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ORIGINAL RESEARCH ARTICLE

Anisotropic Fractional Sobolev Space Restricted on a Bounded Domain

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Abstract

This paper studies the anisotropic fractional Sobolev space restricted on a bounded domain in the Euclidean space $\mathbb{R}^n$ with fractional order $\alpha \in [0, n]$, which complements the previous theory with fractional order $\alpha > n$. We investigate the seminorm of the characteristic function as the anisotropic fractional perimeter restricted on a bounded domain, and systematically establish its metric properties including the upper bound estimation. For application, we prove the embedding law with respect to the anisotropic fractional Sobolev space and the Radon measure based Lebesgue space restricted on a bounded domain by the intrinsic geometric characterization.

Keywords Anisotropic fractional Sobolev space · Anisotropic fractional perimeter · Anisotropic fractional Sobolev embedding

1 Preliminaries

For $p \geqslant 1$, $0 < s < 1$ and $\Omega \subset \mathbb{R}^n$, the fractional Sobolev space is introduced by Gagliardo in [3] including all the functions $f \in L^p(\Omega)$ with the fractional Sobolev s-seminorm

$$\|f\|^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} \, dxdy < +\infty. \tag{1}$$

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The fractional Sobolev space has been widely developed with respect to various aspects in mathematics and applied mathematics. For example, it plays important role in the trace problems of the Sobolev space (see also in [3]). For more applications, we refer to [1, 2, 7, 10–12].

For $s \in (0, 1)$, the fractional s-perimeter of a Borel set $E \subset \mathbb{R}^n$ is defined by

$$P_s(E) = \int_E \int_{E^c} \frac{1}{|x - y|^{n+s}}\, dxdy, \tag{2}$$

where E^c denotes the complement of E in $\mathbb{R}^n$. Fractional perimeter attracts increasing attentions in geometry (see [5] and the references therein), which is closely related to the fractional Sobolev space. Note that, let $p = 1$ and $\Omega = \mathbb{R}^n$, then $\|\mathbf{1}_E\|_{W^{s,1}(\mathbb{R}^n)} = 2P_s(E)$, where $\mathbf{1}_E$ denotes the characteristic function on E.

Recently, both fractional Sobolev space and fractional perimeter have been generalized in an anisotropic way. For this, we need first recall some basic conceptions and results in convex geometry analysis.

A set $K \subsetneqq \mathbb{R}^n$ is called star-shaped with respect to the origin if the intersection of every line through origin with K is a compact line segment. The radial function of K is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\} \;\; \forall \;\; x \in \mathbb{R}^n \setminus o,$$

where o denotes the origin of $\mathbb{R}^n$. If ρ_K is positive and continuous, K is called a star body with respect to the origin and if for any $x \in \mathbb{R}^n \setminus o$, $\rho_K(x) = \rho_K(-x)$, K is called symmetric with respect to the origin. In this paper, we always assume that K is a symmetric star body with respect to the origin.

The Minkowski functional of K, $\|\cdot\|_K$ is defined by:

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\} \;\; \forall \;\; x \in \mathbb{R}^n,$$

where $\lambda K = \{\lambda y : y \in K\}$. Note that $\|x\|_K = \|-x\|_K$ for any $x \in \mathbb{R}^n$ since K is assumed to be symmetric in this paper.

Let $y \in \mathbb{R}^n$, $a > 0$ and

$$B_a^K(y) = \{x \in \mathbb{R}^n : \|x - y\|_K \leq a\}$$

be the K-ball centered at y with radius a. In this paper, let $E \subset \mathbb{R}^n$ be a bounded measurable set, and E^c, $V(E)$ denote the complement of E in $\mathbb{R}^n$ and the n-dimensional volume of E, respectively. Then, it is easy to check that

$$V(B_a^K(y)) = a^n V(K).$$

For more information on convex geometry, we refer to [4] and [9].

The anisotropic fractional Sobolev space and fractional perimeter follows by replacing $|x - y|$ by $\|x - y\|_K$ in respectively (1) and (2), which have been well developed in recent years. For example, Ludwig studies the limiting behavior of the anisotropic fractional Sobolev s-seminorm for both $s \to 1^-$ and $s \to 0^+$ in [6], while the limiting cases of the anisotropic fractional s-perimeter are investigated respectively by also Ludwig in [5] for $s \to 1^-$ and Maz'Ya, Shaposhnikova for $s \to 0^+$ in [8]. The anisotropic Sobolev capacity with fractional order is introduced by Xiao and Ye in [14] with applications to the theory of anisotropic fractional Sobolev space embeddings. Estimation for the anisotropic fractional perimeter is also established in [14], which is optimal in a limiting way.

Note that the fractional orders $n + ps$ in these previous theories are greater than n. As for the fractional orders not more than n, we will study the corresponding theory in this paper. Let Ω be a bounded domain and $\alpha \in [0, n]$ if not specially mentioned in this paper.

Definition 1 The anisotropic fractional Sobolev space restricted on Ω, denoted by $W_K^{\alpha,1}(\Omega)$, is the set of all the functions $f \in L^1(\Omega)$ with the seminorm

$$\|f\|_{W_K^{\alpha,1}(\Omega)} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{\|x - y\|_K^\alpha} \, dxdy < +\infty.$$

Let

$$P_\alpha(E \cap \Omega, K) = \int_{E\cap\Omega} \int_{E^c\cap\Omega} \frac{1}{\|x - y\|_K^\alpha} \, dxdy$$

be the anisotropic fractional perimeter restricted on Ω for a bounded measurable set $E \subset \mathbb{R}^n$ with respect to K. We can check that

$$\|\mathbf{1}_E\|_{W_K^{\alpha,1}(\Omega)} = 2P_\alpha(E \cap \Omega, K), \tag{3}$$

and if $V(E \cap \Omega) = 0$ or $V(E^c \cap \Omega) = 0$, it is easy to check that $\|\mathbf{1}_E\|_{W_K^{\alpha,1}(\Omega)} = 2P_\alpha(E\cap\Omega, K) = 0$, which is trivial. Hence, we will always assume that $V(E\cap\Omega) \neq 0$ and $V(E^c \cap \Omega) \neq 0$ in this paper. Moreover, note that K is symmetric star body with respect to the origin, then by Fubini's theorem, we can check that

$$\begin{aligned} P_\alpha(E \cap \Omega, K) &= \int_{E\cap\Omega} \int_{E^c\cap\Omega} \frac{1}{\|x - y\|_K^\alpha} \, dxdy \\ &= \int_{E^c\cap\Omega} \int_{E\cap\Omega} \frac{1}{\|x - y\|_K^\alpha} \, dydx \\ &= \int_{E^c\cap\Omega} \int_{E\cap\Omega} \frac{1}{\|y - x\|_K^\alpha} \, dydx \\ &= \int_{E^c\cap\Omega} \int_{E\cap\Omega} \frac{1}{\|x - y\|_K^\alpha} \, dxdy. \end{aligned}$$

2 Metric Properties for the Anisotropic Fractional Perimeter Restricted on a Bounded Domain

In this section, we are going to study the metric properties of the anisotropic fractional perimeter restricted on Ω, which induce corresponding properties for the seminorm of characteristic function in the anisotropic fractional Sobolev space restricted on Ω by (3).

Theorem 1 (i) *Homogeneity for K: let $r > 0$, then*

$$P_\alpha\left(E \cap \Omega, rK\right) = r^\alpha P_\alpha\left(E \cap \Omega, K\right).$$

(ii) *Translation: for any $x_0 \in \mathbb{R}^n$,*

$$\begin{cases} P_\alpha\left((x_0 + E) \cap \Omega, K\right) = P_\alpha(E \cap (\Omega - x_0), K), \\ P_\alpha\left(x_0 + E \cap \Omega, K\right) = P_\alpha(E \cap \Omega, K), \end{cases}$$

where $x_0 + E = \{x_0 + y : y \in E\}$, $\Omega - x_0 = \{y - x_0 : y \in \Omega\}$ and

$$x_0 + E \cap \Omega = \{x_0 + y : y \in E \cap \Omega\}.$$

(iii) *Interpolation: let $0 \leqslant \alpha < \beta < \gamma \leq n$, then*

$$\begin{cases} [P_\beta(E \cap \Omega, K)]^{\gamma-\alpha} \leqslant [P_\alpha(E \cap \Omega, K)]^{\gamma-\beta}[P_\gamma(E \cap \Omega, K)]^{\beta-\alpha}, \\ \ln[P_\beta(E \cap \Omega, K)] \leqslant \frac{\gamma-\beta}{\gamma-\alpha} \ln[P_\alpha(E \cap \Omega, K)] + \frac{\beta-\alpha}{\gamma-\alpha} \ln[P_\gamma(E \cap \Omega, K)], \end{cases} \tag{4}$$

and

$$\beta \mapsto \left[\frac{P_\beta(E \cap \Omega, K)}{V(E \cap \Omega)V(E^c \cap \Omega)}\right]^{1/\beta}$$

is increasing on $(0, n)$.

Proof (i) Note that $\|x - y\|_{rK} = r^{-1}\|x - y\|_K$ holds for any $x, y \in \mathbb{R}^n$, then

$$\begin{aligned} P_\alpha\left(E \cap \Omega, rK\right) &= \int_{E\cap\Omega} \int_{E^c\cap\Omega} \frac{1}{\|x - y\|_{rK}^\alpha} dxdy \\ &= r^\alpha \int_{E\cap\Omega} \int_{E^c\cap\Omega} \frac{1}{\|x - y\|_K^\alpha} dxdy \\ &= r^\alpha P_\alpha\left(E \cap \Omega, K\right). \end{aligned}$$

(ii) Note that $(x_0 + E)^c = x_0 + E^c$, then

$$
\begin{aligned}
P_\alpha\left((x_0+E)\cap\Omega, K\right) &= \int_{(x_0+E)\cap\Omega}\left(\int_{(x_0+E)^c\cap\Omega}\frac{1}{\|x-y\|_K^\alpha}\,dx\right)dy\\
&= \int_{x_0+E\cap(\Omega-x_0)}\left(\int_{x_0+E^c\cap(\Omega-x_0)}\frac{1}{\|x-y\|_K^\alpha}\,dx\right)dy\\
&= \int_{x_0+E\cap(\Omega-x_0)}\left(\int_{E^c\cap(\Omega-x_0)}\frac{1}{\|z+x_0-y\|_K^\alpha}\,dz\right)dy\\
&= \int_{x_0+E\cap(\Omega-x_0)}\left(\int_{E^c\cap(\Omega-x_0)}\frac{1}{\|z-(y-x_0)\|_K^\alpha}\,dz\right)dy\\
&= \int_{E\cap(\Omega-x_0)}\left(\int_{E^c\cap(\Omega-x_0)}\frac{1}{\|z-w\|_K^\alpha}\,dz\right)dw\\
&= P_\alpha(E\cap(\Omega-x_0), K),
\end{aligned}
$$

where we let $x=z+x_0$ and $y=w+x_0$. Hence, it follows that

$$
\begin{aligned}
P_\alpha\left(x_0+E\cap\Omega, K\right) &= P_\alpha\left((x_0+E)\cap(x_0+\Omega), K\right)\\
&= P_\alpha(E\cap\Omega, K).
\end{aligned}
$$

(iii) Note that $0<\frac{\gamma-\beta}{\gamma-\alpha}<1$, $0<\frac{\beta-\alpha}{\gamma-\alpha}<1$ and $\frac{\gamma-\beta}{\gamma-\alpha}+\frac{\beta-\alpha}{\gamma-\alpha}=1$. Hence, by Hölder's inequality, it follows that

$$
\begin{aligned}
&P_\beta(E\cap\Omega, K)\\
&= \int_{E\cap\Omega}\int_{E^c\cap\Omega}\frac{1}{\|x-y\|_K^\beta}\,dxdy\\
&= \int_{E\cap\Omega}\int_{E^c\cap\Omega}\left(\frac{1}{\|x-y\|_K^\alpha}\right)^{(\gamma-\beta)/(\gamma-\alpha)}\left(\frac{1}{\|x-y\|_K^\gamma}\right)^{(\beta-\alpha)/(\gamma-\alpha)}dxdy\\
&\leqslant \int_{E\cap\Omega}\left(\int_{E^c\cap\Omega}\frac{dx}{\|x-y\|_K^\alpha}\right)^{(\gamma-\beta)/(\gamma-\alpha)}\left(\int_{E^c\cap\Omega}\frac{dx}{\|x-y\|_K^\gamma}\right)^{(\beta-\alpha)/(\gamma-\alpha)}dy\\
&\leqslant \left(\int_{E\cap\Omega}\int_{E^c\cap\Omega}\frac{dxdy}{\|x-y\|_K^\alpha}\right)^{(\gamma-\beta)/(\gamma-\alpha)}\left(\int_{E\cap\Omega}\int_{E^c\cap\Omega}\frac{dxdy}{\|x-y\|_K^\gamma}\right)^{(\beta-\alpha)/(\gamma-\alpha)}\\
&= \left(P_\alpha(E\cap\Omega, K)\right)^{(\gamma-\beta)/(\gamma-\alpha)}\left(P_\gamma(E\cap\Omega, K)\right)^{(\beta-\alpha)/(\gamma-\alpha)},
\end{aligned}
$$

which implies the desired inequalities (4) by taking power $\gamma-\alpha$ to both sides and applying the logarithmic function to both sides.

Let $\alpha=0$. Then $P_0(E\cap\Omega, K)=V(E\cap\Omega)V(E^c\cap\Omega)$ and it follows from (4) that

$$
P_\beta(E\cap\Omega, K)\leqslant\left(V(E\cap\Omega)V(E^c\cap\Omega)\right)^{(\gamma-\beta)/\gamma}\left(P_\gamma(E\cap\Omega, K)\right)^{\beta/\gamma},
$$

which implies

$$\left[\frac{P_\beta(E\cap\Omega, K)}{V(E\cap\Omega)V(E^c\cap\Omega)}\right]^{1/\beta} \leqslant \left[\frac{P_\gamma(E\cap\Omega, K)}{V(E\cap\Omega)V(E^c\cap\Omega)}\right]^{1/\gamma}.$$

Hence,

$$\beta \mapsto \left[\frac{P_\beta(E\cap\Omega, K)}{V(E\cap\Omega)V(E^c\cap\Omega)}\right]^{1/\beta}$$

is increasing on $(0, n)$. □

We can establish the upper bound estimation for the anisotropic fractional perimeter restricted on Ω and the seminorm of characteristic function in the anisotropic fractional Sobolev space restricted on Ω.

Theorem 2 *Let* $\alpha \in [0, n)$. *Then*

$$\begin{aligned}\|\mathbf{1}_E\|_{W_K^{\alpha,1}(\Omega)} &= 2P_\alpha(E\cap\Omega, K)\\ &\le \frac{2n}{n-\alpha}V(E\cap\Omega)V(E^c\cap\Omega)\left(\frac{V(K)}{\max(V(E\cap\Omega), V(E^c\cap\Omega))}\right)^{\frac{\alpha}{n}}.\end{aligned} \tag{5}$$

Proof It is easy to check that the desired inequality (5) holds trivially if $V(E\cap\Omega) = 0$, or $V(E^c\cap\Omega) = 0$, or $\alpha = 0$. Hence, we will suppose $V(E\cap\Omega) \neq 0$, $V(E^c\cap\Omega) \neq 0$ and $\alpha \in (0, n)$ in the following proof. Let $y \in E^c\cap\Omega$ and $B_r^K(y)$ be the K−ball with center y and radius

$$r = \left(\frac{V(E\cap\Omega)}{V(K)}\right)^{\frac{1}{n}} > 0.$$

Note that $V(B_r^K(y)) = V(\{x : \|x-y\|_K \le r\}) = r^n V(K) = V(E\cap\Omega)$ and hence

$$V((E\cap\Omega)^c \cap B_r^K(y)) = V((B_r^K(y)^c \cap (E\cap\Omega)),$$

which, together with the fact

$$\begin{cases}\|x-y\|_K \le r, & \forall x \in (E\cap\Omega)^c \cap B_r^K(y);\\ \|x-y\|_K > r, & \forall x \in (B_r^K(y))^c \cap (E\cap\Omega),\end{cases}$$

implies

$$\int_{(E\cap\Omega)^c\cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} \geq \frac{V(B_r^K(y)\cap(E\cap\Omega)^c)}{r^\alpha}$$
$$= \frac{V((B_r^K(y))^c\cap(E\cap\Omega))}{r^\alpha}$$
$$\geq \int_{(B_r^K(y))^c\cap(E\cap\Omega)} \frac{dx}{\|x-y\|_K^\alpha}.$$

Hence, it follows that

$$\begin{aligned}\int_{E\cap\Omega} \frac{dx}{\|x-y\|_K^\alpha} &= \int_{(E\cap\Omega)\cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} + \int_{(E\cap\Omega)\cap(B_r^K(y))^c} \frac{dx}{\|x-y\|_K^\alpha}\\ &\leq \int_{(E\cap\Omega)\cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} + \int_{(E\cap\Omega)^c\cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha}\\ &= \int_{B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha}. \end{aligned} \tag{6}$$

Then by Fubini's theorem, we have

$$\begin{aligned}\int_{B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} &= \int_{\{x:\|x-y\|_K\leq r\}} \left(\int_{\|x-y\|_K}^\infty \alpha t^{-1-\alpha}\,dt\right) dx\\ &= \int_r^\infty \alpha t^{-1-\alpha} \left(\int_{\{x:\|x-y\|_K\leq r\}} dx\right) dt\\ &\quad + \int_0^r \alpha t^{-1-\alpha} \left(\int_{\{x:\|x-y\|_K\leq t\}} dx\right) dt\\ &= r^n V(K) \int_r^\infty \alpha t^{-1-\alpha}\,dt + V(K)\int_0^r \alpha t^{n-\alpha-1}\,dt\\ &= r^{n-\alpha}V(K) + \frac{\alpha}{n-\alpha} r^{n-\alpha} V(K)\\ &= \frac{n}{n-\alpha} V(E\cap\Omega)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}, \end{aligned}$$

which, together with (6), implies

$$\begin{aligned}P_\alpha(E\cap\Omega, K) &= \int_{E\cap\Omega}\int_{E^c\cap\Omega} \frac{1}{\|x-y\|_K^\alpha}\,dxdy\\ &= \int_{E^c\cap\Omega}\int_{E\cap\Omega} \frac{1}{\|x-y\|_K^\alpha}\,dxdy\\ &\leq \int_{E^c\cap\Omega} \frac{n}{n-\alpha} V(E\cap\Omega)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}\,dy\\ &= \frac{n}{n-\alpha} V(E\cap\Omega)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} V(E^c\cap\Omega). \end{aligned} \tag{7}$$

On the other hand, similar way can be applied to get

$$\int_{E^c\cap\Omega}\frac{dx}{\|x-y\|_K^\alpha}\leq\frac{n}{n-\alpha}V(E^c\cap\Omega)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}},$$

and hence

$$\begin{aligned}P_\alpha(E\cap\Omega,K)&=\int_{E\cap\Omega}\int_{E^c\cap\Omega}\frac{1}{\|x-y\|_K^\alpha}\,dxdy\\&\leq\int_{E\cap\Omega}\frac{n}{n-\alpha}V(E^c\cap\Omega)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}dy\\&=\frac{n}{n-\alpha}V(E^c\cap\Omega)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}V(E\cap\Omega),\end{aligned}$$

which, together with (7), implies (5) holds. □

By Theorem 2, we can explore more metric properties of the anisotropic fractional perimeter restricted on Ω, including the uniform continuity and regularity, which induce corresponding metric properties of the seminorm of characteristic function in the anisotropic fractional Sobolev space restricted on Ω, and contribute to the anisotropic fractional Sobolev embedding restricted on Ω in the next section.

Theorem 3 *Let $0\leq\alpha<n$ and $E, G\subset\mathbb{R}^n$ be bounded measurable sets with $V(E\Delta G)=0$, where $E\Delta G=(E^c\cap G)\cup(E\cap G^c)$, then*

$$P_\alpha(E\cap\Omega,K)=P_\alpha(G\cap\Omega,K).$$

Proof For any $x\in\mathbb{R}^n$, it follows by Theorem 2 that

$$\begin{aligned}\left|\int_{E\cap\Omega}\frac{dy}{\|x-y\|_K^\alpha}-\int_{G\cap\Omega}\frac{dy}{\|x-y\|_K^\alpha}\right|&=\int_{(E\Delta G)\cap\Omega}\frac{dy}{\|x-y\|_K^\alpha}\\&\leq\frac{n}{n-\alpha}V((E\Delta G)\cap\Omega)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}\\&=0,\end{aligned}$$

which implies $\int_{E\cap\Omega}\frac{dy}{\|x-y\|_K^\alpha}\equiv\int_{G\cap\Omega}\frac{dy}{\|x-y\|_K^\alpha}$, and hence

$$\begin{aligned}P_\alpha(E\cap\Omega,K)&=\int_{E\cap\Omega}\int_{E^c\cap\Omega}\frac{1}{\|x-y\|_K^\alpha}\,dxdy\\&=\int_{E^c\cap\Omega}\int_{E\cap\Omega}\frac{1}{\|x-y\|_K^\alpha}\,dydx\end{aligned}$$

$$
\begin{aligned}
&= \int_{G^c\cap\Omega}\int_{G\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dydx\\
&= \int_{G\cap\Omega}\int_{G^c\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dxdy\\
&= P_\alpha(G\cap\Omega, K).
\end{aligned}
$$

□

Theorem 4 *Let $0 \le \alpha < n$. $P_\alpha(\cdot\cap\Omega, K)$ is uniformly continuous in the following way: for any $\varepsilon > 0$, there exists $\delta > 0$, such that, for any bounded measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$ with $V(E_1\Delta E_2) < \delta$, it follows that*

$$
|P_\alpha(E_1\cap\Omega, K) - P_\alpha(E_2\cap\Omega, K)| < \varepsilon.
$$

Proof For any $\varepsilon > 0$, let

$$
\delta = \min\left\{\frac{\varepsilon(n-\alpha)}{4n}V(\Omega)^{\frac{\alpha-n}{n}}V(K)^{\frac{-\alpha}{n}}, \left(\frac{\varepsilon(n-\alpha)}{4n}V(\Omega)^{-1}V(K)^{\frac{-\alpha}{n}}\right)^{\frac{n}{n-\alpha}}\right\}.
$$

Then, for any bounded measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$ with $V(E_1\Delta E_2) < \delta$, by Fubini's theorem, we have

$$
\begin{aligned}
&|P_\alpha(E_1\cap\Omega, K) - P_\alpha(E_2\cap\Omega, K)|\\
&= \left|\int_{E_1\cap\Omega}\int_{E_1^c\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dxdy - \int_{E_2\cap\Omega}\int_{E_2^c\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dxdy\right|\\
&= \left|\int_{E_1^c\cap\Omega}\int_{E_1\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dydx - \int_{E_2^c\cap\Omega}\int_{E_2\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dydx\right|\\
&\le \left|\int_{(E_1^c\Delta E_2^c)\cap\Omega}\int_{E_1\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dydx\right| + \left|\int_{(E_1^c\Delta E_2^c)\cap\Omega}\int_{E_2\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dydx\right|\\
&\quad + \left|\int_{(E_1^c\cap E_2^c)\cap\Omega}\left(\int_{E_1\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dy - \int_{E_2\cap\Omega} \frac{1}{\|x-y\|_K^{\alpha}}\,dy\right)dx\right|\\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

Note that $E_1\Delta E_2 = E_1^c\Delta E_2^c$ and hence $V(E_1\Delta E_2) = V(E_1^c\Delta E_2^c) < \delta$. Then, by Theorem 2, it follows that

$$
\begin{aligned}
I_1 &\le \frac{n}{n-\alpha}V((E_1^c\Delta E_2^c)\cap\Omega)V(E_1\cap\Omega)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}\\
&\le \frac{n\delta}{n-\alpha}V(\Omega)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}\\
&< \frac{\varepsilon}{3}.
\end{aligned}
$$

Estimation for I_2 follows in a similar way:

$$
\begin{aligned}
I_2 &\leq \frac{n}{n-\alpha} V((E_1^c \Delta E_2^c) \cap \Omega) V(E_2 \cap \Omega)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} \\
&\leq \frac{n\delta}{n-\alpha} V(\Omega)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} \\
&< \frac{\varepsilon}{3}.
\end{aligned}
$$

For I_3, by Theorem 2 again, we have

$$
\begin{aligned}
I_3 &\leq \int_{(E_1^c \cap E_2^c)\cap\Omega} \left| \int_{E_1\cap\Omega} \frac{dy}{\|x-y\|_K^\alpha} - \int_{E_2\cap\Omega} \frac{dy}{\|x-y\|_K^\alpha} \right| dx \\
&= \int_{(E_1^c \cap E_2^c)\cap\Omega} \left| \int_{(E_1\Delta E_2)\cap\Omega} \frac{dy}{\|x-y\|_K^\alpha} \right| dx \\
&\leq \frac{n}{n-\alpha} V((E_1^c \cap E_2^c) \cap \Omega) V((E_1 \Delta E_2) \cap \Omega)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} \\
&\leq \frac{n}{n-\alpha} V(\Omega) \delta^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} \\
&< \frac{\varepsilon}{3}.
\end{aligned}
$$

In conclusion,

$$
|P_\alpha(E_1 \cap \Omega, K) - P_\alpha(E_2 \cap \Omega, K)| = I_1 + I_2 + I_3 < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
$$

□

By theorem 4, we have the following corollary for the regularity of the anisotropic fractional perimeter restricted on Ω, which induces corresponding regularity of the seminorm of characteristic function in the anisotropic fractional Sobolev space restricted on Ω.

Corollary 5 *Let* $0 \leq \alpha < n$.
(i) *For any bounded measurable set E and any open set sequence* $\{O_n\}_{n\in\mathbb{N}_+}$ *decreasing to E, which means* $O_n \supseteq O_{n+1} \supseteq E$ *for any* $n \in \mathbb{N}_+$ *and for any* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}_+$ *such that* $V(O_n \setminus E) < \varepsilon$ *for* $n > N$*, it follows that*

$$
P_\alpha(E \cap \Omega, K) = \lim_{n\to\infty} P_\alpha(O_n \cap \Omega, K).
$$

(ii) *For any bounded measurable open set O and any compact set sequence* $\{L_n\}_{n\in\mathbb{N}_+}$ *increasing to O, which means* $L_n \subseteq L_{n+1} \subseteq O$ *for any* $n \in \mathbb{N}_+$ *and for any* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}_+$ *such that* $V(O \setminus L_n) < \varepsilon$ *for* $n > N$*, it follows that*

$$
P_\alpha(O \cap \Omega, K) = \lim_{n\to\infty} P_\alpha(L_n \cap \Omega, K).
$$

Proof (i) By theorem 4, for any $\varepsilon > 0$, there exists $\delta > 0$, such that, for any bounded measurable sets O with $V(E\Delta O) < \delta$, it follows that

$$|P_\alpha(E \cap \Omega, K) - P_\alpha(O \cap \Omega, K)| < \varepsilon.$$

For this $\delta > 0$ and the open set sequence $\{O_n\}_{n\in\mathbb{N}_+}$ decreasing to E, there exists $N \in \mathbb{N}_+$ such that $V(O_n \setminus E) = V(O_n \Delta E) < \delta$ for $n > N$, and hence

$$|P_\alpha(E \cap \Omega, K) - P_\alpha(O_n \cap \Omega, K)| < \varepsilon \text{ for } n > N,$$

which implies

$$P_\alpha(E \cap \Omega, K) = \lim_{n\to\infty} P_\alpha(O_n \cap \Omega, K).$$

(ii) The proof is similar with (i) and we omit the details here. □

3 Anisotropic Fractional Sobolev Inequality Restricted on a Bounded Domain

In this section, we will establish the embedding from anisotropic fractional Sobolev space restricted on Ω to the Radon measure based Lebesgue space restricted on Ω by the intrinsic geometric characterization. Before this, we need the following lemma with respect to the coarea formula for the anisotropic fractional Sobolev space restricted on Ω.

Lemma 6 *Let $f \in W_K^{\alpha,1}(\Omega)$ and $O_t(f) = \{x \in \mathbb{R}^n : |f(x)| > t\}$ for $t \geq 0$. Then*

$$\|f\|_{W_K^{\alpha,1}(\Omega)} = 2\int_0^\infty P_\alpha\left(O_t(f) \cap \Omega, K\right) dt.$$

Proof Note that $f \in L^1(\Omega)$ since $f \in W_K^{\alpha,1}(\Omega)$ and Visintin in [13] pointed out that as a consequence of Fubini's theorem, a generalized coarea formula for the anisotropic fractional Sobolev space restricted on Ω can be established:

$$\begin{aligned}
\|f\|_{W_K^{\alpha,1}(\Omega)} &= \int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{\|x-y\|_K^\alpha}\,dxdy \\
&= 2\int_{-\infty}^{+\infty} P_\alpha(\{x \in \mathbb{R}^n : f(x) > t\} \cap \Omega, K)dt \\
&= 2\int_0^{+\infty} P_\alpha(\{x \in \mathbb{R}^n : f(x) > t\} \cap \Omega, K)dt \\
&\quad + 2\int_{-\infty}^0 P_\alpha(\{x \in \mathbb{R}^n : f(x) > t\} \cap \Omega, K)dt
\end{aligned}$$

$$= 2\int_0^{+\infty} P_\alpha(\{x \in \mathbb{R}^n : f(x) > t\} \cap \Omega, K)dt$$
$$+ 2\int_0^{+\infty} P_\alpha(\{x \in \mathbb{R}^n : f(x) > -s\} \cap \Omega, K)ds, \tag{8}$$

where the variable changing $t = -s$ is applied in the last equality. Note that K is origin symmetric, then, for any $s \in (0, +\infty)$, it follows that

$$\begin{aligned} P_\alpha(\{x \in \mathbb{R}^n : f(x) > -s\} \cap \Omega, K) &= P_\alpha(\{x \in \mathbb{R}^n : f(x) > -s\}^c \cap \Omega, K) \\ &= P_\alpha(\{x \in \mathbb{R}^n : f(x) \le -s\} \cap \Omega, K) \\ &= P_\alpha(\{x \in \mathbb{R}^n : f(x) < -s\} \cap \Omega, K), \end{aligned}$$

where the last equality holds by Theorem 3 since $V(\{x \in \mathbb{R}^n : f(x) = -s\}) = 0$ almost everywhere. This, together with (8), implies

$$\begin{aligned} \|f\|_{W_K^{\alpha,1}(\Omega)} &= 2\int_0^{+\infty} P_\alpha(\{x \in \mathbb{R}^n : f(x) > t\} \cap \Omega, K)dt \\ &\quad + 2\int_0^{+\infty} P_\alpha(\{x \in \mathbb{R}^n : f(x) < -s\} \cap \Omega, K)ds \\ &= 2\int_0^{+\infty} P_\alpha(\{x \in \mathbb{R}^n : |f(x)| > t\} \cap \Omega, K)dt \\ &= 2\int_0^{\infty} P_\alpha\left(O_t(f) \cap \Omega, K\right) dt. \end{aligned}$$

□

Theorem 7 *Let μ be a nonnegative Radon measure on $\mathbb{R}^n$, $p \ge 1$ and $0 \le \alpha < n$. The following two inequalities are equivalent.*
(i) *The anisotropic fractional Sobolev inequality restricted on Ω: there is a constant $c > 0$ such that*

$$\|f\|_{L^p_\mu(\Omega)} \le c\|f\|_{W_K^{\alpha,1}(\Omega)} \ \ \forall \ \ f \in W_K^{\alpha,1}(\Omega). \tag{9}$$

(ii) *The anisotropic fractional isoperimetric inequality restricted on Ω: there is a constant $c > 0$, such that for any bounded measurable set $E \subset \mathbb{R}^n$,*

$$\mu(E \cap \Omega)^{\frac{1}{p}} \le 2cP_\alpha(E \cap \Omega, K). \tag{10}$$

Proof (i) $\Rightarrow$ (ii) Suppose (9) holds true, then for any compact set $L \subseteq \mathbb{R}^n$ and any $\epsilon \in (0, 1)$, let

$$f_\epsilon(x) = \begin{cases} 1 - \epsilon^{-1}\mathrm{dist}(x, L), & \text{if } \mathrm{dist}(x, L) < \epsilon, \\ 0, & \text{if } \mathrm{dist}(x, L) \ge \epsilon, \end{cases}$$

where $\mathrm{dist}(x, L) = \inf\{|x - y| : y \in L\}$ denotes the Euclidean distance of the point x and the set L. Let L_{f_ϵ} be the support set of f_ϵ. Note that $0 \leq f_\epsilon \leq \mathbf{1}_{L_\epsilon}$ and $f_\epsilon \in L^1(\Omega)$, then by a similar estimation as in Theorem 2, it follows that

$$\begin{aligned}
\|f_\epsilon(x)\|_{W_K^{\alpha,1}(\Omega)} &= \int_\Omega\int_\Omega \frac{|f_\epsilon(x) - f_\epsilon(y)|}{\|x-y\|_K^\alpha}\,dxdy \\
&\leq \int_\Omega\int_\Omega \frac{|f_\epsilon(x)|}{\|x-y\|_K^\alpha}\,dxdy + \int_\Omega\int_\Omega \frac{|f_\epsilon(y)|}{\|x-y\|_K^\alpha}\,dxdy \\
&\leq 2\int_\Omega\int_\Omega \frac{\mathbf{1}_{L_\epsilon}(x)}{\|x-y\|_K^\alpha}\,dxdy \\
&\leq \frac{2n}{n-\alpha} V(L_\epsilon \cap \Omega)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} V(\Omega) \\
&< +\infty,
\end{aligned}$$

which implies $f_\epsilon \in W_K^{\alpha,1}(\Omega)$. Then, by (9), it follows that

$$\begin{aligned}
\mu(L\cap\Omega)^{\frac{1}{p}} &= \left(\int_\Omega \mathbf{1}_L d\mu(x)\right)^{\frac{1}{p}} \leq \left(\int_\Omega f_\epsilon(x)^p d\mu(x)\right)^{\frac{1}{p}} \\
&= \|f_\epsilon\|_{L_\mu^p(\Omega)} \leq c\|f_\epsilon\|_{W_K^{\alpha,1}(\Omega)}.
\end{aligned}$$

Let $\epsilon \to 0^+$, then by the dominated convergence theorem, it follows that

$$\begin{aligned}
\mu(L\cap\Omega)^{\frac{1}{p}} &\leq \lim_{\epsilon\to 0^+} c\|f_\epsilon\|_{W_K^{\alpha,1}(\Omega)} = c\,\|\mathbf{1}_L\|_{W_K^{\alpha,1}(\Omega)} \\
&= 2cP_\alpha(L\cap\Omega, K).
\end{aligned} \tag{11}$$

For any open set $O \subseteq \mathbb{R}^n$, there exists a sequence of compact sets $\{L_n\}_{n\in\mathbb{N}_+}$ increasing to O. By Corollary 5 and (11), it follows that

$$\begin{aligned}
\mu(O\cap\Omega)^{\frac{1}{p}} &= \lim_{n\to\infty} \mu(L_n\cap\Omega)^{\frac{1}{p}} \leq \lim_{n\to\infty} 2cP_\alpha(L_n\cap\Omega, K) \\
&= 2cP_\alpha(O\cap\Omega, K).
\end{aligned} \tag{12}$$

For any bounded measurable set $E \subseteq \mathbb{R}^n$, there exists a sequence of open sets $\{O_n\}_{n\in\mathbb{N}_+}$ decreasing to E. By Corollary 5 and (12), it follows that

$$\begin{aligned}
\mu(E\cap\Omega)^{\frac{1}{p}} &= \lim_{n\to\infty} \mu(O_n\cap\Omega)^{\frac{1}{p}} \leq \lim_{n\to\infty} 2cP_\alpha(O_n\cap\Omega, K) \\
&= 2cP_\alpha(E\cap\Omega, K).
\end{aligned}$$

(ii) $\Rightarrow$ (i) Assume (10) holds. Let $f \in W_K^{\alpha,1}(\Omega)$. Obviously, $\mu\left(O_t(f)\right)$ is a decreasing function on $t \in [0, \infty)$, and hence for $p \geq 1$, by Fubini's theorem, it follows that

$$
\begin{aligned}
\|f\|_{L^p_\mu(\Omega)} &= \left(\int_\Omega |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \\
&= \left(\int_\Omega \left[\int_0^{|f(x)|} pt^{p-1}\, dt\right] d\mu(x)\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left[\int_{O_t(f)\cap\Omega} pt^{p-1}\, d\mu(x)\right] dt\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \mu\left(O_t(f)\cap\Omega\right) dt^p\right)^{\frac{1}{p}} \\
&= \int_0^\infty \frac{d}{dt}\left(\int_0^t \mu\left(O_s(f)\cap\Omega\right) ds^p\right)^{\frac{1}{p}} dt \\
&= \int_0^\infty \left(\int_0^t \mu\left(O_s(f)\cap\Omega\right) ds^p\right)^{\frac{1}{p}-1} \mu\left(O_t(f)\cap\Omega\right) t^{p-1} dt \\
&\le \int_0^\infty \left(\int_0^t \mu\left(O_t(f)\cap\Omega\right) ds^p\right)^{\frac{1}{p}-1} \mu\left(O_t(f)\cap\Omega\right) t^{p-1} dt \\
&= \int_0^\infty \left(\mu\left(O_t(f)\cap\Omega\right)\right)^{\frac{1}{p}} dt,
\end{aligned}
$$

which, together with (10) and lemma 6, implies, for any $f \in C_0^\infty$,

$$
\begin{aligned}
\|f\|_{L^p_\mu} &\le \int_0^\infty \left(\mu\left(O_t(f)\cap\Omega\right)\right)^{\frac{1}{p}} dt \\
&\le 2c \int_0^\infty P_\alpha\left(O_t(f)\cap\Omega, K\right) dt \\
&= c\|f\|_{W^{\alpha,1}_K(\Omega)}.
\end{aligned}
$$

□

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Declarations

Conflict of interest The authors declare that there is no Conflict of interest.

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