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REVIEW ARTICLE

Positive Solutions and Estimates for the Poisson and Martin Kernels for the Time-Independent Schrödinger Equation

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Abstract

In this expository paper, we describe a sequence of earlier papers presenting applications of a general theorem regarding pointwise estimates for kernels of Neumann series operators $\sum_{j=0}^{\infty} T^j$. Here T is an integral operator with a quasi-metric kernel on a measure space (Ω, ω), with $\|T\|_{L^2(\omega)\to L^2(\omega)} < 1$. Applications are made to the study of non-negative solutions u to the time-independent Schrödinger equation $-\triangle u = qu$ on a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, with $u = f$ on $\partial\Omega$, where $q \in L^1_{loc}(\Omega)$ and q and f are non-negative. We obtain a balayage condition on the potential q measuring how rapidly q can blow up at $\partial\Omega$ and still allow for an almost everywhere finite solution. We also derive bilateral estimates for the Green's function and Poisson kernel for the Schrödinger operator $-\triangle - q$ in terms of q and the Green's function and Poisson kernel for the Laplacian. These results are first described for a C^2 domain. They are later extended to analogues involving the Martin kernel and harmonic measure on a uniform domain.

Keywords Schrödinger equation · Uniform domain · Harmonic measure · Martin's kernel · Gauge

Mathematics Subject Classification Primary 35R11 · 31B35 · Secondary 35J10

Contents

For David Adams, with gratitude for his encouragement and support.

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1 Introduction

This paper is an exposition of a sequence of papers [9–11]. We hope to obtain a perspective in retrospect that gives a greater clarity to the progression of these papers. This article is meant for non-specialists. The reader will be referred to the original papers for the more technical aspects of the proofs, and some context and background will be presented in more detail than in an article for specialists.

We consider an open, bounded, connected domain $\Omega \subseteq \mathbb{R}^n$ where, for simplicity, $n \geq 3$ (some modifications are necessary for $n = 2$). Let $q \in L^1_{loc}(\Omega)$ with $q \geq 0$ and suppose $f : \partial\Omega \to [0, \infty)$ is Borel measurable. We consider the problem of finding solutions u to

$$\begin{cases} -\triangle u = qu & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ u \geq 0. \end{cases} \tag{1.1}$$

The function q is the potential for the time-independent Schrödinger operator $-\triangle - q$. One can generalize without major difficulties by replacing q with a measure ω, but again for expository simplicity, we consider only $q \in L^1_{loc}$ here. If $q \in L^\infty(\Omega)$, this problem fits into the standard elliptic theory, as, for example, in Chapter 6 of [6]. Our main interest is in relaxing the condition on q, to see, for example, how large q can be at $\partial\Omega$ and still allow the existence of a solution to (1.1). In the case where $f = 1$, the solution of (1.1) is called the Feynman–Kac gauge. It is considered extensively in the probability literature, where it has an interpretation in terms of Brownian motion. At the start we will consider C^2 domains, but eventually we will generalize to the class of uniform domains.

For a sufficiently nice domain Ω, let $P(x, z)$ be the Poisson kernel (for $x \in \Omega$, $z \in \partial\Omega$) and let σ be surface measure on $\partial\Omega$. Then

$$Pf(x) = \int_{\partial\Omega} P(x, z) f(z)\, d\sigma(z) \tag{1.2}$$

is harmonic on Ω and extends to be continuous on $\overline{\Omega}$, with boundary values f, provided f is continuous on $\partial\Omega$. Let $G(x, y)$ be the Green's function for $-\triangle$ on Ω, for $x, y \in \Omega$. The Green's function is always symmetric ($G(x, y) = G(y, x)$) and strictly positive for $x, y \in \Omega$. Then for u nice enough on Ω,

$$Gu(x) = \int_{\Omega} G(x, y) u(y)\, dy \tag{1.3}$$

satisfies $-\triangle(Gu) = u$ on Ω and Gu extends continuously to $\overline{\Omega}$ with boundary values 0 on $\partial\Omega$. Hence if u satisfies

$$u = G(qu) + Pf, \tag{1.4}$$

then formally u satisfies (1.1). We define T by

$$Tu(x) = G(qu)(x) = \int_\Omega G(x, y)u(y)q(y)\, dy. \tag{1.5}$$

Then Eq. (1.4) becomes

$$u = Tu + Pf, \tag{1.6}$$

or $(I - T)u = Pf$. Working formally again for the moment, $I - T$ has an inverse given by the Neumann series $\sum_{j=0}^{\infty} T^j$, and the solution of (1.1) is

$$u = \sum_{j=0}^{\infty} T^j(Pf). \tag{1.7}$$

Notice that all terms in the series are non-negative, since $f, q \geq 0$. The main issue is whether the series $\sum_{j=0}^{\infty} T^j(Pf)$ converges a.e. on Ω, or whether it is forced to be $+\infty$ on a set of positive measure in Ω, in which case we do not regard it as a solution of (1.1). This approach of treating Eq. (1.1) as a perturbation of the case $q = 0$, i.e., the case where u is harmonic, is common in this subject.

Solutions of $-\triangle u = qu$ on Ω, $u = f$ on $\partial\Omega$ are not unique in general (see e.g., [20] for some discussion). However, if $w \geq 0$ satisfies $w = Tw + Pf$, then substituting $Tw + Pf$ for w on the right-hand side yields $w = T^2 w + TPf + Pf$. Iterating this process, we obtain $w = T^3 w + T^2 Pf + TPf + Pf$. After n steps, one has

$$w = T^{n+1} w + \sum_{j=0}^{n} T^j Pf \geq \sum_{j=0}^{n} T^j Pf.$$

Letting $n \to \infty$, we see that $w \geq u$. Hence if there is a non-negative solution of (1.4), then u defined by (1.7) is the minimal non-negative solution, and if $\sum_{j=0}^{\infty} T^j(Pf)$ is infinite on a set of positive measure in Ω, then (1.4) has no non-negative solution.

As long as f is not 0 a.e. with respect to surface measure σ, then Pf is strictly positive on Ω, since f is assumed to be non-negative. So (1.4) implies that u is strictly positive on Ω, and Eq. (1.6) implies that $Tu(x) < u(x)$ for all $x \in \Omega$. It follows from Schur's Lemma that T is a bounded operator on $L^2(q) = L^2(\Omega, q\, dy)$ with operator norm $\|T\|_{L^2(q)\to L^2(q)}$ at most 1. Here is the elementary proof, for convenience. For $x \in \Omega$, applying the Cauchy-Schwarz inequality gives

$$\begin{aligned}(Tf(x))^2 &= \left(\int_\Omega G(x, y) \frac{f(y)}{\sqrt{u(y)}} \sqrt{u(y)} q(y)\, dy \right)^2 \\ &\leq \int_\Omega G(x, y) \frac{f^2(y)}{u(y)} q(y)\, dy \cdot \int_\Omega G(x, y)u(y)q(y)\, dy.\end{aligned}$$

The second integral is $Tu(x)$. Using the condition $Tu \leq u$ and applying Fubini's theorem gives

$$\int_\Omega (Tf(x))^2 q(x)\,dx \leq \int_\Omega \int_\Omega G(x,y)u(x)\,q(x)\,dx \frac{f^2(y)}{u(y)} q(y)\,dy.$$

Because G is symmetric, the inner integral is $Tu(y)$, so the estimate $Tu \leq u$ gives $\|T\|_{L^2(q)\to L^2(q)} \leq 1$.

With this last fact in mind, in order to deal rigorously with the formal solution given by Eq. (1.7), we make the assumption

$$\|T\|_{L^2(q)\to L^2(q)} < 1. \tag{1.8}$$

The estimate (1.8) is equivalent (see Lemma 3.1 in [11]) to the condition that there exists $\beta \in (0,1)$ such that

$$\int_\Omega h^2 q\,dx \leq \beta^2 \int_\Omega |\nabla h|^2\,dx \quad \text{for all} \quad h \in C_0^\infty(\Omega). \tag{1.9}$$

Inequality (1.9) has been considered earlier in the literature, starting with [19], in more general settings. The critical case when $\|T\|_{L^2(q)\to L^2(q)} = 1$ is of interest but is not treated by the methods considered here. Under assumption (1.8), $(I-T)^{-1} = \sum_{j=0}^\infty T^j$ is a bounded operator on $L^2(q)$. Hence if $Pf \in L^2(q)$, then u given by (1.7) belongs to $L^2(q)$. As long as $q > 0$ a.e. with respect to Lebesgue measure, then $u < \infty$ a.e., and hence is a solution of (1.6). In the special case where f is identically 1, so that $Pf = 1$, the assumption $Pf \in L^2(q)$ just means that $\int_\Omega q\,dy < \infty$, i.e., $q \in L^1(\Omega, dy)$. Our purpose is to relax this assumption, requiring only $q \in L^1_{loc}(\Omega)$, to see how quickly q can blow up at $\partial\Omega$ and still allow for the series (1.7) to converge a.e.

The sense in which our solution u of the integral equation (1.6) is a solution of the original differential equation (1.1) is a bit technical: u is a "very weak solution" of (1.1) as in [4]. Very weak solutions are defined similarly to classical weak solutions, except that 2 derivatives are moved to the test function. We refer to [9], §2 for a full discussion.

Our conclusions are based on a pointwise analysis of the kernel of the Neumann series operator $\sum_{j=0}^\infty T^j$, rather than the global $L^2(q)$ analysis as above. To give an example of the type of issues we will consider, let $G_j(x,y)$ be the kernel of T^j, so that

$$T^j u(x) = \int_\Omega G_j(x,y)u(y)q(y)\,dy.$$

Then $G_j(x,y) = \int_\Omega G(x,z)G_{j-1}(z,y)q(z)dz$. If we consider the inhomogeneous problem

$$\begin{cases} -\triangle v = qv + g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where $g : \Omega \to \mathbb{R}$ is measurable, then applying the Green's operator G defined by (1.3) to both sides yields the integral equation

$$v = G(qv) + Gg = Tv + Gg, \tag{1.11}$$

with formal solution $v = (I-T)^{-1}Gg = \sum_{j=0}^{\infty} T^j Gg$. Note that

$$\begin{aligned} T^j Gg(x) &= \int_\Omega G_j(x,z) \int_\Omega G(z,y)g(y)\,dy\, q(z)dz \\ &= \int_\Omega \int_\Omega G_j(x,z)G(z,y)q(z)\,dz\, g(y)\,dy = \int_\Omega G_{j+1}(x,y)g(y)\,dy. \end{aligned}$$

Thus

$$v(x) = \int_\Omega \sum_{j=0}^{\infty} G_{j+1}(x,y)g(y)\,dy = \int_\Omega \mathcal{G}(x,y)g(y)\,dy, \tag{1.12}$$

where

$$\mathcal{G}(x,y) = \sum_{j=1}^{\infty} G_j(x,y). \tag{1.13}$$

Thus $\mathcal{G}$ is the kernel of the solution operator for (1.10), with respect to the measure dy instead of $q\,dy$, just as G is the kernel of the solution operator for Poisson's equation $-\triangle v = g$ on Ω, with $v = 0$ on $\partial\Omega$ (the special case $q = 0$ of (1.10)). Therefore we call $\mathcal{G}$ the Green's kernel for the perturbed operator $-\triangle - q$, or the q-perturbed Green's kernel. Theorem 3.3 gives pointwise estimates for $\mathcal{G}$ in terms of G and q. Theorem 3.5 gives a sufficient condition for (1.1) to have a solution $u \in L^1(\Omega, dx)$, and Theorem 4.2 gives an analogous result for $u \in L^1_{loc}(\Omega)$ on a uniform domain.

Related arguments will give estimates for the q-perturbed Poisson kernel (Theorem 3.9) in terms of q and the standard Green's and Poisson kernels on a C^2 domain. Estimates of the same type are obtained for the q-perturbed Martin's kernel on uniform domains in Theorem 4.3.

All of these results depend on some estimates (see Theorem 2.2) on general measure spaces for the kernels of operators $\sum_{j=1}^{\infty} T^j$ associated to the Neumann series derived from an integral operator T with a quasi-metric kernel.

2 Quasi-Metric Kernels

To motivate the definition of a quasi-metric kernel, consider the simplest analogue of (1.1), where Ω is replaced by $\mathbb{R}^n$ and f is replaced by the function 1 as a "boundary value" at infinity in the sense that $\liminf_{x\to\infty} f(x) = 1$ (which turns out to be as much as one can obtain). That is, for $q \in L^1_{loc}(\mathbb{R}^n)$, $q \geq 0$ (where we assume $n \geq 3$ for simplicity), we consider a solution $u : \mathbb{R}^n \to \mathbb{R}$ to:

$$\begin{cases} -\Delta u = qu & \text{in } \mathbb{R}^n, \\ \liminf_{x\to\infty} u(x) = 1, \\ u \geq 0. \end{cases} \tag{2.1}$$

Here the role of the Green's function G is played by the Riesz potential I_2, the integral operator on $\mathbb{R}^n$ with kernel $c_n/|x-y|^{n-2}$. Since the harmonic function with value 1 at infinity in $\mathbb{R}^n$ is the constant function 1, our formal solution in (1.7) is

$$u = \sum_{j=0}^{\infty} T^j 1, \tag{2.2}$$

where $Tu = I_2(qu)$.

The denominator of the Riesz kernel is a power of the metric $|x-y|$. The power of any metric is a quasi-metric, and that turns out to be the critical property of the Riesz kernel that is needed for our analysis.

Definition 2.1 Let (Ω, ω) be a metric space. A function $K : \Omega \times \Omega \to (0, +\infty]$ is a quasi-metric kernel with quasi-metric constant $\kappa > 0$ if K is $\omega \times \omega$ measurable, symmetric ($K(y,x) = K(x,y)$), and $d = 1/K$ satisfies the quasi-triangle inequality

$$d(x,y) \leq \kappa(d(x,z) + d(z,y)) \quad \text{for all} \quad x, y, z \in \Omega. \tag{2.3}$$

We do not assume that $d(x,x) = 0$, or that $d(x,y) > 0$ for $x \neq y$, so d is not necessarily a quasi-metric in the usual sense. Quasi-metric kernels have been considered by several authors; see, for example, [13, 17]. Given a quasi-metric kernel K on (Ω, ω), we define the iterates K_j for $j \in \mathbb{N}$ by $K_1 = K$ and, inductively,

$$K_j(x,y) = \int_\Omega K(x,z) K_{j-1}(z,y)\, d\omega(z), \tag{2.4}$$

for $j > 1$.

More explicitly,

$$K_j(x,y) = \int_\Omega \cdots \int_\Omega K(x,z_1)K(z_1,z_2)\cdots K(z_{j-1},y)\, d\omega(z_{j-1})\, d\omega(z_{j-2}) \cdots d\omega(z_1). \tag{2.5}$$

Each K_j is non-negative and symmetric. If we define the integral operator

$$Tf(x) = \int_{\Omega} K(x, y) f(y)\, d\omega(y), \tag{2.6}$$

for $x \in \Omega$, then T^j, the jth iterate of T, is the integral operator with kernel K_j. Hence $\sum_{j=1}^{\infty} K_j(x, y)$ is the kernel of the integral operator $\sum_{j=1}^{\infty} T^j$. The foundation for our results is the following estimate for $\sum_{j=1}^{\infty} K_j(x, y)$.

Theorem 2.2 *Suppose* (Ω, ω) *is a* σ*-finite measure space, and* K *is a quasi-metric kernel on* Ω *with quasi-metric constant* κ*. Suppose*

$$\|T\| = \|T\|_{L^2(\Omega,\omega)\to L^2(\Omega,\omega)} < 1. \tag{2.7}$$

Then there exists a constant C *depending only on* κ *and* $\|T\|$ *such that*

$$\sum_{j=1}^{\infty} K_j(x, y) \le K(x, y) e^{C K_2(x,y)/K(x,y)}, \quad \textit{for all} \quad x, y \in \Omega. \tag{2.8}$$

See [11], Theorem 1.1, for the proof, which is somewhat technical. Although the kernels involved are non-negative, the rough idea is similar to estimates for Calderón–Zygmond operators: use pointwise estimates for $K(z_m, z_{m+1}) = \frac{1}{d(z_m, z_{m+1})}$ when $d(z_m, z_{m+1})$ is relatively large, and L^2 estimates for T when $d(z_m, z_{m+1})$ is relatively small. The region of integration Ω^{j-1} is broken into sets determined by "paths" $(x, z_1, z_2, \dots, z_{j-1}, y)$ for which the same terms satisfy the criterion for being relatively large, and at the end the sum is taken over all such sets.

The estimate $\|T\| < 1$ is close to being sharp in the sense that if $\|T\| > 1$, then $\sum_{j=1}^{\infty} K_j(x, y) = +\infty$ for every $x, y \in \Omega$. The proof (see [11, Lemma 2.1]) is an application of Schur's Lemma.

There is a much easier lower estimate of the same form. If (Ω, ω) is a σ-finite measure space, and K is a quasi-metric kernel on Ω with quasi-metric constant κ, then there exists a constant $c > 0$, depending only on κ, such that

$$\sum_{j=1}^{\infty} K_j(x, y) \ge K(x, y) e^{c K_2(x,y)/K(x,y)}, \quad \text{for all} \quad x, y \in \Omega. \tag{2.9}$$

See [11], Lemma 2.3 for the proof. Note that the norm condition (2.7) is not required for the lower estimate, although it is not meaningful if $\|T\| > 1$ because $\sum_{j=1}^{\infty} K_j$ is identically ∞ in this case, as noted above.

Estimates of the form (2.8) and (2.9) were proved under stronger assumptions in [12], which was in turn motivated by similar results for a specific example of a quasi-metric kernel considered in [8] as a discrete model for (1.1).

Notice that (2.8) shows that the kernel $\sum_{j=1}^{\infty} K_j(x, y)$ associated to the Neumann series for $(I - T)^{-1}$ is equivalent to the original kernel $K(x, y)$ of T if and only if there exists a constant $C_1 > 0$ such that $K_2(x, y) \le C_1 K(x, y)$ for all $x, y \in \Omega$. See [11], Theorem 3.2, for related results.

The connection between (2.8) and the quantity $\sum_{j=0}^{\infty} T^j 1$ in (2.2) is not immediately clear. However, suppose there is a point $z \in \Omega$ such that $\frac{1}{K(x,z)} = d(x, z) = A$, where $A > 0$ is constant, for all $x \in \Omega$. Then

$$\frac{K_j(x, z)}{K(x, z)} = A \int_{\Omega} K_{j-1}(x, y) K(y, z)\, d\omega(y) = T^{j-1} 1(x),$$

since $K(y, z) = 1/A$. Hence (2.8) becomes

$$\sum_{j=0}^{\infty} T^j 1(x) = \sum_{j=1}^{\infty} T^{j-1} 1(x) = \frac{1}{K(x, z)} \sum_{j=1}^{\infty} K_j(x, z) \le e^{C K_2(x,z)/K(x,z)} = e^{C T 1(x)}.$$

In general, we cannot expect such a point z to exist. However, if d is bounded, and if we choose A sufficiently large compared to the bound on d, we can add a point z to Ω and define $d(x, z) = A$ for all $x \in \Omega$, with $d(z, z) = 0$ (which is not an issue because we set $\omega(\{z\}) = 0$). Then we obtain a quasi-metric kernel $K = 1/d$ on $\Omega \cup \{z\}$ which has quasi-metric constant equal to the maximum of 1 and the original quasi-metric constant. We can then obtain the required estimate in the case where d is bounded. Using that estimate, in the general case of an unbounded d, we can exhaust Ω by a sequence of increasing domains on which d is bounded, and obtain the general result by a monotone convergence argument. See [11], Theorem 3.1 for the details of the proof of the following result.

Theorem 2.3 *Let (Ω, ω) be a σ-finite measure space. Let K be a quasi-metric kernel on Ω with quasi-metric constant κ and corresponding integral operator T.*

(a) *There exists $c > 0$, depending only on κ, such that*

$$\sum_{j=0}^{\infty} T^j 1(x) \ge e^{c T 1(x)}, \quad \text{for all} \quad x \in \Omega. \tag{2.10}$$

(b) *If, in addition, $\|T\| < 1$, then there exists $C > 0$, depending only on κ and $\|T\|$, such that*

$$\sum_{j=0}^{\infty} T^j 1(x) \le e^{C T 1(x)}, \quad \text{for all} \quad x \in \Omega. \tag{2.11}$$

Applying this result to (2.1) leads quickly to the following result (Theorem 1.4 in [12]):

Theorem 2.4 *Suppose $n \ge 3$, $q \in L^1_{loc}(\mathbb{R}^n)$, and define $Tu = I_2(qu)$.*

(a) *Suppose* $\|T\| = \|T\|_{L^2(\mathbb{R}^n, q\,dx) \to L^2(\mathbb{R}^n, q\,dx)} < 1$ *and*

$$\int_{\mathbb{R}^n} \frac{q(y)}{(1+|y|)^{n-2}}\,dy < \infty. \tag{2.12}$$

Then $u = \sum_{j=0}^{\infty} T^j 1 < \infty$ *a.e.,* u *satisfies* (2.1) *in the distributional sense, and there exists* $C > 0$*, depending on* n *and* $\|T\|$*, such that*

$$u \le e^{C I_2 q} \tag{2.13}$$

on $\mathbb{R}^n$.

(b) *Conversely, if* u *is a distributional solution of* (2.1)*, then* $\|T\| \le 1$ *and there exists* $c > 0$*, depending only on* n*, such that* $u \ge e^{c I_2 q}$.

The essence of the proof is to use Theorem 2.3 to write

$$u(x) \le e^{CT1(x)} = e^{C I_2 q(x)},$$

and show that inequality (2.12) implies that $I_2 q < \infty$ a.e.

As an example, the function $q : \mathbb{R}^n \to \mathbb{R}$ defined by $q(x) = A|x|^{-\alpha}$, for $2 < \alpha < n$ and A sufficiently small, satisfies the conditions of the theorem (by the multi-dimensional fractional Hardy inequality, see e.g. [7]), yet $q \notin L^\infty$ (the uniformly elliptic case) and $q \notin L^1(\mathbb{R}^n)$.

3 Bounded Domains and Modifiable Kernels

We return to the setting of (1.1), where Ω is a bounded, connected open set in $\mathbb{R}^n$, where $n \ge 3$. When Ω is a C^2 domain (or even a $C^{1,1}$ domain), the behavior of the Green's function G on Ω is well understood: by [22, 24],

$$G(x,y) \approx \frac{\delta(x)\delta(y)}{|x-y|^{n-2}(|x-y| + \delta(x) + \delta(y))^2}, \tag{3.1}$$

where $\delta(x)$ is the distance of $x \in \Omega$ to $\partial\Omega$. G is not a quasi-metric kernel on Ω, but (3.1) shows that $\frac{G(x,y)}{\delta(x)\delta(y)}$ is. It turns out that our main general results, Theorems 2.2 and 2.3, have elementary modifications that cover situations like this.

Definition 3.1 Let (Ω, ω) be a measure space, and let $K : \Omega \times \Omega \to (0, \infty]$ be $\omega \times \omega$ measurable and symmetric. K is quasi-metrically modifiable (or q–m modifiable, for short) with constant κ if there exists a measurable function $m : \Omega \to (0, \infty)$ such that

$$H(x,y) = \frac{K(x,y)}{m(x)m(y)} \tag{3.2}$$

is a quasi-metric kernel with quasi-metric constant κ. The function m is called the q–m modifier for K.

In particular δ is a q–m modifier for G on a bounded C^2 domain.

We can obtain estimates for a q–m modifiable kernel K by applying Theorem 2.2 to the modified kernel H defined by (3.2), but with respect to the measure space (Ω, ν), where $d\nu = m^2 d\omega$. We define the integral operator S by

$$Sf(x) = \int_\Omega H(x, y) f(y)\, d\nu(y). \tag{3.3}$$

Then $\|f\|_{L^2(\nu)} = \|fm\|_{L^2(\omega)}$, and

$$Sf(x) = \int_\Omega \frac{K(x, y)}{m(x)m(y)} f(y)m^2(y)\, d\omega(y) = \frac{T(fm)(x)}{m(x).}$$

Hence

$$\int_\Omega |Sf|^2\, d\nu = \int_\Omega \frac{|T(fm)|^2}{m^2} m^2\, d\omega = \int_\Omega |T(fm)|^2\, d\omega.$$

Therefore

$$\|S\|_{L^2(\nu)\to L^2(\nu)} = \|T\|_{L^2(\omega)\to L^2(\omega)}. \tag{3.4}$$

Define $H_j(x, y)$ inductively by $H_1 = H$ and $H_j(x, y) = \int_\Omega H(x, z)H_{j-1}(z, y)\, d\nu(z)$, analogously to (2.4). We observe that

$$H_j(x, y) = \frac{K_j(x, y)}{m(x)m(y)} \tag{3.5}$$

for all $j \in \mathbb{N}$: by definition for $j = 1$, and for $j > 1$,

$$H_j(x, y) = \int_\Omega H(x, z)H_{j-1}(z, y)\, d\nu(z) = \int_\Omega \frac{K(x, z)}{m(x)m(z)} \frac{K_{j-1}(z, y)}{m(z)m(y)} m^2(z)\, d\omega(z),$$

which gives (3.5). We obtain the following.

Corollary 3.2 *Suppose (Ω, ω) is a σ-finite measure space and K is a q–m modifiable kernel on Ω with constant κ. Define T by (2.6) and suppose $\|T\|_{L^2(\omega)\to L^2(\omega)} < 1$. Then (2.8) holds with C a constant depending only on κ and $\|T\|$. The estimate (2.9) also holds, with a constant depending on κ only, without any requirement on $\|T\|$.*

The proof of the upper estimate is to apply Theorem 2.2 to the quasi-metric kernel H, with notation as above (using (3.4)) to obtain

$$\sum_{j=1}^{\infty} \frac{K_j(x, y)}{m(x)m(y)} \leq \frac{K(x, y)}{m(x)m(y)} e^{C\left(\frac{K_2(x,y)}{m(x)m(y)}\right)/\left(\frac{K(x,y)}{m(x)m(y)}\right)},$$

and then cancel all of the $m(x)$ and $m(y)$ terms.

When applied to the Green's kernel on a C^2 domain, Corollary 3.2 and the fact that G has a q–m modifier (namely, δ) yields the following estimates (Theorem 1.2 in [11]) on the q-perturbed Green's kernel $\mathcal{G} = \sum_{j=1}^{\infty} G_j$.

Theorem 3.3 *Let $n \geq 3$ and let $\Omega \subseteq \mathbb{R}^n$ be a C^2 bounded, connected open set. Suppose $q \in L^1_{loc}(\Omega)$ with $q \geq 0$. Define $\mathcal{G}$ by (1.13). Then there exist a constant $c = c(\Omega)$ such that*

$$\mathcal{G}(x, y) \geq G(x, y) e^{cG_2(x,y)/G(x,y)}. \tag{3.6}$$

If also the operator T defined by (1.5) satisfies $\|T\|_{L^2(q)\to L^2(q)} < 1$, then there exists $C = C(\Omega, \|T\|)$ such that

$$\mathcal{G}(x, y) \leq G(x, y) e^{CG_2(x,y)/G(x,y)}. \tag{3.7}$$

The lower estimate (3.6) is known to hold with $c = 1$ from the probabilistic interpretation (see [11], p. 906 for a discussion), but (3.7) seems to have been new at the time of [11]. Note the consequence of (3.7) that $\mathcal{G} \approx G$ (which we interpret as meaning that the potential q is sufficiently mild that the Green's kernel for the Schrödinger operator $-\triangle - q$ behaves like the Green's kernel when $q = 0$, i.e., the Laplacian) if and only if there exists some constant $C_1 > 0$ such that $G_2 \leq C_1 G$.

Just as Theorem 2.2 has a useful generalization to the q–m modifiable case, so does Theorem 2.3 (Corollary 3.5 in [11]), as follows.

Corollary 3.4 *Suppose (Ω, ω) is a σ-finite measure space, and K is a q–m modifiable kernel on Ω with modifier m and constant κ. Define T by (2.6). Then there exists $c = c(\kappa)$ such that*

$$\sum_{j=0}^{\infty} T^j m \geq m e^{cTm/m}. \tag{3.8}$$

If also $\|T\|_{L^2(\Omega,\omega)\to L^2(\Omega,\omega)} < 1$, then there exists $C = C(\kappa, \|T\|)$ such that

$$\sum_{j=0}^{\infty} T^j m \leq m e^{CTm/m}. \tag{3.9}$$

For the proof, define H as in (3.2), let $d\nu = m^2 d\omega$, and define S by (3.3). Note that

$$S^j 1 = \int_\Omega H_j(x, y)\, d\nu(y) = \int_\Omega \frac{K_j(x, y)}{m(x)m(y)} m^2(y)\, d\omega(y) = \frac{T^j m(x)}{m(x)}.$$

Hence, recalling (3.4), (2.11) becomes $\sum_{j=0}^{\infty} \frac{T^j m(x)}{m(x)} \leq e^{CTm(x)/m(x)}$, or (3.9). The proof of the lower estimate (3.8) is similar.

Corollary 3.4 suggests an enhanced role for the q–m modifier m. For example, let $K = G$ on a C^2 domain $\Omega \subseteq \mathbb{R}^n$, where $n \geq 3$, $d\omega = q\,dx$ and $Tu = G(qu)$ as in (1.5). Then (3.1) shows that $\delta(x)$ is a q–m modifier for G. Hence if $\|T\| < 1$, Corollary 3.4 yields the upper estimate $\sum_{j=0}^{\infty} T^j\delta \leq \delta e^{CT\delta/\delta}$, with a lower estimate of the same type with a different constant in the exponent.

This last estimate becomes relevant to problem (1.10) for the case $g = 1$ because it is relatively easy to see that $G1 \approx \delta$ (see (2.4) in [9]). Our formal solution to (1.10) for $g = 1$ is $v = \sum_{j=0}^{\infty} T^j G1$. Hence there exists $C_1 > 0$ such that

$$v = \sum_{j=0}^{\infty} T^j G1 \leq C_1 \sum_{j=0}^{\infty} T^j \delta \leq C_1 \delta e^{CT\delta/\delta} = C_1 \delta e^{CG(\delta q)/\delta}. \tag{3.10}$$

With a bit of work (see [9, pp. 1409–10]), one can show, using the equivalence of (1.9) and the condition that $\|T\| < 1$, that $G(\delta q) \in L^1(\Omega, dx)$ and hence is finite a.e. Then inequality (3.10) shows that $v < \infty$ a.e., and hence v is a solution of (1.10) for $g = 1$. There is also a lower estimate of the form in (3.10) with different constants. See Theorem 1.1 in [9] for the full statement of these results.

Returning to Eq. (1.1), with the solution u given by (1.7), we have

$$\int_\Omega u\,dx = \int_\Omega \sum_{j=0}^{\infty} T^j Pf\,dx = \int_\Omega Pf\,dx + \int_\Omega \sum_{j=1}^{\infty} T^j Pf\,dx. \tag{3.11}$$

Using Fubini's theorem, the symmetry of the kernels G_j, and Eqs. (1.12) and (1.13) (for v defined with $g = 1$),

$$\begin{aligned}
\int_\Omega \sum_{j=1}^{\infty} T^j Pf\,dx &= \int_\Omega \int_\Omega \sum_{j=1}^{\infty} G_j(x,y) Pf(y) q(y)\,dy\,dx \\
&= \int_\Omega \int_\Omega \sum_{j=1}^{\infty} G_j(x,y)\,dx\; Pf(y) q(y)\,dy = \int_\Omega v(y)\, Pf(y) q(y)\,dy \\
&= \int_\Omega \int_{\partial\Omega} P(y,z) f(z)\,d\sigma(z) v(y) q(y)\,dy \\
&= \int_{\partial\Omega} \int_\Omega P(y,z) v(y) q(y)\,dy f(z)\,d\sigma(z).
\end{aligned}$$

The Poisson kernel $P(y,z)$ is the outward normal derivative of the Green's function $G(y,x)$ as $x \to z$. If we take a sequence (x_j) of points of Ω which converge to $z \in \partial\Omega$ normally, then

$$\int_\Omega P(y,z) v(y) q(y)\,dy = \lim_{j\to\infty} \int_\Omega \frac{G(y,x_j)}{\delta(x_j)} v(y) q(y)\,dy = \lim_{j\to\infty} \frac{G(qv)(x_j)}{\delta(x_j)}. \tag{3.12}$$

See Lemma 3.3 in [9] for the full justification of this fact. By Eq. (1.11), $G(qv) < G(qv) + G1 = v$. Hence, using estimate (3.10),

$$\lim_{j\to\infty} \frac{G(qv)(x_j)}{\delta(x_j)} \leq \lim_{j\to\infty} \frac{v(x_j)}{\delta(x_j)} \leq \lim_{j\to\infty} C_1 e^{CG(\delta q)(x_j)/\delta(x_j)}. \tag{3.13}$$

By the identity (3.12) with v replaced by δ,

$$\lim_{j\to\infty} \frac{G(\delta q)(x_j)}{\delta(x_j)} = \int_\Omega P(y,z)\delta(y)q(y)\,dy = P^*(\delta q)(z),$$

where P^*, defined by

$$P^*(h)(z) = \int_\Omega P(y,z)h(y)\,dy, \tag{3.14}$$

for $z \in \partial\Omega$, is the formal adjoint of the Poisson operator. P^*h is known as the balayage, or sweep, of h, because the integral sweeps h from Ω to $\partial\Omega$. Putting these estimates together gives

$$\int_\Omega P(y,z)v(y)q(y)\,dy \leq C_1 e^{CP^*(\delta q)(z)}.$$

Substituting this estimate above gives

$$\int_\Omega u\,dx \leq \int_\Omega Pf\,dx + C_1 \int_{\partial\Omega} e^{CP^*(\delta q)(z)} f(z)\,d\sigma(z). \tag{3.15}$$

We summarize with this statement (see Theorem 1.2 in [9] for the statement when $q = 1$, i.e., the case of the gauge).

Theorem 3.5 *Suppose Ω is a bounded C^2 domain in $\mathbb{R}^n$ for $n \geq 3$, $q \in L^1_{loc}(\Omega)$, $q \geq 0$, $f : \partial\Omega \to [0,\infty)$ is Borel measurable, and T is defined by (1.5). Let $\delta(x)$ be the distance of $x \in \Omega$ to $\partial\Omega$.*

(a) *If $\|T\|_{L^2(q)\to L^2(q)} < 1$ and $Pf \in L^1(\Omega, dx)$, then there exists $C > 0$, depending on Ω and $\|T\|$, such that if*

$$\int_{\partial\Omega} e^{CP^*(\delta q)} f(z)\,d\sigma < \infty, \tag{3.16}$$

then $u = \sum_{j=0}^\infty T^j Pf$ satisfies inequality (3.15), and hence $u \in L^1(\Omega, dx)$ is a very weak solution of (1.1).

(b) *Conversely, suppose $f \in L^1(\partial\Omega, d\sigma)$. If there is a very weak solution w of (1.1), then $\|T\| \leq 1$. If $w \in L^1(\Omega, dx)$, then $Pf \in L^1(\Omega)$, and (3.16) holds with C replaced by some constant c depending only on Ω.*

The proof of the converse only requires a few modifications to the argument above. We noted above that the estimate $\|T\| \leq 1$ follows from Schur's Lemma. Equation (3.11) and the minimality of u imply that $Pf \in L^1(\Omega, dx)$. In place of (3.13) we use the converse estimate to (3.10), the equation $v = G(qv) + G1$, and the equivalence of $G1$ and δ, to obtain

$$e^{cG(\delta q)(x_j)/\delta(x_j)} \leq c_1 \lim_{j\to\infty} \frac{v(x_j)}{\delta(x_j)}$$
$$= c_1 \lim_{j\to\infty} \frac{G(qv)(x_j) + G1(x_j)}{\delta(x_j)} \leq c_1 \lim_{j\to\infty} \frac{G(qv)(x_j)}{\delta(x_j)} + c_2.$$

The constant c_2 ultimately results in adding a term $c_2 \int_{\partial\Omega} f(z)\, d\sigma(z)$, which is assumed to be finite, to the estimate for $\int_{\partial\Omega} e^{CP^*(\delta q)} f(z)\, d\sigma$.

Condition (3.16) is a measure of how rapidly q can blow up on $\partial\Omega$ and still allow for a solution to (1.1). For $f = 1$ (the case of the gauge), (3.16) holds if $P^*(\delta q) \in BMO$ with sufficiently small norm, which in turn holds if δq is a Carleson measure on Ω with small enough Carleson norm—see [9], Corollary 1.3.

Theorem 3.5 is based on the estimate (3.10), which comes from the general estimate (3.9) and the observation that δ is a q–m modifier for Green's kernel. If we can find other q–m modifiers for T, we can derive further results. The task of identifying q–m modifiers is made easier by the following lemmas. A quasi-metric d with constant κ on a set X is a symmetric function $d : X \times X \to [0, \infty)$ which is non-degenerate ($d(x, y) = 0$ if and only if $x = y$) and satisfies the quasi-triangle inequality (2.3). The following is Lemma 2.2 in [11], where it is used in the proof of Theorem 2.2 above.

Lemma 3.6 (Ptolemy inequality) *Let d be a quasi-metric with constant κ on a set X. Suppose $x, y, z, w \in X$. Then*

$$d(x, y)d(z, w) \leq 4\kappa^2[d(x, w)d(y, z) + d(x, z)d(y, w)]. \tag{3.17}$$

Proof Suppose that $d(x, z) = \min\{d(x, z), d(y, z), d(y, w), d(x, w)\}$. Then

$$d(x, y) \leq \kappa(d(x, z) + d(z, y)) \leq 2\kappa d(z, y),$$

and

$$d(z, w) \leq \kappa(d(z, x) + d(x, w)) \leq 2\kappa d(w, x).$$

Hence

$$d(x, y)d(z, w) \leq 4\kappa^2(d(z, y)d(w, x)) \leq 4\kappa^2[d(x, w)d(y, z) + d(x, z)d(y, w)].$$

Inequality (3.17) is invariant under interchanging x and y, so it holds if $d(y, z)$ is the minimum of the 4 distances above. It is also invariant under interchanging z and w, so it holds if $d(x, w)$ is the minimum. Finally, the result holds in the case where the minimum is $d(y, w)$ by interchanging x and y, and also z and w. □

The following lemma originated with [21], Lemma A.1 in the context of normed vector spaces, and appears in generality in [14], Proposition 8.1 and Corollary 8.2.

Lemma 3.7 *Suppose d is a quasi-metric with constant κ on X, and let $z \in X$. Define $\tilde{d} : X\backslash\{z\} \times X\backslash\{z\} \to [0, \infty)$ by*

$$\tilde{d}(x, y) = \frac{d(x, y)}{d(x, z) \cdot d(y, z)}. \tag{3.18}$$

Then $\tilde{d}$ is a quasi-metric on $X\backslash\{z\}$ with constant $4\kappa^2$.

Proof The non-degeneracy and symmetry of $\tilde{d}$ are trivial. For the quasi-triangle inequality, dividing both sides of (3.17) by the non-zero quantity $d(x, z)d(y, z)d(w, z)$ gives

$$\frac{d(x, y)}{d(x, z)d(y, z)} \leq 4\kappa^2 \left(\frac{d(x, w)}{d(x, z)d(w, z)} + \frac{d(y, w)}{d(y, z)d(w, z)} \right),$$

or

$$\tilde{d}(x, y) \leq 4\kappa^2 \left(\tilde{d}(x, w) + \tilde{d}(w, y) \right).$$

□

The following result appears to be fortuitous.

Lemma 3.8 *Let $n \geq 3$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded C^2 domain with Green's function $G(x, y)$ and Poisson kernel $P(x, z)$ for $x, y \in \Omega$, $z \in \partial\Omega$. For each $z \in \partial\Omega$, define $m_z : \Omega \to (0, \infty)$ by*

$$m_z(x) = P(x, z).$$

Then m_z is a quasi-metric modifier for G with constant independent of $z \in \partial\Omega$.

Proof Define $d : \overline{\Omega} \times \overline{\Omega} \to [0, \infty)$ by

$$d(x, y) = |x - y|^{n-2}(|x - y|^2 + \delta(x)^2 + \delta(y)^2),$$

where δ is the distance to the boundary as usual. One can show that d is a quasi-metric on $\overline{\Omega}$. Let the quasi-metric constant be κ, which depends only on Ω. For $x \in \Omega$ and $z \in \partial\Omega$, $\delta(z) = 0$ and $\delta(x) \leq |x - z|$, so $d(x, z) \approx |x - z|^n$. The Poisson kernel satisfies the equivalence

$$m_z(x) = P(x, z) \approx \frac{\delta(x)}{|x - z|^n}$$

(see, for example, [5]). Hence, using (3.1),

$$\frac{G(x, y)}{m_z(x)m_z(y)} \approx \frac{\delta(x)\delta(y)}{d(x, y)} \Big/ \left(\frac{\delta(x)}{|x - z|^n} \cdot \frac{\delta(y)}{|y - z|^n} \right) \approx \frac{d(x, z)d(y, z)}{d(x, y)},$$

which is the reciprocal of a quasi-metric on $\Omega\backslash\{z\}$ (hence on Ω), with constant $4\kappa^2$, by Lemma 3.7. ◻

To apply these observations to the study of (1.1), we make the following formal calculation based on our solution u given by (1.7):

$$\begin{aligned} u(x) &= \sum_{j=0}^{\infty} T^j Pf(x) = Pf(x) + \sum_{j=1}^{\infty} \int_{\Omega} G_j(x,y) Pf(y) q(y)\, dy \\ &= Pf(x) + \sum_{j=1}^{\infty} \int_{\Omega} G_j(x,y) \int_{\partial\Omega} P(y,z) f(z)\, d\sigma(z)\, q(y)\, dy \\ &= Pf(x) + \int_{\partial\Omega} \sum_{j=1}^{\infty} \int_{\Omega} G_j(x,y) P(y,z)\, q(y)\, dy f(z)\, d\sigma(z). \end{aligned}$$

We define

$$\mathcal{P}(x,z) = P(x,z) + \sum_{j=1}^{\infty} \int_{\Omega} G_j(x,y) P(y,z)\, q(y)\, dy, \tag{3.19}$$

for $x \in \Omega$ and $z \in \partial\Omega$, so that

$$u(x) = \int_{\partial\Omega} \mathcal{P}(x,z) f(z)\, d\sigma(z). \tag{3.20}$$

This equation is analogous to the standard Poisson integral formula (1.2), which it reduces to in the case $q = 0$. Hence, we call $\mathcal{P}$ the q-perturbed Poisson kernel. We are led to the following estimate for $\mathcal{P}$.

Theorem 3.9 *Let $n \geq 3$ and let $\Omega \subseteq \mathbb{R}^n$ be a C^2 bounded, connected open set. Suppose $q \in L^1_{loc}(\Omega)$ with $q \geq 0$. Then there exists a constant $c = c(\Omega)$ such that*

$$\mathcal{P}(x,z) \geq P(x,z) e^{c \int_{\Omega} G(x,y) P(y,z) q(y)\, dy / P(x,z)}. \tag{3.21}$$

If also the operator T defined by (1.5) satisfies $\|T\|_{L^2(q)\to L^2(q)} < 1$, then there exists $C = C(\Omega, \|T\|)$ such that

$$\mathcal{P}(x,z) \leq P(x,z) e^{C \int_{\Omega} G(x,y) P(y,z) q(y)\, dy / P(x,z)}. \tag{3.22}$$

For the proof of the upper estimate, $P(x,z) = m_z(x)$ is a q–m modifier for G by Lemma 3.8, so by Corollary 3.4,

$$\mathcal{P}(x,z) = \sum_{j=0}^{\infty} T^j m_z(x)$$
$$\le m_z(x) e^{C(Tm_z(x))/m_z(x)} = P(x,z) e^{C \int_\Omega G(x,y)P(y,z)q(y)\,dy/P(x,z)},$$

where C is independent of $z \in \partial\Omega$ because it depends only on $\|T\|$ and the constant for the q–m modifier m_z, which is independent of z. The lower estimate is similar.

Of course the estimates (3.21) and (3.22) yield pointwise estimates for the solution u of (1.1): if $\|T\| < 1$, then

$$u(x) \le \int_{\partial\Omega} P(x,z) e^{C \int_\Omega G(x,y)P(y,z)q(y)\,dy/P(x,z)} f(z)\, d\sigma(z),$$

with a similar lower estimate with C replaced by c, which holds without any assumption on $\|T\|$. See equations (1.12) and (1.14) in [9] for the case of the gauge, when $f = 1$. Notice that these estimates reduce in the case $q = 0$ to the standard Poisson formula (1.2). Also note that they yield the result that $\mathcal{P} \approx P$ if and only if $\int_\Omega G(x,y)P(y,z)q(y)\,dy \le C_1 P(x,z)$.

Theorems 3.3 and 3.9 yield estimates for the q-perturbed Green's and Poisson kernels in terms of q and the classical ($q = 0$) Green's and Poisson kernels. Ultimately these estimates derive from Theorem 2.2 about general quasi-metric kernels on general measure spaces, but only because of the robustness of Theorem 2.2 in its consequences: Theorem 2.3 dealing with estimates of $T^j 1$, and Corollaries 3.2 and 3.4 where q–m modifiers are introduced. This robustness is further exhibited in the following section, where these topics are considered on uniform domains.

4 Uniform Domains, Harmonic Measure, and Martin's Kernel

Since the 1970s, substantial attention has been paid to the study of partial differential equations on domains that are less smooth than C^2 domains, starting with Lipschitz domains (domains whose boundary is, after rotation, locally the graph of a Lipschitz function). An extensive theory was developed by Dahlberg, Jerison, Kenig, and others. In 1982, Jerison and Kenig introduced the more general class of nontangentially accessible, or NTA, domains in [16].

Definition 4.1 Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain (connected open set). For $M > 1$, an M-tangential ball in Ω is a ball $B(x,r)$ with $B(x,r) \subseteq \Omega$, with $\frac{r}{M} \le d(B(x,r), \partial\Omega) \le Mr$, where $d(B(x,r), \partial\Omega)$ denotes the distance of $B(x,r)$ to $\partial\Omega$. For $x, y \in \Omega$, a Harnack chain from x to y is a finite sequence of M-tangential balls such that x is in the first ball, y is in the last ball, and consecutive balls intersect. The number of balls in the Harnack chain is called its length. Ω satisfies the Harnack chain condition if, for every $\epsilon > 0, C > 0$, and $x, y \in \Omega$ satisfying $d(x, \partial\Omega) > \epsilon, d(y, \partial\Omega) > \epsilon$, and $|x - y| < C\epsilon$, there exists a Harnack chain from x to y of length depending only on C, but not on ϵ.

Ω satisfies the interior corkscrew condition if there exist $M > 0$ and $r_0 > 0$ such that for any point $z \in \partial\Omega$, and any $0 < r < r_0$, there exists a point $x \in \Omega$ such that $M^{-1}r < |x - z| < r$ and dist $(x, \partial\Omega) > M^{-1}r$.

Ω satisfies the exterior corkscrew condition if there exist $M > 0$ and $r_0 > 0$ such that for any point $z \in \partial\Omega$, and any $0 < r < r_0$, there exists a point $x \in \mathbb{R}^n \backslash \Omega$ such that $M^{-1}r < |x - z| < r$ and dist $(x, \partial\Omega) > M^{-1}r$.

A non-tangentially accessible, or NTA, domain is a bounded domain that satisfies the Harnack chain condition, the interior corkscrew condition, and the exterior corkscrew condition.

A uniform domain is a bounded domain that satisfies the interior corkscrew condition and the Harnack chain condition.

The class of NTA domains includes Lipschitz domains, but is more general, and the class of uniform domains includes, but is more general, than the class of NTA domains. An example of a uniform domain that is not an NTA domain is a ball in $\mathbb{R}^n$ with $n \geq 3$, with an interior line seqment deleted. The exterior corkscrew condition fails at points of the boundary on the deleted segment. A ball in $\mathbb{R}^2$ with a deleted interior line segment is not a uniform domain because the Harnack chain condition fails for points x, y close together on opposite sides of the deleted segment. In $\mathbb{R}^3$, there is a Harnack chain going around the segment in a direction normal to the segment.

The exterior corkscrew condition guarantees that an NTA domain is regular for the Dirichlet problem, which means that if $f : \partial\Omega \to \mathbb{R}$ is continuous on $\partial\Omega$, the there exists a function $u : \overline{\Omega} \to \mathbb{R}$ such that u is harmonic on Ω, $u = f$ on $\partial\Omega$, and u is continuous on $\overline{\Omega}$. A uniform domain, however, is not necessarily regular for the Dirichlet problem.

In generalizing Theorem 3.3 to uniform domains, the first difficulty is that δ, the distance to the boundary, is not necessarily a q–m modifier for the Green's function $G(x, y)$ on a uniform domain. However, estimates [2, 13] show that for any $x_0 \in \Omega$, the function $m(x) = \min(1, G(x, x_0))$ is a q–m modifier for G, with constant independent of $x_0 \in \Omega$. The existence of a q–m modifier then implies, by Corollary 3.2, that the Green's Function estimates (3.6) and (3.7) of Theorem 3.3 hold when Ω is a uniform domain (and $\|T\| < 1$ for (3.7)).

The boundary of an NTA domain Ω can be quite wild, so that in general, surface measure $d\sigma$ is not defined on $\partial\Omega$. In particular, then, there is no direct analogue of the Poisson integral formula (1.2).

However, for an arbitrary bounded domain in $\mathbb{R}^n$, given a function $f : \partial\Omega \to \mathbb{R}$, one can follow the Perron process and consider $\overline{P}f$, which is (roughly) the infimum of the superharmonic functions on Ω which dominate f on $\partial\Omega$, and $\underline{P}f$, the supremum of the subharmonic functions lying under f on $\partial\Omega$. If $\overline{P}f = \underline{P}f$, and $\overline{P}f$ is harmonic on Ω (which Wiener proved to be true for any continuous f on any bounded domain in $\mathbb{R}^n$ in [23]), we define $Pf = \overline{P}f$ to be the generalized solution to the Dirichlet problem. Then by the maximum principle, for any fixed $x \in \Omega$, the map taking f to $Pf(x)$ is a bounded linear functional on $C(\partial\Omega)$ (the continuous functions on $\partial\Omega$). By the Riesz representation theorem, there exists a measure dH^x on $\partial\Omega$ such that

$$Pf(x) = \int_{\partial\Omega} f(z)dH^x(z). \tag{4.1}$$

See, for example, [16, p. 83], for a more precise explanation. H^x is called the harmonic measure on Ω for the base point x.

The Martin boundary and Martin kernel for a domain Ω are defined somewhat abstractly by considering certain extremal positive harmonic functions on Ω. These extremal functions form the Martin boundary, and general positive harmonic functions on Ω can be obtained as an integral over the Martin boundary with respect to a certain kernel. However, for uniform domains, the Martin boundary can be identified with the topological boundary ([1], Corollary 3, for uniform domains; see [15] and [3] for Lipschitz domains, and [16, 18] for NTA domains). To define the Martin kernel $M(x, z)$ on a uniform domain, for $x \in \Omega$ and $z \in \partial\Omega$, fix a reference point $x_0 \in \Omega$ and set

$$M(x, z) = \lim_{y \to z, y \in \Omega} \frac{G(x, y)}{G(x_0, y)},$$

where that limit exists. For $x, x_0 \in \Omega$, then dH^x and dH^{x_0} are mutually absolutely continuous, and it turns out (see [16], p. 104 and 115 for NTA domains, and [10], §2 for uniform domains) that

$$dH^x(z) = M(x, z) dH^{x_0}(z), \tag{4.2}$$

where x_0 is the reference point in the definition of $M(x, z)$. Therefore (4.1) becomes

$$Pf(x) = \int_{\partial\Omega} M(x, z) f(z)\, dH^{x_0}(z). \tag{4.3}$$

This formula is an analogue of the classical Poisson formula (1.2).

If Ω is a C^2 domain, then $dH^x = P(x, z)\, d\sigma(z)$. Taking a sequence y_j of points of Ω converging normally to $z \in \partial\Omega$,

$$M(x, z) = \lim_{j \to \infty} \frac{G(x, y_j)/\delta(y_j)}{G(x_0, y_j)/\delta(y_j)} = \frac{P(x, z)}{P(x_0, z)}.$$

Hence (4.3) just becomes

$$Pf(x) = \int_{\partial\Omega} \frac{P(x, z)}{P(x_0, z)} f(z) P(x_0, z) d\sigma(z) = \int_{\partial\Omega} P(x, z) f(z) d\sigma(z),$$

as usual.

To obtain an analogue of Theorem 3.5 in a uniform domain, the first issue, as noted above, is that δ is not a q–m modifier for G, although $m(x) = \min(1, G(x, x_0))$ is. Second, in proving Theorem 3.5, we used the fact that $G1 \approx \delta$ on a C^2 domain. The analogous result for the modifier m on a uniform domain would be that $G1 \approx m$, but that may not be true. However, if K is a compact subset of Ω and χ_K is the characteristic function of K, then $G\chi_K \le C_K m$, with the converse estimate $G\chi_K \ge c_K m$ holding as long as $|K| > 0$. As a result, instead of obtaining that the solution u of (1.1) belongs to $L^1(\Omega, dx)$, we can only conclude that $u \in L^1_{loc}(\Omega, dx)$. Instead of working with the solution v of $-\triangle v = qv + 1$ on Ω, $v = 0$ on $\partial\Omega$, we work with v_K, the solution of $-\triangle v = qv + \chi_K$ on Ω, $v = 0$ on $\partial\Omega$. We obtain analogous estimates to those described above for Theorem 3.5, such as equation (3.19) in [10]:

$$\int_K u\,dx = \int_K Pf\,dx + \int_\Omega v_K(y)Pf(y)q(y)\,dy.$$

We define the analogue of the balayage for the Martin kernel in place of the Poisson kernel analogously to (3.14):

$$M^*h(z) = \int_\Omega M(y,z)h(y)\,dy$$

for $z \in \partial\Omega$. The result (Theorem 1.2 in [10]) is the following.

Theorem 4.2 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a uniform domain, $n \geq 3$, $q \in L^1_{loc}(\Omega)$, $q \geq 0$ and let $x_0 \in \Omega$ be the fixed reference point for the Martin kernel M. Suppose $f : \partial\Omega \to [0,\infty)$ is a Borel measurable function which is not a.e. 0 with respect to dH^{x_0}. Let $Tu(x) = \int_\Omega G(x,y)u(y)q(y)\,dy$, and let $m(x) = \min(1, G(x,x_0))$.*

(a) *If $\|T\| < 1$, then there exists a constant $C = C(\Omega, \|T\|) > 0$ such that if*

$$\int_{\partial\Omega} e^{CM^*(mq)} f\,dH^{x_0} < \infty,$$

then $u = \sum_{j=0}^\infty T^j Pf \in L^1_{loc}(\Omega, dx)$ and hence is a (generalized) solution of (1.1).

(b) *Conversely, if $u = \sum_{j=0}^\infty T^j Pf \in L^1_{loc}(\Omega, dx)$, then $\|T\| \leq 1$ and*

$$\int_{\partial\Omega} e^{M^*(mq)} f\,dH^{x_0} < \infty.$$

We refer to the proof of Theorem 1.2 in [10] for more detail.

Martin's kernel was developed to obtain a representation formula for general positive harmonic functions on a domain, not just those of the form Pf for a boundary function f. There is an analogue of Theorem 4.2 for solutions to $u = G(qu) + h$, $u \geq 0$, for $h \geq 0$ on Ω, with h harmonic, in place of (1.4). Here Martin's representing measure takes the place of harmonic measure, and care must be paid to the irregular points of the boundary. See Theorem 3.5 in [10] for the precise statement and proof of this result.

Finally, we look for an analogue on uniform domains of Theorem 3.9. With P defined by (4.1), we can proceed as above to obtain the formal solution (1.7) to (1.4) and hence (1.1). Then, just as in the derivation of (3.20),

$$u(x) = \sum_{j=0}^\infty T^j Pf(x) = Pf(x) + \sum_{j=1}^\infty \int_\Omega G_j(x,y)Pf(y)q(y)\,dy$$

$$= Pf(x) + \sum_{j=1}^{\infty} \int_{\Omega} G_j(x, y) \int_{\partial\Omega} M(y, z) f(z) dH^{x_0}(z)\, q(y)\, dy$$

$$= Pf(x) + \int_{\partial\Omega} \sum_{j=1}^{\infty} \int_{\Omega} G_j(x, y) M(y, z)\, q(y)\, dy\; f(z)\, dH^{x_0}(z).$$

We define

$$\mathcal{M}(x, z) = M(x, z) + \sum_{j=1}^{\infty} \int_{\Omega} G_j(x, y) M(y, z)\, q(y)\, dy, \tag{4.4}$$

for $x \in \Omega$ and $z \in \partial\Omega$, so that we have

$$u(x) = \int_{\partial\Omega} \mathcal{M}(x, z) f(z)\; dH^{x_0}(z). \tag{4.5}$$

This formula is analogous to (4.3) in the same way that (3.20) is analogous to (1.2), so we call $\mathcal{M}$ the q-perturbed Martin kernel.

The key for the proof of Theorem 3.9 was Lemma 3.8, which stated that as a function of x, the Poisson kernel $P(x, z)$ is a q–m modifier for $G(x, y)$ with constant independent of $z \in \partial\Omega$. The analogue here is that the function

$$m_z(x) = M(x, z),$$

where M is Martin's kernel with fixed reference point x_0, is a q–m modifier for G with constant independent of $z \in \partial\Omega$. The proof uses the fact that $m(x) = \min(1, G(x, x_0))$ is a q–m modifier for G, Lemma 3.7, and some additional calculations (see Lemma 2.4 in [10]). With the fact that m_z is a q–m modifier for G, the following result follows from Corollary 3.4.

Theorem 4.3 *Let $n \geq 3$ and let $\Omega \subseteq \mathbb{R}^n$ be a uniform domain. Suppose $q \in L^1_{loc}(\Omega)$ with $q \geq 0$. Then there exists a constant $c = c(\Omega)$ such that*

$$\mathcal{M}(x, z) \geq M(x, z) e^{c \int_{\Omega} G(x,y) M(y,z) q(y)\, dy / M(x,z)}.$$

If also the operator defined by (1.5) satisfies $\|T\|_{L^2(q) \to L^2(q)} < 1$, then there exists $C = C(\Omega, \|T\|)$ such that

$$\mathcal{M}(x, z) \leq M(x, z) e^{C \int_{\Omega} G(x,y) M(y,z) q(y)\, dy / M(x,z)}.$$

Hence, by Eq. (4.5), we obtain the pointwise estimate for the solution of (1.1) on a uniform domain (under the assumption $\|T\| < 1$):

$$u(x) \leq \int_{\partial\Omega} e^{C \int_{\Omega} G(x,y) M(y,z) q(y)\, dy / M(x,z)} f(z)\, dH^{x}(z),$$

using Eq. (4.2). There is also the lower bound

$$u(x) \geq \int_{\partial\Omega} e^{\int_\Omega G(x,y)M(y,z)q(y)\,dy/M(x,z)} f(z)\,dH^x(z),$$

which holds without any requirement on $\|T\|$. See Theorem 1.1 in [10] and its proof for details.

Hence, we have full analogues for uniform domains of Theorem 3.3, 3.5, and 3.9 for C^2 domains. The key points are to identify appropriate q–m modifiers and apply Corollaries 3.2 and 3.4 of Theorem 2.2.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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