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Unitarily invariant norms on finite von Neumann algebras

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Abstract. We give a characterization of all the unitarily invariant norms on a finite von Neumann algebra acting on a separable Hilbert space. The characterization is analogous to von Neumann's characterization for the $n \times n$ complex matrices and the characterization in Fang et al. (J Funct Anal 255(1):142–183, 2008) for II_1 factors.

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1. Introduction

Since John von Neumann's beautiful characterization of the unitarily invariant norms for the $n \times n$ complex matrices $\mathbb{M}_n(\mathbb{C})$, there have been over four hundred papers related to this subject. In the 1930's von Neumann [24] showed that there is a natural one-to-one correspondence between the unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$ and the normalized symmetric gauge norms on $\mathbb{C}^n$. As pointed out by the referee, these norms have been generalized and utilized in several contexts (see [20] or [21]). More recently, J. Fang, D. Hadwin, E. A. Nordgren and J. Shen [10] showed that there is an analogous correspondence between the unitarily invariant norms on a II_1 factor von Neumann algebra $\mathcal{M}$ and the normalized symmetric gauge norms on $L^\infty[0,1]$. Although the proofs of both results relied on s-numbers, the proof of the latter result was different from von Neumann's proof. We provide a new proof of the II_1 factor result that more closely parallels the proof for $\mathbb{M}_n(\mathbb{C})$. The key ingredient is an "approximate" version of the Ky Fan Lemma that is used in the finite-dimensional case.

It is our goal to find a similar characterization of all the unitarily invariant norms on a finite von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space H. A von Neumann algebra on H is a unital subalgebra of the algebra $B(H)$ of all operators on H that is closed under the adjoint operation and is closed in the weak operator topology. A von Neumann algebra is a *factor* if it cannot be written as the direct sum of two von Neumann algebras; equivalently, if its center contains no projection P with $0 \neq P \neq 1$. A von Neumann algebra on a separable Hilbert space can be written as a direct integral (i.e., continuous direct sum) of factor von Neumann algebras, which is called the *central decomposition*. A finite factor is a factor that has a faithful normal tracial state. The finite-dimensional finite factors are all isomorphic to $\mathbb{M}_n(\mathbb{C})$ for some positive integer n. An infinite-dimensional finite factor is called a II_1 factor. A finite von Neumann algebra on a separable Hilbert space is a direct integral of finite factors. General references for von Neumann algebras are [16] and [23].

Thus the results in [24] and [10] characterize the unitarily invariant norms on finite factors. To make these two examples look the same, we want to view $\mathbb{C}^n$ as $L^\infty(J_n, \delta_n)$, where (J_n, δ_n) is a probability space. We also want to have $J_n \subset [0,1]$. Our choice is $J_n = \left\{\frac{1}{n}, \ldots, \frac{n}{n}\right\}$ and δ_n is normalized counting measure, i.e.,

$$\delta_n(E) = \frac{1}{n}\mathrm{Card}(E).$$

We define $J_\infty = [0,1]$ and δ_∞ to be Lebesgue measure.

A finite von Neumann algebra $\mathcal{R}$ on a separable Hilbert space can be decomposed into a direct integral of factors that are either isomorphic to $\mathbb{M}_n(\mathbb{C})$ or are II_1 factors. Each finite factor von Neumann algebra has a unique tracial state. From the central decomposition we can define a tracial state τ on $\mathcal{R}$. The problem is to identify the corresponding measure space (Λ, λ). A key observation is that every maximal abelian selfadjoint subalgebra (*masa*) of $\mathbb{M}_n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n = L^\infty(J_n, \delta_n)$ and with the normalized trace corresponding to integration with respect to δ_n, and each masa in a II_1 factor is isomorphic to $L^\infty[0,1] = L^\infty(J_\infty, \delta_\infty)$ with the unique tracial state τ corresponding to integration with respect to δ_∞. It is known [22] that any two masas on a finite von Neumann algebra are isomorphic. If $\mathcal{A}$ is a masa in $\mathcal{R}$, then the central decomposition of $\mathcal{R}$ decomposes $\mathcal{A}$ to a direct integral of algebras that are masas in the corresponding factors. We must analyze this decomposition carefully to see that the masas are all isomorphic, *in a very special way*, to $L^\infty(\Lambda, \lambda)$ for some measure space (Λ, λ). Once we find the measure space, we find a certain group $\mathbb{G}(\mathcal{R})$ of invertible measure-preserving transformations. We then have to show how the unitarily invariant norms on $\mathcal{R}$ correspond to the normalized $\mathbb{G}(\mathcal{R})$-symmetric gauge norms on $L^\infty(\Lambda, \lambda)$. We will see that $\mathcal{R}$ is a factor if and only if $\mathbb{G}(\mathcal{R})$ is the group of all invertible measure-preserving transformations. This involves defining the analogue of the "s-numbers" and proving a general approximate Ky Fan Lemma. To show that things are independent of the choices of the masas used, we need a result on approximate unitary equivalence [4].

In Sect. 2 we discuss the basic properties of unitarily invariant norms, give a brief description of von Neumann's characterization for $\mathbb{M}_n(\mathbb{C})$ and give our new proof of the characterization of [10] for II_1 factors.

In Sect. 3, we prove a reformulation of a result of Ding and Hadwin [4] for approximate unitary equivalence of representations of a (not necessarily separable) abelian C*-algebra into a finite von Neumann algebra $\mathcal{R}$ in terms of the center-valued trace on $\mathcal{R}$.

In Sect. 4 we discuss the definitions and techniques of direct integrals a a canonical way to represent the center valued trace of a finite von Neumann in terms of its central decomposition.

In Sect. 5 we describe, for a given finite von Neumann algebra $\mathcal{R}$, two measure spaces with measures μ and λ so that $L^{\infty}(\mu)$ is isomorphic to the center $\mathcal{Z}(\mathcal{R})$ of $\mathcal{R}$ and, such that every masa $\mathcal{A}$ of $\mathcal{R}$ is isomorphic to $L^{\infty}(\lambda)$. We prove (Theorem 6) the isomorphism from $L^{\infty}(\lambda)$ to a masa $\mathcal{A}$ of $\mathcal{R}$ can be chosen in a special way so that a certain commutative diagram holds.

In Sect. 4 we discuss invertible measure-preserving transformations, non-increasing rearrangements, a function version of s-numbers, and define our group $\mathbb{G}(\mathcal{R})$ of measure-preserving transformations on the measure space (Λ, λ). We prove (Theorem 8) a general "approximate" version of the Ky Fan Lemma.

In Sect. 5 we put things together and prove our main theorem, which characterizes the unitarily invariant norms on $\mathcal{R}$ in terms of $\mathbb{G}(\mathcal{R})$-symmetric normalized gauge norms on $L^{\infty}(\lambda)$.

2. Preliminaries

2.1. Unitarily invariant norms

If $\mathcal{A}$ is a unital C*-algebra, $\mathcal{U}(\mathcal{A})$ denotes the set of all unitary elements of $\mathcal{A}$. If $T \in \mathcal{A}$ we define $|T| = (T^*T)^{1/2}$.

Lemma 1. *Suppose $\mathcal{A}$ is a unital C*-algebra and α is a norm on $\mathcal{A}$ such that $\alpha(1) = 1$. The following are equivalent.*

(1) For every $T \in \mathcal{A}$ and for every $U \in \mathcal{U}(\mathcal{A})$,

$$\alpha(T) = \alpha(|T|) = \alpha(U^*TU).$$

(2) For all U, V in $\mathcal{U}(\mathcal{A})$,

$$\alpha(T) = \alpha(UTV).$$

Proof. Suppose $T \in \mathcal{A}$ and for every $U \in \mathcal{U}(\mathcal{A})$, we have $\alpha(T) = \alpha(|T|) = \alpha(U^*TU)$. Then

$$\begin{aligned}\alpha(UTV) &= \alpha(|UTV|) = \alpha\left([(UTV)^*(UTV)]^{1/2}\right)\\ &= \alpha\left(V^*(T^*T)^{1/2}V\right) = \alpha(|T|) = \alpha(T).\end{aligned}$$

Suppose $T \in \mathcal{A}$ and $\alpha(T) = \alpha(UTV)$ for every $U, V \in \mathcal{U}(\mathcal{A})$. It is clear that $\alpha(T) = \alpha(U^*TU)$. To prove $\alpha(T) = \alpha(|T|)$, we can assume, using the universal representation [23, Chapter 3: Sect. 2], that $\mathcal{A} \subset B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that the second dual $\mathcal{A}^{\#\#}$ of $\mathcal{A}$ is isomorphic to $\mathcal{A}''$, the weak operator closure of $\mathcal{A}$. Furthermore we have that the weak operator topology on $\mathcal{A}^{\#\#}$ coincides with the weak*-topology, in fact, if φ

is a continuous linear functional on $\mathcal{A}$, then there are vectors $e, f \in \mathcal{H}$ such that $\varphi = e \otimes f$, i.e., for every $A \in \mathcal{A}$,

$$\varphi (A) = (e \otimes f)(A) = \langle Ae, f \rangle .$$

Now suppose $A, B \in \mathcal{A}$, $W \in \mathcal{A}''$, $\|W\| \leq 1$ and $A = WB$. We claim

$$A \in \overline{\text{co}}^{\|\|} (\{UB : U \in \mathcal{U}(\mathcal{A})\}),$$

where $\overline{\text{co}}^{\|\|}$ represents the norm-closed convex hull. Since $\{e^{it}U : t \in \mathbb{R}, U \in \mathcal{U}(\mathcal{A})\} = \mathcal{U}(\mathcal{A})$, we see that $\overline{\text{co}}^{\|\|} (\{UB : U \in \mathcal{U}(\mathcal{A})\})$ is absolutely convex and norm closed. It follows from the Hahn Banach theorem that $\text{co}^{-\|\|}(\{UB : U \in \mathcal{U}(\mathcal{A})\})$ closed in the weak topology on $\mathcal{A}$. If we assume, via contradiction, that the claim is false, there are $e, f \in \mathcal{H}$ and a number $r \in \mathbb{R}$ such that, for all $C \in \text{co}^{\|\|} (\mathcal{U}(\mathcal{A}))$,

$$|\langle CBe, f \rangle| \leq r < \text{Re}(\langle Ae, f \rangle) \leq |\langle Ae, f \rangle| .$$

We know from the Russo-Dye Theorem [7] that $\text{co}^{\|\|} (\mathcal{U}(\mathcal{A}))$ is $\{C \in \mathcal{A} : \|C\| \leq 1\}$. It follows from the Kaplansky density theorem [16] that there is a net $\{C_j\}$ in the closed unit ball of $\mathcal{A}$ such that $C_\lambda \to W$ in the weak operator topology. Thus,

$$r < |\langle Ae, f \rangle| = |\langle WBe, f \rangle| = \lim_{\lambda} |\langle C_j Be, f \rangle| \leq r,$$

which is the desired contradiction. We know from the Russo-Dye Theorem [7] that if $\|S\| < 1$ then $S \in \text{co}\ (\mathcal{U}(\mathcal{A}))$, which implies $\alpha(S) \leq 1$. Hence $\alpha(S) \leq \|S\|$ for every $S \in \mathcal{A}$. Since the claim is true, and it follows that $\alpha(A) \leq \alpha(B)$. Since $\mathcal{A}''$ is a von Neumann algebra and $T \in \mathcal{A}$, there is a partial isometry $W \in \mathcal{A}''$ such that $T = W|T|$ and $|T| = W^*T$, from which we can conclude $\alpha(T) = \alpha(|T|)$. □

Definition 1. If $\mathcal{A}$ is a unital C^*-algebra and α is a norm on $\mathcal{A}$ satisfying $\alpha(1) = 1$ and either of the two conditions in Lemma 1, we say that α is a *unitarily invariant norm* on $\mathcal{A}$. It is clear that when A is commutative, a unitarily invariant norm α need only satisfy $\alpha(1) = 1$ and $\alpha(T) = \alpha(|T|)$.

Below are some properties about unitarily invariant norms.

Proposition 1. *If $\mathcal{A}$ is a unital C^*-algebra and α is a unitarily invariant norm on $\mathcal{A}$, and $T, A, B \in \mathcal{A}$, we have the following:*

(1) $\alpha(T) \leq \|T\|$,
(2) $\alpha(T) = \alpha(T^*)$,
(3) $\alpha(ATB) \leq \|A\| \alpha(T) \|B\|$,
(4) $0 \leq A \leq B$ *implies* $\alpha(A) \leq \alpha(B)$.

Note: Whenever we discuss a measure space (Ω, Σ, μ) we always assume that the space is complete in the sense that, whenever $F \in \Sigma$, $E \subset F$ and $\mu(F) = 0$, we have $E \in \Sigma$. The following lemma is an immediate consequence of Proposition 1.

Lemma 2. *If α is a unitarily invariant norm on a unital C^*-algebra $\mathcal{R}$, $S, T \in \mathcal{R}$, and $\{U_j\}$ is a net of unitary operators in $\mathcal{R}$ such that*

$$\lim_j \left\| S - U_j^* T U_j \right\| = 0,$$

then

$$\alpha(S) = \alpha(T).$$

Definition 2. If (Ω, μ) is a probability space, then $L^\infty(\mu)$ is a von Neumann algebra, and a unitarily invariant norm α on $L^\infty(\mu)$ is called a *normalized gauge norm on* $L^\infty(\mu)$. In this case all we need require of α is that $\alpha(1) = 1$ and $\alpha(f) = \alpha(|f|)$ for every $f \in L^\infty(\mu)$. We let $\mathbb{MP}(\Omega, \mu)$ denote the group (under composition) of all invertible measure-preserving transformations from Ω to Ω. We say that a gauge norm α on $L^\infty(\mu)$ is *symmetric* if, for every $\gamma \in \mathbb{MP}(\Omega, \mu)$ and every $f \in L^\infty(\mu)$, we have

$$\alpha(f \circ \gamma) = \alpha(f).$$

In [24], J. von Neumann characterized all of the unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$, which is the $n \times n$ full matrix algebra with entries in $\mathbb{C}$. Also [10] characterizes the unitarily invariant norms on a II_1 factor von Neumann algebra. The goal of this paper is to give a characterization of all unitarily invariant norms of a finite von Neumann algebra acting on a separable Hilbert space. Along the way we give a new proof of the characterization of unitarily invariant norms on a II_1 factor in [10].

2.1.1. Unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$. In this section we give a brief description of von Neumann's characterization [24] of unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$. Let τ_n be the normalized trace on $\mathbb{M}_n(\mathbb{C})$, i.e., $\tau_n = \frac{1}{n} Trace$.

Lemma 3. *Suppose $T \in \mathbb{M}_n(\mathbb{C})$, then there exists a unitary matrix $U \in \mathcal{U}(\mathbb{M}_n(\mathbb{C}))$ and numbers $s_T\left(\frac{1}{n}\right) \geqslant s_T\left(\frac{2}{n}\right) \geqslant \cdots s_T\left(\frac{n}{n}\right) \geq 0$ such that*

$$U^*|T|U = \begin{pmatrix} s_T\left(\frac{1}{n}\right) & 0 & \cdots & 0 \\ 0 & s_T\left(\frac{2}{n}\right) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & s_T\left(\frac{n}{n}\right) \end{pmatrix}.$$

The numbers $s_T\left(\frac{1}{n}\right), s_T\left(\frac{1}{n}\right), \ldots, s_T\left(\frac{n}{n}\right)$ are unique and are called the s-numbers *of the matrix T. Define*

$$s(T) = \left(s_T\left(\frac{1}{n}\right), s_T\left(\frac{2}{n}\right), \cdots, s_T\left(\frac{n}{n}\right)\right).$$

If α is a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$, then

$$\alpha(T) = \alpha(|T|) = \alpha(U^*|T|U) = \alpha \begin{pmatrix} s_T\left(\frac{1}{n}\right) & 0 & \cdots & 0 \\ 0 & s_T\left(\frac{2}{n}\right) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & s_T\left(\frac{n}{n}\right) \end{pmatrix},$$

and thus $\alpha(T)$ depends only on the s-numbers of T.
Note that $s(T) \in \mathbb{C}^n$, and in classical matrix theory [2] the standard notation is $s_k(T)$ instead of our $s_T\left(\frac{k}{n}\right)$ for $1 \leq k \leq n$. We know that $\mathbb{C}^n$ is isomorphic to $L^\infty(\delta_n)$, where δ_n is normalized counting measure on $\left\{\frac{1}{n}, \ldots, \frac{n}{n}\right\}$. Let $\mathbb{S}_n$ be the permutation group (i.e., all the bijective functions on $J_n = \left\{\frac{1}{n}, \ldots, \frac{n}{n}\right\}$). It is clear that $\mathbb{S}_n = \mathbb{MP}(J_n, \delta_n)$.

In this case a normalized gauge norm β on $\mathbb{C}^n = L^\infty(\delta_n)$ is *symmetric* if, for every $f \in L^\infty(\delta_n)$ and every $\sigma \in \mathbb{S}_n$,

$$\beta(f) = \beta(f \circ \sigma),$$

that is

$$\beta((a_1, \ldots, a_n)) = \beta\left(\left(a_{\sigma(1)}, \ldots, a_{\sigma(n)}\right)\right).$$

We know that, for each $x = (x_1, \ldots, x_n)$ in $\mathbb{C}^n$ with $|x| = (|x_1|, \ldots, |x_n|)$, there is a $\sigma \in \mathbb{S}_n$ such that

$$\sigma(|x|) = \left(\left|x_{\sigma(1)}\right|, \cdots, \left|x_{\sigma(n)}\right|\right) \underset{\text{def}}{=} \left(s_{|x|}\left(\frac{1}{n}\right), s_{|x|}\left(\frac{2}{n}\right), \cdots, s_{|x|}\left(\frac{n}{n}\right)\right) = s_{|x|},$$

where $s_x\left(\frac{1}{n}\right) \geq s_x\left(\frac{2}{n}\right) \geq \cdots \geq s_x\left(\frac{n}{n}\right) \geq 0$. We call $s_{|x|}$ the *nonincreasing rearrangement* of $|x|$. Note that, although σ may not be unique, $s_{|x|}$ is unique.

Given a unitarily invariant norm α on $\mathbb{M}_n(\mathbb{C})$, define β_α on $\mathbb{C}^n$ by

$$\beta_\alpha(x) = \beta_\alpha(x_1, ..., x_n) = \alpha\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} = \alpha\begin{pmatrix} s_{|x|}\left(\frac{1}{n}\right) & & \\ & \ddots & \\ & & s_{|x|}\left(\frac{n}{n}\right) \end{pmatrix}.$$

Clearly, permutations on J_n corresponds to unitary conjugations by permutation matrices in $\mathbb{M}_n(\mathbb{C})$. Hence β_α is a normalized gauge norm on $L^\infty(\delta_n) = \mathbb{C}^n$.
Conversely, given a symmetric normalized gauge norm β on $\mathbb{C}^n$, we would like to define α_β on $\mathbb{M}_n(\mathbb{C})$ by

$$\alpha_\beta(T) = \beta\left(s_T\left(\frac{1}{n}\right), s_T\left(\frac{2}{n}\right), \cdots, s_T\left(\frac{n}{n}\right)\right).$$

We need to check that α_β is a norm. Clearly, $s_{\lambda T}\left(\frac{1}{n}\right) = |\lambda| s_T\left(\frac{1}{n}\right)$, so

$$\alpha_\beta(\lambda T) = \beta\left(s_{\lambda T}\left(\frac{1}{n}\right), s_{\lambda T}\left(\frac{2}{n}\right), \cdots, s_{\lambda T}\left(\frac{n}{n}\right)\right) = |\lambda| \alpha_\beta(T).$$

Also, $\alpha_\beta(T) \geq 0$ and $\alpha_\beta(T) = 0$ implies $T = 0$. The big problem is the triangle inequality: $s_{A+B}\left(\frac{k}{n}\right) \leq s_A\left(\frac{k}{n}\right) + s_B\left(\frac{k}{n}\right)$ can fail if $k > 1$. When $k = 1$, $s_T\left(\frac{k}{n}\right) = \|T\|$.

Example 1. $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$.
In this example, $s_{A+B}\left(\frac{2}{n}\right) = \frac{3}{2}, s_A\left(\frac{2}{n}\right) + s_B\left(\frac{2}{n}\right) = \frac{5}{4}$.

In order to prove the triangle inequality of α_β, the *Ky Fan Norms* play a central role. For $1 \le k \le n$ we define $KF_{\frac{k}{n}} : \mathbb{M}_n(\mathbb{C}) \to [0,\infty)$ and $KF_{\frac{k}{n}} : \mathbb{C}^n \to [0,\infty)$, by

$$KF_{\frac{k}{n}}(T) = \frac{s_T\left(\frac{1}{n}\right) + \cdots + s_T\left(\frac{k}{n}\right)}{k} \text{ and } KF_{\frac{k}{n}}(x) = \frac{s_{|x|}\left(\frac{1}{n}\right) + \cdots + s_{|x|}\left(\frac{k}{n}\right)}{k}.$$

To prove $KF_{\frac{k}{n}}$ is a norm on $\mathbb{M}_n(\mathbb{C})$ and on $\mathbb{C}^n$, we use the following Lemma whose proof can be found in [7]. Once we know $\alpha = KF_{\frac{k}{n}}$ is a norm on $\mathbb{M}_n(\mathbb{C})$, it easily follows that $KF_{\frac{k}{n}} = \beta_\alpha$ is a symmetric gauge norm on $\mathbb{C}^n$.

Lemma 4. *For $T \in \mathbb{M}_n(\mathbb{C})$, $KF_{\frac{k}{n}}(T) = \sup\{|\mathrm{Tr}(UTP)| : U$ is unitary, P is a projection of rank $k\}$.*

We easily obtain the following corollary.

Corollary 1. *$\sum_{m=1}^{k} s_{A+B}(\frac{m}{n}) \le \sum_{m=1}^{k}\left[s_A(\frac{m}{n}) + s_B(\frac{m}{n})\right]$ for $A, B \in \mathbb{M}_n(\mathbb{C})$ and $1 \le k \le n$.*

The key result relates the Ky Fan norms to arbitrary unitarily invariant norms. The proof can be found in [9].

Lemma 5. *Suppose $n \in \mathbb{N}$, $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \mathbb{C}^n$, $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, and $b_1 \ge b_2 \ge \cdots b_n \ge 0$. If $KF_{\frac{k}{n}}(a) \le KF_{\frac{k}{n}}(b)$ for $1 \le k \le n$, then there exists $N \in \mathbb{N}$, $\sigma_1, \cdots, \sigma_N \in \mathbb{S}_n$, $0 \le t_j \le 1$, with $\sum_{j=1}^{N} t_j = 1$ such that $a \le \sum_{j=1}^{N} t_j(\sigma_j(b))$, i.e.,*

$$(a_1, \ldots, a_n) \le \sum_{j=1}^{N} t_j\left(b_{\sigma_j(1)}, \ldots, b_{\sigma_j(n)}\right).$$

Corollary 2. *Suppose $a, b \in \mathbb{C}^n$ with $KF_{\frac{k}{n}}(a) \le KF_{\frac{k}{n}}(b)$ for $1 \le k \le n$, then, for every symmetric gauge norm β on $\mathbb{C}^n$, $\beta(a) \le \beta(b)$.*

Proof. $\beta(a) \le \beta\left(\sum_{j=1}^{N} t_j \sigma_j(b)\right) \le \sum_{j=1}^{N} t_j \beta(\sigma_j(b)) = \left(\sum_{j=1}^{N} t_j\right)\beta(b) = \beta(b)$. ☐

Lemma 6. *If β is a symmetric normalized gauge norm on $\mathbb{C}^n$, then α_β is a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$.*

Proof. We just need to prove the triangle inequality. Suppose $A, B \in \mathbb{M}_n(\mathbb{C})$. If

$$a = \left(s_{A+B}\left(\frac{1}{n}\right), s_{A+B}\left(\frac{2}{n}\right), ..., s_{A+B}\left(\frac{n}{n}\right)\right) \text{ and,}$$
$$b = \left(s_A\left(\frac{1}{n}\right) + s_B\left(\frac{1}{n}\right), s_A\left(\frac{2}{n}\right) + s_B\left(\frac{2}{n}\right), \ldots, s_A\left(\frac{n}{n}\right) + s_B\left(\frac{n}{n}\right)\right),$$

then, by Corollary 1, we know that $KF_{\frac{k}{n}}(a) \leq KF_{\frac{k}{n}}(b)$ for $1 \leq k \leq n$. It follows from Corollary 2 that $\beta(a) \leq \beta(b)$. However,

$$\begin{aligned}\alpha_\beta(A+B) &= \beta(a) \leq \beta(b) = \beta(s_A + s_B) \\ &\leq \beta(s_A) + \beta(s_B) = \alpha_\beta(A) + \alpha_\beta(B).\end{aligned}$$

This completes the proof. □

It is easy to see that $\alpha_{\beta_\alpha} = \alpha$ and $\beta_{\alpha_\beta} = \beta$ always hold. This gives us von Neumann's characterization of unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$.

Theorem 1 [24]. *There is a one to one correspondence between symmetric gauge norms on* $\mathbb{C}^n$ *and unitarily invariant norms on* $\mathbb{M}_n(\mathbb{C})$.

2.1.2. Approximate discrete Ky-Fan Lemma. We want to prove an approximate version of the Ky-Fan Lemma. Suppose $n \in \mathbb{N}$, $a = (1, 0, ..., 0), b = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}) \in \mathbb{C}^n$. The Ky-Fan Lemma says that a convex combination of permutations of a is greater than or equal to b. It is clear that each permutation of a has only one nonzero entry, so the number of permutations must be at least n. We want, given a positive number ε, to find a number k that is independent of n, so that when $0 \leq a, b \in \mathbb{C}^n$ and $KF_j(a) \geq KF_j(b)$ for $1 \leq j \leq n$, there is an *average* of k permutations of a that is greater than or equal to $(b_1 - \varepsilon, b_2 - \varepsilon, \ldots, b_n - \varepsilon)$.

Suppose $n \in \mathbb{N}$, $f : \{1, \ldots, n\} \to \mathbb{C}$ and $\gamma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is bijective. Let $\mathbb{S}_n$ denote the set of all bijective functions $\gamma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with the identity map $\mathrm{id} \in \mathbb{S}_n$ defined by

$$\mathrm{id}(k) = k \text{ for all } 1 \leq k \leq n.$$

Let $\mathbb{C}^n$ denote the set of all functions $f : \{1, \ldots, n\} \to \mathbb{C}$. For each $\gamma \in \mathbb{S}_n$, define the map $C_\gamma : \mathbb{C}^n \to \mathbb{C}^n$ by

$$C_\gamma(f) = f \circ \gamma.$$

If N is a positive integer, we define

$$\mathcal{C}_N = \left\{ \frac{1}{N} \sum_{k=1}^{N} C_{\gamma_k} : \gamma_1, \ldots, \gamma_N \in \mathbb{S}_N \right\}.$$

It is easily seen that if $\varphi_1 \in \mathcal{C}_{N_1}$ and $\varphi_2 \in \mathcal{C}_{N_2}$, then $\varphi_2 \circ \varphi_1 \in \mathcal{C}_{N_1 N_2}$. Also if $m_1, \ldots m_k \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_k \in \mathbb{S}_n$, and if $\sum_{j=1}^{k} m_j = N$, then

$$\sum_{j=1}^{k} \frac{m_j}{N} C_{\gamma_j} \in \mathcal{C}_N.$$

Suppose $(X_1, <_1), (X_2, <_2)$ are strictly linearly ordered sets. We let $< = (<_1, <_2)$ denote the *lexicographical order* on $X_1 \times X_2$, i.e.,

$$(a_1, b_1) < (a_2, b_2) \Leftrightarrow a_1 <_1 a_2, \text{ or } a_1 = a_2 \text{ and } b_1 <_2 b_2.$$

Then $(<_1, <_2)$ is a strict linear order on $X_1 \times X_2$.

Suppose $m \in \mathbb{N}$. Let $E_m = \{1, \ldots, m\} \times \{1, \ldots, m\}$, linearly ordered by $\prec = (<, <)$ (where $<$ is the usual order). Let $\prec' = (\prec, \prec)$ on $E_m \times E_m$. This makes $E_m \times E_m$ order-isomorphic to $\{1, 2, \ldots, m^4\}$.

Lemma 7. *Suppose $m, n \in \mathbb{N}$ and $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ and $h : \{1, \ldots, n\} \to \{0, 1, \ldots, m\}$ are nonincreasing and, for every $k \in \{1, \ldots, n\}$*

$$\sum_{j=1}^{k} h(j) \leq \sum_{j=1}^{k} f(j).$$

If $N = (m!)^{m^5}$, then there are permutations $\gamma_1, \ldots, \gamma_N$ of $\{1, \ldots, n\}$ such that

$$\frac{1}{N} \sum_{k=1}^{N} f \circ \gamma_k \geq h.$$

Proof. Suppose $g : \{1, \ldots, n\} \to \{0, 1, \ldots, m\}$ and $g \ngeq h$. We define

$$q_g = \min \{k : g(k) < h(k)\} \text{ and}$$
$$q'_g = \max \{k : (g(k), h(k)) = (g(q_g), h(q_g))\}.$$

We say g is *nice* if $g \ngeq h$ and

(1)

$$\sum_{j=1}^{k} h(j) \leq \sum_{j=1}^{k} g(j)$$

for $q'_g \leq k \leq n$,

(2) g is nonincreasing on $\{k : q_g \leq k \leq n\}$.

Suppose now that g is "nice". It is clear from the definition of q_g and the fact that h is nonincreasing that $(g(q_g), h(q_g))$ is the largest element (with respect to $\prec$) of

$$\{(g(k), h(k)) : 1 \leq k \leq n,\ g(k) < h(k)\}.$$

It follows from (2) that $g(k) \leq g(q_g)$ whenever $q_g \leq k \leq n$. Since g and h are nonincreasing on $\{k : q_g \leq k \leq n\}$, we know that

$$\{k : (g(k), h(k)) = (g(q_g), h(q_g))\} = \left\{k : q_g \leq k \leq q'_g\right\}.$$

Thus we have

(3) for $q_g \leq k \leq n$.

$$\sum_{j=1}^{k} h(j) \leq \sum_{j=1}^{k} g(j).$$

Let $F = \left\{k : q_g \leq k \leq q'_g\right\}$, and let

$$b = Card(F) = q'_g - q_g + 1.$$

It follows from (3), with $k = q_g$, that there is an integer k_0 with $1 \leq k_0 < q_g$ such that $g(k_0) > h(k_0)$. Let p_g be the smallest positive integer such that

$$g(p_g) = \max \{g(k) : 1 \leq k \leq n,\ g(k) > h(k)\}.$$

If $g(k) = g(p_g)$, then

$$g(k) = g(p_g) \geq g(k_0) > h(k_0) \geq h(q_g) > g(q_g),$$

which means, from (2), that $k < q_g$. In particular, $p_g < q_g$.

It is now clear that $(g(p_g), h(p_g))$ is the largest element (with respect to $\prec$) of $\{(g(k), h(k)) : g(k) > h(k)\}$.

Let

$$E = \{k : 1 \leq k \leq n, (g(k), h(k)) = (g(p_g), h(p_g))\}$$

and let $a = Card(E)$. Since, for each $k \in E$, $g(k) = g(p_g) > g(q_g)$, so $E \subset [1, q_g)$.

A simple computation shows that if $t_g = \frac{g(p_g)-h(p_g)}{g(p_g)-g(q_g)}$ and $1 - t_g = \frac{h(p_g)-g(q_g)}{g(p_g)-g(q_g)}$, then

$$\begin{aligned}
&(1-t_g)\, g(p_g) + t_g g(q_g) = h(p_g), \text{ and} \\
&(1-t_g)\, g(q_g) + t_g g(p_g) = g(q_g) + [g(p_g) - h(p_g)].
\end{aligned}$$

Note that $1 \leq g(p_g) - g(q_g) \leq m$. Also

$$\begin{aligned}
&g(q_g) + [g(p_g) - h(p_g)] = g(p_g) + [g(q_g) - h(p_g)] \\
&g(p_g) + [g(q_g) - h(q_g)] < g(p_g).
\end{aligned}$$

We now set up some notation. Suppose $D \subset E$ and $d = Card(D) \leq Card(F)$. Let $\alpha_{D,g} : D \to \{k : q_g \leq k < q_g + d\}$ be the unique order-preserving bijection. Define $\gamma_{D,g} \in \mathbb{S}_n$ by

$$\gamma_{D,g}(k) = \begin{cases} \alpha_{D,g}(k) & \text{if } k \in D \\ \alpha_{D,g}^{-1}(k) & \text{if } k \in \alpha_{D,g}(D) \\ k & \text{otherwise} \end{cases}.$$

Define $\varphi_{D,g} \in \mathcal{C}_{m!}$ by

$$\varphi_{D,g} = (1-t_g)\, C_{\gamma_{D,g}} + t_g C_{\mathrm{id}}.$$

Thus

$$\varphi_{D,g}(g)(k) = \begin{cases} h(k) & \text{if } k \in D \\ g(k) + (g(p_g) - h(p_g)) = g(q_g) + (g(p_g) - h(p_g)) & \text{if } k \in \alpha_{D,g}(D) \\ g(k) & \text{otherwise} \end{cases}.$$

Also

$$\begin{aligned}
g(q_g) < \varphi_{D,g}(g)(q_g) &= g(q_g) + [g(p_g) - h(p_g)] = g(p_g) + [g(q_g) - h(p_g)] \\
&\leq g(p_g) + [g(q_g) - h(q_g)] < g(p_g).
\end{aligned}$$

Since $g(p_g) - h(p_g) > 0$, it is clear that $\varphi_{D,g}(g)$ is nonincreasing on $\{k : q_g \leq k \leq n\}$. If $\varphi_{D,g}(g) \not\geq h$, then $q'_{\varphi_{D,g}(g)} \geq q'_g$. Since

$$\sum_{j=1}^{q_g} \varphi_{D,g}(g)(j) = \sum_{j=1}^{q_g} g(j),$$

it follows from (1) that, whenever $k \geq q'_{\varphi_{D,g}(g)} \geq q'_g$,

$$\sum_{j=1}^{k} h(j) = \sum_{j=1}^{k} \varphi_{D,g}(g)(j).$$

Thus if $\varphi_{D,g}(g) \not\geq h$, then $\varphi_{D,g}(g)$ is nice.

For any nice g we define $\rho(g) \in E_m \times E_m$ by

$$\rho(g) = ((g(p_g), h(p_g)), (g(q_q), h(q_g))).$$

CLAIM: There is a $\hat{\varphi} \in \mathcal{C}_{(m!)^m}$ such that $\hat{g} = \hat{\varphi}(g)$ satisfies $\hat{g} \geq h$ or $\hat{g}$ is nice and $\rho(\hat{g}) < \rho(g)$.

We now consider a few cases.

Case 1: $a \leq b$. Let $D = E$. If $\varphi_\alpha(g) \ngeq h$, then $\varphi_\alpha(g)$ is nice and

$$\left(\varphi_\alpha(g)\left(p_{\varphi_\alpha(g)}\right), h\left(p_{\varphi_\alpha(g)}\right)\right) \prec (g(p_g), h(p_g)).$$

It follows that

$$\rho(\varphi_\alpha(g)) \prec' \rho(g).$$

In this case we define $\hat{g} = \varphi_\alpha(g)$ and $\hat{\varphi} = \varphi_\alpha$.

We now can assume that $a > b$. There is a smallest positive integer w such that

$$g(q_g) + w(g(p_g) - h(p_g)) \geq h(q_g),$$

and there is a positive integer v and a disjoint collection $\{E_0, E_1, \ldots E_v\}$ whose union is E and such that

$$Card(E_k) = b \text{ for } 1 \leq k \leq v.$$

Let $u = \min(v, w)$, and define bijections $\alpha_k : E_k \to F$ for $1 \leq k \leq u$. We let

$$t_k = \frac{g(p_g) - h(p_g)}{g(p_g) - [g(q_g) + k(g(q_g) + (k-1)[g(p_g - h(p_g))])]},$$

and

$$\varphi_{\alpha_k} = (1 - t_k) C_{\gamma_k} + t_k \mathrm{id}.$$

For $1 \leq s \leq u$, let

$$\varphi_s = \varphi_{\alpha_s} \circ \cdots \circ \varphi_{\alpha_1}.$$

We see that

$$\varphi_s(g)(k) = \begin{cases} h(k) & \text{if } k \in \cup_{1 \leq j \leq s} E_j \\ g(q_g) + s(g(p_q) - h(p_g)) & \text{if } k \in F \\ g(k) & \text{otherwise} \end{cases}.$$

We see that $\varphi_{u-1}(g) \ngeq h$ and $\varphi_{u-1}(g)$ is nice. Thus $\varphi_u(g) \geq h$ or $\varphi_u(g)$ is nice.

We now consider more cases.

Case 2: $v < w$. We know that $\varphi_v(g)$ is nice. We let $D = E_0$ and let $\alpha : E_0 \to \{k : q_g \leq k < q_q + Card(E_0)\}$ be a bijection and let $\hat{g} = \varphi_\alpha(g_v)$. Then $\hat{g} \geq h$ or $\hat{g}$ is nice and

$$(\hat{g}(p_{\hat{g}}), h(p_{\hat{g}})) \prec (g(p_g), h(p_g)).$$

Thus if $\hat{g} \not\geq h$, then

$$\rho(\hat{g}) \prec' \rho(g).$$

$v > w$ or $v = w$ and $E_0 \neq \varnothing$. In this case g_w is nice and if $g_w \not\geq h$, then g_ω is nice,

$$(g_\omega(p_{g_w}), h(p_{g_w})) = (g(p_g), h(p_g))$$

and

$$(g_\omega(q_{g_w}), h(p_{g_w})) = (g(p_g), h(p_g)).$$

Thus if $\hat{g} = g_w$ and $\hat{\varphi} = \varphi_{\alpha_w} \circ \cdots \circ \varphi_1$, then $\hat{g} = \hat{\varphi}(g)$, either $\hat{g} \geq h$ or $\hat{g}$ is nice, and

$$\rho(\hat{g}) \prec' \rho(g).$$

Case 3: $v = w$ and $E = \varnothing$. In this case $\varphi_w(g) \geq h$ or $\varphi_w(g)$ is nice. If $\varphi_w(g)$ is nice, then

$$(g_\omega(p_{g_w}), h(p_{g_w})) \prec (g(p_g), h(p_g))$$

and

$$(g_\omega(q_{g_w}), h(p_{g_w})) \prec (g(p_g), h(p_g)).$$

Thus if $\hat{g} = g_w$ and $\hat{\varphi} = \varphi_{\alpha_w} \circ \cdots \circ \varphi_1$, then $\hat{g} = \hat{\varphi}(g)$ and either $\hat{g} \geq h$ or $\hat{g}$ is nice, and

$$\rho(\hat{g}) \prec' \rho(g).$$

$\hat{\varphi}_{\hat{g}} \in \mathcal{C}_{(m!)^m}$ such that $\hat{g} = \hat{\varphi}_g(g)$ is nice and $\rho(\hat{g}) \prec' \rho(g)$.

It follows from Cases $1, 2, 3$ that the claim is proved.

If we let $f_0 = f$ and if f_k is nice and $f_k \not\geq h$, then $f_{k+1} = \hat{f}_k$ and $\hat{f}_{k+1} = \varphi_{k+1}(f_k)$ with $\varphi_{k+1} \in \mathcal{C}_{(m!)^m}$. It follows from the Claim that there is a smallest $k \leq m^4$ such that $f_k \geq h$. Then

$$\varphi = \varphi_k \circ \cdots \circ \varphi_1 \in \mathcal{C}_{((m!)^m)^k} \subset \mathcal{C}_{(m!)^{m^5}}$$

and $f_k = \varphi(f)$. Thus the lemma is proved. □

2.1.3. Unitarily invariant norms on a II_1 factor. In this section we give a new proof of the characterization in [10] of unitarily invariant norms on a II_1 factor von Neumann algebra $\mathcal{M}$. If $\mathcal{M}$ is a type II_1 factor von Neumann algebra. then $\mathcal{M}$ has a unique faithful normal tracial state τ with the property that if P and Q are projections in $\mathcal{M}$, then P and Q are unitarily equivalent in $\mathcal{M}$ if and only if $\tau(P) = \tau(Q)$. In this case the measure space (J_n, δ_n) is replaced with the measure space $(J_\infty, \delta_\infty)$, where $J_\infty = [0,1]$ and δ_∞ is Lebesgue measure. A normalized gauge norm β on $L^\infty[0,1] = L^\infty(\delta_\infty)$ is *symmetric* if, for every $\gamma \in \mathbb{MP}(J_\infty, \delta_\infty)$ and every $f \in L^\infty(\delta_\infty)$, we have $\beta(f) = \beta(f \circ \gamma)$.

The main result in [10] is that there is a one-to-one correspondence between the unitarily invariant norms on $\mathcal{M}$ and the symmetric normalized gauge norms on $L^\infty(\delta_\infty)$. This looks just like von Neumann's result for $\mathbb{M}_n(\mathbb{C})$.

The definition of the s-numbers for a function in $L^{\infty}[0,1]$ can be obtained from nonincreasing rearrangements in measure theory. The proof in [10] doesn't use a version of the Ky Fan Lemma (Lemma 5); we present a new proof here using an "approximate" version of the Ky Fan Lemma (Theorem 2).

The first result we need is nonincreasing rearrangements from [11, Chapter 37].

Lemma 8. *Suppose $f:[0,1]\to\mathbb{C}$ is measurable. Then there is a $\gamma\in\mathbb{MP}(J_{\infty},\delta_{\infty})$ such that $s_f \underset{\text{def}}{=} |f|\circ\gamma$ is nonincreasing on $[0,1]$. The transformation γ may not be unique, but s_f is unique (a.e.). It therefore follows that if $f_1, f_2:[0,1]\to\mathbb{C}$ are measurable, then*

$$s_{f_1}=s_{f_2} \text{ if and only if } |f_1|=|f_2|\circ\gamma \text{ for some } \gamma\in\mathbb{MP}(J_{\infty},\delta_{\infty}).$$

The function s_f is called the *nonincreasing rearrangement* of $|f|$.

For $0<t\le 1$, we define the Ky Fan norm KF_t on $L^{\infty}[0,1]$ by

$$KF_t(f)=\frac{1}{t}\int_0^t s_f\,d\delta_{\infty}.$$

For an operator $T\in\mathcal{M}$ and $0\le t\le 1$, the t^{th} s-number of T, denoted by $s_T(t)$, was defined by Fack and Kosaki in [8] as

$$s_T(t)=\inf\{\|TE\|: E \text{ is a projection in } \mathcal{M} \text{ with } \tau(E^{\perp})\le t\}.$$

It is clear that the map $t\mapsto s_T(t)$ is nonincreasing on $[0,1]$. The t^{th} Ky Fan norm $KF_t(T)$ is defined as

$$KF_t(T)=\begin{cases}\|T\| & \text{if } t=0\\ \frac{1}{t}\int_0^t s_T(t)\,d\delta_{\infty} & \text{if } 0<t\le 1\end{cases}.$$

In the matrix case $|T|$ is unitarily equivalent to a diagonal matrix, which naturally corresponds to an element of $\mathbb{C}^n$. In the II_1 factor case we need a more complicated approach.

Definition 3. A normal $*$-isomorphism $\pi:L^{\infty}(\delta_{\infty})\to\mathcal{M}$ such that, for every $f\in L^{\infty}(\delta_{\infty})$,

$$(\tau\circ\pi)(f)=\int_{J_{\infty}} f\,d\delta_{\infty}.$$

is called a tracial embedding.

In the matrix case, the assertion that $|T|$ is unitarily equivalent to a diagonal matrix can be rephrased as $|T|$ is contained in a maximal abelian selfadjoint algebra (i.e., masa) of $\mathbb{M}_n(\mathbb{C})$, and every masa in $\mathbb{M}_n(\mathbb{C})$ is unitarily equivalent to the algebra of diagonal $n\times n$ matrices. Here is the analogue for a II_1 factor. This can be found in [22].

Lemma 9. *Suppose $\mathcal{A}$ is a masa in a type II_1 factor $\mathcal{M}$. Then there is a* surjective *tracial embedding $\pi:L^{\infty}(\delta_{\infty})\to\mathcal{A}$.*

Lemma 10. *Suppose $\mathcal{A}$ is a masa in a type II_1 factor $\mathcal{M}$. If $f \in L^\infty[0,1]$ and $\pi(f) = T$, then, for almost every $t \in [0,1]$,*

$$s_f(t) = s_{\pi(f)}(t) = s_T(t).$$

The following lemma is a consequence of Hadwin-Ding in [4].

Lemma 11. *If π and ρ are tracial embeddings into a II_1 factor $\mathcal{M}$, then π and ρ are approximately unitarily equivalent in $\mathcal{M}$, i.e., there is a net $\{U_j\}$ of unitary operators in $\mathcal{M}$ such that, for every $f \in L^\infty(\delta_\infty)$,*

$$\left\| U_j^* \pi(f) U_j - \rho(f) \right\| \to 0.$$

Corollary 3. *If $\pi : L^\infty(\delta_\infty) \to \mathcal{M}$ is a tracial embedding and $\gamma \in \mathbb{MP}(J_\infty, \delta_\infty)$, then $\rho : L^\infty(\delta_\infty) \to \mathcal{M}$ defined by $\rho(f) = \pi(f \circ \gamma)$ is also a tracial embedding. Hence, there is a net $\{U_j\}$ of unitary operators in $\mathcal{M}$ such that, for every $f \in L^\infty(\delta_\infty)$,*

$$\left\| U_j^* \pi(f) U_j - \pi(f \circ \gamma) \right\| \to 0.$$

As in the matrix case we need to prove KF_t is a norm on $\mathcal{M}$ by giving an alternate characterization given in [10, Lemma 5.1]

Lemma 12. *If $T \in \mathcal{M}$ and $0 < t \le 1$, then*

$$KF_t(T) = \sup\{|\tau(UTP)| : U \in \mathcal{U}(\mathcal{M}), P \text{ is a projection in } \mathcal{M}, \tau(P) = t\}.$$

Suppose α is a unitarily invariant norm on $\mathcal{M}$. We can choose a tracial embedding $\pi : L^\infty(J_\infty, \delta_\infty) \to \mathcal{M}$ and define a norm β_α on $L^\infty(J_\infty, \delta_\infty)$ by

$$\beta_\alpha(f) = \alpha(\pi(f)).$$

We need to show that the definition does not depend on the embedding π. If $\rho : L^\infty(J_\infty, \delta_\infty) \to \mathcal{M}$ is another tracial embedding, then by Lemma 11, there is a net $\{U_j\}$ of unitary operators in $\mathcal{M}$ such that, for every $f \in L^\infty(J_\infty, \delta)$

$$\left\| U_j^* \pi(f) U_j - \rho(f) \right\| \to 0.$$

Since

$$\begin{aligned} |\beta(\pi(f)) - \beta(\rho(f))| &= \left|\beta\left(U_j^* \pi(f) U_j\right) - \beta(\rho(f))\right| \\ &\le \beta\left(U_j^* \pi(f) U_j - \rho(f)\right) \le \left\| U_j^* \pi(f) U_j - \rho(f) \right\| \to 0, \end{aligned}$$

we see that $\beta(\pi(f)) = \beta(\rho(f))$. Moreover, it follows from Corollary 3 that, the gauge norm β_α is symmetric. A simple consequence is that $KF_t = \beta_{KF_t}$ is a symmetric gauge norm on $L^\infty(J_\infty, \delta_\infty)$.

Next suppose β is a symmetric gauge norm on $L^\infty(J_\infty, \delta_\infty)$. We want to define α_β on $\mathcal{M}$. If $T \in \mathcal{M}$, we can choose a masa $\mathcal{A}$ in $\mathcal{M}$ such that $|T| \in \mathcal{A}$. We then choose a surjective tracial embedding $\pi : L^\infty(J_\infty, \delta_\infty) \to \mathcal{A}$ and choose $f \in L^\infty(J_\infty, \delta_\infty)$ such that $\pi(f) = |T|$ and then define

$$\alpha_\beta(T) = \beta(f) = \beta(s_f).$$

Since

$$s_f(t) = s_{\pi(f)}(t) = s_{|T|}(t),$$

we see that the definition is independent of $\mathcal{A}$ and π. As in the matrix case, the main difficulty is proving that α_β satisfies the triangle inequality. In [10] this was done using an approach that avoids proving an analogue of the matrix Ky Fan Lemma (Lemma 5). Here we prove a general "continuous" version of the approximate Ky Fan Lemma that we will need later in our paper.

Lemma 13. *Suppose $f, h \in L^\infty[0,1]$, and $0 \le f, h \le 1, \|f\|_\infty = 1$. Suppose f, h are non-increasing, then there exist step functions $s_f^{[m]} \ge f$ and $s_h^{[m]} \le h$ with ranges contained in $\left\{\frac{k}{m} : 0 \le k \le m\right\}$ such that $\frac{1}{m} \le s_f^{[m]} \le 1$ and $0 \le s_h^{[m]} \le \frac{m-1}{m}$ and $f \le s_f^{[m]} \le f + \frac{1}{m}$ and $\max\left(h - \frac{1}{m}, 0\right) \le s_h^{[m]} \le h$. It follows that $KF_t\left(s_h^{[m]}\right) \le KF_t(h)$ and $KF_t(f) \le KF_t\left(s_f^{[m]}\right)$ for every $t \in (0,1]$.*

Proof. For every $m \in \mathbb{N}$, let $p_i = \sup f^{-1}\left((1 - \frac{i}{m}, 1 - \frac{i-1}{m}]\right)$,
$q_i = \inf h^{-1}\left((1 - \frac{i}{m}, 1 - \frac{i-1}{m}]\right)$, $i = 1, ..., m$. Let $p_0 = q_0 = 0$. Then define

$$s_f^{[m]}(x) = \sum_{i=0}^{m-1}\left(1 - \frac{i}{m}\right)\chi_{[p_i, p_{i+1})}(x) \text{ for } i = 0, ..., m-1.$$

$$s_h^{[m]}(x) = \sum_{i=0}^{m-1}\left(1 - \frac{i+1}{m}\right)\chi_{[q_i, q_{i+1})}(x) \text{ for } i = 0, ..., m-1.$$

It is easy to see that $f \le s_f^{[m]} \le f + \frac{1}{m}$; thus $\left\|f - s_f^{[m]}\right\|_\infty \le \frac{1}{m}$. Also $\max\left(h - \frac{1}{m}, 0\right) \le s_h^{[m]} \le h$; so $\left\|h - s_h^{[m]}\right\|_\infty \le \frac{1}{m}$.

Therefore, $KF_t\left(s_h^{[m]}\right) \le KF_t(h)$ and $KF_t(f) \le KF_t\left(s_f^{[m]}\right)$ for every $t \in (0,1]$ □

Lemma 14. *Suppose f is a step function on $[a,b]$ and $k \in \mathbb{N}$, then there exists an invertible measure preserving map $\varphi_k : [a,b] \to [a,b]$ such that*

$$\left\|\frac{1}{k}\sum_{j=1}^{k} f \circ \varphi_k^{(j)} - \frac{1}{b-a}\int_a^b f(x)\, d\delta_\infty\right\|_\infty \le \eta \|f\|_\infty \frac{4}{k},$$

where $\eta = card\ f([a,b])$, $\varphi_k^{(j)}$ is the composition of j of the φ_k's, i.e., $\varphi_k \circ \varphi_k \circ \cdots \circ \varphi_k$.

Proof. Define $\varphi_k : [a,b] \to [a,b]$ by

$$\varphi_k(x) = \begin{cases} x + \frac{b-a}{k} & \text{if } a \le x \le b - \frac{b-a}{k} \\ x + \frac{b-a}{k} - b + a & \text{if } b - \frac{b-a}{k} < x \le b. \end{cases}$$

Then $\varphi_k^{(k)}$ is the identity map.
If we define $\rho_k(f) = \frac{1}{k}\sum_{j=1}^{k} f \circ \varphi_k^{(j)} - \frac{1}{b-a}\int_a^b f d\delta_\infty$, then ρ_k is linear and $\|\rho_k\| \le 2$ (with ρ_k acting as an operator on $L^\infty(J_\infty, \delta_\infty)$). Suppose $0 \le j < k$. Then $\rho_k\left(\chi_{[a+j\frac{b-a}{k}, a+(j+1))\frac{b-a}{k})}\right) = 0$ a.e. (δ_∞). However, ρ_k is linear; therefore

$\rho_k\left(\chi_{[a+j_1\frac{b-a}{k},a+(j_2))\frac{b-a}{k})}\right)=0$ whenever $0\leq j_1<j_2\leq k$. Suppose $a\leq\alpha<\beta\leq b$. We choose j_1 and j_2 such that j_1 is the largest j, $1\leq j\leq k$ for which $a+j_1\frac{b-a}{k}\leq\alpha$ and choose j_2 to be the smallest j, $1\leq j\leq k$ for which $\beta\leq a+j_2\frac{b-a}{k}$. Then

$$\chi_{[a+j_1\frac{b-a}{k},a+(j_2))\frac{b-a}{k})}-\chi_{[\alpha,\beta)}=\chi_{[a+j_1\frac{b-a}{k},\alpha)}-\chi_{[\beta,a+(j_2))\frac{b-a}{k})}.$$

Hence

$$\rho_k\left(\chi_{[\alpha,\beta)}\right)=\rho_k\left(\chi_{[a+j_1\frac{b-a}{k},\alpha)}\right)-\rho_k\left(\chi_{[\beta,a+(j_2))\frac{b-a}{k})}\right).$$

However, if $E\in\left\{[a+j_1\frac{b-a}{k},\alpha),[\beta,a+(j_2))\frac{b-a}{k})\right\}$ and $f=\chi_E$ then, since $f\circ\varphi_k^{(j)}=\chi_{\left(\varphi_k^{(j)}\right)^{-1}(E)}$ and the collection $\left\{\left(\varphi_k^{(j)}\right)^{-1}(E):1\leq j\leq k\right\}$ is disjoint, we have

$$\left\|\frac{1}{k}\sum_{j=1}^{k}f\circ\varphi_k^{(j)}\right\|_\infty\leq\frac{1}{k}$$

and

$$\frac{1}{b-a}\int_a^b\chi_E d\delta_\infty\leq\frac{1}{b-a}\frac{b-a}{k}=\frac{1}{k},$$

we have $\|\rho_k(E)\|_\infty\leq\frac{2}{k}$. Hence

$$\rho_k\left(\chi_{[\alpha,\beta)}\right)\leq\frac{4}{k}.$$

Suppose f is a step function, then $f=\sum_{j=1}^{n}a_j\chi_{[\alpha_j,\alpha_{j+1})}$ for some $n\in\mathbb{N}$. Denote $f_j=\chi_{[\alpha_j,\alpha_{j+1})}$. Then

$$f=\sum_{j=1}^{n}a_jf_j\int_a^b f(x)\,d\delta_\infty=\sum_{j=1}^{n}a_j\int_a^b\chi_{[\alpha_j,\alpha_{j+1})}d\delta_\infty.$$

Thus

$$\begin{aligned}\|\rho_k(f)\|_\infty&\leq\sum_{j=1}^{n}|a_j|\,\|\rho_k(f_j)\|_\infty\\&\leq\left(\sum_{j=1}^{n}|a_j|\right)\frac{2}{k}\leq\eta\,\|f\|_\infty\left(\frac{4}{k}\right).\end{aligned}$$

□

We call the following the *approximate Ky Fan Lemma* for $L^\infty(\delta_\infty)$.

Theorem 2. *Suppose m is a positive integer. Then whenever $0\leq f,h\leq 1$ in $L^\infty(\delta_\infty)$ satisfy*

$$KF_t(h)\leq KF_t(f)\ \textit{for all rational numbers}\ 0<t\leq 1,$$

there are, $\gamma_1, \ldots, \gamma_{m^{m^2}} \in \mathbb{MP}\left(J_\infty, \delta_\infty\right)$, *such that*

$$s_h \leq \frac{1}{m^{m^2}} \sum_{i=1}^{m^{m^2}} s_f \circ \gamma_i + \frac{2}{m}.$$

Hence $\beta\left(h\right) \leq \beta\left(f\right)$ *for every symmetric gauge norm* β *on* $L^\infty\left(\delta_\infty\right)$.

Proof. If $f \in L^\infty\left(J_\infty, \delta_\infty\right)$, then the map $t \mapsto KF_t\left(f\right)$ is continuous on $(0,1]$. Hence we have $KF_t\left(h\right) \leq KF_t\left(f\right)$ for all $0 < t \leq 1$. We know that $KF_t\left(f\right) = KF_t\left(s_f\right)$ and $\beta\left(f\right) = \beta\left(s_f\right)$ for every $f \in L^\infty\left(\delta_\infty\right)$. We may assume that f, h are nonincreasing, and we let u, w be the step functions defined in the proof of Lemma 13. Then u, w satisfy $f \leq u \leq f + \frac{1}{m}$and $\max\left(h - \frac{1}{m}, 0\right) \leq w \leq h$. Recall that

$$u = \left(1 - \frac{i}{m}\right) \chi_{[p_i, p_{i+1})}\left(x\right) \text{ for } i = 0, ..., m-1.$$
$$w = \left(1 - \frac{i+1}{m}\right) \chi_{[q_i, q_{i+1})}\left(x\right) \text{ for } i = 0, ..., m-1.$$

and it is easy to see that

$$\int_0^t f d\delta_\infty + \frac{t}{m} \geq \int_0^t u d\delta_\infty \geq \int_0^t w d\delta_\infty \geq \int_0^t h d\delta_\infty - \frac{t}{m},$$

for all $0 \leq t \leq 1$.

By Lemma 14, for each $m \in \mathbb{N}$, there exists a measure preserving map $\varphi_m : [0,1] \to [0,1]$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} u \circ \varphi_m^{(j)} - \int_0^1 u d\delta_\infty \right\|_\infty \leq \eta \left\|u\right\|_\infty \frac{4}{m}.$$

where $\eta = card\left(Ran\left(u\right)\right)$

Let $l(t) = \frac{1}{t} \int_0^t u d\delta_\infty$, then $l : [0,1] \longrightarrow [0, \infty)$ is a continuous function. There are 2 cases to consider:

Case 1: If $l(1) = \int_0^1 u d\delta_\infty \geq b_1 = \max\left\{w\left(t\right) : 0 < t \leq 1\right\}$, then by Lemma 14, for $\forall k = m^2 \in \mathbb{N}$, there exists $\varphi_k \in \mathbb{MP}[0,1]$ such that

$$\left\| \frac{u \circ \varphi_k^{(1)} + \cdots + u \circ \varphi_k^{(m^2)}}{m^2} - \int_0^1 u d\delta_\infty \right\|_\infty \leq \frac{4\eta \left\|u\right\|_\infty}{m^2} \leq \frac{4}{m},$$

where $\eta = \text{card}\left(u\right) \leq m$.Denote $\varphi_k^{(i)}$ by γ_j Then we have

$$\frac{1}{m^2} \sum_{j=1}^{m^2} u \circ \gamma_j \geq w - \frac{6}{m}.$$

Therefore $\frac{1}{m^2}\sum_{j=1}^{m^2} f\circ\varphi_{(j)} + \frac{1}{m} \geq h - \frac{4}{m}$ follows from Lemma 13. That is

$$\frac{1}{m^2}\sum_{j=1}^{m^2} f\circ\varphi_{(j)} \geq h - \frac{3}{m}.$$

We can view it as

$$\frac{1}{m^{2m}}\sum_{j=1}^{m^{2m}} f\circ\varphi_{(j)} \geq h - \frac{3}{m},$$

where $\varphi_{(i+m^2t)} = \varphi_{(i)}$ for $1\leq i\leq m^2$ and $0\leq t\leq m^{2\,m-2}-1$.

Case 2: $l(1) = \int_0^1 ud\delta_\infty < b_1$.

Then there must exist $p_1^{'} \in (0,1)$, so that

$$l(p_1^{'}) = \frac{1}{p_1^{'}}\int_0^{p_1^{'}} ud\delta_\infty = b_1.$$

Define $u^{(1)}$ in the following way

$$u^{(1)}(x) = \begin{cases} b_1 & 0\leq x\leq p_1' \\ u(x) & p_1' < x \leq 1. \end{cases}$$

Then for every $t > p_1^{'}$,

$$l(t) = \int_0^t ud\delta_\infty \geqslant \int_0^t wd\delta_\infty \implies \int_0^{p_1'} ud\delta_\infty + \int_{p_1'}^t ud\delta_\infty \geqslant \int_0^{p_1'} wd\delta_\infty$$
$$+\int_{p_1'}^t wd\delta_\infty.$$

Thus we have $b_1p_1' + \int_{p_1'}^t u^{(1)}d\delta_\infty \geqslant b_1p_1' + \int_{p_1'}^t wd\delta_\infty$; therefore

$$\int_{q_1}^t u^{(1)}d\delta_\infty \geqslant \int_{q_1}^t wd\delta_\infty.$$

Therefore, for every $0 < t \leq 1$, we have

$$u - \frac{1}{m} \leq u^{(1)} \leq u,$$
$$KF_t\left(u^{(1)}\right) \geq KF_t(w),$$

and for every $t\leq t_1$, $\left\|u^{(1)}\right\|_t = b_1 = \|h\|_t$

By Lemma 14 again, for $k = m^2 \in \mathbb{N}$, there exist $\varphi_{(1)},\ldots,\varphi_{(m^2)}$: $[0,1] \longrightarrow [0,1]$ such that

$$\left\|\frac{1}{m^2}\sum_{i=1}^{m^2} u\circ\varphi_{(i)} - \int_0^1 ud\delta_\infty\right\|_\infty \leq \eta\,\|u\|_\infty \frac{4}{m^2} \leq \frac{4}{m}.$$

Let $\varphi_{(r)}^{(1)}(t) = \begin{cases} \varphi_{(r)}(t) & t \leq q_1 \\ t & t > q_1 \end{cases}$, $r = 1, \ldots, m^2$. Then $\varphi_{(r)}^{(1)} \in \mathbb{MP}[0,1]$ for all $1 \leq r \leq m^2$ and

$$\left\| \frac{1}{m^2} \sum_{r=1}^{m^2} u \circ \varphi_{(r)}^{(1)} - u^{(1)} \right\|_\infty \leq \frac{2}{m}.$$

That is $u^{(1)} \approx \frac{1}{m^2} \sum_{r=1}^{m^2} u \circ \varphi_{(r)}^{(1)}$ and $\mathrm{Ran}(u^{(1)}) \subseteq \{b_1, a_2, ..., a_m\}$.

If $\frac{1}{q_1} \int_{q_1}^1 u^{(1)} d\delta_\infty \geqslant b_2$, go to case 1.

If $\frac{1}{q_1} \int_{q_1}^1 u^{(1)} d\delta_\infty < b_2$, do the similar process as case 2 above, we have $u^{(2)}$ and

$$\left\| \frac{1}{m^2} \sum_{i=1}^{m^2} u^{(1)} \circ \varphi_i^{(2)} - u^{(2)} \right\|_\infty \leq \frac{2}{m}.$$

That is

$$\begin{aligned} u^{(2)} &\approx \frac{1}{m^2} \sum_{i=1}^{m^2} u^{(1)} \circ \varphi_i^{(2)} \\ &= \frac{1}{m^2} \sum_{i_1=1}^{m^2} \left(\sum_{i_2=1}^{m^2} \frac{1}{m^2} \left(u \circ \varphi_{i_1}^{(1)} \right) \right) \circ \varphi_{i_2}^{(2)} \\ &= \frac{1}{m^4} \sum_{i=1}^{m^2} \sum_{j=1}^{m^2} (u \circ \varphi_{i_1}^{(1)} \circ \varphi_{i_2}^{(2)}). \end{aligned}$$

and $\mathrm{Ran}\Big((u)^{(2)}\Big) \subseteq \{b_1, b_2, a_3, ..., a_m\}$.

Finally, after r steps (at most m), we will have

$$u^{(r)} \approx \frac{1}{m^{2r}} \sum_{i_1=1}^{m^2} \cdots \sum_{i_r=1}^{m^2} (u \circ \varphi_{i_1}^{(1)} \circ \varphi_{i_2}^{(2)} \cdots \circ \varphi_{i_r}^{(r)}),$$

and thus $u^{(r)} \geqslant w$.

Since $m^{2r} | m^{m^2}$, as in case 1, we can view this as

$$\frac{1}{m^{m^2}} \sum_{j=1}^{m^{m^2}} u \circ \varphi_{(j)} \geq w - \frac{2}{m}.$$

In conclusion, for every m, there is an integer $N = m^{m^2}$, and there are $\gamma_1, \ldots \gamma_N \in \mathbb{MP}(J_\infty, \delta_\infty)$ such that

$$\frac{1}{N} \sum_{i=1}^{N} u \circ \gamma_i \geq w - \frac{2m}{N}.$$

By Lemma 13, we know that $f \geq u - \frac{1}{m}$ and $h \leq w + \frac{1}{m}$

Thus, $\frac{1}{N} \sum_{i=1}^{N} s_f \circ \gamma_i + \frac{2m}{N} + \frac{1}{m} \geq s_h$.

Therefore, $\beta(f) \geq \beta(h)$ as $m \to \infty$. □

Corollary 4. *If β is a symmetric gauge norm on $L^\infty(J_\infty, \delta_\infty)$, then α_β is a norm on $\mathcal{M}$.*

Proof. We need only prove the triangle inequality. If $A, B \in \mathcal{M}$, we define $h(t) = s_{A+B}(t)$ and $f(t) = s_A(t) + s_B(t)$. Then $KF_t(h) = KF_t(A+B)$ and $KF_t(f) = KF_t(A) + KF_t(B)$, so Lemma 2 applies, and we get

$$\begin{aligned}\alpha_\beta(A+B) &= \beta(h) \leq \beta(f) = \beta(s_A(t) + s_B(t)) \\ &\leq \beta(s_A(t)) + \beta(s_B(t)) = \alpha_\beta(A) + \alpha_\beta(B).\end{aligned}$$

□

Since it is easily seen that $\alpha = \alpha_{\beta_\alpha}$ and $\beta = \beta_{\alpha_\beta}$, we obtain the characterization [10] of the unitarily invariant norms on a II_1 factor von Neumann algebra.

Theorem 3. *Let $\mathcal{M}$ be a type II_1 factor von Neumann algebra, then there is a one-to-one correspondence between unitarily invariant norms on $\mathcal{M}$ and symmetric gauge norms on $L^\infty(J_\infty, \delta_\infty)$.*

2.2. Approximate unitary equivalence

The following is a consequence of a result of Hadwin and Ding [4]. Suppose $\mathcal{R}$ is a von Neumann algebra and $T \in \mathcal{R}$. $\mathcal{Z}(\mathcal{R}) = \mathcal{R} \cap \mathcal{R}'$ is the center. In [12] the $\mathcal{R}$*-rank* of T was defined to be the Murray-von Neumann equivalence class of the projection P_T onto the closure of the range of T. We let (SOT) and (WOT) denote, respectively, the strong and weak operator topologies. Note that

$$P_T = \lim_{n\to\infty} (TT^*)^{1/n} \ (SOT),$$

so $P_T \in \mathcal{M}$.

The following theorem can be found in [13]. The center-valued trace Φ is described in Sect. 2.3.6.

Theorem 4. *Suppose $\mathcal{R}$ is a finite von Neumann algebra acting on a separable Hilbert space H. Let $\Phi : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ be the unique center-valued trace on $\mathcal{R}$. Suppose $\mathcal{A}$ is a unital commutative C^* algebra and $\pi, \rho : \mathcal{A} \to \mathcal{R}$ are unital $*$-homomorphisms. The following are equivalent:*

(1) *There is a net $\{U_j\}$ of unitary operators in $\mathcal{R}$ such that, for every $a \in \mathcal{A}$,*

$$\left\| U_j^* \pi(a) U_j - \rho(a) \right\| \to 0.$$

(2) $\Phi \circ \pi = \Phi \circ \rho$.

2.3. The central decomposition

We refer the reader to [16] for the theory of direct integrals and the central decomposition of a von Neumann algebra acting on a separable Hilbert space. Since we are only interested in the von Neumann algebra $\mathcal{R}$ and not how it acts on a Hilbert space, we can ignore multiplicities when using the central decomposition [16]. Suppose $\mathcal{R}$ is a finite von Neumann algebra acting on a separable Hilbert space. Then we can write

$$\mathcal{R} = [\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots] \oplus \mathcal{R}_\infty,$$

where $\mathcal{R}_k$ is of type I_k for $1 \leq k < \infty$ and $\mathcal{R}_\infty$ is a type II_1 von Neumann algebra.

2.3.1. Measurable families. Suppose $\mathcal{M}$ is a type II_1 von Neumann algebra with a faithful tracial state acting on a separable Hilbert space $H{=}l^2_\infty$. We will associate with $\mathcal{M}$ a probability space (Ω, μ) and a unitary operator $U : H \longrightarrow L^2(\mu, H)$ that transforms $\mathcal{M}$ into a certain von Neumann algebra of operators on $L^2(\mu, H)$ that will be described next.

We let $\mathcal{M}'$ denote the *commutant* of $\mathcal{M}$, i.e., the set of all operators that commute with every operator in $\mathcal{M}$.

For each $\omega \in \Omega$, there is a type II_1 von Neumann algebra $\mathcal{M}_\omega$ in $B(H)$ that is determined by two sequences of SOT (strong operator topology) measurable operator-valued functions f_n and g_n from Ω into the unit ball of $B(H)$ so that $\mathcal{M}_\omega$ is generated by the set $\{f_n(\omega) : n \in \mathbb{N}\}$, $\mathcal{M}_\omega^{'}$ is generated by the set $\{g_n(\omega) : n \in \mathbb{N}\}$, and each of those sets is SOT dense in the unit ball of the von Neumann algebra it generates. Suppose $\varphi : \Omega \to B(H)$ is an SOT-measurable function, and define $|\varphi| = \|\cdot\| \circ \varphi$, that is $|\varphi|(\omega) = \|\varphi(\omega)\|$ for $\omega \in \Omega$. If $|\varphi| \in L^\infty(\mu)$, then let $\|\varphi\|_\infty = \||\varphi|\|_\infty$. We will assume that (Ω, μ), U, and the $f_n, g_n, \mathcal{M}_\omega$ have been chosen so that

$$U^*\mathcal{M}U = \{\varphi : \Omega \longrightarrow B(H) | \varphi \text{ is SOT-measurable, } \varphi(\omega) \in \mathcal{M}_\omega \text{ a.e.} (\mu), |\varphi| \in L^\infty(\mu)\}.$$

As usual, $\varphi_1 = \varphi_2$ will mean $\varphi_1(\omega) = \varphi_2(\omega)$ a.e. (μ), and each φ in $U^*\mathcal{M}U$ is viewed as the operator on $L^2(\mu, H)$ defined for $f \in L^2(\mu, H)$ by

$$(\varphi f)(\omega) = \varphi(\omega) f(\omega).$$

2.3.2. Measurable cross-sections.

Definition 4. Suppose (X, d) is a metric space and $\mu : \text{Bor}(X) \longrightarrow [0, \infty)$ is a finite measure. A subset B of X is called *μ-measurable* if there are $A, F \in \text{Bor}(X)$ such that $B \backslash A \subset F$ and $\mu(F) = 0$. The σ-algebra of all μ-measurable sets is denoted by $\mathcal{M}_\mu$. A subset D of X is *absolutely measurable* if D is μ-measurable for every finite measure μ on $Bor(X)$. The σ-algebra of all absolutely measurable subsets of X is denoted by $\mathbb{AM}(X)$. Clearly we have

$$\mathbb{AM}(X) = \bigcap \{\mathcal{M}_\mu : \mu \text{ is a finite Borel measure on } X\}.$$

It is obvious that each $\mathcal{M}_\mu$ contains $\text{Bor}(X)$, so $\text{Bor}(X) \subset \mathbb{AM}(X)$. However, it is often the case that $\text{Bor}(X) \neq \mathbb{AM}(X)$. If Y is another metric space, we say that a function $f : X \to Y$ is *absolutely measurable* if f is $\mathbb{AM}(X)$-$\text{Bor}(Y)$ measurable, i.e., for every Borel set $E \subseteq Y$, $f^{-1}(E) \in \mathbb{AM}(X)$. Recall that a finite measure space $(\Lambda, \Sigma, \lambda)$ is *complete* if, $E \in \Sigma$ whenever $E \subset F, F \in \Sigma$ and $\lambda(F) = 0$, i.e., all subsets of sets of measure 0 are in Σ. Note that statement (4) in Lemma 15 shows how, in the presence of a complete measure space, absolute measurability turns into measurability.

Lemma 15. *Suppose X, Y and Z are metric spaces and $f : X \longrightarrow Y$, and $g : Y \to Z$. Then*

(1) *f is absolutely measurable if and only if f is $\mathbb{AM}(X)$-$\mathbb{AM}(Y)$ measurable.*

(2) *If f and g are absolutely measurable, then $g \circ f : X \to Z$ is absolutely measurable.*
(3) *For every Borel set $E \subseteq Y$, $f^{-1}(E)$ is absolutely measurable.*
(4) *If $(\Lambda, \Sigma, \lambda)$ is a complete finite measure space and $\varphi : \Lambda \to X$ is Borel measurable, then*
  (a) φ is Σ-$\mathbb{AM}(X)$ measurable, and,
  (b) If f is absolutely measurable, then $f \circ \varphi : X \to Y$ is measurable.

Definition 5. If $f : X \to Y$ and $g : f(X) \longrightarrow X$ satisfy, for every $y \in f(X)$,

$$f(g(y)) = y,$$

then g is called a cross-section for f.

The following Theorem is from Theorem 3.4.3 in [1] and is the key to dealing with direct integrals.

Theorem 5. *Suppose X is a Borel subset of a complete separable metric space, and Y is a separable metric space. If $f : X \longrightarrow Y$ is a continuous function, then*
(1) *$f(X)$ is an absolutely measurable subset of Y, and*
(2) *f has an absolutely measurable cross-section $g : f(X) \longrightarrow X$.*

Here is a simple result proved using measurable cross-section.

Lemma 16. *Suppose n is a positive integer and $\mathbb{M}_n(\mathbb{C})^+$ is the set of $n \times n$ matrices A such that $A \geq 0$. Let $\mathcal{U}_n$ be the set of unitary $n \times n$ matrices and let $\mathcal{D}_n$ be the set of all diagonal $n \times n$ matrices in $\mathbb{M}_n(\mathbb{C})^+$ of the form $\operatorname{diag}\left(s_{\frac{1}{n}}, \ldots s_1\right)$ with $s_{\frac{1}{n}} \geq s_{\frac{2}{n}} \geq \cdots \geq s_1 \geq 0$. Then there is an absolutely measurable function $u : \mathbb{M}_n(\mathbb{C})^+ \to \mathcal{U}_n$ such that, for every $A \in \mathbb{M}_n(\mathbb{C})^+$,*

$$u(A)^* A u(A) \in \mathcal{D}_n,$$

i.e.,

$$u(A)^* A u(A) = \begin{pmatrix} s_A\left(\frac{1}{n}\right) & & & \\ & s_A\left(\frac{2}{n}\right) & & \\ & & \ddots & \\ & & & s_A\left(\frac{n}{n}\right) \end{pmatrix}.$$

Hence, for every $T \in \mathbb{M}_n(\mathbb{C})$,

$$u(|T|)^* |T| u(|T|) = \begin{pmatrix} s_T\left(\frac{1}{n}\right) & & & \\ & s_T\left(\frac{2}{n}\right) & & \\ & & \ddots & \\ & & & s_T\left(\frac{n}{n}\right) \end{pmatrix}.$$

Proof. Let $X =$
$\left\{(A, U_A) : A \in \mathbb{M}_n(\mathbb{C})^+, U_A \in \mathcal{U}_n, U_A^* A U_A = \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \ldots, s_A\left(\frac{n}{n}\right)\right)\right\}$,
which is a subset of $\mathbb{M}_n(\mathbb{C})^+ \times \mathcal{U}_n$. For every $(A_\lambda, U_{A_\lambda}) \in X$, and $(A_\lambda, U_{A_\lambda}) \longrightarrow (A, U_A)$, we have $A_\lambda \longrightarrow A, U_{A_\lambda} \longrightarrow U_A$, Thus

$$\|U_A^* A U_A - U_{A_\lambda} A_\lambda U_{A_\lambda}\| \to 0,$$

We also know that $\frac{1}{i}\sum_{j=1}^{i} s_A\left(\frac{j}{n}\right) = KF_i(A)$ for all $1 \le i \le n$ and $s_A\left(\frac{1}{n}\right) = KF_1(A) \le \|A\|$. We can get $s_{A_\lambda}\left(\frac{i}{n}\right) \xrightarrow{\|\cdot\|} s_A\left(\frac{i}{n}\right)$ for all $1 \le i \le n$. Thus

$$U_{A_\lambda}^* A_\lambda U_{A_\lambda} = \operatorname{diag}\left(s_{A_\lambda}\left(\frac{1}{n}\right), \ldots, s_{A_\lambda}\left(\frac{n}{n}\right)\right) \xrightarrow{\|\cdot\|} \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \ldots, s_A\left(\frac{n}{n}\right)\right)$$

Therefore $U_A^* A U_A = \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \ldots, s_A\left(\frac{n}{n}\right)\right)$, and X is a closed subset of a $\mathbb{M}_n(\mathbb{C})^+ \times \mathcal{U}_n$, which is a complete separable metric space.

Define $\pi_1 : X \longrightarrow \mathbb{M}_n(\mathbb{C})^+$ and $\pi_2 : X \longrightarrow \mathcal{U}_n$ by

$$\pi_1(A, U) = A, \ \pi_2(A, U) = U.$$

It is easy to see that $\pi_1(X) = \mathbb{M}_n(\mathbb{C})^+$.

Since we know for every $A \in \mathbb{M}_n(\mathbb{C})^+$, there exists a unitary U_A such that

$$U_A^* A U_A = \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \ldots, s_A\left(\frac{n}{n}\right)\right).$$

Thus by Theorem 5, there exists an absolutely measurable function $g : \mathbb{M}_n(\mathbb{C})^+ \longrightarrow X$ such that $\pi_1 \circ g = id$ on $\mathbb{M}_n(\mathbb{C})^+$, for every $A \in \mathbb{M}_n(\mathbb{C})^+$, $g(A) = (A, U_A)$. Then we define $u = \pi_2 \circ g : \mathbb{M}_n(\mathbb{C})^+ \longrightarrow \mathcal{U}_n$, it is absolutely measurable.

Therefore, for every $A \in \mathbb{M}_n(\mathbb{C})^+$,

$$u(A) = U_A \text{ and } u(A)^* A u(A) = \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \ldots, s_A\left(\frac{n}{n}\right)\right) \in \mathcal{D}_n.$$

Hence, for every $T \in \mathbb{M}_n(\mathbb{C})$,

$$u(|T|)^* |T| u(|T|) = \begin{pmatrix} s_T\left(\frac{1}{n}\right) & & & \\ & s_T\left(\frac{2}{n}\right) & & \\ & & \ddots & \\ & & & s_T\left(\frac{n}{n}\right) \end{pmatrix}.$$

□

2.3.3. Direct integrals. Suppose $\Omega \subseteq \mathbb{R}$ is compact, μ is a probability Borel measure, H is a separable Hilbert space. Define $\int_\Omega^\oplus H d\mu = L^2(\mu, H)$ to be the set of all measurable functions $f : \Omega \to H$ such that

$$\|f\|_2^2 \underset{\text{def}}{=} \int_\Omega \|f(\omega)\|^2 d\mu(\omega) < \infty.$$

We define an inner product $\langle \cdot, \cdot \rangle$ on $L^2(\mu, H)$ by

$$\langle f, h \rangle = \int_\Omega \langle f(\omega), h(\omega) \rangle d\mu(\omega).$$

In this way $L^2(\mu, H)$ is a Hilbert space.

We define $L^\infty(\mu, B(H))$ to be the set of all bounded functions $\varphi : \Omega \to B(H)$ that are measurable with respect to the weak operator topology (WOT) on $B(H)$. Although the weak operator topology, strong operator topology (SOT) and $*$-strong operator topology ($*$-SOT) on $B(H)$ are different, the Borel sets with respect to these topologies are all the same. Suppose the map $\omega \mapsto T_\omega$ is in $L^\infty(\mu, B(H))$. We define an operator $T = \int_\Omega^\oplus T_\omega d\mu(\omega)$ by

$$(Tf)(\omega) = T_\omega(f(\omega)).$$

If $\varphi \in L^\infty(\mu, B(H))$ and $T_\omega = \varphi(\omega)$ for $\omega \in \Omega$, we also use the notation M_φ to denote $\int_\Omega^\oplus T_\omega d\mu(\omega)$. In this way we can view $L^\infty(\mu, B(H)) \subseteq B(L^2(\mu, H))$, and we can write $L^\infty(\mu, B(H)) = \int_\Omega^\oplus B(H)\, d\mu(\omega)$.

We have that $L^\infty(\mu)$ can be viewed as the subalgebra $\mathcal{D}$ of $L^\infty(\mu, B(H))$ of all functions φ such that $\varphi(\omega) \in \mathbb{C} \cdot 1$ a.e. (μ), that is, by identifying $h \in L^\infty(\mu)$ with the function $\omega \mapsto h(\omega) 1$. We denote $\mathcal{D}$ by

$$\mathcal{D} = \int_\Omega^\oplus \mathbb{C} \cdot 1 d\mu(\omega).$$

We have $\mathcal{D}' = L^\infty(\mu, B(H))$ and $L^\infty(\mu, B(H))' = \mathcal{D}$, therefore $\mathcal{D} = \mathcal{Z}(L^\infty(\mu, B(H)))$.

Suppose, for each $\omega \in \Omega$, $\mathcal{R}_\omega \subset B(H)$ is a von Neumann algebra. We say that the family $\{\mathcal{R}_\omega\}_{\omega \in \Omega}$ is a *measurable family* if there is a countable set $\{\varphi_1, \varphi_2, \ldots\} \subset L^\infty(\mu, B(H))$ such that

$$\text{ball}(\mathcal{R}_\omega) = \{\varphi_1(\omega), \varphi_2(\omega), \ldots\}^{-SOT} \text{ a.e. } (\mu).$$

It is known that if $\{\mathcal{R}_\omega\}_{\omega \in \Omega}$ is a measurable family, then so is $\{\mathcal{R}'_\omega\}_{\omega \in \Omega}$. Moreover, if $\{\mathcal{R}'_\omega\}_{\omega \in \Omega}$ is a measurable family, then there is a sequence $\{\psi_1, \psi_2, \ldots\} \subset L^\infty(\mu, B(H))$ such that

$$\text{ball}(\mathcal{R}'_\omega) = \{\psi_1(\omega), \psi_2(\omega), \ldots\}^{-SOT} \text{ a.e. } (\mu).$$

If $\{\mathcal{R}_\omega\}_{\omega \in \Omega}$ is a measurable family, then we define the *direct integral* $\int_\Omega^\oplus \mathcal{R}_\omega d\mu(\omega)$ to be the set of all $T = \int_\Omega^\oplus T_\omega d\mu(\omega) \in L^\infty(\mu, B(H))$ such that

$$T_\omega \in \mathcal{R}_\omega \text{ a.e. } (\mu).$$

It is known [16] that a von Neumann algebra $\mathcal{R} \subset B(L^2(\mu, H))$ can be written as

$$\mathcal{R} = \int_\Omega^\oplus \mathcal{R}_\omega d\mu(\omega)$$

for a measurable family $\{\mathcal{R}_\omega\}_{\omega \in \Omega}$ if and only if

$$\mathcal{D} = \int_\Omega^\oplus \mathbb{C} \cdot 1 d\mu(\omega) \subset \mathcal{R} \subset \int_\Omega^\oplus B(H)\, d\mu(\omega) = \mathcal{D}',$$

equivalently,

$$\mathcal{D} \subset \mathcal{Z}(\mathcal{R}).$$

In particular, since $\mathcal{Z}(\mathcal{R}) = \mathcal{Z}(\mathcal{R}') = \mathcal{R} \cap \mathcal{R}'$ for every von Neumann algebra $\mathcal{R}$, we see that $\mathcal{R}$ can be decomposed as a direct integral if and only if $\mathcal{R}'$ can be decomposed as a direct integral.

Suppose $1 \leq n \leq \infty = \aleph_0$. We define ℓ_n^2 to be the space of square summable sequences with the inner product $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, where $x, y \in H$ and H is a Hilbert space with dimension n.

Lemma 17. *Suppose $\mathcal{A}$ is an abelian von Neumann algebra on a separable Hilbert space H. Then there are compact subsets $\Omega_n \subset \mathbb{R}$ for $1 \leq n \leq \infty$ and a Borel measure μ_n on Ω_n such that $\mu_n(\Omega_n) \in \{0, 1\}$ and $\mathcal{A}$ is unitarily equivalent to $\sum_{1\leq n\leq\infty}^{\oplus} L^\infty(\mu_n, \mathbb{C}\cdot 1)$ acting on $\sum_{1\leq n\leq\infty}^{\oplus} L^2(\mu_n, \ell_n^2)$.*

Suppose $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space H. Then the center $\mathcal{Z}(\mathcal{R})$ of $\mathcal{R}$ is an abelian von Neumann algebra on H. From Lemma 17 we can write

$$H = \sum_{1\leq n\leq\infty}^{\oplus} L^2(\mu_n, \ell_n^2)$$

and

$$\mathcal{Z}(\mathcal{R}) = \sum_{1\leq n\leq\infty}^{\oplus} L^\infty(\mu_n, \mathbb{C}\cdot 1).$$

Since $\mathcal{R}$ commutes with $\mathcal{Z}(\mathcal{R})$, we can write

$$\mathcal{R} = \sum_{1\leq n\leq\infty}^{\oplus} \mathcal{R}_n,$$

where $\mathcal{R}_n \subset B(L^2(\mu_n, \ell_n^2))$. It is clear, for $1 \leq n \leq \infty$, that

$$\mathcal{Z}(\mathcal{R}_n) = L^\infty(\mu_n, \mathbb{C}\cdot 1),$$

which implies

$$\mathcal{R}_n \subset \mathcal{Z}(\mathcal{R}_n)' = L^\infty(\mu_n, \mathbb{C}\cdot 1)' = L^\infty(\mu_n, B(\ell_n^2)).$$

Hence, for each n, $1 \leq n \leq \infty$, there is a measurable family $\{\mathcal{R}_n(\omega)\}_{\omega\in\Omega_n}$ such that

$$\mathcal{R}_n = \int_{\Omega_n}^{\oplus} \mathcal{R}_n(\omega)\, d\mu_n(\omega).$$

We therefore have

$$\mathcal{R} = \sum\nolimits_{1\leq n\leq\infty}^{\oplus} \int_{\Omega_n}^{\oplus} \mathcal{R}_n(\omega)\, d\mu_n(\omega).$$

This is called the *central decomposition* of $\mathcal{R}$.

The following Lemma is a well-known result [16].

Lemma 18. *In the central decomposition of $\mathcal{R}$, almost every $\mathcal{R}_n(\omega)$ is a factor von Neumann algebra.*

The following lemma can be obtained from [5, Chapter 3 of Part II]; we include a short proof.

Lemma 19. *Suppose $\mathcal{A}_n$ is a masa of a $\mathcal{R}_n$ for $1 \leq n \leq \infty$, then there is a measurable family $\{\mathcal{A}_n(\omega)\}_{\omega \in \Omega_n}$ such that*

$$\mathcal{A}_n = \int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega),$$

where $\mathcal{A}_n(\omega)$ is a masa in $\mathcal{R}_n(\omega)$.

Proof. Suppose

$$\mathcal{W} = \left(B\left(l_n^2\right)\right) \times A \times B \times C \times E \times \mathbb{N} \times \mathbb{N},$$

where $A = B = C = \prod_{i=1}^{\infty} \text{ball}\left(B\left(l_n^2\right)\right)$ and $E = \left\{x \in l_n^2 : \|x\| = 1\right\}$. Then $\mathcal{W}$ is a complete separable metric space with product topology.

Define $\mathcal{X}_{m,k}$ to be the set of elements $(T, \{A_i\}_{i=1}^{\infty}, \{B_i\}_{i=1}^{\infty}, \{C_i\}_{i=1}^{\infty}, e, m, k)$ in $\mathcal{W}$ satisfying

$$TA_i = A_i T,\ TB_i = B_i T,\ \|(TC_m - C_m T)\, e\| \geq \frac{1}{k}, \text{ for every } i \in \mathbb{N}.$$

Then $\mathcal{X}_{m,k}$ is a closed subset of $\mathcal{W}$. We define $\mathcal{X} = \cup_{m,k=1}^{\infty} \mathcal{X}_{m,k}$, then $\mathcal{X}$ is a Borel subset of $\mathcal{W}$.

Let $\pi_{2,3,4} : \mathcal{X} \to A \times B \times C$ be the projection map. Then $\pi_{2,3,4}(\mathcal{X})$ consists of elements $(\{A_i\}_{i=1}^{\infty}, \{B_i\}_{i=1}^{\infty}, \{C_i\}_{i=1}^{\infty})$ so that there exists $T \in \text{ball}\left(B\left(l_n^2\right)\right)$ such that

$$T \in \{A_1, A_2, \ldots\}' \cap \{B_1, B_2, \ldots\}' \text{ and } T \notin \{C_1, C_2, \ldots\}'.$$

Suppose there are sequences $\{f_1, f_2, \ldots\}$, $\{\psi_1, \psi_2, \ldots\}$ and $\{g_1, g_2, \ldots\}$ contained in $L^\infty\left(\mu_n, B\left(l_n^2\right)\right)$ such that

$$\begin{aligned} &\text{ball}\mathcal{A}_n(\omega) = \{f_1(\omega), f_2(\omega), \ldots\}^{-SOT}, \\ &\text{ball}\mathcal{R}_n(\omega)' = \{\psi_1(\omega), \psi_2(\omega), \ldots\}^{-SOT}, \\ &\text{ball}\mathcal{A}_n(\omega)' = \{g_1(\omega), g_2(\omega), \ldots\}^{-SOT}. \end{aligned}$$

By Theorem 5, we know there exists an absolutely measurable function $\Upsilon : \pi_{2,3,4}(\mathcal{X}) \longrightarrow \mathcal{X}$ such that $\pi_{2,3,4} \circ \Upsilon$ is the identity function on $\pi_{2,3,4}(\mathcal{X})$.

Define $F : \Omega_n \to A \times B \times C$ by

$$F(\omega) = \{f_i(\omega)\}_{i=1}^{\infty} \times \{\psi_i(\omega)\}_{i=1}^{\infty} \times \{g_i(\omega)\}_{i=1}^{\infty}.$$

Let

$$G = F^{-1}(\pi_{2,3,4}(\mathcal{X})) = \left\{\omega : \text{there exists } T \in B\left(l_n^2\right) \text{ with } T(\omega) \in \mathcal{A}_n(\omega)' \cap \mathcal{R}_n(\omega), T(\omega) \notin \mathcal{A}_n(\omega)\right\}.$$

We know from Lemma 15 and the completeness of (Ω_n, μ_n) that G is measurable. We need to prove $\mu_n(G^c) = 0$. Suppose not, and let $\pi_1 : \mathcal{X} \to B\left(l_n^2\right)$ be the projection map (into the first coordinate). Then, by Lemma 15, $\pi_1 \circ \Upsilon \circ F|_G$ is a measurable function from G to $B\left(l_n^2\right)$. We define T by

$$T(\omega) = \begin{cases} (\pi_1 \circ \Upsilon \circ F|_G)(\omega) & \text{if } \omega \in G \\ 0 & \text{if } \omega \notin G \end{cases}.$$

Thus

$$T=\int_{G}^{\oplus}T(\omega)\,d\mu_{n}(\omega)\oplus\int_{\Omega_{n}\backslash G}^{\oplus}0d\mu_{n}(\omega),$$

then $T\in\mathcal{A}_{n}^{\prime}\cap\mathcal{R}_{n}$ and $T\notin\mathcal{A}_{n}$, which contradicts to the assumption that $\mathcal{A}_{n}$ is a masa. Therefore $\mu_{n}(G)=0$ and

$$\mathcal{A}_{n}=\int_{\Omega_{n}}^{\oplus}\mathcal{A}_{n}(\omega)\,d\mu_{n}(\omega),$$

$\mathcal{A}_{n}(\omega)$ is a masa a.e.(μ_{n}). This completes the proof. □

2.3.4. Multiplicities for type $\mathbf{I}_n$ factors. A type I factor von Neumann algebra is isomorphic to $B(H)$ for some Hilbert space H. However, if m is a cardinal, we can let $H^{(m)}$ denote a direct sum of m copies of H and, for each $T\in B(H)$ write $T^{(m)}$ be a direct sum of m copies of T acting on $H^{(m)}$, and let $B(H)^{(m)}=\{T^{(m)}:T\in B(H)\}$. Clearly, $B(H)^{(m)}$ is isomorphic to $B(H)$. The number m is called the *multiplicity of* the factor $B(H)^{(m)}$ and it is the minimal rank of a nonzero projection in $B(H)^{(m)}$. If we consider a type I von Neumann algebra acting on a separable Hilbert space as a direct integral of factors, we can change the factors so that they all have multiplicity 1. This gives another von Neumann algebra that is isomorphic to the original one. Since we are interested in finite von Neumann algebras, the type I_n algebras, with $1\leq n<\infty$, can be written as direct integrals of copies of $\mathbb{M}_{n}(\mathbb{C})$, i.e., $\int_{\Omega_{n}}^{\oplus}\mathbb{M}_{n}(\mathbb{C})\,d\mu_{n}(\omega)$ acting on $L^{2}(\mu_{n},\ell_{n}^{2})$ for some probability space (Ω_{n},μ_{n}) where μ_{n} is a Borel measure on a compact subset Ω_{n} of $\mathbb{R}$. In this case, $\int_{\Omega_{n}}^{\oplus}\mathbb{M}_{n}(\mathbb{C})\,d\mu_{n}(\omega)$ is naturally isomorphic to $\mathbb{M}_{n}(L^{\infty}(\mu_{n}))$ acting on $L^{2}(\mu_{n})^{(n)}$. When we write the type I_n part of a von Neumann algebra this way, we have an isomorphic copy, but maybe not a unitarily equivalent copy of the algebra, since we changed all of the multiplicities to be 1. Note that the center $\mathcal{Z}\left(\int_{\Omega_{n}}^{\oplus}\mathbb{M}_{n}(\mathbb{C})\,d\mu_{n}(\omega)\right)=\int_{\Omega_{n}}^{\oplus}\mathbb{C}\cdot 1\mu_{n}(\omega)$ acts on $L^{2}(\mu_{n},\ell_{n}^{2})$.

For example, if a von Neumann algebra is $\int_{E_1}^{\oplus}\mathbb{M}_{2}(\mathbb{C})\,d\eta_{1}(\omega)\oplus\int_{E_2}^{\oplus}\mathbb{M}_{2}(\mathbb{C})^{(3)}\,d\eta_{2}(\omega)$, then it is isomorphic to $\int_{\Omega}^{\oplus}\mathbb{M}_{2}(\mathbb{C})\,d\mu(\omega)$ where Ω is the disjoint union of E_1 and E_2 and $\mu(A)=\eta_{1}(A\cap E_{1})+\eta_{2}(A\cap E_{2})$.

Thus in the central decomposition, we can assume, for each positive integer n (i.e., $1\leq n<\infty$), that

$$\mathcal{R}_{n}=\int_{\Omega_{n}}^{\oplus}\mathcal{R}_{n}(\omega)\,d\mu_{n}(\omega)=\int_{\Omega_{n}}^{\oplus}\mathbb{M}_{n}(\mathbb{C})\,d\mu_{n}(\omega),$$

and

$$\mathcal{Z}(\mathcal{R}_{n})=\int_{\Omega_{n}}^{\oplus}\mathbb{C}\cdot 1d\mu_{n}.$$

For $1\leq n<\infty$ we have that the map is a normal faithful tracial state on $\mathcal{R}_{n}$.

2.3.5. II_1 von Neumann algebras. Once we have changed the multiplicities of the type I_n parts of $\mathcal{R}$, in the decomposition

$$\mathcal{R}_\infty = \int_{\Omega_\infty}^{\oplus} \mathcal{R}_\infty(\omega)\, d\mu_\infty(\omega),$$

we have

$$\mathcal{Z}(\mathcal{R}_\infty) = \int_{\Omega_\infty}^{\oplus} \mathbb{C} \cdot 1 d\mu_\infty(\omega).$$

where each $\mathcal{R}_\infty(\omega)$ must be an infinite dimensional finite factor; this means it must be a type II_1 factor, and we can assume it acts on ℓ^2. In this case making the multiplicity infinite can make things more convenient.

We let $\mathcal{R}_\infty^{(\infty)} = \left\{T^{(\infty)} = T \oplus T \oplus \cdots : T \in \mathcal{R}_\infty\right\}$. Clearly, $\mathcal{R}_\infty^{(\infty)}$ is isomorphic to $\mathcal{R}_\infty$, and we have

$$\mathcal{R}_\infty^{(\infty)} = \int_{\Omega_\infty}^{\oplus} \mathcal{R}_\infty^{(\infty)}(\omega)\, d\mu_\infty(\omega)$$

acting on $L^2\left(\mu_\infty, \left(\ell^2\right)^{(\infty)}\right)$. The nice thing about $\mathcal{R}_\infty^{(\infty)}(\omega)$ is that every normal state φ on $\mathcal{R}_\infty^{(\infty)}(\omega)$ can be written as

$$\varphi\left(T^{(\infty)}\right) = \left\langle T^{(\infty)} e, e\right\rangle$$

for some unit vector $e \in \left(\ell^2\right)^{(\infty)}$. Since $\left(\ell^2\right)^{(\infty)}$ is isomorphic to $\ell^2 = \ell^2_\infty$, we can, by replacing $\mathcal{R}_\infty$ with $\mathcal{R}_\infty^{(\infty)}$, assume that every normal state φ on $\mathcal{R}_\infty(\omega)$ can be written as

$$\varphi(T) = \langle Te, e\rangle$$

for some unit vector e. In particular, since $\mathcal{R}_\infty(\omega)$ is a II_1 factor, there is a unique normal tracial state $\tau_{\infty,\omega}$ on $\mathcal{R}_\infty(\omega)$. Hence there is a unit vector $e(\omega) \in \ell^2_\infty$ such that, for every $T \in \mathcal{R}_\infty(\omega)$,

$$\tau_{\infty,\omega}(T) = \langle Te(\omega), e(\omega)\rangle.$$

Using the measurable cross-section theorem, Theorem 5, we can choose $e(\omega)$ so that the map $e : \Omega_\infty \to \ell^2_\infty$ is absolutely measurable.

Lemma 20. *Suppose $\mathcal{R}_\infty$ is a type II_1 von Neumann algebra with*

$$\mathcal{R}_\infty = \int_{\Omega_\infty}^{\oplus} \mathcal{R}_\infty(\omega)\, d\mu_\infty(\omega).$$

Then there exists a map $e \in L^2\left(\mu_\infty, \ell^2_\infty\right)$ with $\|e(\omega)\| = 1$ a.e., such that for every $T = \int_{\Omega_\infty}^{\oplus} T_\omega d\mu_\infty(\omega) \in \mathcal{R}_\infty$, $\langle T_{\infty,\omega} e(\omega), e(\omega)\rangle = \tau_{\infty,\omega}(T_\omega)$, where $\tau_{\infty,\omega}$ is the unique normal tracial state on $\mathcal{R}_\infty(\omega)$.

Proof. Suppose

$$\mathcal{W} = \text{ball}\left(B\left(l^2_\infty\right)\right) \times \prod_{i=1}^{\infty} \text{ball}\left(B\left(l^2_\infty\right)\right) \times E,$$

where $E = \left\{x \in l^2_\infty : \|x\| = 1\right\}$. Then $\mathcal{W}$ is a complete separable metric space with product topology.

Let $\mathcal{X}$ be the set of elements $(T, \{A_i\}_{i=1}^{\infty}, e)$ in $\mathcal{W}$ satisfying

$$TA_i = A_i T, \langle A_i A_j e, e\rangle = \langle A_j A_i e, e\rangle \text{ for every } i, j \in \mathbb{N}.$$

It is easy to verify that $\mathcal{X}$ is closed.

Let $\pi_2 : \mathcal{X} \to \prod_{i=1}^{\infty} \text{ball}\left(B\left(l_{\infty}^2\right)\right)$, $\pi_3 : \mathcal{X} \to E$ be the projection maps. Then $\pi_2(\mathcal{X})$ is the set of elements $\{A_i\}_{i=1}^{\infty}$ so that there exists $T \in \text{ball}\left(B\left(l_{\infty}^2\right)\right)$ such that

$$T \in \{A_1, A_2, \dots\}' \cap \{B_1, B_2, \dots\}' \text{ and } \langle A_i A_j e, e\rangle = \langle A_j A_i e, e\rangle \text{ for all } i, j \in \mathbb{N}.$$

There is sequence $\{\psi_1, \psi_2, \dots\}$ contained in $L^{\infty}\left(\mu_{\infty}, B\left(l_{\infty}^2\right)\right)$ such that

$$\text{ball}\mathcal{R}_{\infty}(\omega)' = \{\psi_1(\omega), \psi_2(\omega), \dots\}^{-SOT}$$

By Theorem 5, we know there exists an absolutely measurable function $\Upsilon : \pi_2(\mathcal{X}) \longrightarrow \mathcal{X}$ such that $\pi_2 \circ \Upsilon$ is the identity function on $\pi_2(\mathcal{X})$.

Define $F : \Omega_{\infty} \to \Pi_{i=1}^{\infty} \text{ball}\left(B\left(l_{\infty}^2\right)\right)$ by

$$F(\omega) = \{\psi_i(\omega)\}_{i=1}^{\infty},$$

which is measurable, thus, by Lemma 15, $\pi_3 \circ \Upsilon \circ F$ is a measurable function from Ω_{∞} to l_{∞}^2. We define e by

$$e(\omega) = (\pi_3 \circ \Upsilon \circ F)(\omega).$$

Thus e is a measurable function with $e = \int_{\Omega_{\infty}}^{\oplus} e(\omega)\, d\mu_{\infty}(\omega)$, that is $e \in L^2\left(\mu_{\infty}, \ell_{\infty}^2\right)$ and

$$\|e\|_2^2 = \int_{\Omega_{\infty}} \|e(\omega)\|^2 d\mu_{\infty}(\omega) = \int_{\Omega_{\infty}} 1 d\mu_{\infty}(\omega) = \mu_{\infty}(\Omega_{\infty}) = 1.$$

□

The map

$$\tau_{\infty} : \mathcal{R}_{\infty} \to \mathbb{C}$$

defined by

$$\tau_{\infty}(T) = \langle Te, e\rangle = \int_{\Omega_{\infty}} \langle T_{\omega} e(\omega), e(\omega)\rangle\, d\mu_{\infty}(\omega) \underset{\text{def}}{=} \int_{\Omega_{\infty}} \tau_{\infty,\omega}(T_{\omega})\, d\mu_{\infty}(\omega)$$

is a faithful normal trace on $\mathcal{R}_{\infty}$. Since $\tau_{\infty,\omega}$ is a faithful normal trace on $\mathcal{R}_{\infty}(\omega)$ and the trace on a type II_1 factor is unique, it follows that $\tau_{\infty,\omega}$ is the usual trace.

2.3.6. The center-valued trace. Suppose $\mathcal{R}$ is an arbitrary finite von Neumann algebra, possibly not acting on a separable Hilbert space. There is (see [23]) a *unique* map $\Phi_{\mathcal{R}} : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ satisfying

(1) $\Phi_{\mathcal{R}}$ is linear and completely positive,
(2) $\Phi_{\mathcal{R}}(1) = 1$,
(3) $\Phi_{\mathcal{R}}(AB) = \Phi_{\mathcal{R}}(BA)$ for all $A, B \in \mathcal{R}$,
(4) $\Phi_{\mathcal{R}}$ is weak*-weak* continuous, and
(5) $\Phi_{\mathcal{R}}(ATB) = A\Phi_{\mathcal{R}}(T)B$ for all $T \in \mathcal{R}$ and all $A, B \in \mathcal{Z}(\mathcal{R})$.

The map $\Phi_{\mathcal{R}}$ is called the *center-valued trace* on $\mathcal{R}$.

When $\mathcal{R}$ acts on a separable Hilbert space, we have

$$\mathcal{R} = \sum\nolimits_{1 \leq n \leq \infty}^{\oplus} \mathcal{R}_n,$$

we have

$$\mathcal{Z}(\mathcal{R}) = \sum\nolimits_{1 \leq n \leq \infty}^{\oplus} \mathcal{Z}(\mathcal{R}_n),$$

and we have

$$\Phi_{\mathcal{R}} = \sum\nolimits_{1 \leq n \leq \infty}^{\oplus} \Phi_{\mathcal{R}_n}.$$

We can write each $\Phi_{\mathcal{R}_n}$ explicitly in terms of the central decomposition, i.e.,

$$\Phi_{\mathcal{R}_n}(T) = \int_{\Omega_n}^{\oplus} \tau_n(T_\omega) \cdot 1 d\mu_n(\omega)$$

when $1 \leq n < \infty$, and

$$\Phi_{\mathcal{R}_\infty}(T) = \int_{\Omega_\infty}^{\oplus} \tau_\omega(T_\omega) \cdot 1 d\mu_\infty(\omega).$$

It is clear that these maps satisfy the defining properties (1)-(5) and the uniqueness tells us that these formulas are correct.

2.3.7. Two simple relations. Suppose $1 \leq n \leq \infty$. There is a normal $*$-isomorphism $\gamma_n : L^\infty(\mu_n) \to \mathcal{Z}(\mathcal{R}_n)$ defined by

$$\gamma_n(f) = \int_{\Omega_n}^{\oplus} f(\omega) \cdot 1 d\mu_n(\omega).$$

Recall $\rho_n : \mathcal{R}_n \to \mathbb{C}$ is defined by

$$\rho_n(T) = \int_{\Omega_n} \tau_{n,\omega}(T_\omega) d\mu_n(\omega).$$

The map $f \mapsto \int_{\Omega_n} f d\mu_n$ is a state on $L^\infty(\mu_n)$. The simple relation between this state and the $*$-isomorphism γ_n and the state ρ_n is given by

$$(\rho_n \circ \gamma_n)(f) = \int_{\Omega_n} f d\mu_n$$

for every $f \in L^\infty(\mu_n)$.

Another simple relationship between ρ_n and $\Phi_{\mathcal{R}_n}$ is

$$\rho_n = \rho_n \circ \Phi_{\mathcal{R}_n}.$$

2.3.8. Putting things together. We let Ω be the disjoint union of $\{\Omega_n : 1 \leq n \leq \infty\}$, which can be represented as a Borel subset of $\mathbb{R}$. We define a probability Borel measure μ on Ω by

$$\mu(E) = \frac{1}{2}\mu_\infty(E \cap \Omega_\infty) + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}\mu_n(E \cap \Omega_n).$$

Then the von Neumann algebra $L^\infty(\mu)$ can be written as

$$L^\infty(\mu) = L^\infty(\mu_\infty) \oplus \sum\nolimits_{1 \leq n \leq \infty}^{\oplus} L^\infty(\mu_n)$$

We define an isomorphism

$$\gamma : L^\infty(\mu) \to \mathcal{Z}(\mathcal{R}),$$

by

$$\gamma(f_\infty \oplus f_1 \oplus f_2 \oplus \cdots) = \gamma_\infty(f_\infty) \oplus \gamma_1(f_1) \oplus \gamma_2(f_2) \cdots.$$

We can define a faithful normal tracial state $\rho : \mathcal{R} \to \mathbb{C}$ by

$$\rho\left(\sum\nolimits_{1 \leq n \leq \infty}^{\oplus} T_n\right) = \frac{1}{2}\rho_\infty(T_\infty) + \sum_{1 \leq n < \infty} \frac{1}{2^{n+1}}\rho_n(T_n).$$

We have

1. $\rho = \rho \circ \Phi_{\mathcal{R}}$,
2. $(\rho \circ \gamma)(f) = \int_\Omega f d\mu$ for every $f \in L^\infty(\mu)$, and, as we stated above,
3. $\Phi_{\mathcal{R}}(T) = \sum_{1 \leq n \leq \infty}^{\oplus} \Phi_{\mathcal{R}_n}(T_n)$
$= \left[\sum_{1 \leq n < \infty}^{\oplus} \int_{\Omega_n}^{\oplus} \tau_n(T_n(\omega)) \cdot 1 d\mu_n(\omega)\right] \oplus \int_{\Omega_\infty}^{\oplus} \tau_\omega(T_\infty(\omega)) \cdot 1 d\mu_\infty(\omega).$

3. MASAS in finite von Neumann algebras

A *masa* in a C*-algebra is a maximal abelian selfadjoint subalgebra. In $B(H)$ where H is a separable infinite-dimensional Hilbert space there are many different masas. For example, the set of all diagonal operators with respect to some fixed orthonormal basis is a discrete masa. On the other hand $L^\infty[0,1] = L^\infty(\delta_\infty)$ acting as multiplications on $L^2[0,1]$ with Lebesgue measure is also a masa that is not isomorphic to the diagonal masa, since it is *diffuse* (i.e., has no minimal nonzero projections). It was show by A. Sinclair and R. Smith [22] that in a finite von Neumann algebra acting on a separable Hilbert space all masas are isomorphic. We will prove that all masas are isomorphic in a very special way.

Theorem 6. *Suppose $\mathcal{A}$ is a masa in a finite von Neumann algebra $\mathcal{R}$. Then there is an tracial embedding $\pi_{\mathcal{A}} : L^\infty(\lambda) \to \mathcal{A}$ such that the following diagram commutes*

$$\begin{array}{ccc} L^\infty(\lambda) & \overset{\pi_{\mathcal{A}}}{\to} & \mathcal{A} \\ \downarrow \eta & & \downarrow \Phi_{\mathcal{R}} \\ L^\infty(\mu) & \overset{\gamma}{\to} & Z(\mathcal{R}) \end{array}$$

Moreover, if $\mathcal{B}$ is another masa in $\mathcal{R}$, then $\mathcal{B}$ is isomorphic to $\mathcal{A}$. In fact, $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are approximately equivalent in $\mathcal{R}$.

We first need to prove this theorem when $\mathcal{R}$ is a finite factor. When $\mathcal{R}$ is a type I_n factor, i.e., $\mathcal{R} = \mathbb{M}_n(\mathbb{C})$, the result is obvious.

Lemma 21. *Suppose $\mathcal{A} \subset \mathbb{M}_n(\mathbb{C})$ is a masa. Then there exists a unitary $U \in \mathcal{U}(\mathbb{M}_n(\mathbb{C}))$ such that $U\mathcal{A}U^* = \mathcal{D}_n$, the $n \times n$ complex diagonal matrices. Hence there is a *-isomorphism $\pi_{\mathcal{A}} : L^\infty(\delta_n) \to \mathcal{A}$ such that, for every $f \in L^\infty(\delta_n)$, which is isometrically isomorphic to $\mathbb{C}^n$.*

$$\tau_n(\pi_{\mathcal{A}}(f)) = \int_{J_n} f d\delta_n.$$

When $\mathcal{R}$ is a type II_1 factor the result is well-known [22].

Lemma 22. *Suppose $\mathcal{M}$ is a type II_1 factor von Neumann algebra acting on a separable Hilbert space with a (unique) faithful normal tracial state τ, and suppose $\mathcal{A}$ is a masa in $\mathcal{M}$. Then there is an isomorphism $\pi_{\mathcal{A}} : L^\infty(\delta_\infty) \to \mathcal{A}$ such that, for every $f \in L^\infty(\delta_\infty)$,*

$$\tau(\pi_{\mathcal{A}}(f)) = \int_0^1 f(t)\, d\delta_\infty(t).$$

Corollary 5. *Suppose $\mathcal{A}$ is an abelian von Neumann algebra on a separable Hilbert space with a faithful (tracial) state τ. The following are equivalent:*

(1) There is a tracial embedding $\pi : L^\infty(\delta_\infty) \to \mathcal{A}$ such that, for every $f \in L^\infty(\delta_\infty)$,
$$\tau(\pi(f)) = \int_0^1 f(t)\, d\delta_\infty(t).$$
(2) There is a $T \in \mathcal{A}$ such that
  (a) $W^*(T) = \mathcal{A}$
  (b) $T = T^*$
  (c) $\tau(T^n) = \frac{1}{n+1}$ for $n \in \mathbb{N}$

Moreover, if (2) holds, then $0 \le T \le 1$, then the map
$$\pi(f) = f(T)$$
is the required map in (1).

Proof. (1) $\Rightarrow$ (2). Suppose π exists as in (1). Define $f(t) = t$ in $L^\infty(\delta_\infty)$ and let $T = \pi(f)$. Then $0 \le T \le 1$,
$$\mathcal{A} = \pi(L^\infty(\delta_\infty)) = \pi(W^*(f)) = W^*(\pi(f)) = W^*(T),$$
and, for each $n \in \mathbb{N}$,
$$\tau(T^n) = \tau(\pi(f^n)) = \int_0^1 t^n dt = \frac{1}{n+1}.$$

(2) $\Rightarrow$ (1). Define the state $\rho : L^\infty(\delta_\infty) \to \mathbb{C}$ by
$$\rho(f) = \int_0^1 f(t)\, d\delta_\infty(t).$$
Letting $f \in L^\infty(\delta_\infty)$ be as above, we have $\tau(T^n) = \rho(f^n) = \frac{1}{n+1}$ for each $n \in \mathbb{N}$. It follows from Lemma 1 in [6] that there is a normal (i.e., weak*-weak*

continuous) $*$-isomorphism $\pi : L^{\infty}(\delta_{\infty}) \to \mathcal{A}$ such that $\pi(f) = T$ and such that $\tau \circ \pi = \rho$. It is clear that, for any polynomial $p(t)$, $\pi(p) = p(T)$. Suppose $f \in L^{\infty}(\delta_{\infty})$. By changing f on a set of measure 0, we can assume that f is Borel measurable. Then there is a sequence $\{p_n\}$ of polynomials such that $p_n \to f$ weak*. Thus

$$f(T) = (\text{weak*}) \lim_{n\to\infty} p_n(T) = (\text{weak*}) \lim_{n\to\infty} \pi(p_n) = \pi(f).$$

□

From this Lemma, we can see that $\pi(f) = f(T)$ and $\tau(T^n) = \tau(\pi_A(x^n)) = \int_0^1 x^n d\delta_{\infty} = \frac{1}{n+1}$ for $n = 1, 2, \cdots$. Let $\mathbb{C}_{\mathbb{Q}} = \mathbb{Q} + i\mathbb{Q}$ denote the set of all complex numbers z such that $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are rational. Since the set $\mathbb{C}_{\mathbb{Q}}[z]$ of all polynomials with coefficients in $\mathbb{C}_{\mathbb{Q}}$ is countable, we can write

$$\mathbb{C}_{\mathbb{Q}}[z] = \{p_1, p_2, \ldots\}.$$

Lemma 23. *Suppose $A = A^* \in B(H)$. It follows that*

$$W^*(A) = \{p_1(A), p_2(A), \cdots\}^{-WOT}$$

Lemma 24. *Suppose $\mathcal{A}_{\infty}$ is a masa of $\mathcal{R}_{\infty}$. Then there exists an operator $T = \int_{\Omega_{\infty}}^{\oplus} T_{\omega} d\mu_{\infty}(\omega)$ such that $W^*(T_{\omega}) = \mathcal{A}_{\infty}(\omega)$, and $\tau_{\omega,\infty}(T_{\omega}^n) = \langle T_{\omega}^n e(\omega), e(\omega)\rangle = \frac{1}{n+1}$ for $n \geq 1$, $\mathcal{A}_{\infty} = W^*(T)$.*

Proof. Let

$$\mathcal{Y} = B(l_{\infty}^2) \times \prod_{i=1}^{\infty} \text{ball}(B(l_{\infty}^2)) \times \prod_{i=1}^{\infty} \text{ball}(B(l_{\infty}^2)) \times \prod_{i=1}^{\infty} \text{ball}(B(l_{\infty}^2)) \times E,$$

where $E = \{x \in l_{\infty}^2 : \|x\| = 1\}$. It is clear that $\mathcal{Y}$ is a complete separable metric space with product topology. Let $\mathcal{X}$ be the set of tuples

$$(S, \{A_i\}_{i=1}^{\infty}, \{B_i\}_{i=1}^{\infty}, \{C_i\}_{i=1}^{\infty}, x) \in \mathcal{Y}$$

satisfying

$$SA_i = A_i S, SB_i = B_i S, \langle S^n x, x\rangle = \frac{1}{n+1} \text{ for } n \geq 1.$$

From Lemma 23, we know there exists a sequence $\{p_n\}$ of polynomials such that $W^*(S) = W^*(p_1(T), p_2(T), \ldots)$. Define $\mathcal{W}_{i,k,n}$ to be the subset of $\mathcal{X}$ satisfying

$$S = S^*, d(A_i, p_n(S)) \geq \frac{1}{k} \text{ for } n \geq 1.$$

Let $\mathcal{W}_{i,k} = \bigcap_{n=1}^{\infty} \mathcal{W}_{i,k,n}$ and $\mathcal{W} = \bigcap_{i=1}^{\infty} \bigcap_{k=1}^{\infty} \mathcal{W}_{i,k}$, then $\mathcal{W}$ is a subset of $\mathcal{X}$ satisfying

$$A_i \notin W^*(p_1(T), p_2(T), \cdots), \text{ for } i \geq 1.$$

Then $\mathcal{X} \backslash \mathcal{W} = \bigcap_{i=1}^{\infty} \bigcap_{i=1}^{\infty} \mathcal{X} \backslash \mathcal{W}_{i,k}$ is a subset of $\mathcal{X}$ satisfying

$$W^*(A_1, A_2, \cdots) \subseteq W^*(p_1(T), p_2(T), \cdots),$$

which is a G_{δ} set. By [3], there exists an equivalent metric on $\mathcal{X} \backslash \mathcal{W}$ that makes $\mathcal{X} \backslash \mathcal{W}$ a complete separable metric space. If $\pi_{2,3,4}$ is the projection map into the second, third, fourth coordinates, then there exists an absolute measurable function $\Upsilon : \pi_{2,3,4}(\mathcal{X}) \rightarrow \mathcal{X}$ such that $\pi_{2,3,4} \circ \Upsilon$ is the identity on $\pi_{2,3,4}(\mathcal{X})$.

Suppose there are sequences $\{f_1, f_2, \cdots\}$, $\{\psi_1, \psi_2, \cdots\}$, $\{\varphi_1, \varphi_2, \cdots\}$ contained in $L^\infty\left(\mu_\infty, B\left(l^2_\infty\right)\right)$ such that, for almost every ω,

$$\begin{aligned} \text{ball}\mathcal{A}_\infty(\omega) &= \{f_1(\omega), f_2(\omega), \dots\}^{-SOT}, \\ \text{ball}\mathcal{R}_\infty(\omega)' &= \{\psi_1(\omega), \psi_2(\omega), \dots\}^{-SOT}, \\ \text{ball}\mathcal{R}_\infty(\omega) &= \{\varphi_1(\omega), \varphi_2(\omega), \dots\}^{-SOT}. \end{aligned}$$

Define $F : \Omega_\infty \rightarrow \Pi_{i=1}^\infty \text{ball}\left(B\left(l^2_\infty\right)\right) \times \Pi_{i=1}^\infty \text{ball}\left(B\left(l^2_\infty\right)\right) \times \Pi_{i=1}^\infty \text{ball}\left(B\left(l^2_\infty\right)\right)$ by

$$F(\omega) = \{f_i(\omega)\}_{i=1}^\infty \times \{\psi_i(\omega)\}_{i=1}^\infty \times \{\varphi_i(\omega)\}_{i=1}^\infty.$$

Clearly F is measurable. Thus, by Lemma 15, if π_1 is the projection from $\mathcal{X} \backslash \mathcal{W}$ into its first coordinate, then $T = \pi_1 \circ \Upsilon \circ F : \omega \longmapsto T_\omega$ is the desired measurable function from Ω_∞ to $B\left(l^2_\infty\right)$ such that $\text{ball}(\mathcal{A}_\infty(\omega)) = W^*(T_\omega)$. □

Lemma 25. *Suppose $\mathcal{A}_n$ is a masa of $\mathcal{R}_n$ for every $1 \leq n \leq \infty$. Then there is an isomorphism $\pi_{\mathcal{A}_n} : L^\infty(\Omega_n \times J_n, \mu_n \times \delta_n) \rightarrow \mathcal{A}_n = \int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega)$.*

Proof. Suppose $1 \leq n < \infty$. We know that $\mathcal{R}_n$ is isomorphic to $\int_{\Omega n}^{\oplus} \mathbb{M}_n(\mathbb{C})\, d\mu_n(\omega)$, so if $\mathcal{A}_n$ is a masa in $\mathcal{R}_n$, then $\mathcal{A}_n = \int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega)$ where each $\mathcal{A}_n(\omega)$ is a masa in $\mathbb{M}_n(\mathbb{C})$. There is a unitary operator $U_\omega \in \mathbb{M}_n(\mathbb{C})$ such that $\mathcal{A}_n(\omega) = U_\omega^* \mathcal{D}_n(\mathbb{C}) U_\omega$. An easy measurable cross-section proof allows us to choose the U_ω's measurably. However, $\mathcal{D}_n$ is isomorphic to $L^\infty(J_n, \delta_n)$. Define

$$\pi_{\mathcal{A}_n} : L^\infty(\Omega_n \times J_n) \rightarrow \int_{\Omega_n}^{\oplus} L^\infty(\delta_n)\, d\mu_n(\omega)$$

by

$$\pi_{\mathcal{A}_n}(f) = \int_{\Omega_n}^{\oplus} U_\omega^* \begin{pmatrix} f\left(\omega, \frac{1}{n}\right) & & \\ & \ddots & \\ & & f\left(\omega, \frac{n}{n}\right) \end{pmatrix} U_\omega d\mu_n(\omega).$$

Now suppose $n = \infty$. We choose $\{T_\omega\}$ as in Lemma 24, and we define

$$\pi_{\mathcal{A}_\infty}(f) = \int_{\Omega_\infty}^{\oplus} f_\omega(T_\omega)\, d\mu_\infty(\omega),$$

where $f_\omega(t) = f(\omega, t)$. □

Suppose now that $\mathcal{R}$ is a finite von Neumann algebra acting on a separable Hilbert space H,

$$\mathcal{R} = [\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots] \oplus \mathcal{R}_\infty.$$

For $1 \leq n < \infty$, $\mathcal{R}_n$ is a type I_n von Neumann algebra acting on H_n, $\mathcal{R}_\infty$ is a type II_1 von Neumann algebra acting on H_∞,

$$H = [H_1 \oplus H_2 \oplus \cdots] \oplus H_\infty.$$

If $\mathcal{A}$ is a masa in $\mathcal{R}$, then, we can write

$$\mathcal{A} = [\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots] \oplus \mathcal{A}_\infty,$$

where, for $1 \leq n \leq \infty$, $\mathcal{A}_n$ is a masa in $\mathcal{R}_n$. Clearly, since $\mathcal{A}_n$ is a masa in $\mathcal{R}_n$, we know that $\mathcal{D}_n = \mathcal{Z}(\mathcal{R}_n) \subseteq \mathcal{A}_n \subseteq \mathcal{R}_n \subseteq L^\infty(\mu_n, B(H_n))$. It follows from Lemma 19 that there is a measurable family $\{\mathcal{A}_n(\omega) : \omega \in \Omega_n\}$ of von Neumann algebras such that

$$\mathcal{A}_n = \int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega).$$

If $1 \leq n < \infty$, then almost every $\mathcal{A}_n(\omega)$ must be a masa in $\mathbb{M}_n(\mathbb{C})$. If $n = \infty$, then almost every $\mathcal{A}_n(\omega)$ must be a masa in the II_1 factor $\mathcal{R}_\infty(\omega)$. Since throwing away a set of measure 0 from Ω_n doesn't change anything, we can assume that, when $1 \leq n < \infty$ every $\mathcal{A}_n(\omega)$ is a masa in $\mathbb{M}_n(\mathbb{C})$, and when $n = \infty$, every $\mathcal{A}_\infty(\omega)$ is a masa in $\mathcal{R}_\infty(\omega)$.

If $1 \leq n \leq \infty$, then each $\mathcal{A}_n(\omega)$ is isomorphic to $L^\infty(\delta_n)$ (see Lemmas 21 and 22). And $\int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega)$ is isomorphic to $\int_{\Omega_n}^{\oplus} L^\infty(\delta_n)\, d\mu_n(\omega)$, which is isomorphic to $L^\infty(\Omega_n \times J_n, \mu_n \times \delta_n)$. The isomorphism sends a function $f(\omega, t) \in L^\infty(\Omega_n \times J_n, \mu_n \times \delta_n)$ to $\int_{\Omega_n}^{\oplus} f_\omega(t)\, d\mu_n(\omega)$, where $f_\omega(t) = f(\omega, t)$.

For each n, $1 \leq n \leq \infty$, we define $\Lambda_n = \Omega_n \times J_n$ and we define $\lambda_n = \mu_n \times \delta_n$. We let Λ denote the disjoint union of the Λ_n's for $1 \leq n \leq \infty$, and we can choose Λ to be a Borel subset of $\mathbb{R}$, and we define a probability Borel measure λ on Λ by

$$\lambda(F) = \frac{1}{2}\lambda_\infty(F \cap \Lambda_\infty) + \sum_{n=1}^{\infty} \frac{1}{2^n}\lambda_n(F \cap \Lambda_n).$$

We then have

$$L^\infty(\lambda) = L^\infty(\lambda_\infty) \oplus \prod_{1 \leq n < \infty} L^\infty(\lambda_n).$$

For each n, $1 \leq n \leq \infty$, there is a mapping

$$\eta_n : L^\infty(\lambda_n) = L^\infty(\mu_n \times \delta_n) \to L^\infty(\mu_n),$$

defined by

$$\eta_n(f)(\omega) = \int_{J_n}^{\oplus} f(\omega, t)\, d\delta_n(t).$$

We define $\eta : L^\infty(\lambda) \to L^\infty(\mu)$ by

$$\eta(f) = \eta(f_\infty \oplus f_1 \oplus f_2 \oplus \cdots) = \eta_\infty(f_\infty) \oplus \eta_1(f_1) \oplus \eta_2(f_2) \oplus \cdots.$$

Lemma 26. *For $1 \leq n \leq \infty$, if $\mathcal{A}_n$ is a masa in $\mathcal{R}_n$, then there exists a tracial embedding $\pi_{\mathcal{A}_n} : L^\infty(\lambda_n) = L^\infty(\mu_n \times \delta_n) \to \mathcal{A}_n$ such that the following diagram commutes*

$$\begin{array}{ccc} L^\infty(\lambda_n) & \overset{\pi_{\mathcal{A}_n}}{\to} & \mathcal{A}_n \\ \downarrow \eta_n & & \downarrow \Phi_n \\ L^\infty(\mu_n) & \overset{\gamma_n}{\to} & Z(\mathcal{R}_n) \end{array}.$$

$$\Phi_n \circ \pi_{\mathcal{A}_n} = \gamma_n \circ \eta_n.$$

where

$$\gamma_n(f) = \int_{\Omega_n}^{\oplus} f(\omega)\, I d\mu_n(\omega),$$

$$\eta_n(f)(\omega,t) = \int_{J_n} f(\omega,t)\, d\delta_n(t) \ \ \textit{and,}$$

$$\Phi_n\left(\int_{\Omega_n}^{\oplus} T(\omega)\, d\mu_n(\omega)\right) = \int_{\Omega_n}^{\oplus} \tau_{\omega,n}(T(\omega))\, I d\mu_n(\omega).$$

Moreover, if $\mathcal{B}_n$ *is a masa in* $\mathcal{R}_n$*, and there is a tracial embedding* $\pi_{\mathcal{B}_n} : L^\infty(\lambda_n) \to \mathcal{B}_n$ *such that* $\Phi_n \circ \pi_{\mathcal{B}_n} = \gamma_n \circ \eta_n$*, then,*
if $1 \le n < \infty$*, then there exists a unitary* $U \in \mathcal{U}(\mathcal{R}_n)$ *such that*

$$U\pi_{\mathcal{A}_n}(L^\infty(\lambda_n))\, U^* = \pi_{\mathcal{B}_n}(L^\infty(\lambda_n)),$$

if $n = \infty$*, then* $\pi_{\mathcal{A}_n}$ *is approximately equivalent to* $\pi_{\mathcal{B}_n}$ *in* $\mathcal{R}_n$*.*

Proof. For $1 \le n < \infty$, we have

$$\gamma_n \circ \eta_n(f)(\omega) = \gamma_n\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\omega, \frac{k}{n}\right)\right) I = \frac{1}{n}\sum_{k=1}^{n}\int_{\Omega_n}^{\oplus} f\left(\omega,\frac{k}{n}\right) I d\mu_n(\omega),$$

and

$$\Phi_n(\pi_{\mathcal{A}_n}(f)) = \Phi_n\left(\int_{\Omega_n}^{\oplus} U_\omega^* \begin{pmatrix} f\left(\omega,\frac{1}{n}\right) & & \\ & \ddots & \\ & & f\left(\omega,\frac{n}{n}\right)\end{pmatrix} U_\omega d\mu_n(\omega)\right)$$

$$= \int_{\Omega_n}^{\oplus} \tau_n\left(U_\omega^* \begin{pmatrix} f\left(\omega,\frac{1}{n}\right) & & \\ & \ddots & \\ & & f\left(\omega,\frac{n}{n}\right)\end{pmatrix} U_\omega\right) d\mu_n(\omega)$$

$$= \int_{\Omega_n}^{\oplus} \frac{1}{n}\sum_{k=1}^{n} f\left(\omega,\frac{k}{n}\right) I d\mu_n(\omega).$$

Thus the diagram commutes. For $n = \infty$, by Lemma 24, we know there exists an operator $T = \int_{\Omega_\infty}^{\oplus} T_\omega d\mu_\infty(\omega)$ such that T_ω generates $\mathcal{A}_\infty(\omega)$ in weak operator topology with $0 \le T_\omega \le 1$ and $\tau_{\omega,\infty}(T_\omega^n) = \frac{1}{n+1}$ for $n \ge 1$. The map $\pi_{\mathcal{A}_\infty} : L^\infty(\delta_\infty) \to W^*(T) = \mathcal{A}_\infty$ is defined by $\pi_{\mathcal{A}_\infty}(f) = \int_{\Omega_\infty}^{\oplus} f_\omega(T_\omega)\, d\mu_\infty(\omega)$. Thus $\gamma_\infty \circ \eta_\infty(f)(\omega) = \left[\int_{J_n} f(\omega,t)\, d\delta_n(t)\right] I$ and $\Phi_\infty \circ \pi_{\mathcal{A}_\infty}(f)(\omega) = \tau_{\omega,\infty}(f_\omega(T_\omega))\, I = \left[\int_{J_n} f(\omega,t)\, d\delta_n(t)\right] I$. Therefore the diagram commutes. □

Combining all of these results we obtain Theorem 6.

And we also have the following corollary.

Corollary 6. *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are masas in* $\mathcal{R}$*, then the tracial embeddings* $\pi_{\mathcal{A}}, \pi_{\mathcal{B}}$ *are approximately unitarily equivalent in* $\mathcal{R}$*.*

Proof. If $\mathcal{A}$ and $\mathcal{B}$ are masas in $\mathcal{R}$, then there are tracial embeddings $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ as in Theorem 6. Thus $\Phi \circ \pi_{\mathcal{A}} = \Phi \circ \pi_{\mathcal{B}}$. By Theorem 4, we have $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are approximately unitarily equivalent in $\mathcal{R}$. □

4. Measure-preserving transformations

4.1. Basic facts

A Borel measurable map $\sigma : [0,1] \rightarrow [0,1]$ is *measure-preserving* if and only if, for every Borel set $E \subseteq [0,1]$,

$$\delta_\infty\left(\sigma^{-1}(E)\right) = \delta_\infty(E).$$

We say that $\sigma : [0,1] \rightarrow [0,1]$ is an invertible measure-preserving map if there are measure-preserving measurable maps $\sigma_1, \sigma_2 : [0,1] \rightarrow [0,1]$ such that

$$(\sigma \circ \sigma_1)(x) = x \text{ and } (\sigma_2 \circ \sigma)(x) = x, \text{ almost everywhere } (\delta_\infty).$$

In this case, let $E = \{y \in J_\infty : \sigma \circ \sigma_1(y) \neq y \text{ or } \sigma_2 \circ \sigma(y) \neq y\}$ and let S be the semigroup generated by $\sigma, \sigma_1, \sigma_2, id_{[0,1]}$. Then S is countable, thus denoted by $S = \{\widehat{\sigma}_n : n \in \mathbb{N}\}$. Suppose $F = \left(\underset{n\in\mathbb{N}}{\cup} \widehat{\sigma}_n(E)\right) \cup \left(\underset{n\in\mathbb{N}}{\cup} \widehat{\sigma}_n^{-1}(E)\right)$, it follows that $\delta_\infty(F) = 0$. and $\sigma(F) = \sigma_1(F) = \sigma_2(F) = F$. Therefore, on $J_\infty \backslash F$, the maps $\sigma, \sigma_1, \sigma_2 : J_\infty \backslash F \rightarrow J_\infty \backslash F$ are bijective, also $\sigma \circ \sigma_1 = \sigma_2 \circ \sigma$. Define $\widetilde{\sigma}$ on J_∞ by

$$\widetilde{\sigma}(y) = \begin{cases} \sigma(y) & y \in J_\infty / F \\ y & y \in F \end{cases}$$

Then $\widetilde{\sigma}$, $\widetilde{\sigma}^{-1}$ are bijective, measurable, and $\widetilde{\sigma} = \sigma$ a.e.(δ_∞). We can change σ and σ_1, σ_2 on sets of measure 0 so that $\sigma : J_\infty \rightarrow J_\infty$ is bijective and $\sigma_1 = \sigma_2 = \sigma^{-1}$ a.e.(δ_∞). In the following sections, whenever we talk about an invertible measure-preserving transformation σ on J_∞, we will mean a bijective map $\sigma : J_\infty \rightarrow J_\infty$ such that σ and σ^{-1} are measurable and measure-preserving.

Let $\mathbb{MP}[0,1] =$

$$\{\sigma | \sigma : [0,1] \rightarrow [0,1] \text{ is an invertible measurable preserving transformation}\}.$$

Clearly $(\mathbb{MP}[0,1], \circ)$ is a group.

Let $\mathcal{V}$ be all unitaries U in $\mathcal{U}\left(B\left(L^2([0,1])\right)\right)$ with $U(1) = 1$, and for all $f, g \in L^\infty[0,1]$, $U(fg) = U(f)U(g)$.

Lemma 27. *$\mathcal{V}$ is $*$-SOT closed.*

Proof. Suppose $\{U_n\} \subseteq \mathcal{V}$, and $U_n \overset{SOT}{\rightarrow} U$, $U_n^* \overset{SOT}{\rightarrow} U^*$. It is easy to see $U^*U = UU^* = 1$ and $U(1) = 1$. And we know that $U_n \overset{SOT}{\rightarrow} U$ if and only if

$$\overline{sp}\left\{f \in L^2[0,1] : \|U_n f - Uf\|_2^2 \rightarrow 0\right\} = L^2[0,1].$$

Thus there exists a subsequence $\{U_{n_k}\}$ such that for all $f, g \in L^\infty[0,1]$, $Ufg = \lim_{k\rightarrow\infty} U_{n_k}(fg) = \lim_{k\rightarrow\infty}(U_{n_k} f)(U_{n_k} g) = UfUg$, thus $U \in \mathcal{V}$. □

Corollary 7. *$\mathcal{V}$ is a complete separable, metric space in the $*$-SOT.*

Proof. Since $\mathcal{V}$ is a $*$-SOT closed subset of $\mathcal{U}\left(B\left(L^2[0,1]\right)\right)$ and $\mathcal{U}\left(B\left(L^2[0,1]\right)\right)$ is a complete separable metric space. It follows that $\mathcal{V}$ is a complete separable metric space. □

Lemma 28. *There exists a group isomorphism* $\sigma \to U_\sigma$ *from* $\mathbb{MP}[0,1]$ *onto* $\mathcal{V}$.

Proof. If $\sigma \in \mathbb{MP}[0,1]$, define $U_\sigma : L^2[0,1] \to L^2[0,1]$ by $U_\sigma f = f \circ \sigma^{-1}$. Since, for every $f \in L^2[0,1]$,

$$\|U_\sigma f\|_2^2 = \int_Y \left(f \circ \sigma^{-1}\right)^2 d\delta_\infty = \int_Y |f|^2 \circ \sigma^{-1} d\delta_\infty = \int_Y |f|^2 d\delta_\infty = \|f\|_2^2,$$

U_σ is an isometry. Since $U_{\sigma^{-1}} = U_\sigma^{-1}$, U_σ is unitary. Also $U_\sigma(fg) = (fg)\circ\sigma = (f\circ\sigma)(g\circ\sigma) = (U_\sigma f)(U_\sigma g)$ when $f, g \in L^\infty[0,1]$. Thus $U_\sigma \in \mathcal{V}$.

To prove that the map $\sigma \to U_\sigma$ is onto, we suppose $U \in \mathcal{V}$. Define $x \in L^2[0,1]$ by $x(t) = t$, and define $\gamma = U(x)$. We will show that $\gamma \in \mathbb{MP}[0,1]$. Then $U(x^n) = \gamma^n$ for all $n \geq 1$. Thus

$$\|\gamma\|_\infty = \lim \|\gamma\|_{2^n} = \lim \left[\left\|\gamma^{2^{n-1}}\right\|_2\right]^{1/2^{n-1}} = \lim \left[\left\|Ux^{2^{n-1}}\right\|_2\right]$$
$$= \left[\left\|x^{2^{n-1}}\right\|_2\right]^{1/2^{n-1}} = \|x\|_\infty = 1.$$

Also if $\gamma = u + iv$, then

$$4\int v^2 d\delta_\infty = \int \|\gamma - \bar{\gamma}\|_2^2 d\delta_\infty = \|\gamma\|_2 + \|\bar{\gamma}\|_2^2 - 2\operatorname{Re}\langle\gamma, \bar{\gamma}\rangle$$
$$= 2\|\gamma\|_2^2 - 2\left\langle\gamma^2, 1\right\rangle = 2\|x\|_2^2 - 2\int x^2 d\delta_\infty = 0.$$

Thus $\gamma = \bar{\gamma}$. Since

$$\int_0^1 \gamma^n d\delta_\infty = \int_0^1 x^n d\delta_\infty = \frac{1}{n+1}$$

for each $n \geq 1$. It follows from Corollary 5, using $\tau(f) = \int_0^1 f d\delta_\infty$, that $0 \leq \gamma \leq 1$. And the map $\pi(f) = f \circ \gamma$ is a weak*-continuous automorphism on $L^\infty([0,1])$ such that, for every $f \in L^\infty[0,1]$,

$$\int_0^1 f d\delta_\infty = \tau(\pi(f)) = \int_0^1 f \circ \gamma d\delta_\infty.$$

Thus

$$\delta_\infty\left(\gamma^{-1}(E)\right) = \int_0^1 \chi_E \circ \gamma d\delta_\infty = \delta_\infty(E).$$

Hence γ is a measure-preserving transformation on $[0,1]$. Furthermore, $U_\gamma f = f \circ \gamma$ is an isometry on $L^2([0,1])$ and equals U on the dense subset of polynomials. Thus $U = U_\gamma$. Since U_γ is unitary, $\gamma \in \mathbb{MP}[0,1]$.

Since $\mathcal{V}$ is closed in the $*$-strong operator topology ($*$-SOT), and the closed unit ball of $B\left(L^2[0,1]\right)$ is a $*$-SOT complete metric space, we know that $\mathbb{MP}[0,1]$ is a complete separable metric space with the topology $\gamma_n \to \gamma$ if and only if $U_{\gamma_n} \to U_\gamma$ in the $*$-SOT. On $\mathbb{MP}[0,1]$ this topology is called the *weak topology.* [14] The metric for the unit ball of $B\left(L^2[0,1]\right)$ is rather complicated.

For $\mathbb{MP}[0,1]$ we have a simpler metric. □

Lemma 29. $\mathbb{MP}[0,1]$ *is a complete separable metric space with the metric* d *on* $\mathbb{MP}[0,1]$ *defined by*

$$d(\gamma_1,\gamma_2) = \|\gamma_1-\gamma_2\|_2 + \left\|\gamma_1^{-1}-\gamma_2^{-1}\right\|_2$$

Proof. Suppose $d(\gamma_n, r) \to 0$, then $\|\gamma_n-\gamma\|_2 \to 0$ and $\left\|\gamma_n^{-1}-\gamma^{-1}\right\|_2 \to 0$. Thus $\left\|\gamma_n^k-\gamma^k\right\|_2 \to 0$ and $\left\|\left(\gamma_n^{-1}\right)^k-\left(\gamma^{-1}\right)^k\right\|_2 \to 0$ for every $k \geq 0$. Thus $\left\|U_{\gamma_n}x^k - U_\gamma x^k\right\|_2 \to 0$ which implies $U_{\gamma_n} \to U$ in SOT and $U_{\gamma_n}^* = U_{\gamma_n^{-1}} \to U_{\gamma^{-1}} = U_\gamma^*$ in SOT. The converse is obvious. To prove completeness, a similar argument to the one above shows that if $\{\gamma_n\}$ is d-Cauchy, then $\{U_{\gamma_n}\}$ is $*$-SOT Cauchy, so there is a $\gamma \in \mathbb{MP}[0,1]$ such that $U_{\gamma_n} \to U_\gamma$ in the $*$-SOT. Hence $\gamma_n \to \gamma$ in d. □

We now turn to our measure space (Λ,λ). We want to describe a subgroup $\mathbb{G}(\mathcal{R})$ of $\mathbb{MP}(\Lambda,\lambda)$.

Definition 6. Suppose $\sigma \in \mathbb{MP}(\Lambda_n,\lambda_n)$. Then $\sigma \in \mathbb{G}_n(\mathcal{R})$ if and only if, for every measurable $E \subset \Omega_n$,

$$\sigma(E\times J_n) \subset E\times J_n, \text{ a.e.,}$$

i.e.,

$$\lambda_n(\sigma(E\times J_n)\setminus(E\times J_n)) = 0.$$

Since it is known that

$$\sigma((\Omega_n\backslash E)\times J_n) \subset (\Omega_n\backslash E)\times J_n, \text{ a.e.,}$$

it follows that

$$\sigma(E\times J_n) = E\times J_n, \text{ a.e..}$$

This implies that $\sigma^{-1} \in \mathbb{G}_n(\mathcal{R})$. Clearly, $\mathbb{G}_n(\mathcal{R})$ is a subgroup of $\mathbb{MP}(\Lambda_n,\gamma_n)$.

Definition 7. We define $\mathbb{G}(\mathcal{R})$ to be all $\sigma \in \mathbb{MP}(\Lambda,\lambda)$ such that, for $1 \leq n \leq \infty$, $\sigma(\Lambda_n) = \Lambda_n$ and $\sigma|_{\Lambda_n} \in \mathbb{G}_n(\mathcal{R})$. We see that we can view

$$\mathbb{G}(\mathcal{R}) = \prod_{1\leq n\leq\infty} \mathbb{G}_n(\mathcal{R}),$$

as a product space.

We can express the following lemma as:

$$\mathbb{G}(\mathcal{R}) = \sum_{1\leq n\leq\infty}^{\oplus} \int_{\Omega_n}^{\oplus} \mathbb{MP}(J_n,\delta_n)\, d\mu_n(\omega) \subset \mathbb{MP}(\Lambda,\lambda).$$

In other words, every element of $\mathbb{G}(\mathcal{R})$ is a direct integral of invertible measure preserving transformations.

Lemma 30. *Suppose* $\sigma \in \mathbb{G}_n$, $1 \leq n \leq \infty$. *Then there is a measurable family* $\{\sigma_\omega : \omega \in \Omega_n\}$ *in* $\mathbb{MP}(J_n,\delta_n)$ *such that, for every* $f \in L^\infty(\Lambda_n,)$

$$(f\circ\sigma)(\omega,t) = f(\omega,\sigma_\omega(t)).$$

We write this as

$$\sigma = \int_{\Omega_n}^{\oplus} \sigma_\omega d\mu_n(\omega).$$

Proof. We can view $L^2(\Lambda_n, \lambda_n) = L^2(\Omega_n \times J_n, \mu_n \times \delta_n)$ as

$$\int_{\Omega_n}^{\oplus} L^2(J_n, \delta_n)\, d\mu_n(\omega)$$

by identifying $f \in L^2(\Omega_n \times J_n, \mu_n \times \delta_n)$ with

$$\int_{\Omega_n}^{\oplus} f_\omega d\mu_n(\omega),$$

where $f_\omega(t) = f(\omega, t)$. Fubini's theorem shows that this is an isomorphism, i.e.,

$$\|f\|_2^2 = \int_{\Omega_n \times J_n} |f(\omega,t)|^2\, d(\mu_n \times \delta_n) = \int_{\Omega_n} \int_{J_n} |f_\omega|^2\, d\delta_n(t) = \int_{\Omega_n} \|f_\omega\|^2\, d\mu_n(\omega).$$

Clearly, $U(f) = f \circ \sigma$ is a unitary operator on $L^2(\Lambda_n, \lambda_n)$. Suppose $E \subset \Omega_n$ is measurable. Then

$$P_E \underset{\text{def}}{=} \int_{\Omega_n}^{\oplus} \chi_E(\omega)\, 1 d\mu(\omega) \in \int_{\Omega_n}^{\oplus} B\left(L^2(J_n, \delta_n)\right) d\mu_n(\omega),$$

and the definition of $\sigma^{-1} \in \mathbb{G}_n(\mathcal{R})$ implies that $P_E U = U P_E$. Since the linear span of $\{\chi_E : E \subset \Omega_n, E \text{ measurable}\}$ is dense in $L^\infty(\Omega_n, \mu_n)$, we see that U is in the commutant of

$$\left\{ \int_{\Omega_n}^{\oplus} \varphi(\omega)\, 1 d\mu_n(\omega) : \varphi \in L^\infty(\Omega_n, \mu_n) \right\}.$$

Thus there is a measurable family $\{U_\omega : \omega \in \Omega_n\}$ of unitary operators in $B\left(L^2(J_n, \delta_n)\right)$ such that

$$U = \int_{\Omega_n}^{\oplus} U_\omega d\mu(\omega).$$

If $h \in L^2(J_n, \delta_n)$, we define $\hat{h} \in L^2(\Omega_n \times J_n, \mu_n \times \delta_n)$ by

$$\hat{h}(\omega, t) = h(t),$$

i.e.,

$$\hat{h} = \int_{\Omega_n}^{\oplus} h d\mu_n(\omega).$$

If $h, k \in L^\infty(J_n, \delta_n)$, then $U\left(\hat{h}\hat{k}\right) = U\left(\hat{h}\right) U\left(\hat{k}\right)$, so, for almost every $\omega \in \Omega_n$,

$$U_\omega(hk) = U_\omega(h)\, U_\omega(k).$$

Since $L^2(J_n, \delta_n)$ is separable, there is a countable set $\mathcal{E}$ whose closure in $\|\cdot\|_2$ is

$$\{h \in L^\infty(J_n, \delta_n) : \|h\|_\infty \le 1\}$$

(which is $\|\cdot\|_2$-closed). We now have for almost every $\omega \in \Omega_n$ and $h, k \in \mathcal{E}$,

$$U_\omega(hk) = U_\omega(h) U_\omega(k).$$

We can change U_ω on a set of measure 0 and assume that the above relation holds for all $\omega \in \Omega_n$. Suppose $h, g \in L^\infty(J_n, \delta_n)$ and $\|h\|_\infty, \|g\|_\infty \le 1$ and suppose $\omega \in \Omega_n$. We can choose sequences $\{h_k\}$ and $\{g_k\}$ in $\mathcal{E}$ such that $\|h_k - h\|_2 \to 0$ and $\|g_k - g\|_2 \to 0$. By replacing these sequences with appropriate subsequences, we can assume that $h_k(t) \to h(t)$, $(U_\omega h_k)(t) \to (U_\omega h)(t)$, $g_k(t) \to g(t)$, $(U_\omega g_k)(t) \to (U_\omega g)(t)$ a.e. (δ_n). It follows that $\|h_k g_k - hg\|_2 \to 0$. Thus

$$\begin{aligned} U_\omega(hg)(t) &= \lim_{k\to\infty} U_\omega(h_k g_k)(t) = \lim_{k\to\infty} (U_\omega h_k)(t)(U_\omega g_k)(t) \\ &= (U_\omega h)(t)(U_\omega g)(t). \end{aligned}$$

It follows from Lemma 28 that, for each $\omega \in \Omega_n$, there is a (unique) $\sigma_\omega \in \mathbb{MP}(J_n, \delta_n)$ such that, for every $h \in L^2(J_n, \delta_n)$,

$$U_\omega h = h \circ \sigma_\omega.$$

Our measurable cross-section theorems can be used to show that there is a measurable choice of the σ_ω's, but the uniqueness implies that the map $\omega \mapsto \sigma_\omega$ is already measurable on Ω_n. □

4.2. Nonincreasing rearrangement functions, s-functions, and Ky Fan functions

Theorem 7. *Suppose $f : \Lambda \to [0, \infty)$ is measurable. Then there is a $\sigma \in \mathbb{G}(\mathcal{R})$ such that, for $1 \le n \le \infty$, the mapping $t \mapsto (f \circ \sigma)(\omega, t)$ is nonincreasing on J_n a.e. (μ_n).*

Proof. Choose $R > \|f\|_\infty$. Suppose $1 \le n \le \infty$. Let

$$\mathcal{X} = \{(h, \sigma) \in L^\infty(\delta_n) \times \mathbb{MP}(J_n) : 0 \le h \le R, h \circ \sigma \text{ is nonincreasing on } J_n\},$$

where $\{f : 0 \le f \le R\}$ is given the $\|\cdot\|_{2,\delta_n}$-topology, $\mathbb{MP}(J_n)$ is given the weak topology, and $L^\infty(\delta_n) \times \mathbb{MP}(J_n)$ is given the product topology. (Note that if $n < \infty$, $\mathbb{MP}(J_n)$ corresponds to the set of $n \times n$ permutation matrices and has the discrete topology.) Since $\|\cdot\|_2$ convergence implies subsequential convergence almost everywhere, it follows that $\mathcal{X}$ is a complete separable metric space. Since every measurable h has a nonincreasing rearrangement, the map

$$\pi_1 : \mathcal{X} \to \{h : 0 \le h \le R\}$$

is onto, so, by Lemma 5, there is an absolutely measurable cross-section $\gamma_n : Y \to \mathcal{X}$ for π_1. Let $\eta_n = \pi_2 \circ \gamma_n : Y \to \mathbb{MP}(J_n)$.

We now define $s_n : \Omega_n \to \mathbb{MP}(J_n)$ by

$$s_n(\omega) = \eta_n(f_\omega) \in \mathbb{MP}(J_n).$$

It is clear from the construction that that $f_\omega \circ s_n(\omega)$ is a nonincreasing function of t, i.e., $f(\omega, s_n(\omega)(t))$ is a nonincreasing function of t for each $\omega \in \Omega_n$.

We define

$$\sigma_n(\omega, t) = (\omega, s_n(\omega)(t)).$$

Then $\sigma = \{\sigma_n\}_{1\leq n\leq\infty} \in \mathbb{G}(\mathcal{R})$ has the desired properties. □

Note that the function σ is not necessarily unique, but the function $f\circ\sigma$ is unique. It is called the *nonincreasing rearrangement function* for f, and we denote it by s_f. If f and h are nonnegative measurable functions on Λ, we say that f and h are $\mathbb{G}(\mathcal{R})$*-equivalent* if and only if $s_f = s_h$ a.e. (λ). This holds if and only if there is a $\sigma_1 \in \mathbb{G}(\mathcal{R})$ such that $h = f\circ\sigma_1$.

For each $\omega \in \Omega_n$ and $t \in J_n$, $s_f(\omega, t)$ is call *the* t^{th} *s-number of* f *at* ω.

Definition 8. Suppose $T \in \mathcal{R}$. We can write $T = \sum_{1\leq n\leq\infty}\int_{\Omega_n}^{\oplus} T(\omega)\, d\mu_n(\omega)$. We define $s_T \in L^\infty(\Lambda, \lambda)$ by

$$s_T(\omega, t) = s_{T(\omega)}(t)$$

when $1 \leq n \leq \infty$, $\omega \in \Omega_n$ and $t \in J_n$.

Definition 9. Suppose $f \in L^\infty(\Lambda, \lambda)$ and $0 \leq f$. For each $1 \leq n \leq \infty$, and each $\omega \in \Omega_n$, we define $f_\omega \in L^\infty(J_n, \delta_n)$ by

$$f_\omega(t) = f(\omega, t).$$

We view

$$f = \sum_{1\leq n\leq\infty}^{\oplus}\int_{\Omega_n}^{\oplus} f_\omega d\mu_n(\omega).$$

We then define $s_f \in L^\infty(\Lambda, \lambda)$ by

$$s_f(\omega, t) = s_{f_\omega}(t).$$

Lemma 31. *Suppose* $0 \leq f \in L^\infty(\Lambda, \lambda)$*. Then there is a* $\sigma \in \mathbb{G}$ *such that,* $f\circ\sigma = s_f$*.*

Proof. For $1 \leq n \leq \infty$, the map $\omega \mapsto f_\omega$ from Ω_n to $L^\infty(J_n, \delta_n)$ is measurable. For each $\omega \in \Omega_n$, there is a $\sigma_\omega \in \mathbb{MP}(J_n, \delta_n)$ such that $f_\omega \circ \sigma_\omega = s_{f_\omega}$. Using measurable cross-sections, we can choose the σ_ω's so that $\{\sigma_\omega : \omega \in \Omega\}$ is measurable. Thus $\sigma = \sum_{1\leq n\leq\infty}\int_{\Omega_n}^{\oplus}\sigma_\omega \in G$ and

$$(f\circ\sigma)(\omega, t) = f(\omega, \sigma_\omega(t)) = (f_\omega\circ\sigma_\omega)(t) = s_{f_\omega}(t) = s_f(\omega, t).$$

□

Lemma 32. *Suppose* $T \in \mathcal{R}$*,* $\mathcal{A}$ *is a masa in* $\mathcal{R}$*,* $|T| \in \mathcal{A}$*,* $\pi_{\mathcal{A}} : L^\infty(\Lambda, \lambda) \to \mathcal{A}$ *is a tracial embedding as in Theorem 6, and* $f \in L^\infty(\Lambda, \lambda)$ *satisfies* $\pi_{\mathcal{A}}(f) = |T|$*. Then* $s_T = s_f$*.*

Proof. We can write

$$\mathcal{A} = \sum_{1\leq n\leq\infty}\int_{\Omega_n}^{\oplus}\mathcal{A}_\omega d\mu_n(\omega),$$

where, for $1 \leq n \leq \infty$ and $\omega \in \Omega_n$, A_ω is a masa in $\mathcal{R}_\omega$. We can also write

$$\pi_{\mathcal{A}} = \sum_{1 \leq n \leq \infty} \int_{\Omega_n}^{\infty} \pi_\omega d\mu_n(\omega),$$

where, for each $\omega \in \Omega_n$, $\pi_\omega : L^\infty(J_n, \delta_n) \to \mathcal{A}_\omega$ is a tracial embedding. If $\pi_{\mathcal{A}}(f) = |T|$, then, for almost every ω,

$$\pi_\omega(f_\omega) = |T|(\omega) = |T_\omega|.$$

Thus, for almost every $\omega \in \Omega$,

$$s_{f_\omega} = s_{T_\omega}.$$

Thus $s_f = s_T$. □

Lemma 33. *Suppose $\mathcal{A}_1, \mathcal{A}_2$ are masas in $\mathcal{R}$, $0 \leq A_k \in \mathcal{A}_k$, $\pi_k : L^\infty(\Lambda, \lambda) \to \mathcal{A}_k$ are the isomorphisms in Theorem 6 and $f_1, f_2 \in L^\infty(\Lambda, \lambda)$ satisfy $\pi_k(f_k) = A_k$ for $k = 1, 2$. The following are equivalent:*

(1) $s_{f_1} = s_{f_2}$,
(2) *There is a $\gamma \in \mathbb{G}(\mathcal{R})$ such that $f_2 = f_1 \circ \gamma$,*
(3) *There is a sequence $\{U_n\}$ of unitary operators in $\mathcal{R}$ such that*

$$\|U_n A_1 U_n^* - A_2\| \to 0,$$

(4) *For every unitarily invariant norm α on $\mathcal{R}$*

$$\alpha(A_1) = \alpha(A_2),$$

(5) *For every rational number $t \in (0, 1]$ $KF_t(A_1) = KF_t(A_2)$.*

Proof. (1) ⇒ (2). There are $\gamma_1, \gamma_2 \in \mathbb{G}(\mathcal{R})$ such $s_{f_k} = f_k \circ \gamma_k$ for $k = 1, 2$. By (1) we have $f_2 = f_1 \circ (\gamma_1 \circ \gamma_2^{-1})$.

(2) ⇒ (3). Define $\pi_3 : L^\infty(\Lambda, \lambda) \to \mathcal{A}_2$ by

$$\pi_3(f) = \pi_2(f \circ \gamma).$$

Thus $\pi_3(f_1) = A_2$. By Theorem 4, $\pi_1 \sim_a \pi_3$. Thus there is a net $\{U_j\}$ of unitary operators in $\mathcal{R}$ such that

$$\lim_j \left\|U_j A_1 U_j^* - A_2\right\| = \lim_j \left\|U_j \pi_1(f_1) U_j^* - \pi_3(f_1)\right\| = 0.$$

Hence, for every $n \in \mathbb{N}$, there is a unitary U_n such that

$$\|U_n A_1 U_n^* - A_2\| < 1/n.$$

(3) ⇒ (4), (4) ⇒ (5) are trivial.

(5) ⇒ (1). We know that $KF_t(A_1) = KF_t(s_{f_1})$ and $KF_t(s_{f_2})$. Let

$$E_t = \{\omega \in \Omega : KF_t(s_{f_1})(\omega) \neq KF_t(s_{f_2})(\omega)\},$$

and let $E = \cup E_t$, then $\lambda(E) = 0$. Therefore $\int_0^t f_1(x)\,dx = \int_0^t f_2(x)\,dx$ for every $0 < t \leq 1$. Thus $f_1(x) = f_2(x)$ except on a countable set. Therefore $f_1 = f_2$ a.e. (δ_∞). □

Corollary 8. *Suppose $\mathcal{A}_1, \mathcal{A}_2$ are masas in $\mathcal{R}$, $0 \leq A \in \mathcal{A}_k$, $\pi_k : L^\infty(\Lambda, \lambda) \to \mathcal{A}_k$ are the isomorphisms in Theorem 6 and $f_1, f_2 \in L^\infty(\Lambda, \lambda)$ satisfy $\pi_k(f_k) = A$ for $k = 1, 2$. Then $s_{f_1} = s_{f_2}$.*

If $T \in \mathcal{R}$, we define

$$KF_t(T) = KF_t(s(f_T))$$

We need to define t^{th} *Ky Fan function* $KF_t(T)$ solely in terms of T and $\mathcal{R}$. (See Lemma 12.)

Note that when $n = \infty$, KF_t is defined on $L^\infty(J_n, \delta_n)$ for all $0 < t \leq 1$. For $1 \leq n < \infty$, KF_t is only defined when $t \in \left\{\frac{1}{n}, \ldots, \frac{n}{n}\right\}$. The next definition extends this concept.

Definition 10. Suppose $1 \leq n < \infty$ and $0 < t \leq 1$. We choose an integer k, $1 \leq k \leq n$ such that

$$\frac{k-1}{n} < t \leq \frac{k}{n}.$$

We define KF_t on $L^\infty(J_n, \delta_n)$ by

$$KF_t = KF_{\frac{k}{n}}.$$

For $f \in L^\infty(\Lambda)$ and $1 \leq n \leq \infty$ and $\omega \in \Omega_n$ and $t \in J_n$, we define

$$KF_t(f)(\omega, t) = KF_t(s_{f_\omega}),$$

and we define, for $T \in \mathcal{R}$,

$$KF_t(T) = KF_t(s_T).$$

We easily have that for $S, T \in \mathcal{R}$

$$KF_t(S+T) \leq KF_t(S) + KF_t(T)$$

always holds.

4.3. $\mathbb{G}(\mathcal{R})$-symmetric normalized Gauge norms on $L^\infty(\Lambda, \lambda)$

Suppose (Y, ν) is a probability space, and $\mathbb{G}$ is a subgroup of $\mathbb{MP}(Y, \nu)$. A norm β on $L^\infty(Y, \nu)$ is called a $\mathbb{G}$-*symmetric normalized gauge norm* if and only if

(1) $\beta(1) = 1$
(2) $\beta(f) = \beta(|f|)$ for every $f \in L^\infty(Y, \nu)$,
(3) $\beta(f \circ \sigma) = \beta(f)$ for every $f \in L^\infty(Y, \nu)$ and every $\sigma \in \mathbb{G}$.

The examples that interest us here are for $Y = \Lambda$, $\nu = \lambda$, and $\mathbb{G} = \mathbb{G}(\mathcal{R})$, i.e., the $\mathbb{G}(\mathcal{R})$-symmetric normalized gauge norms on $L^\infty(\Lambda, \lambda)$.

Suppose β is a $\mathbb{G}(\mathcal{R})$-symmetric normalized gauge norm on $L^\infty(\Lambda, \lambda)$. For every $f \in L^\infty(\Lambda, \lambda)$, we see that

$$\beta(f) = \beta(s_f).$$

4.4. Approximate Ky Fan Lemma

If $T \in \mathcal{R}$, we define

$$KF_t(T) = KF_t(s(f_T))$$

We can show that KF_t satisfies the triangle inequality on $\mathcal{R}$ by describing $KF_t(T)$ directly in terms of T. The Ky Fan Lemma is more complicated. We will apply the Ky Fan Lemmas we have throughout the direct integral. However, this is impossible to do directly as the next examples show.

Example 2. In $\mathbb{C}^n$, if $f = (1, 0, \ldots, 0)$ and $g = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, we have $KF_{\frac{k}{n}}(f) \geq KF_{\frac{k}{n}}(g)$ for $1 \leq k \leq n$, But the number N of permutations $\gamma_1, \ldots, \gamma_N$ for

$$\sum_{j=1}^{N} f \circ \gamma_j \geq g$$

must be at least n since each $f \circ \gamma_j$ is nonzero in exactly one coordinate.

Example 3. Suppose $\mathcal{R} = \mathcal{R}_2 = \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C})$ and

$$A = \sum_{1 \leq k < \infty}^{\oplus} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$$

and

$$B = \sum_{1 \leq k < \infty}^{\oplus} \begin{pmatrix} \frac{1}{2} + \frac{1}{2^k} & \\ & \frac{1}{2} - \frac{1}{2^k} \end{pmatrix}$$

Then there are no $\sigma_1, \ldots, \sigma_N \in \mathbb{G}(\mathcal{R})$ and $t_1, \ldots, t_N \in [0, 1]$ such that

$$\sum_{k=1}^{N} t_k (s_A \circ \sigma_k) \geq s_B.$$

This forces us to prove an approximate version of the Ky Fan Lemma that works universally.

Theorem 8. *Suppose m is a positive integer. Then, for $1 \leq n \leq \infty$ and for all $0 \leq f, g \leq 1$ in $L^\infty(J_n, \delta_n)$ with*

$$KF_t(f) \geq KF_t(g) \;\; \text{for all } t \in J_n$$

there are $\left\{\gamma_j : 1 \leq j \leq m^{2m}\right\} \subset \mathbb{MP}(J_n, \mu_n)$ such that

$$\frac{2}{m} + \frac{1}{m^{2m}} \sum_{j=1}^{m^{2m}} s_f \circ \gamma_j \geq s_g.$$

Proof. For $1 \leq n < \infty$, it follows from Lemma 2. For $n = \infty$, it is proved in Theorem 2. □

Corollary 9. *For $1 \leq n \leq \infty$, if $KF_t(f) \geq KF_t(g)$ for all $t \in J_n$, then $\beta(f) \geq \beta(g)$ for all symmetric gauge norms β.*

To prove the approximate Ky Fan Lemma, we need the following Lemmas.

Lemma 34. *Suppose* m, n *are positive integers.* $f = (f_1, \cdots, f_n), h = (h_1, \cdots, h_n)$*, where* $f_1, \ldots, f_n$ *and* $h_1, \ldots, h_n$ *are integers with* $1 \leq f_{i+1} \leq f_i \leq m, 1 \leq h_{i+1} \leq h_i \leq m$*.and* $\sum_{i=1}^{k} f_i \geq \sum_{i=1}^{k} h_i$*, for* $1 \leq k \leq n$*. Then there exists a positive integer* $N \leq m^{m^2}$ *and* $\gamma_1, \ldots, \gamma_N \in \mathbb{S}_n$ *such that*

$$\frac{1}{N} \sum_{i=1}^{N} f \circ \gamma_i \geq h$$

Proof. Suppose $\mathcal{S} = \left\{ \begin{pmatrix} f_k \\ h_k \end{pmatrix}, 1 \leq k \leq n \right\}$, and define an order on $\mathcal{S}$ by

$$\begin{pmatrix} f_i \\ h_i \end{pmatrix} \geq \begin{pmatrix} f_j \\ g_j \end{pmatrix} \text{ if } f_i > f_j \text{ or, } f_i = f_j \text{ and } h_i \geq h_j.$$

Then $\mathcal{S}$ is a linearly ordered set.
We say $\mathcal{S}$ is trivial if for every $\begin{pmatrix} f_k \\ h_k \end{pmatrix} \in \mathcal{S}$, $f_k \geq h_k$. If $\mathcal{S}$ is trivial, we are done, so we may assume $\mathcal{S}$ is nontrivial. Denote $\mathcal{S}_0 = \mathcal{S} \backslash \left\{ \begin{pmatrix} f_k \\ f_k \end{pmatrix}, f_k \in \{1, \cdots, m\} \right\}$.
Define $p(\mathcal{S}_0) = \max(f_k)$, $q(\mathcal{S}_0) = \max\{f_k, \text{ with } h_k > f_k\}$, where $p(\mathcal{S}_0), q(\mathcal{S}_0) \in \{f_1, \cdots, f_n\}$, we may assume $p(\mathcal{S}_0) = f_p, q(\mathcal{S}_0) = f_q$. Then denote $l(\mathcal{S}_0) = p(\mathcal{S}_0) - q(\mathcal{S}_0)$. It is not hard to see that $f_p > h_p \geq h_q > f_q$, so $f_p - f_q \geq 2$.
Let $\gamma_{p,q}$ be the permutation that permute f_p with f_q and leave all other f_i's fixed,
define $f^{(1)} = \left(f_1^{(1)}, \cdots, f_n^{(1)}\right) = \frac{1}{l(\mathcal{S}_0)} \left[(h_p - f_q) f + (f_p - h_p) f \circ \gamma_{p,q}\right]$, where $f^{(1)} \in \mathbb{N}^n$. Then denote $\mathcal{S}^{(1)} = \left\{ \begin{pmatrix} f_k^{(1)} \\ h_k \end{pmatrix}, 1 \leq k \leq n \right\}$, $\mathcal{S}_0^{(1)} = \mathcal{S}^{(1)} \backslash \left\{ \begin{pmatrix} f_k^{(1)} \\ f_k^{(1)} \end{pmatrix} \right\}$, we form linear convex combination of f_i's this way and update f with $f^{(1)}, \cdots, f^{(r)}$ until $l\left(\mathcal{S}_0^{(r)}\right) < l(\mathcal{S}_0)$. We can also see that $l(\mathcal{S}_0) < m$, and $r < m$, so we need at most m^m permutations to reduce $l(\mathcal{S}_0)$ for 1. Repeating this process, we need at most $\left(m^{m^2}\right)$ permutations to reduce $\mathcal{S}_0$ to a trivial set. Note that we can make the number of permutations is exactly $\left(m^{m^2}\right)!$, some permutations are duplicate.
Therefore, there exists a positive integer $N = \left(m^{m^2}\right)!, \gamma_1, \cdots, \gamma_N \in \mathbb{S}_n$ such that

$$\frac{1}{N} \sum_{i=1}^{N} f \circ \gamma_i \geq h.$$

□

Lemma 35. *Suppose* m, n *are positive integers, then there exists a positive integer* $N \leq m^{m^2}$ *such that for all* $f = (f_1, f_2, \ldots, f_n)$ *and* $h = (h_1, \ldots, h_n)$

with $1 \geq f_1 \geq \cdots \geq f_n \geq 0$, $1 \geq h_1 \geq \cdots \geq h_n \geq 0$, *and* $\sum_j^{i=1} f_i \geq \sum_j^{i=1} h_i$ *for all* $1 \leq j \leq n$, *there exist* $\gamma_1, \ldots, \gamma_N \in \mathbb{S}_n$ *such that*

$$\frac{1}{N}\sum_{i=1}^{N} f \circ \gamma_i + \frac{2}{m} \geq h.$$

Proof. For all $1 \leq i \leq n$, if $\frac{k-1}{m} < f_i \leq \frac{k}{m}$ for some $k \in \mathbb{N}$, then define $\widetilde{f_i} = \frac{k}{m}$ and if $\frac{k-1}{m} \leq h_i < \frac{k}{m}$ for some $k \in \mathbb{N}$, then define $\tilde{h}_i = \frac{k-1}{m}$. Let $\widetilde{f} = \left(\widetilde{f}_1, \cdots, \widetilde{f}_n\right)$ and $\widetilde{h} = \left(\widetilde{h}_1, \cdots, \widetilde{h}_n\right)$. It is easy to check that $f_i \leq \widetilde{f_i} \leq f_i + \frac{1}{m}$ and $\max\left(h_i - \frac{1}{m}, 0\right) \leq \widetilde{h}_i \leq h_i$ for all $1 \leq i \leq n$. From Lemma 34, we know there exists a positive integer N and $\gamma_1, \cdots, \gamma_N \in \mathbb{S}_n$ such that $\frac{1}{N}\sum_{i=1}^{N}\left(m\widetilde{f}\right) \circ \gamma_i \geq \left(m\widetilde{h}\right)$. Therefore, $\frac{1}{N}\sum_{j=1}^{N} f \circ \gamma_j + \frac{2}{m} \geq h$. ☐

The following is the *Approximate Ky Fan Lemma.*

Theorem 9. *If* $f, g \in L^\infty(\Lambda, \lambda)$, $m \in \mathbb{N}$, $m \geq 2$, *and* $0 \leq f, g \leq 1$ *and* $KF_t(f) \geq KF_t(g)$ *a.e.* (μ) *for each rational number* $t \in (0, 1]$, *then there are* $\sigma_1, \ldots, \sigma_{(m!)^{m^5} m^{m^2}} \in \mathbb{G}(\mathcal{R})$ *such that*

$$\frac{1}{(m!)^{m^5} m^{m^2}} \sum_{k=1}^{(m!)^{m^5} m^{m^2}} f \circ \sigma_k + \frac{1}{m} \geq g.$$

Thus, for every $\mathbb{G}(\mathcal{R})$*-symmetric normalized gauge norm* β *on* $L^\infty(\Lambda, \lambda)$,

$$\beta(f) \geq \beta(g).$$

Proof. Suppose $f, g \in L^\infty(\Lambda, \lambda)$. Since there are $\sigma_1, \sigma_2 \in \mathcal{G}(\mathcal{R})$ such that $s_f = f \circ \sigma_1$ and $s_g = g \circ \sigma_2$, we can assume $f = s_f$ and $g = s_g$. We know f, g can be viewed as $f = \sum_{1 \leq n \leq \infty}^{\oplus} f_n = \sum_{1 \leq n \leq \infty}^{\oplus} \int_{\Omega_n}^{\oplus} f_{n,\omega} d\mu_n(\omega)$ and $g = \sum_{1 \leq n \leq \infty}^{\oplus} \int_{\Omega_n}^{\oplus} g_{n,\omega} d\mu_n(\omega)$. Suppose $m \in \mathbb{N}$ and $m \geq 2$. For $1 \leq n \leq \infty$, let $\mathcal{X}_n$ be the set of tuples $\left(F, G, \sigma_1, \sigma_2, \cdots, \sigma_{m^{m^2}}\right)$ satisfying $\frac{1}{m^{m^2}} \sum_{k=1}^{m^{m^2}} F \circ \sigma_k + \frac{1}{m} \geq G$, where $0 \leq f, g \leq 1$. Then $\mathcal{X}$ is a closed subset of ball $(L^\infty(J_n, \delta_n)) \times \text{ball}(L^\infty(J_n, \delta_n)) \times \Pi_{i=1}^{m^{m^2}} \mathbb{MP}(\Lambda, \lambda)$, which is a complete separable metric space with the $\|\cdot\|_2$ on ball$(L^\infty(J_n, \delta_n))$. Then by Theorem 5 the projection onto ball$(L^\infty(J_n, \delta_n)) \times$ball$(L^\infty(J_n, \delta_n))$ has an absolutely measurable range $\mathcal{Y}_n$ and an absolutely measurable cross-section ψ and we let ψ_k be the composition of projection onto the coordinate of σ_k with ψ for $1 \leq k \leq \left(m^{m^2}\right)!$. If $1 \leq n < \infty$, it follows from Lemma 35 and Theorem 2 that

$$(s_{f_\omega}, s_{g_\omega}) \in \mathcal{Y}_n$$

for almost all $\omega \in \Omega_n$. We define, for $1 \leq k \leq \left(m^{m^2}\right)!$, $\sigma_k(\omega) \in \mathbb{MP}(J_n, \delta_n)$ by

$$\sigma_k(\omega) = \psi_k(s_{f_\omega}, s_{g_\omega}).$$

This gives $\sigma_1, \ldots \sigma_{(m^{m^2})!} \in \mathbb{G}(\mathcal{R})$ such that

$$\frac{1}{(m^{m^2})!}\sum_{k=1}^{(m^{m^2})!} s_f \circ \sigma_k + \frac{1}{m} \geq s_g.$$

If follows that, for any $\mathbb{G}(\mathcal{R})$-symmetric normalized gauge norm β on $L^\infty(\Lambda, \lambda)$ that

$$\begin{aligned}\beta(g) = \beta(s_g) &\leq \frac{1}{(m^{m^2})!}\sum_{k=1}^{(m^{m^2})!} \beta(s_f \circ \sigma_k) + \beta\left(\frac{1}{m}\right) \\ &= \frac{1}{(m^{m^2})!}\sum_{k=1}^{(m^{m^2})!} \beta(f) + \frac{1}{m} = \beta(f) + \frac{1}{m}.\end{aligned}$$

Since $m \geq 2$ was arbitrary, it follows that $\beta(g) \leq \beta(f)$. □

5. Main theorem

We are finally ready to prove our main theorem.

Theorem 10. *Suppose $\mathcal{R}$ is a finite von Neumann algebra acting on a separable Hilbert space H. Let the probability space $(\Lambda, \Sigma, \lambda)$ and the group $\mathbb{G} \leq \mathbb{MP}(\Lambda, \Sigma, \lambda)$ be as above. Then there is a natural 1-1 correspondence between the normalized unitarily invariant norms on $\mathcal{R}$ and the normalized $\mathbb{G}$-symmetric gauge norms on $L^\infty(\Lambda, \lambda)$.*

Proof. Suppose α is a normalized unitarily invariant norm on $\mathcal{R}$, choose any masa $\mathcal{A}$ in $\mathcal{R}$, and choose a tracial embedding $\pi_{\mathcal{A}} : L^\infty(\Lambda, \lambda) \to \mathcal{A}$ as in Theorem 6. Define $\beta_\alpha : L^\infty(\lambda) \to \mathbb{R}$ by

$$\beta_\alpha(f) = \alpha(\pi_{\mathcal{A}}(f)),$$

If $\mathcal{B}$ is another masa in $\mathcal{R}$ and $\pi_{\mathcal{B}} : L^\infty(\Lambda, \lambda) \to \mathcal{B}$ is as in Theorem 6, we see from Theorem 6 that, if $\Phi : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ is the center-valued trace on $\mathcal{R}$, then

$$\Phi \circ \pi_{\mathcal{A}} = \Phi \circ \pi_{\mathcal{B}}.$$

Thus, by Theorem 6, $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are approximately equivalent in $\mathcal{R}$. Hence, there is a net $\{U_j\}$ in $\mathcal{U}(\mathcal{R})$ such that, for every $f \in L^\infty(\Lambda, \lambda)$,

$$\left\| U_j^* \pi_{\mathcal{A}}(f) U_j - \pi_{\mathcal{B}}(f) \right\| \to 0.$$

It follows from Lemma 2 that, for every $f \in L^\infty(\Lambda, \lambda)$,

$$\alpha(\pi_{\mathcal{A}}(f)) = \alpha(\pi_{\mathcal{B}}(f)).$$

Thus the definition of β_α is independent of choice of the masa $\mathcal{A}$ and tracial embedding $\pi_{\mathcal{A}}$. It is easy to check that β_α is norm. To prove β_α is $\mathbb{G}(\mathcal{R})$-symmetric, suppose $\sigma \in \mathbb{G}(\mathcal{R})$. Then, by Lemma 30, there is a measurable family $\{\sigma_\omega : \omega \in \Omega\}$ with each $\omega \in \Omega_n$, such that $\sigma_\omega \in \mathbb{MP}(J_n, \mu_n)$ and

$$\sigma = \int_\Lambda^\oplus \sigma_\omega d\lambda(\omega).$$

Thus, by Theorem 6,

$$\Phi_n\left(\pi_{\mathcal{A}}\left(f\circ\sigma\right)\right)=\gamma\circ\eta\left(f\circ\sigma\right),$$

but

$$\begin{aligned}\eta\left(f\circ\sigma\right)\left(\omega\right)&=\int_{J_n}\left(f\circ\sigma\right)\left(t,\omega\right)d\delta_n\left(t\right)\\&=\int_{J_n}f_\omega\left(\sigma_\omega\left(t\right)\right)d\delta_n\left(t\right)=\int_{J_n}f_\omega\left(t\right)d\delta_n\left(t\right)=\eta\left(f\right)\left(\omega\right).\end{aligned}$$

Thus, for every $f\in L^\infty\left(\Lambda,\lambda\right)$,

$$\Phi\circ\pi_{\mathcal{A}}\left(f\right)=\Gamma\left(\eta\left(f\right)\right)=\Gamma\left(\eta\left(f\circ\sigma\right)\right)=\Phi\circ\pi_{\mathcal{A}}\left(f\circ\sigma\right).$$

Thus, $\rho\left(f\right)=\pi_{\mathcal{A}}\left(f\circ\sigma\right)$ is a tracial embedding as in Theorem 6, which implies ρ is approximately equivalent to $\pi_{\mathcal{A}}$. Hence, by Lemma 2, for every $f\in L^\infty\left(\Lambda,\lambda\right)$, we have

$$\beta_\alpha\left(f\right)=\alpha\left(\pi_{\mathcal{A}}\left(f\right)\right)=\alpha\left(\pi_{\mathcal{A}}\left(f\circ\sigma\right)\right)=\beta_\alpha\left(f\circ\sigma\right).$$

Thus β_α is a normalized $\mathbb{G}\left(\mathcal{R}\right)$-invariant gauge norm on $L^\infty\left(\Lambda,\lambda\right)$.

Conversely, suppose β is a normalized $\mathbb{G}\left(\mathcal{R}\right)$-symmetric gauge norm on $L^\infty\left(\Lambda,\lambda\right)$. If $T\in\mathcal{R}$, then $W^*\left(|T|\right)$ is abelian and is contained in a masa $\mathcal{A}$ of $\mathcal{R}$. By Theorem 6 there is a tracial embedding $\pi_{\mathcal{A}}:L^\infty\left(\Lambda,\lambda\right)\to\mathcal{A}$ such that, for every $f\in L^\infty\left(\Omega,\mu\right)$,

$$\tau\left(\pi_{\mathcal{A}}\left(f\right)\right)=\int_\Omega fd\mu.$$

Choose $0\leq f\in L^\infty\left(\Lambda,\lambda\right)$ with $\pi_{\mathcal{A}}\left(f\right)=|T|$. Then we define

$$\alpha_\beta\left(T\right)=\beta\left(f\right)=\beta\left(\pi_{\mathcal{A}}^{-1}\left(|T|\right)\right).$$

Suppose $\mathcal{B}$ is another masa in $\mathcal{R}$ with $|T|\in\mathcal{B}$. Then there is a tracial embedding $\pi_{\mathcal{B}}:L^\infty\left(\Lambda,\lambda\right)\to\mathcal{B}$ and an $0\leq h\in L^\infty\left(\Lambda,\lambda\right)$ with $\pi_{\mathcal{B}}\left(h\right)=|T|$. It follows from Lemma 32 that

$$s_f=s_T=s_h.$$

Hence, by Lemma 33, there is a $\sigma\in\mathbb{G}\left(\mathcal{R}\right)$ such that

$$h=f\circ\sigma.$$

Thus

$$\alpha\left(h\right)=\alpha\left(f\right)=\alpha\left(s_T\right).$$

Thus the definition of $\alpha_\beta\left(T\right)=\beta\left(s_T\right)$ is independent of the masa $\mathcal{A}$ or the tracial embedding $\pi_{\mathcal{A}}$. At this point it is easy to see that $\beta_{\alpha_\beta}=\beta$ holds for a $\mathbb{G}\left(\mathcal{R}\right)$-symmetric normalized gauge norm on $L^\infty\left(\Lambda,\lambda\right)$.

If U and V are unitaries in $\mathcal{R}$, then, by Lemma 32,

$$s_{UTV}=s_T.$$

Thus $\alpha_\beta\left(UTV\right)=\alpha_\beta\left(T\right)$ by Lemma 33. Thus α_β is unitarily invariant.

Clearly, $\alpha_\beta\left(1\right)=1$ and $\alpha_\beta\left(zT\right)=|z|\,\alpha_\beta\left(T\right)$. To show α_β is a norm, we just need to check the triangle inequality. Suppose $A,B\in\mathcal{R}$. Let $h=$

$s_A + s_B$. Since, for almost every $\omega \in \Omega$ the functions $s_A(\omega, t)$ and $s_B(\omega, t)$ are nonincreasing in t, we see that

$$s_h = h = s_A + s_B.$$

Thus, we have, if $\omega \in \Omega_n$, $n \in \mathbb{N}$, and $t = k/n$ with $1 \leq k \leq n$, or if $\omega \in \Omega_\infty$ and $0 < t \leq 1$ is rational, then, for almost every ω,

$$\begin{aligned} KF_t(s_h)(\omega) &= KF_t(s_A + s_B)(\omega) = KF_t(s_A)(\omega) + KF_t(s_B)(\omega) \\ &= KF_t(A)(\omega) + KF_t(B)(\omega) \geq KF_t(A+B)(\omega) = KF_t(s_{A+B})(\omega). \end{aligned}$$

It follows from the approximate Ky Fan Lemma (Theorem 9) that

$$\beta(h) \geq \beta(s_{A+B}),$$

which means

$$\alpha_\beta(A+B) \leq \beta(h) = \beta(s_A + s_B) \leq \beta(s_A) + \beta(s_B) = \alpha_\beta(A) + \alpha_\beta(B).$$

This completes the proof. □

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