\tilde{\phi} = J$ on A_{1+} , in particular, on $E(A_1)$. As $\phi = \tilde{\phi}$ on $E(A_1)$, we have the conclusion. \square

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