Original Article

The variationally principle of two-temperature thermoelasticity in the absence of the energy dissipation theorem

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Abstract
Youssef introduced two-temperature thermoelasticity without energy dissipation theorem. This theorem depends upon two distinct temperatures: the conductive temperature and the thermo-dynamical temperature. Through this work, the variational principle theorem will be obtained for an isotropic, homogeneous, and thermoelastic body in the context of two-temperature thermoelasticity without energy dissipation theorem.

Keywords Thermoelasticity · Two-temperature · Energy dissipation · Variational principal

Abbreviations

λ , μ	Lamé's constants
ρ	Density
c_E	Specific heat at constant strain
α_T	Coefficient of linear thermal expansion
δ_{ij}	The Kronecker delta symbol
t	Time
T	Temperature
T_o	Reference temperature
θ	$= \left(T - T_o\right)$ Increment temperature such that $\left\lvert \frac{\theta}{T_o} \right\rvert << 1$
σ_{ij}	Components of the stress tensor
e_{ij}	Components of the strain tensor
u_i	Components of the displacement vector
F_i	Body force vector
q_i	The components of the heat flux
η	The entropy
k^*	The characteristic of the theorem
φ	Conductive temperature
a	$a > 0$ Two- temperature parameter

1 Introduction

Thermoelasticity without energy dissipation is a novel theory in extended thermoelasticity proposed by Green and Naghdi [1]. The main characteristic of this theory is that the heat flow does not entail energy dissipation, in contrast to the traditional thermoelasticity related to Fourier's equation of heat conduction. The constitutive equation for the entropy flow vector is also determined using the same potential function that was constructed to produce the stress tensor. The theory also allows for the limited-speed transport of heat as thermal waves. The linear theory is then provided once the broad discussion is obtained from the nonlinear theory. Green and Naghdi's broad hypothesis is based on the general entropy balance [2]. While the basic developments in [2] are general, the application is confined to the flow of heat in a stationary rigid solid transmitted by conduction and by the heat pulse propagated as thermal waves at a finite speed. Three types of constitutive response functions are suggested. Type I, after linearization of the theory, is the same as the classical heat conduction theory (based on Fourier's law), while types II and III permit

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propagation of thermoelastic disturbances with a finite speed. Although both types II and III theories for heat flow in a stationary rigid solid accommodate finite wave speed, only type II is without energy dissipation [3].

In the context of the Green and Naghdi model, many applications have been found. Chandrasekharaiah and Srinath [4] discussed thermoelastic waves without energy dissipation in an unbounded body with a spherical cavity. Kumar and Deswal [5] studied surface wave propagation in a micropolar thermoelastic medium without energy dissipation. Quintanilla [6] proposed a model of the thermoelastic theory without energy dissipation for materials with affine microstructure. Roychoudhuri and Dutta [7] solved the problem of thermo-elastic interaction without energy dissipation in an infinite solid with distributed periodically varying heat sources. Choudhuri and Bandyopadhyay [7, 8] solved a model of radially symmetric thermo-elastic wave propagation without energy dissipation in an infinitely extended thin plate with a circular hole.

Chen and Gurtin [9], Chen et al. [10] and [11] have formulated a theory of heat conduction in deformable bodies, which depends upon two distinct temperatures, the conductive temperature φ and the thermo-dynamical temperature T. For time-independent situations, the difference between these two temperatures is proportional to the heat supply, and in the absence of any heat supply, the two temperatures are identical Chen and Gurtin [9]. However, regardless of the presence of a heat source, the two temperatures are often different for time-dependent issues and wave propagation difficulties in particular. The two temperatures T, φ and the strain are found to have representations in the form of a travelling wave plus a response, which occurs instantaneously throughout the body [12], and Warren and Chen [13] investigated the wave propagation in the classical theory of two-temperature thermoelasticity while Youssef [14] investigated two-temperature generalized thermoelasticity theory together with a general uniqueness theorem and solved many applications in the context of this theory [15–17].

The two-temperature thermoelasticity without energy dissipation theory is obeyed by a homogeneous, isotropic body, and the variational principle theorem will be proven for this body in this study.

1.1 The basic equations

We will consider an isotropic and homogeneous elastic body in the context of two-temperature thermoelasticity without energy dissipation theorem with zero initial conditions of all state functions. Hence, the governing equations will take the following forms [18, 19]:

The equations of motion

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i \, , \; i,j = 1,2,3 \tag{1}$$

The heat equations

$$k^* \varphi_{,ii} = T_o \, \gamma \, \delta_{ij} \ddot{\varepsilon}_{ij} + \rho \, c_E \ddot{\theta} \, , \; i,j = 1,2,3 \tag{2}$$

The two-temperature equations

$$\varphi - \theta = a \varphi_{,ii} \, , \; i = 1,2,3 \tag{3}$$

The constitutive equations

$$\sigma_{ij} = 2\mu \, e_{ij} + \left(\lambda \, e_{kk} - \gamma \, \theta \right) \delta_{ij} \, , \; i,j,k = 1,2,3 \tag{4}$$

The entropy satisfies the following relation for unit mass

$$T_o \eta = c_E \theta + T_o \gamma e_{ij} \delta_{ij} \, , \; i,j = 1,2,3 \tag{5}$$

The entropy balance equations without internal heat generation take the form

$$q_{i,i} = -T_o \dot{\eta} \, , \; i = 1,2,3 \tag{6}$$

The heat conduction equations take the form

$$\dot{q}_{i,i} = -k^* \varphi_{,ii} \, , \; i = 1,2,3 \tag{7}$$

Formulation of the variational principle.

Under the assumption of small deviations of the thermo-dynamics system from the state of equilibrium, we will consider the statement of virtual external work:

$$\int_v F_i \delta \, u_i \, d\, v + \int_s p_i \delta \, u_i \, d\, s, \tag{8}$$

where v is an arbitrary material volume bounded by a closed and bounded surface s, F_i is the external forces per unit mass and p_i is the components of surface traction applied to the surface s.

We have the relation

$$\sigma_{ji} \, n_j = p_i, \tag{9}$$

where n_i are the normal components to the surfaces.

Using Eq. (2) and Gauss's divergence theorem in the second term of the relation (1), we obtain

$$\int_s p_i \, \delta \, u_i ds = \int_s \sigma_{ji} \, n_j \, \delta \, u_i ds = \int_v \sigma_{ji,j} \, \delta \, u_i dv + \int_v \sigma_{ji} \, \delta \, e_{ij} dv, \tag{10}$$

Since the stress tensor is symmetric, we have $\sigma_{ji} \, \delta \, e_{ij} = \sigma_{ji} \delta u_{i,j}$.

Using the equation of motion (1), Eq. (3) will take the form

$$\int_s p_i\, \delta\, u_i ds + \int_v F_i\, \delta\, u_i dv = \int_v \rho\, \ddot{u}_i\, \delta\, u_i dv + \int_v \sigma_{ji}\, \delta\, e_{ij} dv, \tag{11}$$

Using Eq. (4), the second term on the right-hand side of Eq. (11) takes the form

$$\int_v \sigma_{ji}\, \delta\, e_{ij} dv = \int_v \left(2\,\mu\, e_{ij} + \lambda\, e_{kk}\, \delta_{ij}\right) \delta\, e_{ij}\, dv - \int_v \gamma\, \theta\, \delta\, e_{kk}\, dv \tag{12}$$

We arrive at the theorem of virtual work from Eqs. (8) and (12), we obtain

$$\int_s p_i\, \delta\, u_i ds + \int_v F_i\, \delta\, u_i dv - \int_v \rho\, \ddot{u}_i\, \delta\, u_i dv = \delta\, W - \int_v \gamma\, \theta\, \delta\, e_{kk} dv \tag{13}$$

where

$$\delta W = \int_v \left(2\,\mu\, e_{ij}\, \delta\, e_{ij} + \lambda\, e_{kk} \delta\, e_{ii}\right)\; dv \tag{14}$$

The function W implies the work of the deformation may be expressed by Noda et al. [20] and takes the form

$$W = \int_v \left(\mu\, e_{ij}\, e_{ij} + \frac{\lambda}{2} e_{kk}\, e_{ii}\right)\; dv \tag{15}$$

The three terms on the left-hand side of Eq. (13) express the virtual external work of the body forces, tractions on the boundary and inertia forces, respectively, while the right-hand side expresses the virtual internal work.

We introduce an entropy flux H, which is related to the heat flux through the equation

$$q_i = T_o \dot{H}_i, \tag{16}$$

By eliminating the entropy and the heat flux between Eqs. (5), (6) and (16), we get

$$-T_o H_{i,i} = c_E \theta + T_o \gamma e_{ii} \tag{17}$$

By eliminating q_i Eqs. (7) and (16), we obtain

$$T_o\, \ddot{H}_i = -k^* \varphi\,'_i \tag{18}$$

Without loss of generality, we can put a parameter for the time derivatives, i.e.

$$\frac{\partial^2}{\partial\, t^2} = \beta,$$

hence, Eq. (18) takes the form

$$\frac{T_o\, \beta}{k^*} H_i + \varphi_{,i} = 0 \tag{19}$$

Multiplying by δH_i the above equation and integrating over volume v of the body, we get

$$\int_v \frac{T_o\, \beta}{k^*} H_i \delta\, H_i\, dv + \int_v \varphi_{,i} \delta\, H_i\, dv = 0 \tag{20}$$

The second term of the Eq. (20) by using Gauss's divergence theorem and Eq. (3) reduced to.

$$\int_v \varphi_{,i}\, \delta H_i\, dv = \int_v \left(\varphi\, \delta H_i\right)_{,i} dv - \int_v \varphi\, \delta H_{i,i}\, dv$$
$$= \int_s \varphi\, n_i\, \delta H_i\, ds - \int_v \left(\theta + a\varphi_{,ii}\right)\, \delta H_{i,i}\, dv$$

which gives

$$\int_v \varphi_{,i}\, \delta H_i\, dv = \int_s \varphi\, n_i\, \delta H_i\, ds - \int_v \theta\, \delta H_{i,i}\, dv - \int_v a\, \varphi_{,ii}\, \delta H_{i,i}\, dv \tag{21}$$

From Eq. (17), we have

$$\delta\, H_{i,i} = -\frac{c_E}{T_o}\; \delta\theta - \gamma\; \delta e_{ii} \tag{22}$$

Using Eq. (19) and Eq. (22) in the middle term of the right-hand side of Eq. (21), we get

$$\int_v \varphi_{,i}\, \delta H_i\, dv = \int_s \varphi\, n_i\, \delta H_i\, ds + \frac{c_E}{T_o} \int_v \theta\, \delta\theta\, dv$$
$$+ \gamma \int_v \theta\, \delta e_{ii}\, dv + \frac{T_o\, \beta\, a}{k^*} \int_v H_{i,i}\, \delta H_{i,i}\, dv \tag{23}$$

Now, Eq. (20) takes the form

$$\int_v \left(\frac{T_o \beta}{k^*} H_i + \varphi_{,i}\right) \delta\, H_i\, dv = \frac{T_o \beta}{k^*} \int_v H_i\, \delta\, H_i\, dv + \int_s \varphi\, n_i\, \delta H_i\, ds$$
$$+ \frac{c_E}{T_o} \int_v \theta\, \delta\theta\, dv + \gamma \int_v \theta\, \delta e_{ii}\, dv + \frac{T_o\, \beta\, a}{k^*} \int_v H_{i,i}\, \delta H_{i,i}\, dv = 0 \tag{24}$$

We introduced the heat potential P in the form [20]:

$$P = \frac{c_E}{2\, T_o} \int_v \theta^2\, dv, \tag{25}$$

where

$$\delta\, P = \frac{c_E}{T_o} \int_v \theta\; \delta\theta\, dv \tag{26}$$

and the dissipation function D in the form [20]:

$$D = \frac{T_o \beta}{2\, k^*} \int_v \left(H_i^2 + a\, H_{i,i}^2\right)\; dv = \frac{T_o}{2\, k^*} \frac{\partial^2}{\partial\, t^2} \int_v \left(H_i^2 + a\, H_{i,i}^2\right)\; dv, \tag{27}$$

Hence, we get

$$\delta D = \frac{T_o}{k^*}\frac{\partial^2}{\partial t^2}\int\limits_v \left(H_i\,\delta H_i + a\,H_{i,i}\,\delta H_{i,i}\right)\ dv, \tag{28}$$

The dissipation function D which is defined above is a new dissipation function because it contains a new parameter a. If we let a = 0, we get the old dissipation function D that is defined by [20].

Introducing Eqs. (26) and (28) into Eq. (24), we obtain the variational equation for heat conduction

$$\delta P + \delta D + \gamma\int\limits_v \theta\ \delta e_{ii}\,dv = -\int\limits_s \varphi\ n_i\,\delta H_i\,ds, \tag{29}$$

Elimination of the term $\gamma\int\limits_v \theta\ \delta e_{ii}\,dv$ from Eqs. (13) and (29) leads to

$$\delta W + \delta P + \delta D = \int\limits_s p_i\,\delta\,u_i ds + \int\limits_v F_i\,\delta\,u_i dv - \int\limits_v \rho\,\ddot{u}_i\,\delta\,u_i dv - \int\limits_s \varphi\,n_i\,\delta H_i ds \tag{30}$$

The terms on the right-hand side of Eq. (30) express the virtual external work of the body forces, tractions on the boundary, inertia forces, and heating of the boundary, respectively, while the left-hand side expresses the virtual internal work of deformation, the variation of heat potential, and the variation of the dissipation function, respectively [20].

Introducing the Biot thermoelastic potential ϕ[20]:

$$\phi = W + P = \int\limits_v \left(\mu e_{ij}\,e_{ij} + \frac{\lambda}{2}e_{ii}\,e_{jj} + \frac{c_E}{2\,T_o}\theta^2\right) dv \tag{31}$$

which gives:

$$\delta(\phi + D) = \int\limits_s \left(p_i\,\delta\,u_i - \varphi\,n_i\,\delta H_i\right)\ ds + \int\limits_v \left(F_i - \rho\,\ddot{u}_i\right)\ \delta\,u_i\,dv \tag{32}$$

The above equation is the variational principle for the two-temperature thermoelasticity without energy dissipation theory problem.

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Declarations

Conflict of interests Not Applicable.

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