Acta Scientiarum Mathematicarum (2022) 88:577–580
https://doi.org/10.1007/s44146-022-00041-y

On a result of Fujita and Le

Adel Alahmadi and Florian Luca

Abstract. In this paper, we improve a result of Fujita and Le concerning the Diophantine equation $x^2 + (2c-1)^y = c^z$.

Jeśmanowicz conjectured that if a, b, c form a Pythagorean triple $a^2+b^2 = c^2$ then the only positive integer solution (x, y, z) of the equation $a^x+b^y = c^z$ is $(x, y, z) = (2, 2, 2)$. He proved his conjecture for a few particular Pythagorean triples. Motivated by this conjecture, Terai made additional conjectures about solutions to three terms exponential Diophantine equations with a couple of fixed bases and variable exponents. In [6], he conjectured that, assuming again that a, b, c satisfy $a^2 + b^2 = c^2$, the only solution in positive integers (x, y, z) of the equation $x^2 + b^y = c^z$ is $(a, 2, 2)$. Later, in [7], he conjectured that if $c > 1$ is a positive integer then the only positive integer solution (x, y, z) of the Diophantine equation $x^2 + (2c-1)^y = c^z$ is $(x, y, z) = (c-1, 1, 2)$. Regarding this last conjecture, Fujita and Le [2] put

$$\mathcal{T} := \{c > 1 : \text{Terai's conjecture from [7] is false}\}$$

and $\mathcal{T}(N) := \mathcal{T} \cap [1, N]$ and prove that $\#\mathcal{T}(N) < 0.22N$ holds for all $N > N_0$. Here we improve this as follows.

Theorem 1. *We have*

$$\#\mathcal{T}(N) \ll \frac{N}{\sqrt{\log N}}.$$

Proof. Assume that c is such that $x^2 + (2c-1)^y = c^z$ for some $(y, z) \neq (1, 2)$. If y is even, then c is a sum of two coprime squares. By a result of Landau (see [1], vol 2, p. 649–661 and [8]) the set of such $c \leq N$ has cardinality $O(N/\sqrt{\log N})$. Assume that z is odd and let p be any prime factor of $2c-1$. Then $x^2 \equiv c(c^{(z-1)/2})^2 \pmod{p}$, so $(c|p) = 1$, where the above notation is the Legendre symbol. Since also $c \equiv 2^{-1} \pmod{p}$, it follows that $(2|p) = 1$. Thus, $p \equiv 1, 7 \pmod{8}$. Since this holds for all prime factors p of $2c-1 < 2N$, we deduce, again by Landau's result, that the number of such c is $O(N/\sqrt{\log N})$. So, we assume that z is even so we get $(2c-1)^y = c^z - x^2 = (c^{z/2} - x)(c^{z/2} + x)$

Article history: received 17.9.2021, revised 22.10.2021, accepted 28.10.2021.
AMS Subject Classification: 11D41, 11N37.
Key words and phrases: exponential Diophantine equations.

and the factors on the right are coprime. Hence, there exist divisors d, m of $2c-1$ with $dm = 2c-1$ such that $d^y = c^{z/2} - x$ and $m^y = c^{z/2} + x$. Thus, $2c^{z/2} = d^y + m^y$. Since y is odd, we get that $d+m$ divides $2c^{z/2}$. We distinguish two cases.

Case 1. *One of* d, m *equals* 1.

Then $1 + (2c-1)^y = 2c^{z/2}$. Note that $z > y$ from the original equation. We get

$$\frac{1}{2c^{z/2}} = |(2c-1)^y 2^{-1} c^{-z/2} - 1|.$$

The expression on the left is nonzero. By a linear form in logarithms á la Baker (see, for example, Matveev's version [4]), the right hand side exceeds

$$\exp(-C_1 \log c \log(2c-1) \log z)$$

with some positive absolute constant C_1. Hence, we get that

$$(z/2) \log c < C_1 \log c \log(2c-1) \log z$$

giving $z < 2C_1 \log(2N) \log z$. In particular, $z < C_2 (\log N)^2$ for some absolute constant C_2. Since $y < z$, the pair of exponents $(y, z/2)$ can be fixed in at most $C_3 (\log N)^4$ ways. For each one of them, the expression

$$P(X) := 2X^{z/2} - (2X-1)^y - 1$$

is a fixed polynomial in X which is nonzero. Indeed, if it were 0 then we would get by looking at the largest possible monomials and their coefficients that $2 = 2^y$ and $z/2 = y$, whence $y = z/2 = 1$, but this is excluded. Thus, c is a zero of one of the above polynomials whose degree is at most $C_2(\log N)^2$. Hence, the number of such c is at most $C_4 (\log N)^6$ which is $o(N/\sqrt{\log N})$ as $N \to \infty$.

Case 2. $\min\{d, m\} > 1$.

Clearly, $\min\{d, m\} < \sqrt{2N}$. Assume first that every prime factor of $d + m = d + (2c-1)/d$ is at most $p(N) := (\log N)^{3/2}$. Thus, $d + (2c-1)/d$ belongs to the set $C(N) := \{n \le 2N : P(n) \le p(N)\}$, where $P(n)$ is the largest prime factor of n with the convention that $P(1) = 1$. The cardinality of the set $\{n \le x : P(n) \le y\}$ is denoted $\Psi(x, y)$ and has been intensively studied. A result of de Bruijn (see Theorem 2 in Chapter III.5 in [5]) shows that uniformly in $x \ge y \ge 2$, we get

$$\log \Psi(x, y) = Z\Big(1 + O\Big(\frac{1}{\log y} + \frac{1}{\log\log 2x}\Big)\Big),$$

where

$$Z := \frac{\log x}{\log y} \log\Big(1 + \frac{y}{\log x}\Big) + \frac{y}{\log y} \log\Big(1 + \frac{\log x}{y}\Big).$$

For us $x = 2N,\ y = p(N) = (\log N)^{3/2}$ so $(\log x)/y$ tends to 0 so the second term is

$$\frac{y}{\log y}\log\left(1+\frac{\log x}{y}\right) = O\Big(\frac{\log x}{\log y}\Big) = o(\log N)$$

as N tends to infinity while the first term is

$$\frac{\log(2N)}{(3/2)\log\log N}\log\left(1+\frac{(\log N)^{3/2}}{\log(2N)}\right) = (1/3+o(1))\log N$$

as N tends to infinity. Hence,

$$\Psi(2N, p(N)) = \exp((1/3+o(1))\log N) = N^{1/3+o(1)} \qquad \text{as} \qquad N\to\infty.$$

Since $\min\{d,m\} \le \sqrt{2N}$ and $2c-1$ is uniquely determined by $\min\{m,d\}$ and a number in $C(N)$, it follows that the number of such $c \le N$ is at most $N^{1/2+1/3+o(1)} = o(N/\sqrt{\log N})$ as $N \to \infty$.

Finally, assume now that there exists a prime p such that $p > (\log N)^{3/2}$ and $p \mid d + (2c-1)/d$. Then also $p \mid c$, so $d - d^{-1} \equiv 0 \pmod p$ showing that $d \equiv \pm 1 \pmod p$. Thus, one of d, m is congruent to 1 modulo p and the other is congruent to -1 modulo p. Hence, $2c-1 = (p\ell_1+1)(p\ell_2-1)$ for some positive integers ℓ_1, ℓ_2. Fixing ℓ_2, we get

$$p\ell_1 < p\ell_1 + 1 < \frac{2N}{p\ell_2 - 1} \le \frac{4N}{p\ell_2}.$$

Thus, fixing p and ℓ_2, the number of choices for ℓ_1 is $\le 4N/(p^2\ell_2)$. We now sum the above estimates over $\ell_2 \le N$ and $p > (\log N)^{3/2}$, getting the number of such c is at most

$$4N \sum_{p>(\log N)^{3/2}} \frac{1}{p^2} \sum_{\ell_2 \le 2N} \frac{1}{\ell_2} \ll \frac{N\log N}{(\log N)^{3/2}\log\log N} = o\Big(\frac{N}{\sqrt{\log N}}\Big).$$

This finishes the proof. ∎

Acknowledgments. We thank the referee for suggestions which improved the quality of our manuscript. This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah under grant No. (DF-595-130-1442). The authors gratefully acknowledge DSR technical and financial support. The second author was also supported by the ANR grant JINVARIANT.

References

[1] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen (2 vols.)*, Teubner, Leipzig; 3rd edition: Chelsea, New York 1974.

[2] Y. Fujita and M. H. Le, On a conjecture concerning the generalized Ramanujan–Nagell equation, *J. Combinatorics and Number Theory*, **12** (2020), No. 1.

[3] L. Jeśmanowicz, Kilka uwag o liczbach pitagorejskich [Some remarks on Pythagorean numbers], *Wiadom. Mat.*, **1** (1956), 196–202.

[4] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, *Izv. Math.*, **64** (2000), 1217–1269.

[5] G. Tenebaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press, 1995.

[6] N. Terai, The Diophantine equation $x^2 + q^m = p^n$, *Acta Arith.*, **63** (1993), 351–358.

[7] N. Terai, A note on the Diophantine equation $x^2 + q^m = c^n$, *Bull. Australian Math. Soc.*, **90** (2014), 20–27.

[8] E. Wirsing, Über die Zahlen, deren Primteiler einer gegebenen Menge angehören, *Arch. der Math.*, **7** (1956), 263–272.

A Alahmadi, Research Group in Algebraic Structures and its Applications, King Abdulaziz University, Jeddah, Saudi Arabia

F. Luca, School of Maths, Wits University, 1 Jan Smuts, Brammfontein 2000, Johannesburg, South Africa, Institut Mathematique de Bordeaux, Université de Bordeaux, 351 cours de la Libération, 33 405 Talence cedex, France and Research Group in Algebraic Structures and Applications, King Abdulaziz University, Abdulah Sulayman, Jeddah 22254, Saudi Arabia; *e-mail*: florian.luca@wits.ac.za