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Multipliers and weak multipliers of algebras

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Abstract. We investigate general properties of multipliers and weak multipliers of algebras. We apply the results to determine the (weak) multipliers of associative algebras and zeropotent algebras of dimension 3 over an algebraically closed field.

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1. Introduction

Multipliers of algebras, particularly multipliers of Banach algebras, have been studied in the field of analysis. In this paper we discuss them in a purely algebraic manner.

Let A be a Banach algebra. A mapping $T : A \to A$ is termed a multiplier of A if it satisfies the condition (I) $xT(y) = T(xy) = T(x)y$ for all $x, y \in A$. We denote the collection of all multipliers of A as $M(A)$, and the collection of all bounded linear operators on A as $B(A)$. Notably, $M(A)$ forms an algebra and $B(A)$ constitutes a Banach algebra. A Banach algebra A is referred to as *without order* if it has neither a nonzero left annihilator nor a nonzero right annihilator. If A is without order, the algebra $M(A)$ forms a commutative closed subalgebra of $B(A)$ (see [2], Proposition 1.4.11). In 1952, Wendel [8] proved an important result that the multiplier algebra of $L^1(G)$ on a locally compact group G is isometrically isomorphic to the measure algebra on G. The general theory of multipliers of Banach algebras has been developed by Johnson [1]. For a comprehensive reference on the theory of multipliers of Banach algebras, refer to Larsen [5].

When A is without order, T is a multiplier if it satisfies the condition (II) $xT(y) = T(x)y$ for all $x, y \in A$. Many researchers had been unaware of the difference between conditions (I) and (II) until Zivari-Kazempour [9] (see also [10]) recently articulated the difference. We call a mapping T satisfying (II) a weak multiplier and denote the set of such multipliers of A by $M'(A)$. Then, $M(A)$ is in general a proper subset of $M'(A)$. Furthermore, (weak) multipliers can be defined for an algebra A that is not necessarily associative, and they are not linear mappings in general. We denote the spaces of linear multipliers and linear weak multipliers of A by $LM(A)$ and $LM'(A)$ respectively. $M(A)$ and $LM(A)$ are subalgebras of the algebra A^A consisting of all mappings from A to itself. Meanwhile, $M'(A)$ and $LM'(A)$ are closed under the operation $\circ$ defined by $T \circ S = TS + ST$, and they form a Jordan algebra.

In Sects. 2 to 5, we study general properties of (weak) multipliers. More specifically, in Sects. 3 and 4 we give a decomposition theorem (Theorem 3.1), and a matrix equation (Theorem 4.2) for (weak) multipliers. They play an essential role in our examination of (weak) multipliers.

Complete classifications of associative algebras and zeropotent algebras of dimension 3 over an algebraically closed field of characteristic not equal to 2 were given in Kobayashi et al., [3] and [4]. In Sects. 6 and 7 we undertake a complete determination of the (linear) (weak) multipliers of these algebras.

Some authors have considered other weaker concepts related to multipliers, such as (pseudo-)n-multipliers (for more information, see [6] and [11]).

2. Multipliers and weak multipliers

Let K be a field and A be a (not necessarily associative) algebra over K. The set A^A of all mappings from A to A forms an associative algebra over K in the usual way. Let $L(A)$ denote the subalgebra of A^A of all linear mappings from A to A.

A mapping $T : A \to A$ is a *weak multiplier* of A, if

$$xT(y) = T(x)y \tag{1}$$

holds for any $x, y \in A$, and T is a *multiplier*, if

$$xT(y) = T(xy) = T(x)y \tag{2}$$

for any $x, y \in A$. Let $M(A)$ (resp. $M'(A)$) denote the set of all multipliers (resp. weak multipliers) of A. Define

$$LM(A) \stackrel{\text{def}}{=} M(A) \cap L(A) \text{ and } LM'(A) \stackrel{\text{def}}{=} M'(A) \cap L(A).$$

Proposition 2.1. *$M(A)$ (resp. $LM(A)$) is a unital subalgebra of A^A (resp. $L(A)$), and $M'(A)$ (resp. $LM'(A)$) is a Jordan subalgebra of A^A (resp. $L(A)$).*

Proof. First, the zero mapping is a multiplier because all three terms in (2) are zero. Secondly, the identity mapping is also a multiplier because the three terms in (2) are xy. Let $T, U \in M(A)$. Then we have

$$\begin{aligned} x(T+U)(y) &= xT(y) + xU(y) = T(xy) + U(xy) = T(x)y + U(x)y \\ &= (T+U)(x)y \end{aligned} \tag{3}$$

and

$$x(TU)(y) = xT(U(y)) = T(xU(y)) = TU(xy) = T(U(x)y) = (TU)(x)y \tag{4}$$

for any $x, y \in A$. Hence, $T+U, TU$ belong to $M(A)$. Finally let $k \in K$, then

$$x(kT)(y) = kxT(y) = kT(xy) = kT(x)y = (kT)(x)y, \quad (5)$$

and so $kT \in M(A)$. Therefore, $M(A)$ is a unital subalgebra of A^A, and $LM(A) = M(A) \cap L(A)$ is a unital subalgebra of $L(A)$.

Next, let $T, U \in M'(A)$. Then, the equalities in (3) and (5) hold, removing the center terms $T(xy) + U(xy)$ and $kT(xy)$, respectively. Hence, $M'(A)$ is a subspace of A^A. Moreover, we have

$$x(TU)(y) = xT(U(y)) = T(x)U(y) = U(T(x))y = UT(x)y$$

and similarly $x(UT)(y) = TU(x)y$ for any $x, y \in A$. Hence,

$$x(TU + UT)(y) = (TU + UT)(x)y.$$

It follows that $TU + UT \in M'(A)$.[1] □

Let $\mathrm{Ann}_l(A)$ (resp. $\mathrm{Ann}_r(A)$) be the left (resp. right) annihilator of A and let A_0 be their intersection, that is,

$$\mathrm{Ann}_l(A) = \{a \in A \mid ax = 0 \text{ for all } x \in A\},$$
$$\mathrm{Ann}_r(A) = \{a \in A \mid xa = 0 \text{ for all } x \in A\}$$

and

$$A_0 = \mathrm{Ann}_l(A) \cap \mathrm{Ann}_r(A).$$

They are all subspaces of A, and when A is an associative algebra, they are two-sided ideals. For a subset X of A, $\langle X \rangle$ denotes the subspace of A generated by X.

Proposition 2.2. *A weak multiplier T of A such that $\langle T(A) \rangle \cap A_0 = \{0\}$ is a linear mapping.*

Proof. Let $x, y, z \in A$ and $a, b \in K$, and let T be a weak multiplier. We have

$$\begin{aligned} T(ax+by)z &= (ax+by)T(z) = axT(z) + byT(z) = aT(x)z + bT(y)z \\ &= \left(aT(x) + bT(y)\right) z. \end{aligned}$$

Because z is arbitrary, we have $w = T(ax+by) - aT(x) - bT(y) \in \mathrm{Ann}_l(A)$. Similarly, we can show $w \in \mathrm{Ann}_r(A)$, and so $w \in A_0$. Hence, if $\langle T(A) \rangle \cap A_0 = \{0\}$, then $w = 0$ because $w \in \langle T(A) \rangle$. Since a, b, x, y are arbitrary, T is a linear mapping. □

Corollary 2.3. *If $A_0 = \{0\}$, then any weak multiplier is a linear mapping over K, that is, $M'(A) = LM'(A)$ and $M(A) = LM(A)$.*

Proposition 2.4. *If T is a weak multiplier, then $T(\mathrm{Ann}_l(A)) \subseteq \mathrm{Ann}_l(A)$, $T(\mathrm{Ann}_r(A)) \subseteq \mathrm{Ann}_r(A)$ and $T(A_0) \subseteq A_0$.*

Proof. Let $x \in \mathrm{Ann}_l(A)$, then for any $y \in A$ we have

$$0 = xT(y) = T(x)y.$$

Hence, $T(x) \in \mathrm{Ann}_l(A)$. The other cases are similar. □

[1] In general, for an associative algebra A over a field K of characteristic $\neq 2$, the *Jordan product* $\circ$ on A is defined by $x \circ y = (xy + yx)/2$ for $x, y \in A$.

In this paper we denote the subset $\{xy \mid x, y \in A\}$ of A by A^2.[2]

Proposition 2.5. *Any mapping $T : A \to A$ such that $T(A) \subseteq A_0$ is a weak multiplier. Such a mapping T is a multiplier if and only if $T(A^2) = \{0\}$. In particular, if A is the zero algebra, every mapping T is a weak multiplier, and it is a multiplier if and only if $T(0) = 0$.*

Proof. If $T(A) \subseteq A_0$, then both sides are 0 in (1) and T is a weak multiplier. This T is a multiplier if and only if the term $T(xy)$ in the middle of (2) is 0 for all $x, y \in A$, that is, $T(A^2) = \{0\}$. If A is the zero algebra, then $A = A_0$ and $A^2 = \{0\}$. Hence, any T is a weak multiplier and it is a multiplier if and only if $T(0) = 0$. □

The *opposite* A^{op} of A is the algebra with the same elements and addition as A, but the multiplication $*$ in it is reversed, that is, $x * y = yx$ for all $x, y \in A$.

Proposition 2.6. *A and A^{op} have the same multipliers and weak multiplies, that is,*

$$M(A^{\text{op}}) = M(A) \quad \textit{and} \quad M'(A^{\text{op}}) = M'(A).$$

Proof. Let $T \in A^A$. Then, $T \in M'(A)$ if and only if

$$x * T(y) = T(y)x = yT(x) = T(x) * y$$

for any $x, y \in A$, if and only if $T \in M'(A^{\text{op}})$. Further, $T \in M(A)$ if and only if

$$x * T(y) = T(y)x = T(yx) = T(x * y) = yT(x) = T(x) * y$$

for any $x, y \in A$, if and only if $T \in M(A^{\text{op}})$. □

3. Nihil decomposition

Let A_1 be a subspace of A such that

$$A = A_1 \oplus A_0. \tag{6}$$

Here, A_1 is not unique, but choosing an appropriate A_1 will become important later. When A_1 is fixed, any mapping $T \in A^A$ is uniquely decomposed as

$$T = T_1 + T_0 \tag{7}$$

with $T_1(A) \subseteq A_1$ and $T_0(A) \subseteq A_0$. We call (6) and (7) a *nihil decomposition* of A and T, respectively. Let $\pi : A \to A_1$ be the projection and $\mu : A_1 \to A$ be the embedding, that is, $\pi(x_1 + x_0) = \mu(x_1) = x_1$ for $x_1 \in A_1$ and $x_0 \in A_0$.

Let $M_1(A)$ (resp. $M_0(A)$) denote the set of all multipliers T of A with $T(A) \subseteq A_1$ (resp. $T(A) \subseteq A_0$). Similarly, the sets $M'_1(A)$ and $M'_0(A)$ of weak multipliers of A are defined. Also, set

$$LM_i(A) = M_i(A) \cap L(A) \text{ and } LM'_i(A) = M'_i(A) \cap L(A)$$

[2]Usually A^2 denotes the subspace of A generated by this subset.

for $i = 0, 1$. By Proposition 2.2 we see

$$M_1'(A) = LM_1'(A) \text{ and } M_1(A) = LM_1(A),$$

and by Proposition 2.5 we have

$$M_0'(A) = A_0^A \text{ and } M_0(A) = \{T \in A_0^A \,|\, T(A^2) = \{0\}\}. \tag{8}$$

Theorem 3.1. *Let $A = A_1 \oplus A_0$ and $T = T_1 + T_0$ be nihil decompositions of A and $T \in A^A$ respectively.*

(i) T is a weak multiplier, if and only if T_1 is a weak multiplier. If T is a weak multiplier, T_1 is a linear mapping satisfying $T_1(A_0) = \{0\}$.

(ii) If T_1 is a multiplier and $T_0(A^2) = \{0\}$, then T is a multiplier. If A_1 is a subalgebra of A, the converse is also true.

Suppose that A_1 is a subalgebra of A, and let Φ be a mapping sending $R \in (A_1)^{A_1}$ to $\mu \circ R \circ \pi \in A^A$. Then,

(iii) Φ gives an algebra isomorphism from $M(A_1)$ onto $M_1(A)$ and a Jordan isomorphism from $M'(A_1)$ onto $M_1'(A)$.

Proof. Let $x, y \in A$.

(i) If T is a weak multiplier, then

$$xT_1(y) = x(T(y) - T_0(y)) = xT(y) = T(x)y = T_1(x)y.$$

Thus, T_1 is also a weak multiplier. Moreover, T_1 is a linear mapping by Proposition 2.2 and $T_1(A_0) \subseteq A_1 \cap A_0 = \{0\}$ by Proposition 2.4. Conversely, if T_1 is a weak multiplier, then

$$xT(y) = xT_1(y) = T_1(x)y = T(x)y,$$

and so T is a weak multiplier.

(ii) If T_1 is a multiplier and $T_0(A^2) = 0$, then T is a multiplier because

$$xT(y) = xT_1(y) = T_1(xy) = T(xy) - T_0(xy) = T(xy) = T(x)y.$$

Next suppose that A_1 is a subalgebra. If T is a multiplier, then for any $x, y \in A$ we have

$$T_1(xy) + T_0(xy) = T(xy) = xT(y) = x_1T_1(y), \tag{9}$$

where $x = x_1 + x_0$ with $x_1 \in A_1$ and $x_0 \in A_0$. Here, $x_1T_1(y) \in A_1$ because A_1 is a subalgebra, and thus, we have $T_0(xy) = x_1T_1(y) - T_1(xy) \in A_0 \cap A_1 = \{0\}$. Since x, y are arbitrary, we get $T_0(A^2) = \{0\}$. Moreover, because $T_1(xy) = x_1T_1(y) = xT_1(y)$ by (9) and similarly $T_1(xy) = T_1(x)y$, T_1 is a multiplier. The converse is already proved above.

(iii) Let $S \in (A_1)^{A_1}$ and $x = x_1 + x_0, y = y_1 + y_0 \in A$ with $x_1, y_1 \in A_1$ and $x_0, y_0 \in A_0$. Then, $\pi(x) = \mu(x_1) = x_1$, $\pi(y) = \mu(y_1) = y_1$ and

$$\Phi(S)(x) = \mu(S(\pi(x))) = \mu(S(x_1)) = S(x_1).$$

Thus, if $S \in M'(A_1)$, we have

$$x\Phi(S)(y) = xS(y_1) = x_1S(y_1) = S(x_1)y_1 = \Phi(S)(x)y_1 = \Phi(S)(x)y.$$

Hence, $\Phi(S) \in M_1'(A)$. Moreover, if $S \in M(A_1)$, then because π is a homomorphism, we have

$$\Phi(S)(xy) = S(\pi(xy)) = S(x_1y_1) = x_1S(y_1) = x\Phi(S)(y),$$

and so $\Phi(S) \in M_1(A)$.

Conversely, let $T \in M_1'(A)$, then because T is a linear mapping satisfying $T(A_0) = \{0\}$, there is a linear mapping $S \in L(A_1)$ on A_1 such that $\Phi(S) = T$, that is, $S(x_1) = T(x) = T(x_1)$. We have

$$x_1 S(y_1) = x_1 T(y_1) = T(x_1) y_1 = S(x_1) y_1,$$

and hence $S \in M'(A_1)$. Therefore, Φ induces a linear isomorphism from $M'(A_1)$ to $M_1'(A)$. Similarly, Φ gives a linear isomorphism from $M(A_1)$ to $M_1(A)$. Moreover, for $T, U \in M'(A_1)$, we have

$$\Phi(TU) = \mu \circ T \circ U \circ \pi = \mu \circ T \circ \pi \circ \mu \circ U \circ \pi = \Phi(T)\Phi(U).$$

Thus, Φ gives an isomorphism of algebras from $M(A_1)$ to $M_1(A)$ and a Jordan isomorphism from $M'(A_1)$ to $M_1'(A)$. □

Theorem 3.1 implies

$$M'(A) = M_1'(A) \oplus M_0'(A) \text{ and } M_1(A) \oplus M_0(A) \subseteq M(A),$$

where $M_0'(A)$ and $M_0(A)$ are given as (8). If A_1 is a subalgebra, we have

$$M'(A) \cong M'(A_1) \oplus (A_0)^A \text{ and } M(A) \cong M(A_1) \oplus \left\{T \in (A_0)^A \mid T(A^2) = \{0\}\right\}. \tag{10}$$

Corollary 3.2. *Any weak multiplier T is written as*

$$T = T_1 + R \tag{11}$$

with $T_1 \in LM_1'(A)$ and $R \in (A_0)^A$, and it is a multiplier if and only if

$$R(x_1 y_1) = x_1 T_1(y_1) - T_1(x_1 y_1) \tag{12}$$

for any $x_1, y_1 \in A_1$.

Proof. As stated above T is written as (11). Let $x = x_1 + x_0, y = y_1 + y_0 \in A$ with $x_1, y_1 \in A_1$ and $x_0, y_0 \in A_0$ be arbitrary, then we have

$$xT(y) = x_1(T_1(y) + R(y)) = x_1 T_1(y) = x_1 T_1(y_1) \tag{13}$$

because $R(A) \subseteq A_0$ and $T_1(A_0) = \{0\}$. The last term in (13) is also equal to $T_1(x_1)y_1 = T(x)y$. Hence, T is a multiplier if and only if it is equal to $T(xy) = T(x_1 y_1) = T_1(x_1 y_1) + R(x_1 y_1)$, if and only if (12) holds. □

4. Linear multipliers and matrix equation

In this section, A is a finite-dimensional algebra over K. We derive a matrix equation that characterizes a (weak) multiplier for a linear mapping on A. Suppose that A is n-dimensional with basis $E = \{e_1, e_2, \ldots, e_n\}$.

Lemma 4.1. *A linear mapping $T : A \to A$ is a weak multiplier if and only if*

$$e_i T(e_j) = T(e_i) e_j, \tag{14}$$

and it is a multiplier if and only if

$$T(e_i e_j) = e_i T(e_j) = T(e_i) e_j, \tag{15}$$

for all $e_i, e_j \in E$.

Proof. The necessity of the conditions (14) and (15) is obvious. Let $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n, y = y_1e_1 + y_2e_2 + \cdots + y_ne_n \in A$ with $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in K$. If T satisfies (14), then we have

$$\begin{aligned} xT(y) &= \left(\sum_i x_ie_i\right) T\left(\sum_j y_je_j\right) = \left(\sum_i x_ie_i\right)\left(\sum_j y_jT(e_j)\right) \\ &= \sum_{i,j} x_iy_je_iT(e_j) = \sum_{i,j} x_iy_jT(e_i)e_j \\ &= \left(\sum_i x_iT(e_i)\right)\left(\sum_j y_je_j\right) = T(x)y. \end{aligned}$$

Hence, T is a weak multiplier. Moreover, if T satisfies (15), it is a multiplier in a similar manner. □

Let $\boldsymbol{A}$ (we use the bold character) represent the multiplication table of A on E. $\boldsymbol{A}$ is a matrix whose elements are drawn from A and given by

$$\boldsymbol{A} = \boldsymbol{E}^t\boldsymbol{E}, \tag{16}$$

where $\boldsymbol{E} = (e_1, e_2, \ldots, e_n)$ (we again use the boldface $\boldsymbol{E}$) is the row vector consisting the basis elements and $\boldsymbol{E}^t$ is its transpose. For a linear mapping T on A and a matrix $\boldsymbol{B}$ over A, $T(\boldsymbol{B})$ denotes the matrix obtained by applying T element-wise, that is, the (i,j)-element of $T(\boldsymbol{B})$ is $T(b_{ij})$ for the (i,j)-element b_{ij} of $\boldsymbol{B}$.[3] We employ the same symbol T for the representation matrix of T on E, that is,

$$T(\boldsymbol{E}) = \boldsymbol{E}T. \tag{17}$$

Theorem 4.2. *A linear mapping T is a weak multiplier of A if and only if*

$$\boldsymbol{A}T = T^t\boldsymbol{A}, \tag{18}$$

and T is a multiplier if and only if

$$T(\boldsymbol{A}) = \boldsymbol{A}T = T^t\boldsymbol{A}. \tag{19}$$

Proof. By (16) and (17) we have

$$(e_1, e_2, \ldots, e_n)^t\,(T(e_1), T(e_2), \ldots, T(e_n)) = \boldsymbol{E}^tT(\boldsymbol{E}) = \boldsymbol{E}^t\boldsymbol{E}T = \boldsymbol{A}T \tag{20}$$

and

$$(T(e_1), T(e_2), \ldots, T(e_2))^t\,(e_1, e_2, \ldots, e_n) = T(\boldsymbol{E})^t\boldsymbol{E} = T^t\boldsymbol{E}^t\boldsymbol{E} = T^t\boldsymbol{A}. \tag{21}$$

By Lemma 4.1, T is a weak multiplier if and only if (20) and (21) are equal, if and only if (18) holds. Moreover, T is multiplier if and only if the leftmost sides of (20) and (21) are equal to $(T(e_ie_j))_{i,j=1,2,\ldots,n} = T(\boldsymbol{A})$, if and only if (19) holds. □

The multiplication table of the opposite algebra A^{op} of A is the transpose $\boldsymbol{A}^{\mathrm{t}}$ of $\boldsymbol{A}$. So, the algebras with multiplication tables transposed to each other share the same (weak) multipliers.

[3]This is called a broadcasting (cf. [7]).

5. Associative algebras

In this section, A is an associative algebra over K.

Proposition 5.1. *If $A_0 = \{0\}$, then we have*

$$M(A) = M'(A) = LM(A) = LM'(A).$$

Proof. Let $T \in M'(A)$, then we have

$$T(xy)z = xyT(z) = xT(y)z \quad \text{and} \quad zT(xy) = T(z)xy = zT(x)y$$

for any $x, y, z \in A$. It follows that

$$T(xy) - xT(y) \in \mathrm{Ann}_l(A) \cap \mathrm{Ann}_r(A) = A_0 = \{0\}.$$

Hence, $T(xy) = xT(y)$ and we see $T \in M(A)$. Moreover, $T \in LM(A)$ by Proposition 2.2. □

Let $a \in A$. If $xay = axy$ (resp. $xay = xya$) for any $x, y \in A$, a is called a *left* (resp. *right*) *central element*, and we simply call it a *central element* if $ax = xa$ for any $x \in A$. Let $Z_l(A)$, (resp. $Z_r(A)$, $Z(A)$) denote the set of all left central (resp. right central, central) elements.

For $a \in A$, l_a (resp. r_a) denotes the left (resp. right) multiplication by a, that is,

$$l_a(x) = ax, \quad r_a(x) = xa$$

for $x \in A$. They are linear mappings.

Lemma 5.2. *For $a \in A$ the following statements are equivalent.*
(i) l_a (resp. r_a) is a multiplier,
(ii) l_a (resp. r_a) is a weak multiplier,
(iii) a is left (resp. right) central.

Proof. If l_a is a weak multiplier, then

$$xay = xl_a(y) = l_a(x)y = axy$$

for any $x, y \in A$, which implies that a is left central. Conversely, if a is left central, l_a is a weak multiplier also by the above equalities. Moreover, l_a is a multiplier because $l_a(xy) = axy = l_a(x)y$. The other case is analogous, and we see that these three statements are equivalent. □

As can be easily proved, $Z_l(A)$ (resp. $Z_r(A)$) is a subalgebra of A containing $\mathrm{Ann}_l(A)$ (resp. $\mathrm{Ann}_r(A)$). Hence, we can form the quotient algebras $\bar{Z}_l(A) = Z_l(A)/\mathrm{Ann}_l(A)$ and $\bar{Z}_r(A) = Z_r(A)/\mathrm{Ann}_r(A)$.

Theorem 5.3. *Suppose that A has a left (resp. right) identity e. Then, any weak multiplier is a left (resp. right) multiplication by a left (resp. right) central element and it is a linear multiplier. The mapping $\phi\colon Z_l(A)$ (resp. $Z_r(A)$) $\to M'(A) = LM(A)$ sending $a \in Z_l(A)$ (resp. $Z_r(A)$) to l_a (resp. l_r) induces an isomorphism $\bar{\phi}\colon \overline{Z_l}(A)$ (resp. $\overline{Z_r}(A)$) $\to M(A)$ of algebras. In particular, if A is unital, $M(A)$ is isomorphic to $Z(A)$.*

Proof. Suppose that A has a left identity e. Let $T \in M'(A)$ and set $a = T(e)$. Then we have

$$T(x) = eT(x) = T(e)x = ax$$

for any $x \in A$. Hence, $T = l_a$, where $a \in Z_l(A)$ and T is a linear multiplier by Lemma 5.2. Therefore, $M'(A) = LM(A)$ and ϕ is surjective. Moreover, for $a \in Z_l(A)$, $\phi(a) = 0$ if and only if $ax = 0$ for any $x \in A$, if and only if $a \in \mathrm{Ann}_l(A)$. Thus we have $\mathrm{Ker}(\phi) = \mathrm{Ann}_l(A)$, and ϕ induces the desired isomorphism. Similarly, if A has a right identity, $M(A)$ is isomorphic to $\overline{Z_r}(A)$. Lastly, if A has the identity, then $Z_\ell(A) = Z(A)$ and $\mathrm{Ann}_l(A) = \{0\}$, and hence $M(A)$ is isomorphic to $Z(A)$. □

6. 3-dimensional associative algebras

Over an algebraically closed field K of characteristic not equal to 2, we have, up to isomorphism, 24 families of associative algebras of dimension 3. They consist of 5 unital algebras U_0, U_1, U_2, U_3, U_4 defined on basis $E = \{e, f, g\}$ by the multiplication tables

$$\begin{pmatrix} e & f & g \\ f & 0 & 0 \\ g & 0 & 0 \end{pmatrix}, \begin{pmatrix} e & f & g \\ f & 0 & f \\ g & -f & e \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & f & g \\ 0 & g & 0 \end{pmatrix}, \begin{pmatrix} e & f & g \\ f & g & 0 \\ g & 0 & 0 \end{pmatrix},$$

5 curled algebras C_0, C_1, C_2, C_3, C_4 defined by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & -e & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ e & f & 0 \\ 0 & g & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e & f & g \end{pmatrix}, \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & 0 & g \end{pmatrix},$$

non-unital 4 straight algebras S_1, S_2, S_3, S_4 defined by

$$\begin{pmatrix} f & g & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e & f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and non-unital 10 families of waved algebras W_1, W_2, W_4, W_5, W_6, W_7, W_8, W_9, W_{10} and $\{W_3(k)\}_{k \in H}$ defined by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e & 0 \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f & g \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & 0 & g \end{pmatrix},$$

$$\begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f & g \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & f \\ 0 & 0 & g \end{pmatrix}, \begin{pmatrix} 0 & e & 0 \\ e & f & 0 \\ 0 & g & 0 \end{pmatrix}, \begin{pmatrix} 0 & e & 0 \\ e & f & g \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & ke & e \end{pmatrix},$$

respectively, where H is a subset of K such that $K = H \cup -H$ and $H \cap -H = \{0\}$ (see [3] for details). We determine the (weak) multipliers of them below.

(0) $A = C_0$ is the zero algebra, so by Proposition 2.5, we have

$$M'(A) = A^A, \ \ M(A) = \{T \in A^A \,|\, T(0) = 0\} \text{ and } \ LM(A) = LM'(A) = L(A).$$

(i) The unital algebras U_0, U_2, U_3, U_4 are commutative, so for such A we have

$$M(A) = LM(A) = M'(A) = LM'(A) = \{l_x | x \in A\} \cong A$$

by Theorem 5.3. For $A = U_1$, we have

$$M(A) = LM(A) = M'(A) = LM'(A) \cong Z(A) = Ke.$$

(ii) For $A = C_1$, we observe that $A_0 = \mathrm{Ann}_l(A) = \mathrm{Ann}_r(A) = Ke$, and we have a nihil decomposition $A = A_1 \oplus A_0$ with $A_1 = Kf + Kg$. Let $T_1 \in M_1'(A)$, then by Theorem 3.1, T_1 is a linear mapping such that $T_1(Ke) = \{0\}$. Let

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & r \\ 0 & t & u \end{pmatrix} \tag{22}$$

with $q, r, t, u \in K$ be the representation matrix of T_1 on E. By Theorem 4.2, T_1 is a weak multiplier if and only if

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & te & ue \\ 0 & -qe & -re \end{pmatrix} = \boldsymbol{A} T_1 = T_1^{\mathrm{t}} \boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -te & qe \\ 0 & -ue & re \end{pmatrix},$$

if and only if $r = t = 0$ and $q = u$. Hence, $M_1'(A) = \{T_q \mid q \in K\}$, where $T_q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}$. By Theorem 3.1 we see

$$M'(A) = \{T_q \mid q \in K\} \oplus (Ke)^A$$

and

$$LM'(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \,\middle|\, a, b, c, q \in K \right\}.$$

By examining the multiplication table of A, we find that $\alpha\beta = (xv - yz)e$ for $\alpha = xf + yg, \beta = zf + vg \in A_1$ with $x, y, z, v \in K$. By Corollary 3.2, $T \in M'(A)$ is given by $T = T_q + R$ with $R \in (Ke)^A$ and this T is a multiplier if and only if

$$\begin{aligned} R((xv - yz)e) = R(\alpha\beta) &= \alpha T_q(\beta) - T_q(\alpha\beta) \\ &= \alpha(q\beta) - T_q((xv - yz)e) = q(xv - yz)e \end{aligned}$$

for any α and β, if and only if $R(xe) = qxe$ for all $x \in K$. Let S_q be the scalar multiplication in A by $q \in K$. Then, we see

$$(T - S_q)(A) = (T_q - S_q + R)(A) \subseteq A_0 = Ke$$

and

$$(T - S_q)(xe) = T_q(xe) + R(xe) - S_q(xe) = 0 + qxe - qxe = 0$$

for any $x \in K$, that is, $(T - S_q)(Ke) = \{0\}$. Thus, we conclude

$$M(A) = \{S_q \mid q \in K\} \oplus \left\{R \in (Ke)^A \,\middle|\, R(Ke) = \{0\}\right\},$$

and

$$LM(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a, b, c \in K \right\}.$$

(iii) $A = C_2$: Because $\mathrm{Ann}_l(A) = Ke$ and $\mathrm{Ann}_r(A) = Kg$, we see $A_0 = \{0\}$. Hence, any weak multiplier T is a linear multiplier by Proposition 5.1. By Theorem

4.2,

$$T = \begin{pmatrix} a & b & c \\ p & q & r \\ s & t & u \end{pmatrix} \tag{23}$$

with $a, b, c, p, q, r, s, t, u \in K$ is a (weak) multiplier if and only if

$$\begin{pmatrix} 0 & 0 & 0 \\ ae + pf & be + qf & ce + rf \\ pg & qg & rg \end{pmatrix} = \boldsymbol{A}T = T^{\mathrm{t}}\boldsymbol{A} = \begin{pmatrix} pe & pf + sg & 0 \\ qe & qf + tg & 0 \\ re & rf + ug & 0 \end{pmatrix},$$

if and only if $b = c = p = r = s = t = 0$ and $a = q = u$, that is, T is the scalar multiplication S_a by a. Consequently,

$$M(A) = M'(A) = LM(A) = LM'(A) = \{S_a \,|\, a \in K\} \cong K.$$

(iv) C_3 and C_4 are opposite to each other, and share the same (weak) multipliers by Proposition 2.6. Let $A = C_3$, then, A has a left identity g, $Z_l(A) = A$ and $\mathrm{Ann}_l(A) = Ke + Kf$. Hence, by Theorem 5.3,

$$M(A) = M'(A) = LM(A) = LM'(A) = A/(Ke + Kg) = \{S_a \,|\, a \in K\}.$$

(v) $A = S_1$: We have $A_0 = \mathrm{Ann}_l(A) = \mathrm{Ann}_r(A) = Kg$, and $A = A_1 \oplus A_0$ with $A_1 = Ke + Kf$. Then, $T_1 \in M_1'(A)$ is a linear mapping with $T(Kg) = \{0\}$. Let

$$T_1 = \begin{pmatrix} a & b & 0 \\ p & q & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{24}$$

with $a, b, p, q \in K$ be its representation on E. T_1 is a weak multiplier if and only if

$$\begin{pmatrix} af + pg & bf + qg & 0 \\ ag & bg & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boldsymbol{A}T_1 = T_1^{\mathrm{t}}\boldsymbol{A} = \begin{pmatrix} af + pg & ag & 0 \\ bf + qg & bg & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

if and only if $b = 0$ and $a = q$. Hence,

$$M'(A) = \{T_1^{a,p} \,|\, a, p \in K\} \oplus (Kg)^A \quad \text{with } T_1^{a,p} = \begin{pmatrix} a & 0 & 0 \\ p & a & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, $T \in M'(A)$ is written as $T = T_1^{a,p} + R$ with $R \in (Kg)^A$, and this T is multiplier if and only if

$$\begin{aligned} R(xzf + (xv + yz)g) = R(\alpha\beta) &= \alpha T_1^{a,p}(\beta) - T_1^{a,p}(\alpha\beta) \\ &= (xe + yf)(aze + (pz + av)f) - axzf \\ &= (pxz + a(xv + yz))g \end{aligned}$$

for any $\alpha = xe + yf, \beta = ze + vf \in A_1$ with $x, y, z, v \in K$, if and only if $R(xf + yg) = (px + ay)g$ for all $x, y \in K$. Let $T^{a,p} = \begin{pmatrix} a & 0 & 0 \\ p & a & 0 \\ 0 & p & a \end{pmatrix}$, then $(T - T^{a,p})(A) \subseteq Kg$, and

$$\begin{aligned} (T - T^{a,p})(xf + yg) &= (T_1^{a,p} + R - T^{a,p})(xf + yg) \\ &= axf + (px + ay)g - (axf + pxg + ayg) = 0 \end{aligned}$$

for any $x, y \in K$. Thus, $(T - T^{a,p})(Kf + Kg) = \{0\}$, and hence

$$M(A) = \{T^{a,p} \,|\, a, p \in K\} \oplus \{R \in (Kg)^A \,|\, R(Kf + Kg) = \{0\}\}.$$

By intersecting $M'(A)$ and $M(A)$ with $L(A)$, we obtain

$$LM'(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ p & a & 0 \\ s & t & u \end{pmatrix} \,\middle|\, a, p, s, t, u \in K \right\}$$

$$\text{and } LM(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ p & a & 0 \\ s & p & a \end{pmatrix} \,\middle|\, a, p, s \in K \right\}.$$

(vi) $A = S_2$: We have $A_0 = \mathrm{Ann}_l(A) = \mathrm{Ann}_r(A) = Kg$, and $A = A_1 \oplus A_0$ with $A_1 = Ke + Kf$. Let a linear mapping $T_1 \in M_1'(A)$ be represented as (24), then T_1 is a weak multiplier if and only if

$$\begin{pmatrix} ae & be & 0 \\ pg & qg & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boldsymbol{A}T = T^{\mathrm{t}}\boldsymbol{A} = \begin{pmatrix} ae & pg & 0 \\ be & qg & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

if and only if $b = p = 0$. Hence,

$$M'(A) = \{T_1^{a,q} \,|\, a, q \in K\} \oplus (Kg)^A \text{ with } T_1^{a,q} = \begin{pmatrix} a & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$LM'(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & q & 0 \\ s & t & u \end{pmatrix} \,\middle|\, a, q, s, t, u \in K \right\}.$$

By Corollary 3.2, a weak multiplier T written as $T = T_1^{a,q} + R$ for $a, q \in K$ and $R \in (Kg)^A$ is multiplier if and only if

$$\begin{aligned} R(xze + yvg) = R(\alpha\beta) &= \alpha T_1^{a,q}(\beta) - T_1^{a,q}(xze + yvg) \\ &= (xe + yf)(aze + qvf) - axze = yqvg \end{aligned}$$

for any $\alpha = xe + yf, \beta = ze + vf \in A_1$ with $x, y, z, v \in K$, if and only if $R(xe + yg) = qyg$ for all $x, y \in K$. Let $T^{a,q} = \begin{pmatrix} a & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}$, then we have $(T - T^{a,q})(xe + yg) = 0$ for any $x, y \in K$, following the same calculation as in (v) above. Hence, $(T - T^{a,q})(Ke + Kg) = \{0\}$, and we have

$$M(A) = \{T^{a,p} \,|\, a, p \in K\} \oplus \left\{R \in (Kg)^A \,\middle|\, R(Ke + Kg) = \{0\}\right\}$$

and

$$LM(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & p & 0 \\ 0 & t & 0 \end{pmatrix} \,\middle|\, a, p, t \in K \right\}.$$

(vii) $A = S_3$: We have $A_0 = Kg$ and $A = A_1 \oplus A_0$ with $A_1 = Ke + Kf$. As A_1 is a subalgebra of A, by Theorem 3.1 we obtain the equalities (10) in Sect. 3. Because A_1 is a commutative unital algebra,

$$M(A_1) = M'(A_1) = A_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\middle|\, a, b \in K \right\}$$

by Theorem 5.3. Note that $A^2 = Ke + Kf$. Hence,

$$M'(A) = A_1 \oplus (Kg)^A \text{ and } M(A) = A_1 \oplus \{T \in (Kg)^A \,|\, T(Ke + Kf) = 0\}.$$

Intersecting with $L(A)$ we have

$$LM'(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ s & t & u \end{pmatrix} \,\middle|\, a,b,s,t,u \in K \right\}$$

$$\text{and} \quad LM(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & u \end{pmatrix} \,\middle|\, a,b,u \in K \right\}.$$

(viii) $A = S_4$: We have $A = A_1 \oplus A_0$ with $A_0 = Kg$ and $A_1 = Ke + Kf$. Because A_1 is a commutative unital subalgebra of A, similarly to the previous case, we obtain

$$M'(A) = A_1 \oplus (Kg)^A = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \,\middle|\, a,b \in K \right\} \oplus (Kg)^A,$$

$$M(A) = A_1 \oplus \{T \in (Kg)^A \,|\, T(Ke + Kf) = \{0\}\},$$

$$LM'(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ s & t & u \end{pmatrix} \,\middle|\, a,b,s,t,u \in K \right\} \text{ and } LM(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & u \end{pmatrix} \,\middle|\, a,b,u \in K \right\}.$$

(ix) $A = W_1$: We have $A = A_1 \oplus A_0$ with $A_0 = Ke + Kf$ and $A_1 = Kg$. Let $T_1 \in M_1'(A)$, then T_1 is a linear mapping with $T_1(A_0) = \{0\}$. So T_1 is determined by $T_1(g) = ag$ with $a \in K$. Denoting this T_1 as T_1^a, we have

$$M'(A) = \{T_1^a \,|\, a \in K\} \oplus (Ke + Kf)^A.$$

A weak multiplier $T = T_1^a + R$ with $R \in (Ke + Kf)^A$ is a multiplier if and only if

$$R(xye) = R((xg)(yg)) = xgT_1^a(yg) - T_1^a(xye) = axye$$

for all $x, y \in K$, if and only if $R(xe) = axe$ for any $x \in K$. Let $T_a = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$. Then, $(T - T_a)(Ke) = \{0\}$ and it follows that

$$M(A) = \{T_a \,|\, a \in K\} \oplus \left\{R \in (Ke + Kf)^A \,\middle|\, R(Ke) = \{0\}\right\}.$$

Also we have

$$LM'(A) = \left\{ \begin{pmatrix} a & b & c \\ p & q & r \\ 0 & 0 & u \end{pmatrix} \,\middle|\, a,b,c,p,q,r,u \in K \right\}$$

and

$$LM(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & q & r \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a,b,c,q,r \in K \right\}.$$

(x) $A = W_2$[4]: We have $A = A_1 \oplus A_0$ with $A_0 = Ke$ and $A_1 = Kf + Kg$. Let $T \in M_1'(A)$, then T is a linear mapping with $T(Ke) = \{0\}$ and can be represented as (22). Then, T is a weak multiplier if and only if

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & qe & re \end{pmatrix} = \boldsymbol{A}T = T^{\mathrm{t}}\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & te & 0 \\ 0 & ue & 0 \end{pmatrix},$$

[4]This is the algebra taken up in [9].

if and only if $r = t = 0$ and $q = u$. Hence,

$$M'(A) = \{T_q \mid q \in K\} \oplus (Ke)^A \text{ with } T_q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}.$$

So,

$$LM'(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \,\middle|\, a, b, c, q \in K \right\}.$$

A weak multiplier $T = T_q + R$ with $R \in (Ke)^A$ is a multiplier if and only if

$$R(yze) = R(\alpha\beta) = \alpha T_q(\beta) - T_q(yze) = \alpha(q\beta) = qyze$$

for any $\alpha = xf + yg, \beta = zf + vg \in A_1$ with $x, y, z, v \in K$, if and only if $R(xe) = qxe$ for all $x \in K$. Let S_a be the scalar multiplication by $a \in K$. Then, $(T - S_a)(Ke) = \{0\}$, and hence,

$$M(A) = \{S_a \mid a \in K\} \oplus \left\{R \in (Ke)^A \,\middle|\, R(Ke) = \{0\}\right\}$$

and

$$LM(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a, b, c \in K \right\}.$$

(xi) $A = W_4$: We have $A = A_1 \oplus A_0$ with $A_0 = Kf + Kg$ and $A_1 = Ke$. Because A_1 is a subalgebra isomorphic to the base field K, with the scalar multiplication S_1^a by $a \in K$, we see

$$M'(A) = \{S_1^a \mid a \in K\} \oplus (fK + gK)^A$$

and

$$M(A) = \{S_1^a \mid a \in K\} \oplus \left\{R \in (fK + gK)^A \,\middle|\, R(Ke) = \{0\}\right\}$$

by Theorem 3.1. Taking the intersections with $L(A)$ we have

$$LM'(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ p & q & r \\ s & t & u \end{pmatrix} \,\middle|\, a, p, q, r, s, t, u \in K \right\}$$

and

$$LM(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & q & r \\ 0 & t & u \end{pmatrix} \,\middle|\, a, q, r, t, u \in K \right\}.$$

(xii) W_5 and W_6 are opposite. Let $A = W_5$, then $A = A_1 \oplus A_0$ with $A_0 = Ke$ and $A_1 = Kf + Kg$. Since A_1 is a subalgebra of A and has a left identity g, we have

$$M(A_1) = LM(A_1) = M'(A_1) = LM'(A_1) \cong (A_1)/Kf \cong Kg$$

by Theorem 5.3. So, any element in $M(A_1)$ is a scalar multiplication S_1^q in A_1 by $q \in K$. By Theorem 3.1 we have

$$M'(A) = \{S_1^q \mid q \in K\} \oplus (Ke)^A,$$
$$M(A) = \{S_1^q \mid q \in K\} \oplus \left\{R \in (Ke)^A \,\middle|\, R(Kf + Kg) = \{0\}\right\},$$
$$LM'(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \,\middle|\, a, b, c, q \in K \right\} \text{ and } LM(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \,\middle|\, a, q \in K \right\}.$$

(xiii) W_7 and W_8 are opposite. Let $A = W_7$, then we see $A_0 = \mathrm{Ann}_r(A) = \{0\}$. Hence, any weak multiplier is a linear multiplier by Proposition 5.1, and a linear mapping T represented as (23) is a weak multiplier if and only if

$$\begin{pmatrix} ae & be & ce \\ 0 & 0 & 0 \\ pf+sg & qf+tg & rf+ug \end{pmatrix} = \boldsymbol{A}T = T^{\mathrm{t}}\boldsymbol{A} = \begin{pmatrix} ae & sf & sg \\ be & tf & tg \\ ce & uf & ug \end{pmatrix},$$

if and only if $b = c = p = r = s = t = 0$ and $q = u$. Therefore,

$$M(A) = LM(A) = M'(A) = LM'(A) = LM'(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \,\middle|\, a, q \in K \right\}.$$

(xiv) W_9 and W_{10} are opposite. Let $A = W_9$. Then, because $A_0 = \mathrm{Ann}_l(A) = \{0\}$, any weak multiplier is a linear multiplier and a linear mapping T represented as (23) is a weak multiplier if and only if

$$\begin{pmatrix} pe & qe & re \\ ae+pf & be+qf & ce+rf \\ pg & qg & rg \end{pmatrix} = \boldsymbol{A}T = T^{\mathrm{t}}\boldsymbol{A} = \begin{pmatrix} pe & ae+pf+sg & 0 \\ qe & be+qf+tg & 0 \\ re & ce+rf+ug & 0 \end{pmatrix},$$

if and only if $c = p = r = s = t = 0, a = q = u$. Therefore,

$$LM(A) = M(A) = LM'(A) = M'(A) = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a, b \in K \right\}.$$

(xv) $A = W_3(k)$: We have $A = A_1 \oplus A_0$ with $A_0 = Ke$ and $A_1 = Kf + Kg$. Then, $T \in M_1'(A)$ is a linear mapping with $T(Ke) = \{0\}$ represented as (22). It is a weak multiplier if and only if

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & qe & re \\ 0 & (kq+t)e & (kr+u)e \end{pmatrix} = \boldsymbol{A}T = T^{\mathrm{t}}\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (q+kt)e & te \\ 0 & (r+ku)e & ue \end{pmatrix}. \tag{25}$$

When $k = 0$, (25) holds if and only if $r = t$, and otherwise it holds if and only if $r = t = 0$ and $q = u$. Thus,

$$M'(A) = \{T_1^{q,r,u} \,|\, q, r, u \in K\} \oplus (Ke)^A, \ LM'(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & q & r \\ 0 & r & u \end{pmatrix} \,\middle|\, a, b, c, q, r, u \in K \right\}$$

when $k = 0$, and

$$M'(A) = \{T_1^{q} \,|\, q \in K\} \oplus (Ke)^A, \ LM'(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \,\middle|\, a, b, c, q \in K \right\}$$

when $k \neq 0$, where

$$T_1^{q,r,u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & r \\ 0 & r & u \end{pmatrix} \text{ and } T_1^{q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}.$$

Now, when $k = 0$, $T = T_1^{q,r,u} + R$ with $R \in (Ke)^A$ is multiplier if and only if

$$\begin{aligned} R((xz+yv)e) = R(\alpha\beta) &= \alpha T_1^{q,r,u}(\beta) - T_1^{q,r,u}((xz+yv)e) \\ &= \alpha((qz+rv)f + (rz+uv)g) = (qxz+uyv+r(xv+yz))e \end{aligned}$$

for any $\alpha = xf+yg, \beta = zf+vg \in A_1$ with $x, y, z, v \in K$, if and only if $q = u, r = 0$ and $R(xe) = qxe$ for all $x \in K$. While, when $k \neq 0$, $T = T_1^q + R$ with $R \in (Ke)^A$ is a multiplier if and only if

$$\begin{aligned} R((xz + y(kz + v))e) = R(\alpha\beta) &= \alpha T_1^q(\beta) - T_1^q(xue + y(kz + v)e) \\ = \alpha(q\beta) &= q(xz + y(kz + v))e \end{aligned}$$

for any α, β, if and only if $R(xe) = qxe$ for any $x \in K$. In both cases, with the scalar multiplication S_a by $a \in K$, we have $(T - S_a)(Ke) = \{0\}$. Therefore, for arbitrary k (whether k is zero or nonzero) we conclude that

$$M(A) = \{S_a \,|\, a \in K\} \oplus \{R \in (Ke)^A \,|\, R(Ke) = \{0\}\}$$

and

$$LM(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a, b, c \in K \right\}.$$

7. 3-dimensional zeropotent algebras

An algebra A is *zeropotent* if $x^2 = 0$ for all $x \in A$. A zeropotent algebra A is anti-commutative, that is, $xy = -yx$ for all $x, y \in A$. Thus we see

$$A_0 = \mathrm{Ann}_l(A) = \mathrm{Ann}_r(A).$$

Let A be a zeropotent algebras of dimension 3 over a field K with $\mathrm{char}(K) \neq 2$. Let $E = \{e, f, g\}$ be a basis of A. Because A is anti-commutative, the multiplication table $\boldsymbol{A}$ of A on E is given as

$$\boldsymbol{A} = \begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{pmatrix} \quad \text{with} \quad \begin{cases} \gamma = fg = a_{11}e + a_{12}f + a_{13}g \\ \beta = ge = a_{21}e + a_{22}f + a_{23}g \\ \alpha = ef = a_{31}e + a_{32}f + a_{33}g \end{cases}$$

for $a_{ij} \in K$. We call $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}. \end{pmatrix}$ the *structural matrix* of A. The *rank* of A is defined as the rank of its structural matrix.

Lemma 7.1. *If* $\mathrm{rank}(A) \geq 2$*, then* $A_0 = \{0\}$*.*

Proof. If $\mathrm{rank}(A) \geq 2$, at least two of $\alpha = ef, \beta = ge, \gamma = fg$ are linearly independent. Suppose that α and β are linearly independent (the other cases are similar). If $x = ae+bf+cg$ with $a, b, c \in K$ is in $\mathrm{Ann}_l(A)$, then $xe = -b\alpha+c\beta$ and $xf = a\alpha-c\gamma$ are both zero. It follows that $a = b = c = 0$. Hence, we have $A_0 = \mathrm{Ann}_l(A) = \{0\}$. □

Theorem 7.2. *Let A be a zeropotent algebra of dimension 3 with* $\mathrm{rank}(A) \geq 2$ *over K. Then, any weak multiplier of A is the scalar multiplication S_a for some $a \in K$, that is,*

$$M(A) = M'(A) = LM(A) = LM'(A) = \{S_a \,|\, a \in K\}.$$

Proof. By Lemma 7.1 and Corollary 2.3, any weak multiplier T is a linear mapping. Let $T \in L(A)$ be represented as (23). By Theorem 4.2, T is a weak multiplier if and only if $\boldsymbol{A}T = T^{\mathrm{t}}\boldsymbol{A}$, if and only if

$$\begin{pmatrix} p\alpha - s\beta & q\alpha - t\beta & r\alpha - u\beta \\ -a\alpha + s\gamma & -b\alpha + t\gamma & -c\alpha + u\gamma \\ a\beta - p\gamma & b\beta - q\gamma & c\beta - r\gamma \end{pmatrix} = \begin{pmatrix} -p\alpha + s\beta & a\alpha - s\gamma & -a\beta + p\gamma \\ -q\alpha + t\beta & b\alpha - t\gamma & -b\beta + q\gamma \\ -r\alpha + u\beta & c\alpha - u\gamma & -c\beta + r\gamma \end{pmatrix} \tag{26}$$

holds. Suppose that α, β are linearly independent (the other cases are similar). Then, by comparing the (1,1)-elements of the two matrices in (26), we have $p\alpha - s\beta = -p\alpha + s\beta$, which implies $p = s = 0$. Comparing the (1,2)-elements and (1,3)-elements, we have $q\alpha - t\beta = a\alpha - s\gamma = a\alpha$ and $r\alpha - u\beta = -a\beta + p\gamma = -a\beta$. It follows that $a = q = u$ and $r = t = 0$. Furthermore, comparing (2,2)-elements and (3,3)-elements, we see $b = c = 0$. Consequently, (26) holds if and only if $b = c = p = r = s = t = 0$ and $a = q = u$, that is, $T = S_a$. □

In [4] we classified the zeropotent algebras of dimension 3 over an algebraically field K of characteristic not equal to 2. Up to isomorphism, we have 10 families of zeropotent algebras. They are

$$Z_0, Z_1, Z_2, Z_3, \{Z_4(a)\}_{a\in H}, Z_5, Z_6, \{Z_7(a)\}_{a\in H}, Z_8 \text{ and } Z_9$$

defined respectively by the structural matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Z_0 is the zero algebra, and Z_1 is isomorphic to the 3-dimensional associative algebra C_1, and their (weak) multipliers are already determined in Sect. 6. The algebras Z_3 to Z_9 have rank greater or equal to 2, and they are covered by Theorem 7.2.

Thus, only $A = Z_2$ remains to be analyzed. The multiplication table $\boldsymbol{A}$ of A is $\begin{pmatrix} 0 & g & 0 \\ -g & 0 & g \\ 0 & -g & 0 \end{pmatrix}$. We see $A_0 = \mathrm{Ann}_l(A) = \mathrm{Ann}_r(A) = K(e+g)$, and we have the nihil decomposition $A = A_0 \oplus A_1$ with $A_1 = Ke + Kf$. A weak multiplier $T \in M_1'(A)$ is a linear mapping represented by $\begin{pmatrix} a & b & c \\ p & q & r \\ 0 & 0 & 0 \end{pmatrix}$ satisfying

$$\begin{pmatrix} pg & qg & rg \\ -ag & -bg & -cg \\ -pg & -qg & -rg \end{pmatrix} = \boldsymbol{A}T = T^{\mathrm{t}}\boldsymbol{A} = \begin{pmatrix} -pg & ag & pg \\ -qg & bg & qg \\ -rg & cg & rg \end{pmatrix}.$$

Hence, $a = -c = q$ and $b = p = r = 0$. Let T_a be this linear mapping, then by Theorem 3.1 we have

$$M'(A) = \{T_a \,|\, a \in K\} \oplus (K(f+g))^A$$

and

$$LM'(A) = \left\{ \begin{pmatrix} a+s & t & -a+u \\ 0 & a & 0 \\ s & t & u \end{pmatrix} \,\middle|\, a,s,t,u \in K \right\}.$$

By Corollary 3.2, a weak multiplier $T = T_a + R$ with $R \in (K(e+g))^A$ becomes a multiplier if and only if for any $\zeta = xe + yf$ and $\eta = ze + vf$ with $x, y, z, v \in K$,

$$\begin{aligned} R((xv-yz)g) = R(\zeta\eta) &= \zeta T_a(\eta) - T_a((xv-yz)g) \\ &= \zeta(a\eta) + a(xv-yz)e = a(xv-yz)(e+g) \end{aligned}$$

holds. It follows that $R(xg) = ax(e+g)$ for all $x \in K$. Let S_a be the scalar multiplication by a, then $(T - S_a)(A) \subseteq K(e+g)$ and $(T - S_a)(Kg) = \{0\}$. Hence, we obtain

$$M(A) = \{S_a \,|\, a \in K\} \oplus \left\{R \in (K(e+g))^A \,\middle|\, R(Kg) = \{0\}\right\},$$

and

$$LM(A) = \left\{ \begin{pmatrix} a+s & t & 0 \\ 0 & a & 0 \\ s & t & a \end{pmatrix} \,\middle|\, a,s,t \in K \right\} = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ a-c & b & c \end{pmatrix} \,\middle|\, a,b,c \in K \right\}.$$

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