

ORIGINAL ARTICLE

Open Access

On the renormalization of entanglement entropy



Jin-Yi Pang^{1,2} and Jiunn-Wei Chen^{2,3,4*}

Abstract

The renormalization of entanglement entropy of quantum field theories is investigated in the simplest setting with a $\lambda\phi^4$ scalar field theory. The 3+1 dimensional spacetime is separated into two regions by an infinitely flat 2-dimensional interface. The entanglement entropy of the system across the interface has an elegant geometrical interpretation using the replica trick, which requires putting the field theory on a curved spacetime background. We demonstrate that the theory, and hence the entanglement entropy, is renormalizable at order λ once all the relevant operators up to dimension 4 are included in the action. This exercise has a one-to-one correspondence to entanglement entropy interpretation of the black hole entropy which suggests that our treatment is sensible. Our study suggests that entanglement entropy is renormalizable and is a physical quantity.

Keywords: Entanglement entropy, Quantum field theory, Renormalization, Black hole

1 Introduction

An interesting attempt to understand the Bekenstein-Hawking formula of black hole entropy $S_{BH} = A/4G$, with A the black hole horizon area and G the gravitational constant, is to relate it to the entanglement entropy (S_E) across the black hole horizon [1–5]. S_E quantifies the entanglement between the degrees of freedom inside and outside the black hole. But first attempts to compute S_E from free fields yielded the desired A dependence whereas the prefactor is divergent [2–4]. Susskind and Uglum suggested that the divergence just renormalizes the bare gravitational constant G to the renormalized one, G_R , such that $S_{BH} = S_E = A/4G_R$ [6, 7]. This suggestion was checked by explicit computations in the massive black hole limit for free fields while treating gravity classical [7]. Whether this result is modified by the finite black hole mass or terms beyond Einstein gravity or when gravity is quantized are interesting questions for further exploration [8–16].

In a condensed matter system, S_E can still be defined as the entanglement of degrees of freedom across the interface of area A between two regions [17, 18]. But then how important is gravity to the determination of S_E ? Is S_E still renormalizable without gravity in the theory? If not, why is gravity so special and why is it so important even for a condensed matter system? If yes, how is S_E renormalized? A popular treatment in the computation of S_E is just imposing a UV cut-off without renormalizing it. In this treatment, S_E/A is set by the cut-off scale and when the UV cut-off increases, S_E also increases since there are more degrees of freedom that can entangle. However, this does not always lead to a positive S_E [13–15, 19], which is required by definition. Different ways to fix the problem usually lead to different results. Therefore, it is important to ask a more fundamental question, is S_E a physical quantity? If yes, how do we renormalize it?

In this work, we try to address this issue by renormalizing S_E in the simplest example. We set up the problem using the “replica trick” which provides a geometrical interpretation of S_E [8] which we will review in the following section. Then, for a 3+1 dimensional scalar field theory with quartic interaction (a $\lambda\phi^4$ theory), we compute the entanglement entropy for two regions separated

*Correspondence: jwc@phys.ntu.edu.tw

²Helmholtz-Institut für Strahlen- und Kernphysik and Bethe Center for Theoretical Physics, Universität Bonn, 53115 Bonn, Germany

³Department of Physics, Center for Theoretical Physics, and Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, Taipei, Taiwan

Full list of author information is available at the end of the article

by a flat infinite two-dimensional plane to order λ . The corresponding geometry of the Euclidean spacetime in replica trick is $C_\epsilon \times R^2$, where C_ϵ is a two-dimensional cone with a deficit angle ϵ . And then S_E is the linear response of the effective action to a vanishing deficit angle. Hence, the problem of S_E computation is nothing but a field theory problem in a curve spacetime. This spacetime does not have the Poincare symmetry (or translational and rotational symmetries in Euclidean space) of the flat space time. Hence, to renormalize the theory, we write down all the renormalizable terms with mass dimensions ≤ 4 and with symmetries satisfied by the $C_\epsilon \times R^2$ space. We derived the required scalar propagator in this theory and show that all divergent diagrams at $\mathcal{O}(\epsilon\lambda)$ can all be renormalized and so is S_E at this order. Finally, we do the same exercise to the black hole case by replacing the curve spacetime background by gravity and find that there is a one-to-one correspondence between the non-gravitational theory and the black hole case mathematically. While this result suggests that our formulation is sensible from the point of view of general relativity, the deficit angle in a condensed matter system in flat space remains an illusive concept worth further exploration.

2 Review of the replica trick

Suppose our system occupies an infinitely large and flat $3 + 1$ dimensional spacetime, with the three-dimensional space divided into two regions V and \bar{V} by a time independent, flat 2-dimensional plane. Then, the entanglement entropy, S_E , of a quantum theory between the two sub-regions is defined by the von Neumann entropy of the reduced density matrix $\rho_V = \text{tr}_{\bar{V}}[\rho]$ by tracing out the degrees of freedom in region \bar{V} :

$$S_E = -\text{tr}[\rho_V \ln \rho_V] = -\left. \frac{\partial}{\partial n} \ln \text{tr}[\rho_V^n] \right|_{n \rightarrow 1}, \tag{1}$$

where $\text{tr}[\rho_V]=1$ is used. This expression is called the replica trick because it involves n copies of ρ_V . Equation (1) coincides with the Renyi entropy $S_n = \frac{1}{1-n} \ln \text{tr} \rho_V^n$ in the $n \rightarrow 1$ limit.

An elegant path integral formulation to compute the entanglement entropy using the replica trick was introduced in [8]. In this setup, one recalls that $\rho_{ij} \propto \langle i | e^{-H/T} | j \rangle$ for a thermal equilibrium system with Hamiltonian H and temperature T . $Z = \text{tr}[\rho]$ is the partition function calculated in finite temperature field theory with the range of Euclidean time $\tau = [0, 1/T]$ and with appropriate boundary conditions: $\phi(\tau = 0, \mathbf{x}) = \pm \phi(\tau = 1/T, \mathbf{x})$ with the $+$ ($-$) sign if ϕ is a boson(fermion) field. Then, $\text{tr}[\rho^2] \propto \text{tr}[e^{-2H/T}]$ can be computed by doubling the period in τ such that boundary conditions are imposed at $\tau = 0$ and $2/T$. Similarly, $\text{tr}[\rho_V^2]$ is computed by doubling the period in τ for region V while keeping a single period in region \bar{V} . This can be shown

as a 2-sheeted Riemann surface as in Fig. 1a, where we have the period $\tau = [0, 2/T]$ in region V and periods $\tau = [1/2T, 3/2T]$ and $\tau = [3/2T, 5/2T \sim -1/2T]$ in region \bar{V} . In this figure, only τ (the vertical direction) and the direction perpendicular to the interface (the horizontal direction) are shown while the 2-dimensional interface in the perpendicular direction is not shown. If we circle around point O by contour 1, then it will connect to contour 2 with the total angle for a closed loop to be 4π .

This analysis can be generalized to $\text{tr}[\rho_V^n]$ for an arbitrary integer n for arbitrary sizes of V , \bar{V} , and $1/T$. As a result, $\text{tr}[\rho_V^n]$ becomes the partition function Z_n on the n -sheeted Riemann surface normalized by Z_1^n which follows from imposing the normalization $\text{tr}[\rho_V]=1$:

$$\text{tr}[\rho_V^n] = \frac{Z_n}{Z_1^n}. \tag{2}$$

Then, Eq. (1) yields

$$S_E = \left(-\frac{\partial}{\partial n} + 1 \right) \ln Z_n \Big|_{n \rightarrow 1}. \tag{3}$$

In this work, we concentrate on the simplest case with the sizes of V , \bar{V} , and $1/T$ all become infinite ($T = 0$) and the interface between V and \bar{V} is a flat infinite plane. In this limit, the n -sheeted Riemann surface in Fig. 1a can be redrawn to Fig. 1b which has the $C_\epsilon \times R^2$ topology. The Euclidean time and the direction perpendicular to the interface form the 2-dimensional cone C_ϵ with the

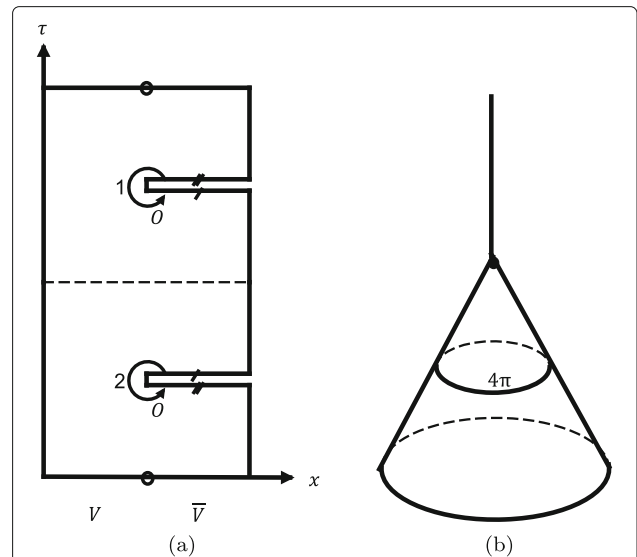


Fig. 1 $\text{tr}[\rho_V^2]$ can be computed by the 2-sheeted Riemann surface shown in **a**. The period of τ in region V is twice of that in region \bar{V} (see the text). A close loop around point O goes through contours 1 and 2 with a total angle 4π . For a system of infinite spacetime volume, **a** can be drawn as **b** with a 2-dimensional cone of angle 4π and 2 co-dimensions. The line connecting to the tip of the cone in **b** denotes the 2-dimensional brane which is the interface. The replica trick requires generalizing this picture to $\text{tr}[\rho_V^n]$ with $n \rightarrow 1$

deficit angle ϵ of the cone satisfying $n = 1 - \epsilon/2\pi$. R^2 is the space parallel to the 2-dimensional interface but transverse to the cone. The line connecting to the tip of the cone in Fig. 1b denotes the 2-dimensional brane which is the interface.

Now Eq. (3) becomes

$$S_E = \left(2\pi \frac{\partial}{\partial \epsilon} + 1 \right) \ln Z(\epsilon) \Big|_{\epsilon \rightarrow 0}, \tag{4}$$

with S_E probing the linear response of the partition function to the deficit angle.

Thermal entropy is a special case of entanglement entropy with vanishing \vec{V} . As shown in Fig. 1a, without \vec{V} , the spacetime topology becomes $S^1 \times R^3$ in flat Euclidean space. n controls the size of S^1 which is $\beta = n/T$. Then, Eq. (3) yields the standard equation for thermal entropy

$$S = \left(-\beta \frac{\partial}{\partial \beta} + 1 \right) \ln Z. \tag{5}$$

The situation changes when it comes to the thermal entropy of a black hole. The Euclidean spacetime geometry just outside the black hole horizon is $R^2 \times S^2$. The period of the polar angle in R^2 is proportional to β . When β is not equal to the inverse Hawking temperature of the black hole β_H , R^2 becomes a cone of deficit angle $\epsilon = 2\pi(\beta_H - \beta)/\beta_H$. Then, Eq. (5) becomes

$$S_{BH} = \left(2\pi \frac{\partial}{\partial \epsilon} + 1 \right) \ln Z(\epsilon) \Big|_{\epsilon \rightarrow 0}, \tag{6}$$

which is very similar to Eq. (4).

3 Renormalized entanglement entropy in the $\lambda\phi^4$ theory

To compute the entanglement entropy S_E using Eq. (3), we need to construct the theory in the $C_\epsilon \times R^2$ space shown in Fig. 1b. The interface is shown as a two-dimensional brane (R^2) which has a 2-dimensional cone C_ϵ outside the brane. The space is locally flat outside of the brane. There could be fields only live on the brane.

A $\lambda\phi^4$ theory in this space has the Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \\ & + \frac{1}{2}Z_\phi(\partial\phi)^2 + \frac{1}{2}Z_m\phi^2 + \frac{1}{4}Z_\lambda\phi^4 + Z_4 \\ & + \epsilon\delta^{(2)}(x_\parallel) (Z_2 + Z_0\phi^2) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{7}$$

The first two lines are the usual Lagrangian in flat space with the full Poincare symmetry. m and λ are renormalized quantities. Terms in the second line are counterterms. Z_4 is the counterterm that renormalizes the cosmological constant. Terms in the third line breaks the Poincare symmetry due to the 2d brane. The Z_2 term is the 2d brane tension while the Z_0 is the brane coupling to the scalar. The third line should vanish when $\epsilon \rightarrow 0$. Hence, under

Taylor expansion, their couplings are proportional to ϵ for small ϵ . (The first derivative of ϵ has been assumed to exist when we apply the replica trick.) This system is nothing but a quantum field theory in a curved spacetime background. We expect that once all the relevant operators (i.e., operators up to dimension 4 for a weakly coupled system) with the symmetries of the problem are included, then the theory should be renormalizable. While the renormalizability is known to all orders in λ at $\mathcal{O}(\epsilon^0)$, we will demonstrate that this is also the case at $\mathcal{O}(\epsilon\lambda)$.

The Green's function $G_n(x, x')$ for the free scalar field on a n -sheeted Riemann surface satisfies

$$(-\partial^2 + m^2) G_n(x, x') = \delta^4(x - x'). \tag{8}$$

The conical singularity breaks translational symmetry such that $G_n(x, x')$ depends on both x and x' instead of $x - x'$ alone. As shown in Refs. [20, 21] and in the Appendix, the Green function has the solution

$$\begin{aligned} G_n(x, x') = & \int \frac{d^2p_\perp}{(2\pi)^2} \int qdq \frac{e^{ip_\perp(x_\perp - x'_\perp)}}{q^2 + p_\perp^2 + m^2} \left[J_0(qr)J_0(qr') \right. \\ & \left. + 2 \sum_{l=1}^{\infty} J_{\frac{l}{n}}(qr)J_{\frac{l}{n}}(qr') \cos \frac{l}{n}(\theta - \theta') \right] \end{aligned} \tag{9}$$

where polar coordinates (r, θ) and (r', θ') are used to describe points on longitudinal plane with respect to the conical singularity.

Under the ϵ expansion, we have

$$G_{1+\epsilon}(x, x') = G(x - x') + \epsilon f(x, x') + \mathcal{O}(\epsilon^2), \tag{10}$$

where we have rewritten the Green's function in flat space $G_1(x, x')$ as $G(x - x')$ since translational symmetry is satisfied for $G_1(x, x')$.

There are two classes of diagrams at $\mathcal{O}(\epsilon)$. The first one is with one $\mathcal{O}(\epsilon)$ coupling insertion and with all the other couplings and propagators of $\mathcal{O}(\epsilon^0)$. The second one is with one $f(x, x')$ propagator but with all the other propagators and couplings of $\mathcal{O}(\epsilon^0)$, which implies if the $f(x, x')$ propagator is removed, then the rest of the diagram is translational invariant. Therefore, a generic diagram of the second class can be expressed as

$$\int dx dx' f(x, x') F(x - x'). \tag{11}$$

With this condition, we show in the Appendix that the $\mathcal{O}(\epsilon)$ propagator in the Fourier space can be effectively written as:

$$f(p) \rightarrow \frac{A_\perp}{12} \frac{\delta^2(p_\parallel)}{p_\perp^2 + m^2}, \tag{12}$$

where A_\perp is the area of the interface. This expression has a mass dimension minus six because $G_n(x, x')$ has a mass

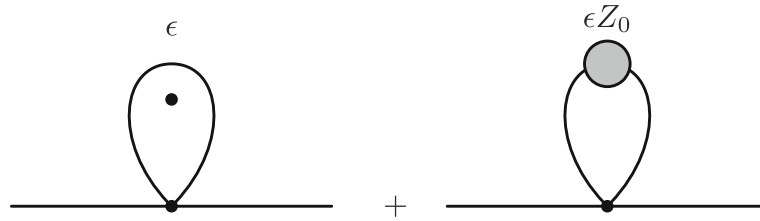


Fig. 2 Two-point functions at $\mathcal{O}(\epsilon\lambda)$. The left diagram has an $\mathcal{O}(\epsilon)$ propagator circling the conical singularity (or the 2d brane) denoted by the dot. The right diagram has one insertion of the Z_0 coupling

dimension two. Now we see all the couplings and propagators at $\mathcal{O}(\epsilon)$ are proportional to A_\perp , so every diagram at $\mathcal{O}(\epsilon)$ is proportional to A_\perp as well.

The $\mathcal{O}(\epsilon)$ propagator of Eq. (12) has two powers of momentum less than the $\mathcal{O}(\epsilon^0)$ propagator. Hence, the standard power counting analysis shows that at $\mathcal{O}(\epsilon)$, only the two-point functions and zero-point functions are divergent and need to be renormalized. The two-point functions could diverge logarithmically while the zero-point function could diverge quadratically at $\mathcal{O}(\epsilon)$.

We will start from the renormalization of two-point functions at $\mathcal{O}(\epsilon\lambda)$ shown in Fig. 2. The sum of the two diagrams is proportional to

$$-\frac{1}{12} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + m^2} + Z_0 \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^2}. \quad (13)$$

Then, using $d^d p = dp p^{d-1} \Omega_d$, $\Omega_4 = 2\pi^2$ and $\Omega_2 = 2\pi$, the two-point function can be renormalized by setting $Z_0 = \pi/3$ such that the divergence between the two diagrams cancel and (13) equals $1/(48\pi)$. Here, we do not demand the first diagram being renormalized by the tree diagram with one insertion of Z_0 because in general Z_0 is of order λ^0 , not λ .

Now it comes to the zero-point function (the logarithm of the partition function) at $\mathcal{O}(\epsilon)$, which is exactly the entanglement entropy using the replica trick. Figure 3a denotes the contribution from the inverse determinant of the free theory. It can be computed by taking the m^2 derivative then integrate m^2 back:

$$S_E^{(3a)} = \frac{1}{12} A_\perp \int_{m^2}^\infty d\mu^2 \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{1}{p_\perp^2 + \mu^2} + A_\perp S_0 \quad (14)$$

S_0 is the contribution when $m \rightarrow \infty$. So S_0 is m independent.

We use the renormalization condition that the tree level mass is the physical mass already, such that the loop corrections to the mass are all canceled by counterterms. Therefore, the one loop correction in Fig. 3b is exactly canceled by the insertion of the mass counterterm Z_m in Fig. 3c:

$$S_E^{(3b)+(3c)} = 0. \quad (15)$$

The Z_2 and Z_0 terms yield

$$S_E^{(3d)} = -A_\perp Z_2, \quad (16)$$

$$S_E^{(3e)} = -A_\perp Z_0 \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2}. \quad (17)$$

And the one loop correction to (3e) vanishes

$$S_E^{(3f)+(3g)} = 0. \quad (18)$$

If we take the mass derivative to Eqs.(14) and (17), the same combination as Eq. (13) arises and yields

$$S_E^{(3a)+(3e)} = -A_\perp \left(\frac{m^2}{48\pi} + C \right), \quad (19)$$

where C is m independent. C could diverge like Λ^2 , with Λ the ultraviolet momentum cut-off.

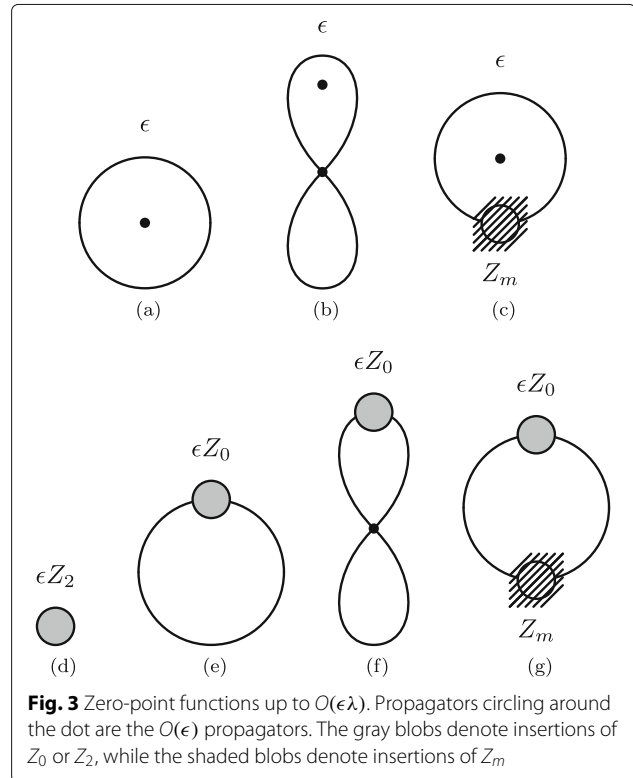


Fig. 3 Zero-point functions up to $\mathcal{O}(\epsilon\lambda)$. Propagators circling around the dot are the $\mathcal{O}(\epsilon)$ propagators. The gray blobs denote insertions of Z_0 or Z_2 , while the shaded blobs denote insertions of Z_m

Putting everything together, we have

$$\frac{S_E}{A_\perp} = -\frac{m^2}{48\pi} - \bar{Z}_2 + \mathcal{O}(\lambda^2), \tag{20}$$

where $\bar{Z}_2 = Z_2 + C$. The divergence in C can be absorbed by Z_2 such that \bar{Z}_2 is finite. Therefore, we have demonstrated that, up to $\mathcal{O}(\lambda)$, with couplings Z_0 and Z_2 living in the two-dimensional brane formed by the interface, S_E could be properly renormalized to be a finite quantity.

Here, we have only discussed the S_E of the ground state of a scalar field theory with an infinitely large flat interface to $\mathcal{O}(\lambda)$. But the conclusion that entanglement entropy is renormalizable could be general. One can generalize our derivation to all orders in λ using the standard techniques and try to work on different interfaces and theories. However, one can already see in our simple example that while Z_0 can be determined by the renormalization of the two-point function, fixing Z_2 (or \bar{Z}_2 of Eq. (20)) requires knowing its dependence on the deficit angle in a condensed matter system which lives in a flat space. This is conceptually illusive. We will have more discussions on this in the next section.

4 The correspondence in the black hole case

It is instructive to reproduce the computation showing the equivalence between the black hole thermal entropy and entanglement entropy which mathematically has a one-to-one correspondence to our case in the previous section.

The Euclidean action of a quantum scalar field in classical gravity is

$$S = \int d^4x \sqrt{g} \mathcal{L}, \tag{21}$$

and

$$\mathcal{L} = \mathcal{L}_\phi - \frac{R}{16\pi G} + \frac{\alpha}{4\pi} \phi^2 R + \mathcal{O}(R^2). \tag{22}$$

\mathcal{L}_ϕ is the first two lines of Eq. (7) written in the general covariant form. R is the Ricci scalar. The dimension four $\mathcal{O}(R^2)$ terms include R^2 , $R^2_{\mu\nu}$, and $R^2_{\mu\nu\rho\sigma}$.

For an infinitely massive black hole, the Hawking temperature is zero and the horizon is a flat infinite plane. The space outside the horizon can be described by the Rindler space which is locally flat except at the origin [6, 7, 14]. Therefore, when one computes the black hole entropy using Eq. (6), or the black hole entanglement entropy across the horizon using Eq. (4), the spacetime geometry is $C_\epsilon \times R^2$ in both cases—the same as the S_E computation in flat space in the previous section. Furthermore, we have $R = 4\pi \epsilon \delta^{(2)}(x_\parallel)$ which makes Eq. (22) have the same form as Eq. (7). These lead to $S_{BH} = S_E$ for the black hole.

We can set $\alpha = Z_0$ to renormalize the two-point function in Fig. 2. As for the terms of $\mathcal{O}(R^2)$, although they are dimension 4, they do not contribute until $\mathcal{O}(\epsilon^2)$ so they do not contribute to S_E or S_{BH} .¹

Following the same procedure as in the previous section, we have

$$\frac{S_{BH}}{A_\perp} = \frac{1}{4G} - C - \frac{m^2}{48\pi} + \mathcal{O}(\lambda^2) = \frac{1}{4G_R} + \mathcal{O}(\lambda^2), \tag{23}$$

where the $1/4G$ term is the Bekenstein-Hawking entropy which is $\mathcal{O}(1/\hbar)$, but interestingly it can be derived with just classical gravity. The $-C - m^2/48\pi$ contribution is from quantum corrections to the black hole entropy starting at $\mathcal{O}(\hbar^0)$. The mass independent term C could diverge like Λ^2 . But its divergence is absorbed by $1/G$ and the combination on the right hand side is matched to the renormalized quantity $1/4G_R$. (Here, G_R is related to the zero-point function of ϕ with one power of R dependence. This quantity is proportional to S_E in this problem.) [6, 7]. Finally, we have

$$S_{BH} = S_E = \frac{A_\perp}{4G_R}. \tag{24}$$

Hence, both S_{BH} and S_E are shown, up to $\mathcal{O}(\lambda)$, to be the same for an infinitely massive black hole. And the entropy per horizon area is set by the Planck scale.

Most of the discussion in this section can be found in [6, 7, 14], except the part that α can be fixed by the two-point function renormalization in Fig. 2. We find that each term in our condensed matter case has a counter part in the gravitational theory. This suggests that our formulation is sensible from the point of view of general relativity. Although the deficit angle is easy to imagine in a gravitational theory, it requires the “off-shell action” to describe a black hole away from its Hawking temperature to generate the deficit angle [22]. This suggests that in the condensed matter system, the determination of the interface term Z_2 will require properties from the “off-shell action” that is not included in its usual “on-shell action” which can possibly be determined by scattering in flat 3+1 dimensions.

5 Conclusion and discussion

Our study suggests that entanglement entropy is renormalizable and is a physical quantity. We have demonstrated the renormalizability of the entanglement entropy of the $\lambda\phi^4$ at order λ when the 3+1 dimensional theory is separated into two regions by an infinitely flat 2-dimensional interface. Using the replica trick, the computation of the entanglement entropy across the interface can be carried out by putting the theory on a curved spacetime background. We have shown by an explicit

¹The $\mathcal{O}(R^2)$ can still contribute at $\mathcal{O}(\epsilon)$ when the curvature at $\mathcal{O}(\epsilon^0)$ is not zero as considered in Ref. [20].

computation at $\mathcal{O}(\lambda)$ that once all the relevant operators are included in the action, the theory and hence the entanglement entropy, can be renormalized.

We also reviewed the computation of black hole entropy and entanglement entropy across the horizon for an infinitely massive black hole and found that our non-gravitational calculation has a one-to-one correspondence to the black hole case. This suggests that our formulation is sensible from the point of view of general relativity.

To renormalize the S_E in a non-gravitational system, one uses the coupling Z_2 which live in the interface to absorb the infinities from loop diagrams. If we wish to make a prediction to S_E , then Z_2 needs to be fixed by other observables. However, this coupling exists in the “off-shell action.” It is not clear what would be a good way to fix it. However, even without knowing Z_2 , one can still make predictions to combinations of entanglement entropies, e.g. mutual entropy [23–25], whose Z_2 dependence cancels.

There are some obvious directions for future work. One could generalize the renormalizability proof to all orders in λ , or generalize it to other theories, for examples, the standard model of particle physics. In the latter case, there are actually no relevant couplings at $\mathcal{O}(\epsilon)$ other than what we have already written down in Eq. (7) and our $\mathcal{O}(\epsilon)$ propagator result in Eq. (12) can be easily generalized to propagators for other fields.

Appendix: The derivation of Eq. (12)

The Laplacian operator in the $C_\epsilon \times R^2$ space is

$$\Delta_n = \partial_\perp^2 + \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{n^2 r^2} \partial_\theta^2, \tag{25}$$

where $\theta = [0, 2\pi)$ and $n = 1 - \epsilon$. Its eigenfunction

$$\phi_n(p, x) = \sqrt{2\pi} J_{\frac{l}{n}}(\rho_{\parallel} r) e^{il\theta} e^{ip_\perp x_\perp} \tag{26}$$

satisfies

$$(-\Delta_n + m^2) \phi_n(p, x) = (p^2 + m^2) \phi_n(p, x). \tag{27}$$

Then, the Green’s function $G_n(x, x')$ can be constructed via

$$\begin{aligned} G_n(x, x') &= \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \phi_n(p, x) \tilde{G}_n(p, p') \phi_n^*(p', x'), \\ \tilde{G}_n(p, p') &= \frac{(2\pi)^4 \delta(p, p')}{p^2 + m^2}, \end{aligned} \tag{28}$$

which yields Eq. (9) and the integral over p_θ , the conjugate momentum of θ , is understood as the sum over all integer l .

In Eq. (11), a general function of $x - x'$ can be written in a similar way as Eq. (28) but in flat ($n = 1$) space:

$$\begin{aligned} F(x - x') &= \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-x')} \tilde{F}(k) \\ &= \int \frac{d^4 k}{(2\pi)^4} \phi_1^*(k, x) \tilde{F}(k) \phi_1(k, x'). \end{aligned} \tag{29}$$

Then, we have

$$\begin{aligned} &\int d^4 x d^4 x' F(x - x') G_n(x, x') \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{F(k)}{p^2 + m^2} \int d^4 x d^4 x' \\ &\quad \times \phi_n(p, x) \phi_n^*(p, x') \phi_1^*(k, x) \phi_1(k, x'). \end{aligned} \tag{30}$$

We only need the $\mathcal{O}(\epsilon)$ contribution of this integral, which is

$$\begin{aligned} I &= \int d^4 x d^4 x' F(x - x') f(x, x') \\ &= - \int \frac{d^4 p d^4 x}{(2\pi)^4} \frac{F(p)}{p^2 + m^2} \partial_{n \rightarrow 1} |\phi_n(p, x)|^2, \end{aligned} \tag{31}$$

where we have used

$$\int d^4 x \phi^*(p, x) \phi(p', x) = (2\pi)^4 \delta^4(p - p'). \tag{32}$$

Using Eq. (26), the x integral in Eq. (31) is

$$\begin{aligned} &\int d^4 x \partial_{n \rightarrow 1} |\phi_n(p, x)|^2 \\ &= A_\perp (2\pi)^2 \partial_{n \rightarrow 1} \int dr r J_{\frac{l}{n}}^2(\rho_{\parallel} r) \\ &= A_\perp (2\pi)^2 \partial_{n \rightarrow 1} \frac{\delta(0)}{p_{\parallel}}. \end{aligned} \tag{33}$$

The last expression vanishes unless $p_{\parallel} = 0$. Since A_\perp and p_{\parallel} are the only scales in this expression, dimensional analysis suggests that it is proportional to $A_\perp \delta(p_{\parallel}^2)$. The proportional constant, $|l|$, can be fixed by considering the integral

$$\begin{aligned} &\int \frac{p dp}{p^2 + m^2} \partial_{n \rightarrow 1} \left[\int r dr J_{\frac{l}{n}}^2(\rho_{\parallel} r) \right] \\ &= \partial_{n \rightarrow 1} \int r dr I_{\frac{l}{n}}(mr) K_{\frac{l}{n}}(mr) \\ &= \frac{1}{m^2} \partial_{n \rightarrow 1} \left[-\frac{|l|}{2n} + \text{const.} \right] = \frac{|l|}{2m^2}. \end{aligned} \tag{34}$$

Now we are ready to rewrite Eq. (31) as

$$\begin{aligned} I &= - \int \frac{d^4 p}{(2\pi)^4} \frac{F(p)}{p^2 + m^2} |l| A_\perp \delta(p_{\parallel}^2) \\ &= - \int \frac{d^2 p_\perp dp_{\parallel}}{(2\pi)^4} \sum_{-\infty}^{\infty} \frac{F(p)}{p^2 + m^2} |l| A_\perp \delta(p_{\parallel}^2) \\ &= \frac{1}{12} \int \frac{d^2 p_\perp}{(2\pi)^4} \frac{F(p_\perp, p_{\parallel} = 0)}{p_\perp^2 + m^2}, \end{aligned} \tag{35}$$

where we have used the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and $\zeta(-1) = -1/12$ via analytic continuation. The final result is summarized in Eq. (12).

Acknowledgements

The authors would like to thank Tatzuma Nishioka, Naoki Yamamoto, Michael Endres, Xing Huang, Feng-Li Lin, Chen-Te Ma, Masahiro Nozaki, Jackson Wu, and Yun-Long Zhang for helpful discussions. JWC is partly supported by the Ministry of Science and Technology, Taiwan, under Grant Nos. 108-2112-M-002-003-MY3, the Kenda Foundation, and DFG and NSFC (CRC 110). JYP is partly supported by DFG through funds provided to the Sino-German CRC 110 "Symmetries and the Emergence of Structure in QCD".

Authors' contributions

JYP and JWC equally contributed to all aspects of the manuscript. Both authors read and approved the final manuscript.

Funding

JWC is partly supported by the Ministry of Science and Technology, Taiwan, under Grant Nos. 108-2112-M-002-003-MY3, the Kenda Foundation, and DFG and NSFC (CRC 110). JYP is partly supported by DFG through funds provided to the Sino-German CRC 110 "Symmetries and the Emergence of Structure in QCD".

Availability of data and materials

Not applicable.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

The authors give the consent for publication of this manuscript to AAPPS Bulletin.

Competing interests

The authors declare that they have no competing interests.

Author details

¹College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China. ²Helmholtz-Institut für Strahlen- und Kernphysik and Bethe Center for Theoretical Physics, Universität Bonn, 53115 Bonn, Germany. ³Department of Physics, Center for Theoretical Physics, and Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, Taipei, Taiwan. ⁴Physics Division, National Center for Theoretical Sciences, Taipei, Taiwan.

Received: 18 May 2021 Accepted: 6 December 2021

Published online: 20 December 2021

References

- G. 't Hooft, On the quantum structure of a black hole. *Nucl. Phys.* **B256**, 727–745 (1985). [https://doi.org/10.1016/0550-3213\(85\)90418-3](https://doi.org/10.1016/0550-3213(85)90418-3)
- L. Bombelli, R. K. Koul, J. Lee, R. D. Sorkin, A quantum source of entropy for black holes. *Phys. Rev.* **D34**, 373–383 (1986). <https://doi.org/10.1103/PhysRevD.34.373>
- M. Srednicki, Entropy and area. *Phys. Rev. Lett.* **71**, 666–669 (1993). <https://doi.org/10.1103/PhysRevLett.71.666>
- V. P. Frolov, I. Novikov, Dynamical origin of the entropy of a black hole. *Phys. Rev.* **D48**, 4545–4551 (1993). <https://doi.org/10.1103/PhysRevD.48.4545>
- S. Das, S. Shankaranarayanan, S. Sur, *Black hole entropy from entanglement: a review*, *Horizons in World Physics*, vol. 268. (M. Everett, L. Pedroza, eds.), (2009). <https://novapublishers.com/shop/classical-and-quantum-gravity-theory-analysis-and-applications/>
- L. Susskind, Some speculations about black hole entropy in string theory (1993). <http://arxiv.org/abs/hep-th/9309145>
- L. Susskind, J. Uglum, Black hole entropy in canonical quantum gravity and superstring theory. *Phys. Rev.* **D50**, 2700–2711 (1994). <https://doi.org/10.1103/PhysRevD.50.2700>
- C. G. Callan Jr, F. Wilczek, On geometric entropy. *Phys. Lett.* **B333**, 55–61 (1994). [https://doi.org/10.1016/0370-2693\(94\)91007-3](https://doi.org/10.1016/0370-2693(94)91007-3)
- D. N. Kabat, M. J. Strassler, A comment on entropy and area. *Phys. Lett.* **B329**, 46–52 (1994). [https://doi.org/10.1016/0370-2693\(94\)90515-0](https://doi.org/10.1016/0370-2693(94)90515-0)
- S. N. Solodukhin, The conical singularity and quantum corrections to entropy of black hole. *Phys. Rev.* **D51**, 609–617 (1995). <https://doi.org/10.1103/PhysRevD.51.609>
- D. V. Fursaev, Black hole thermodynamics and renormalization. *Mod. Phys. Lett.* **A10**, 649–656 (1995). <https://doi.org/10.1142/S0217732395000697>
- J.-G. Demers, R. Lafrance, R. C. Myers, Black hole entropy without brick walls. *Phys. Rev.* **D52**, 2245–2253 (1995). <https://doi.org/10.1103/PhysRevD.52.2245>
- D. N. Kabat, Black hole entropy and entropy of entanglement. *Nucl. Phys.* **B453**, 281–299 (1995). [https://doi.org/10.1016/0550-3213\(95\)00443-V](https://doi.org/10.1016/0550-3213(95)00443-V)
- F. Larsen, F. Wilczek, Renormalization of black hole entropy and of the gravitational coupling constant. *Nucl. Phys.* **B458**, 249–266 (1996). [https://doi.org/10.1016/0550-3213\(95\)00548-X](https://doi.org/10.1016/0550-3213(95)00548-X)
- D. V. Fursaev, G. Miele, Cones, spins and heat kernels. *Nucl. Phys.* **B484**, 697–723 (1997). [https://doi.org/10.1016/S0550-3213\(96\)00631-1](https://doi.org/10.1016/S0550-3213(96)00631-1)
- J. H. Cooperman, M. A. Luty, Renormalization of entanglement entropy and the gravitational effective action. *JHEP.* **1412**, 045 (2014). [https://doi.org/10.1007/JHEP12\(2014\)045](https://doi.org/10.1007/JHEP12(2014)045)
- G. Vidal, J. I. Latorre, E. Rico, A. Kitaev, Entanglement in quantum critical phenomena. *Phys. Rev. Lett.* **90**, 227902 (2003). <https://doi.org/10.1103/PhysRevLett.90.227902>
- J. I. Latorre, E. Rico, G. Vidal, Ground state entanglement in quantum spin chains. *Quant. Inf. Comput.* **4**, 48–92 (2004). <http://arxiv.org/abs/quant-ph/0304098>
- J.-W. Chen, S.-H. Dai, J.-Y. Pang, Strong coupling expansion of the 465 entanglement entropy of Yang-Mills gauge theories. *Nucl. Phys. B.* **951**, 114892 (2020). <https://doi.org/10.1016/j.nuclphysb.2019.114892>
- P. Calabrese, J. L. Cardy, Entanglement entropy and quantum field theory. *J. Stat. Mech.* **0406**, P06002 (2004). <https://doi.org/10.1088/1742-5468/2004/06/P06002>
- M. P. Hertzberg, Entanglement entropy in scalar field theory. *J. Phys.* **A46**, 015402 (2013). <https://doi.org/10.1088/1751-8113/46/1/015402>
- S. Carlip, C. Teitelboim, The off-shell black hole. *Class. Quant. Grav.* **12**, 1699–1704 (1995). <https://doi.org/10.1088/0264-9381/12/7/011>
- H. Liu, M. Mezei, A refinement of entanglement entropy and the number of degrees of freedom. *JHEP.* **162** (2013). [https://doi.org/10.1007/JHEP04\(2013\)162](https://doi.org/10.1007/JHEP04(2013)162)
- H. Casini, M. Huerta, J. A. Rosabal, Remarks on entanglement entropy for gauge fields. *Phys. Rev.* **D89**(8), 085012 (2014). <https://doi.org/10.1103/PhysRevD.89.085012>
- H. Liu, M. Mezei, Probing renormalization group flows using entanglement entropy. *JHEP.* **098** (2014). [https://doi.org/10.1007/JHEP01\(2014\)098](https://doi.org/10.1007/JHEP01(2014)098)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.