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ORIGINAL RESEARCH ARTICLE

Topological Variants of Vector Variational Inequality Problem

Satish Kumar[1] · Ankit Gupta[2] · Pankaj Kumar Garg[3]

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Abstract
In this paper, we provide a couple of solutions for the vector variational inequality problem, adopting a topological approach. We consider here a more general framework, where X and Y are topological vector spaces. Topological concepts including continuity, compactness, closedness, and so on are used for obtaining our results. The condition of admissibility of the function space topology is found to play an important role in achieving the results. It is found that the solution sets so obtained are closed as well as compact.

Keywords Vector variational inequality · KKM-mapping · Set-valued function · Topological vector space · Compactness

Mathematics Subject Classification 49J40 · 54H99 · 58E35

1 Introduction

The notion of variational inequality was initially introduced by Stampacchia [25] and Fichera [10] in 1964. Variational inequality theory has a variety of applications including in mathematics, physics, economics, and in engineering. In 1980, this concept was extended by F. Gianessi [11] for finite dimensional spaces. Since then, the vector

✉ Pankaj Kumar Garg
gargpk08@gmail.com

Satish Kumar
satishataria47@gmail.com

Ankit Gupta
ankitsince1988@yahoo.co.in

1 Department of Mathematics, University of Delhi, Delhi 110007, India

2 Department of Mathematics, Bharati College (University of Delhi), Delhi 110058, India

3 Department of Mathematics, Rajdhani College (University of Delhi), Delhi 110015, India

variational inequalities (VVI) and their generalizations have become an important tool to solve vector optimization problems. In particular, several relations between vector variational inequalities and vector optimization problems have been investigated in [18, 19, 26, 28, 29]. Further, the concept of vector variational inequalities was studied and extended for abstract spaces by Chen along with Cheng and Craven [4–6] and many others [24, 27].

In [3] and in [1, 20], researchers have studied the generalized variational-type inequality and VVI, respectively, in the framework of topological vector spaces. Hung and others studied the concept of generalized quasi-variational inequalities [14–16]. In 2017, Salahuddin [23] provided the existence conditions for the solution of general set-valued vector variational inequalities. In the same year, Li and Yu [21] introduced a class of generalized invex functions, termed as (α-ρ-η)-invex functions and proposed the existence results for two types of vector variational-like inequalities. On the other hand, in 2018, Kim et al. [17] introduced a class of η-generalized operator variational-like inequalities. In 2019, Farajzadeh, Chen, and others [7, 9] studied vector equilibrium problem for set-valued mappings. Recently, Gupta et al. [12] provided existence conditions for the solution of two variants of generalized non-linear vector variational-like inequality problem by adopting topological approach.

In [6, 30], the authors have studied vector variational inequalities for real Banach spaces X and Y by taking a mapping $F : X \to \mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ is the space of all bounded linear operators from X to Y.

In the following, we denote a set-valued map F from X to Y by the notation $F : X \rightrightarrows Y$, where domain of F is X and co-domain is the power set of Y.

The vector variational inequality problem (VVIP) proposed by Chen in [6] can be presented as:

Vector variational inequality problem: Let X and Y be two real Banach spaces and let K be a nonempty closed convex subset of X. Let $T : K \to \mathcal{L}(X, Y)$ be a mapping, where $\mathcal{L}(X, Y)$ denotes the space of all continuous linear mappings from X to Y. Further, let $C : X \rightrightarrows Y$ be a set-valued map such that for each $x \in K$, $C(x)$ is a closed convex pointed cone in Y with int $C(x) \neq \emptyset$, where int $C(x)$ denotes the interior of $C(x)$. Then the vector variational inequality problem (VVIP) is to find $x_0 \in K$ such that

$$\langle T(x_0), x - x_0 \rangle \notin -\text{int}C(x_0) \quad \forall x \in K.$$

Chen gave conditions for existence of solutions for the above-mentioned VVIP by using the monotonicity and hemicontinuity of the map T. In [4], Chen and Cheng have taken a closed convex pointed cone C instead of $C(x)$ to study VVIP. Motivated by their work, we study here vector variational inequalities in a more general framework by taking X, Y as topological vector spaces instead of Banach spaces. In this paper, we consider two variants of VVIP as follows:

Let X and Y be two topological vector spaces and let K be a nonempty closed convex subset of X. Let $T : K \to \mathcal{C}(X, Y)$ be a single-valued mapping, where $\mathcal{C}(X, Y)$ is the space of all continuous linear mappings from X to Y.

VVIP(I) If C is a closed convex pointed cone in Y with int $C \neq \emptyset$, then the vector variational inequality problem is to find $x_0 \in K$ such that

$$T_{x_0}(x - x_0) \notin -\text{int}C \quad \forall\, x \in K.$$

VVIP(II) If $C : K \rightrightarrows Y$ is a set-valued map such that for each $x \in K$, $C(x)$ is a closed convex pointed cone in Y with int $C(x) \neq \emptyset$, then the vector variational inequality problem is to find $x_0 \in K$ such that

$$T_{x_0}(x - x_0) \notin -\text{int}C(x_0) \quad \forall\, x \in K,$$

where T_{x_0} denotes the value of T at x_0, that is $T_{x_0} = T(x_0)$.

In this study, we provide the conditions for existence of solutions for both the variants of the VVIP by using topological approach by assuming the continuity of the map T. The concepts of net theory, topology of the function space (admissible topology) and KKM-Theorem play an important role in obtaining our results.

The remaining part of the paper is organized as follows: In section 2, we recall some preliminaries required in the sequel. In section 3, we provide the existence theorems for the VVIP(I) and VVIP(II). We then give some topological properties of the solution sets so obtained. Also, we give an example to illustrate our results.

2 Preliminaries

In this section, we provide some definitions and basic results which will be used later to obtain our main results.

Definition 2.1 Suppose $F : X \rightrightarrows Y$ is a set-valued map from X to Y. The *graph* of F, denoted by $\mathcal{G}(F)$, is

$$\mathcal{G}(F) = \{(x, y) \in X \times Y \mid x \in X, y \in F(x)\}.$$

Definition 2.2 [8] Suppose S is a nonempty subset of some topological vector space X. A set-valued map $F : S \rightrightarrows X$ is called a *KKM-mapping* if, for every nonempty finite set $\{x_1, x_2, \ldots, x_n\}$ of S, we have

$$\text{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{j=1}^{n} F(x_j),$$

where co$\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $x_1, x_2, \ldots, x_n$.

The following result is taken from [8].

Lemma 2.3 *(KKM-Theorem) Suppose S is a nonempty subset of some topological vector space X and $F : S \rightrightarrows X$ is a KKM-mapping such that for every $x \in S$, $F(x)$*

is a closed subset of X. If there exists a point $x_0 \in S$ such that $F(x_0)$ is compact, then $\bigcap_{x \in S} F(x) \neq \emptyset$.

Definition 2.4 [22] Let (X, τ) be a topological space. Then

(i) a set J is said to be *directed set* with a partial order $\preceq$ such that for each pair α, β in J, there exists an element γ in J such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.
(ii) a *net* in X is a function f from a directed set J into X;
We usually denote $f(\alpha)$ by x_α and the net f itself is represented by $\{x_\alpha\}_{\alpha \in J}$.
(iii) a net $\{x_\alpha\}$ is said to *converge* to be the point $x \in X$ if, for each neighborhood U of x, there exists some $\alpha \in J$ such that for $\alpha \preceq \beta$, we have

$$x_\beta \in U.$$

Definition 2.5 [2, 13] Let (Y, μ_1) and (Z, μ_2) be two topological spaces. Let $\mathcal{C}(Y, Z)$ be the space of all continuous mappings from Y to Z. A topology τ on $\mathcal{C}(Y, Z)$ is called *admissible*, if the *evaluation map* $e : \mathcal{C}(Y, Z) \times Y \to Z$, defined by $e(f, y) = f(y)$, is continuous.

Lemma 2.6 [13] *A function space topology on $\mathcal{C}(X, Y)$ is admissible if and only if for any net $\{f_n\}_{n \in D_1}$ in $\mathcal{C}(X, Y)$, convergence of $\{f_n\}$ to f in $\mathcal{C}(X, Y)$ implies continuous convergence of $\{f_n\}$ to f. That is, if $\{f_n\}_{n \in D_1}$ converges to f in $\mathcal{C}(X, Y)$, then $\{f_n(x_m)\}_{(n,m) \in D_1 \times D_2}$ converges to $f(x)$ in Y and vice-versa, where $\{x_m\}_{m \in D_2}$ is any net in X converging to $x \in X$.*

Here, it may be mentioned that the above characterization of admissibility remains valid for the family of continuous linear mappings from X to Y, where X and Y are topological vector spaces.

3 Existence Theorems for VVI Problems

Theorem 3.1 *Let (X, τ_1) and (Y, τ_2) be any two topological vector spaces. Let $\mathcal{C}(X, Y)$ denote the space of all continuous linear mappings from X to Y, equipped with an admissible topology. Let $K \subseteq X$ be a nonempty closed convex compact subset of X. Let $C \subseteq Y$ be a closed convex pointed cone with* int$C \neq \emptyset$. *Further, let $T : K \to \mathcal{C}(X, Y)$ be a single-valued continuous mapping. Then the vector variational inequality problem* (VVIP(I)) *has a solution. That is, there exists $x_0 \in K$ such that*

$$T_{x_0}(u - x_0) \notin -\text{int}C,$$

for every $u \in K$.

Proof We define a set-valued map $F : K \rightrightarrows K$ as

$$F(u) = \{x \in K : T_x(u - x) \notin -\text{int}C\}.$$

We complete the proof of the theorem in two parts:

(i) F is a KKM-mapping on K:
Let $\{u_1, u_2, \ldots, u_n\} \subseteq K$ be any finite subset of K. We show that $\text{co}\{u_1, u_2, \ldots, u_n\} \subseteq \bigcup_{i=1}^{n} F(u_i)$. Let if possible, $\bar{x} \notin \bigcup_{i=1}^{n} F(u_i)$ for some $\bar{x} \in \text{co}\{u_1, u_2, \ldots, u_n\}$. Then, we have $\bar{x} = \sum_{i=1}^{n} \lambda_i u_i$ where $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.
As $\bar{x} \notin F(u_i)$, we have $T_{\bar{x}}(u_i - \bar{x}) \in -\text{int}C$, for each $i = 1, 2, \ldots, n$.
Since $-\text{int}C$ is convex and $\lambda_i \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, $\sum_{i=1}^{n} \lambda_i (T_{\bar{x}}(u_i - \bar{x})) \in -\text{int}C$.
We have, $\hat{0} = T_{\bar{x}}(\bar{x} - \bar{x}) = T_{\bar{x}}\left(\sum_{i=1}^{n} \lambda_i u_i - \sum_{i=1}^{n} \lambda_i \bar{x}\right) = T_{\bar{x}}\left(\sum_{i=1}^{n} \lambda_i (u_i - \bar{x})\right)$
$= \sum_{i=1}^{n} \lambda_i (T_{\bar{x}}(u_i - \bar{x})) \in -\text{int}C$, where $\hat{0}$ is the zero vector in Y. This implies $\hat{0} \in \text{int}C$, which is a contradiction as C is pointed. Therefore, we have $\text{co}\{u_1, u_2, \ldots, u_n\} \subseteq \bigcup_{i=1}^{n} F(u_i)$. Hence, F is a KKM-mapping on K.

(ii) $F(u)$ is closed:
Let $\{x_n\}$ be a net in $F(u)$ with $x_n \to x$. As K is closed, $x \in K$. We shall show that $x \in F(u)$, that is, $T_x(u - x) \notin -\text{int}C$. Since T is continuous, $x_n \to x$ implies $T_{x_n} \to T_x$. Also, we have, $u - x_n \to u - x$. As $\mathcal{C}(X, Y)$ has an admissible topology, we have, $T_{x_n}(u - x_n) \to T_x(u - x)$. Now, if $T_x(u - x) \in -\text{int}C$, then $T_{x_n}(u - x_n) \in -\text{int}C$ eventually, which leads to a contradiction as $x_n \in F(u)$. Hence, $T_x(u - x) \notin -\text{int}C$, that is, $x \in F(u)$.
Now, $F(u)$ being a closed subset of a compact set K is compact. Therefore by the KKM-Theorem, we have $\bigcap_{u \in K} F(u) \neq \emptyset$. Hence, there exists $x_0 \in K$ such that $x_0 \in \bigcap_{u \in K} F(u)$, that is, $T_{x_0}(u - x_0) \notin -\text{int}C$ for every $u \in K$.

□

In the next theorem, we provide an existence condition for a solution of VVIP(II).

Theorem 3.2 *Let* (X, τ_1) *and* (Y, τ_2) *be two topological vector spaces and* $\mathcal{C}(X, Y)$ *be the space of all continuous linear mappings from* X *to* Y, *equipped with an admissible topology. Let* K *be a nonempty closed convex compact subset of* X. *Let* $C : K \rightrightarrows Y$ *be a set-valued map such that for every* $x \in K$, $C(x)$ *is a closed convex pointed cone with* $\text{int}C(x) \neq \emptyset$. *Also, let* $W : K \rightrightarrows Y$ *be a set-valued map defined by* $W(x) = Y \setminus (-\text{int}C(x))$ *such that the graph of* W, $\mathcal{G}(W)$, *is a closed set in* $X \times Y$. *Let* $T : K \to \mathcal{C}(X, Y)$ *be a single-valued continuous mapping. Then the vector variational inequality problem* (VVIP(II)) *has a solution. That is, there exists* $x_0 \in K$ *such that*

$$T_{x_0}(u - x_0) \notin -\text{int}C(x_0),$$

for every $u \in K$.

Proof Consider a set-valued map $F : K \rightrightarrows K$ defined as

$$F(u) = \{x \in K : T_x(u - x) \notin -\text{int}C(x)\}.$$

The proof of the theorem is divided into two parts:

(i) F is a KKM-mapping on K.
(ii) $F(u)$ is closed for each $u \in K$.

The proof of part (i) is similar to that of the Theorem 3.1. So, we are giving the proof of part (ii) only.

Let $\{x_n\}$ be a net in $F(u)$ with $x_n \to x$. As K is closed, $x \in K$. We have to show that $x \in F(u)$, that is, $T_x(u-x) \notin -\text{int}C(x)$. Since $x_n \in F(u)$, $T_{x_n}(u-x_n) \notin -\text{int}C(x_n)$, which implies $T_{x_n}(u - x_n) \in W(x_n)$, which gives $\{(x_n, T_{x_n}(u - x_n))\} \subseteq \mathcal{G}(W)$. Now, as $x_n \to x$ and T is continuous, we have $T_{x_n} \to T_x$. Also, $x_n \to x$, implies $u-x_n \to u-x$. Since the topology of $\mathcal{C}(X, Y)$ is admissible, we have $T_{x_n}(u-x_n) \to T_x(u-x)$, which gives $\{(x_n, T_{x_n}(u-x_n))\} \to (x, T_x(u-x))$. Since $\mathcal{G}(W)$ is closed, $(x, T_x(u-x)) \in \mathcal{G}(W)$, which implies $T_x(u-x) \notin -\text{int}C(x)$. Hence, $x \in F(u)$ and hence $F(u)$ is closed.

Now, for each $u \in K$, $F(u)$ is a closed subset of K. As K is compact, $F(u)$ is compact. Therefore by KKM-Theorem, we have $\bigcap_{u \in K} F(u) \neq \emptyset$. Hence, there exists $x_0 \in K$ such that $x_0 \in \bigcap_{u \in K} F(u)$, that is, $T_{x_0}(u - x_0) \notin -\text{int}C(x_0)$ for every $u \in K$. □

Here, we provide an example to illustrate our results as well as to show that our result is independent of the result obtained by Chen in [6].

Example 3.3 Consider $X = l^2$, the set of all square summable sequences in $\mathbb{R}$ under the usual norm $\|\cdot\|$ and $Y = \mathbb{R}$, the set of all real numbers. Let $K \subset X$, be the Hilbert cube of l^2, that is, $x \in K$ if and only if $x = \{x_n\}_{n\in\mathbb{N}}$ with $| x_n | \leq \frac{1}{n}$ for $n \in \mathbb{N}$. Clearly, K is nonempty, closed, convex, and compact. Let $C : K \rightrightarrows Y$ be defined by $C(x) = \mathbb{R}^+ \cup \{0\}$, for every $x \in K$. Then $C(x)$ is a closed convex pointed cone with int $C(x) \neq \emptyset$, and $-\text{int}C(x) = (-\infty, 0)$, for each $x \in K$. Let $T : K \to \mathcal{C}(X, Y)$ be defined by $T_x(u) = -\langle x, u\rangle = -\sum x_i u_i$, where $x = \{x_i\}$ in K and $u = \{u_i\}$ is in X. That the induced topology of $\mathcal{C}(X, Y)$ is admissible can be verified by the fact that if $\{x_n\}$ converges to x in X and $\{f_n\}$ converges to f in $\mathcal{C}(X, Y)$, then we have

$$\begin{aligned}\|f_n(x_n) - f(x)\| &= \|f_n(x_n) - f_n(x) + f_n(x) - f(x)\| \\ &\leq \|f_n(x_n) - f_n(x)\| + \|f_n(x) - f(x)\| \\ &\leq \|f_n\|\|x_n - x\| + \|f_n(x) - f(x)\|.\end{aligned}$$

Hence, $f_n(x_n) \to f(x)$.

We take $x_0 = \{-\frac{1}{n}\}$. Then for any $x = \{x_n\}$ in K, we have, $T_{x_0}(x - x_0) = -\langle x_0, x - x_0\rangle = -\sum(-\frac{1}{n})(x_n + \frac{1}{n}) = \sum \frac{1}{n}(x_n + \frac{1}{n}) \geq 0$, as $| x_n | \leq \frac{1}{n}$. Therefore $T_{x_0}(x - x_0) \notin -\text{int}C(x_0)$. Hence, x_0 is a solution for the vector variational inequality problem.

T is not C-*monotone*: Let $T : X \to \mathcal{C}(X, Y)$ be a mapping and let C be a closed convex pointed cone. T is called C-*monotone* [6] if and only if for every pair $x, u \in X$, we have $\langle T(u) - T(x), u - x\rangle \in C$.

Now, $\langle T(u) - T(x), u - x\rangle = \langle T(u - x), u - x\rangle = T_{u-x}(u - x) = -\sum(u_i - x_i)^2 \notin C$, for $u_i \neq x_i$ for some i. Hence, T is not C-monotone.

In the following result, we discuss some topological properties of the solution sets obtained above.

Theorem 3.4 *The solution set for the vector variational inequality problem* VVIP(I) (*resp.* VVIP(II)) *obtained using the method provided in Theorem* 3.1 (*resp. Theorem* 3.2) *is closed as well as compact.*

Proof Let $F : K \rightrightarrows K$, be the set-valued map defined by

$$F(u) = \{x \in K : T_x(u - x) \notin -\text{int}C\}.$$

Then by Theorem 3.1, the solution set $\mathcal{S}$ of the vector variational inequality problem (VVIP(I)) is given by $\mathcal{S} = \bigcap_{u \in K} F(u)$. As shown in Theorem 3.1 (resp. Theorem 3.2) that $F(u)$ is closed for every $u \in K$. Therefore $\bigcap_{u \in K} F(u)$ is closed, that is, $\mathcal{S}$ is closed. Also, $\mathcal{S}$ being a closed subset of a compact set K, is compact. □

Conclusion

Our study here shows that the vector variational inequality problems can be studied from a purely topological point of view. The authors have not come across any such results so far in the literature where function space topology is being used to establish the existence of solution of such problems. It would be interesting to see whether this approach may be used for other variants of variational inequality problems.

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