



# Spectral sets and weak tiling

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## Abstract

A set  $\Omega \subset \mathbb{R}^d$  is said to be spectral if the space  $L^2(\Omega)$  admits an orthogonal basis of exponential functions. Fuglede (1974) conjectured that  $\Omega$  is spectral if and only if it can tile the space by translations. While this conjecture was disproved for general sets, it was recently proved that the Fuglede conjecture does hold for the class of convex bodies in  $\mathbb{R}^d$ . The proof was based on a new geometric necessary condition for spectrality, called “weak tiling”. In this paper we study further properties of the weak tiling notion, and present applications to convex bodies, non-convex polytopes, product domains and Cantor sets of positive measure.

**Keywords** Fuglede’s conjecture · Spectral set · Tiling · Polytopes · Equidecomposability

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## 1 Introduction

### 1.1 Spectral sets

Let  $\Omega \subset \mathbb{R}^d$  be a bounded, measurable set of positive measure. We say that  $\Omega$  is *spectral* if there exists a countable set  $\Lambda \subset \mathbb{R}^d$  such that the system of exponential functions  $\{\exp 2\pi i \langle \lambda, x \rangle\}$ ,  $\lambda \in \Lambda$ , forms an orthogonal basis in  $L^2(\Omega)$ , that is, the system is orthogonal and complete in the space. In this case the set  $\Lambda$  is called a *spectrum* for  $\Omega$ .

The study of spectral sets has a long history, which goes back to Fuglede [3] who conjectured that the spectral sets could be characterized geometrically as the sets which can tile the space by translations. We say that the set  $\Omega$  *tiles the space by translations* if there exists a countable set  $\Lambda \subset \mathbb{R}^d$  such that the translated copies  $\{\Omega + \lambda\}$ ,  $\lambda \in \Lambda$ , constitute a partition of  $\mathbb{R}^d$  up to measure zero.

Fuglede's conjecture inspired extensive research over the years, and a number of interesting results establishing connections between spectrality and tiling had since been obtained. On the other hand, also counterexamples were found to both directions of the conjecture in dimensions  $d \geq 3$ , see [26], [15, Section 4].

It was recently proved in [20] that the Fuglede conjecture holds for the class of *convex bodies* in  $\mathbb{R}^d$  (compact, convex sets with nonempty interior). That is, a convex body  $\Omega \subset \mathbb{R}^d$  is spectral if and only if  $\Omega$  can tile the space by translations.

### 1.2 Weak tiling

To prove the Fuglede conjecture for convex domains, the authors in [20] established a link between the analytic notion of spectrality and a geometric notion which was termed “weak tiling”. It is defined as follows:

**Definition 1.1** We say that the set  $\Omega$  can *weakly tile* another measurable (possibly unbounded) set  $\Sigma \subset \mathbb{R}^d$  by translations, if there exists a positive, locally finite Borel measure  $\nu$  on  $\mathbb{R}^d$  such that  $\mathbb{1}_\Omega * \nu = \mathbb{1}_\Sigma$  a.e.

Note that in the special case where the measure  $\nu$  is the sum of (finitely or countably many) unit masses, the weak tiling becomes a *proper tiling* of  $\Sigma$  by translates of  $\Omega$ .

The following result was proved in [20].

**Theorem 1.2** (see [20, Theorem 1.5]) *Let  $\Omega$  be a bounded, measurable set in  $\mathbb{R}^d$ . If  $\Omega$  is spectral, then it can weakly tile its complement  $\Omega^c = \mathbb{R}^d \setminus \Omega$  by translations.*

This result thus gives a geometric condition necessary for spectrality. It establishes a weak form of the “spectral implies tiling” part of Fuglede's conjecture. We observe that the weak tiling conclusion cannot in general be strengthened to proper tiling since there exist examples of spectral sets which cannot tile by translations.

Several applications of Theorem 1.2 were given in [20]. The main one is the proof that the Fuglede conjecture holds for convex bodies in  $\mathbb{R}^d$ . As another application, it was proved in [20, Theorem 3.6] that if a bounded, open set  $\Omega \subset \mathbb{R}^d$  is spectral then its boundary  $\partial\Omega$  must be a set of Lebesgue measure zero.

In the present paper we study further properties and applications of weak tiling.

### 1.3 Equidecomposability

Let  $A, B$  be two (not necessarily convex) polytopes in  $\mathbb{R}^d$ . We say that  $A$  and  $B$  are *equidecomposable* if  $A$  can be partitioned, up to measure zero, into a finite number of smaller polytopes which can be rearranged using rigid motions to form, again up to measure zero, a partition of  $B$ . If the pieces of the partition can be rearranged using translations only, then  $A$  and  $B$  are said to be *equidecomposable by translations*.

It has long been known that if a polytope  $A \subset \mathbb{R}^d$  can (properly) tile the space by translations, then  $A$  must be equidecomposable by translations to a cube of the same volume. This result was first established by Mürner in [24], and was later rediscovered in [16]. Recently, it was proved in [19] that the same conclusion holds also for the class of spectral polytopes, that is, any spectral polytope  $A \subset \mathbb{R}^d$  must be equidecomposable by translations to a cube of the same volume.

We will prove the following simultaneous strengthening of the latter two results:

**Theorem 1.3** *Let  $A$  be a (not necessarily convex) polytope in  $\mathbb{R}^d$ . Assume that  $A$  can weakly tile its complement by translations. Then  $A$  is equidecomposable by translations to a cube of the same volume.*

### 1.4 Convex domains

It is a classical result due to Mürner, see [25, Section 3.3], that a *convex* polytope  $A \subset \mathbb{R}^d$  is equidecomposable by translations to a cube if and only if  $A$  is centrally symmetric and all the facets of  $A$  (that is, the  $(d - 1)$ -dimensional faces of  $A$ ) are also centrally symmetric. Together with Theorem 1.3 this implies that a convex polytope which can weakly tile its complement by translations must be centrally symmetric and have centrally symmetric facets.

Actually, this conclusion can be significantly strengthened as follows. It was proved in [20, Theorem 4.1] that if  $A$  is a general convex body in  $\mathbb{R}^d$  and if  $A$  can weakly tile its complement by translations, then  $A$  must in fact be a convex polytope. It was also proved [20, Theorem 6.1] that if, in addition,  $A$  is centrally symmetric and has centrally symmetric facets, then each belt of  $A$  must have either 4 or 6 facets. So in the latter case, it follows by the Venkov-McMullen theorem [22, 27] (see also [7, Section 32.2]) that  $A$  can tile its complement not only weakly, but even properly, by translations. We thus arrive at the following result:

**Theorem 1.4** *Let  $A$  be a convex body in  $\mathbb{R}^d$ , and assume that  $A$  can weakly tile its complement by translations. Then  $A$  must be a convex polytope which can also tile the space properly by translations.*

We observe that for general sets (i.e. not assumed to be convex) the result does not hold, since there exist examples of spectral sets which cannot tile by translations.

### 1.5 Product domains

Let  $A \subset \mathbb{R}^n$  and  $B \subset \mathbb{R}^m$  be two bounded, measurable sets. It is known that if  $A$  and  $B$  are both spectral sets in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively, then their cartesian product

$\Omega = A \times B$  is spectral in  $\mathbb{R}^n \times \mathbb{R}^m$ . Indeed, if  $U \subset \mathbb{R}^n$  is a spectrum for  $A$ , and  $V \subset \mathbb{R}^m$  is a spectrum for  $B$ , then the product set  $\Lambda = U \times V$  serves as a spectrum for  $\Omega$  (see e.g. [9, Theorem 3]).

In [12] the question was posed as to whether the converse statement is also true.

**Conjecture 1.5** *Let  $A \subset \mathbb{R}^n$  and  $B \subset \mathbb{R}^m$  be two bounded, measurable sets. Then their product  $\Omega = A \times B$  is spectral if and only if  $A$  and  $B$  are both spectral sets.*

The “only if” part of this conjecture is the non-trivial one. The difficulty lies in that we assume the product set  $\Omega$  to be spectral, but we do not make any a priori assumption that the spectrum  $\Lambda$  also has a product structure, so it is not obvious which sets  $U$  and  $V$  may serve as spectra for the factors  $A$  and  $B$ , respectively. (One can show, see [9, Lemma 2], that if  $\Omega$  happens to admit a spectrum  $\Lambda$  with a product structure,  $\Lambda = U \times V$ , then  $U$  is a spectrum for  $A$  and  $V$  is a spectrum for  $B$ .)

It was proved in [4] that Conjecture 1.5 holds in the case where one of the factors, say  $A$ , is an interval in  $\mathbb{R}$ . In [12] it was established, using a different approach, that the conjecture is true also if the set  $A$  is the union of two intervals in  $\mathbb{R}$ . In [6] the conjecture was proved in the case where the factor  $A$  is a convex polygon in  $\mathbb{R}^2$ .

As an application of the weak tiling method, we will prove the following result:

**Theorem 1.6** *Let  $\Omega = A \times B$  where  $A$  is a convex body in  $\mathbb{R}^n$ , while  $B$  is any bounded, measurable set in  $\mathbb{R}^m$ . If  $\Omega$  is a spectral set then  $A$  must be spectral, or equivalently,  $A$  must be a convex polytope which can (properly) tile the space by translations.*

This significantly strengthens [6, Theorems 2.1 and 2.2] where it was proved that  $A$  cannot have a smooth boundary, and that if  $A$  is assumed a priori to be a convex polytope, then  $A$  must be centrally symmetric and have centrally symmetric facets.

It is known, see [12, Section 1.2], that the product set  $\Omega = A \times B$  can tile the space  $\mathbb{R}^n \times \mathbb{R}^m$  by translations if and only if both  $A$  tiles  $\mathbb{R}^n$  and  $B$  tiles  $\mathbb{R}^m$ . In order to prove Theorem 1.6 we shall use an analogous result for weak tiling:

**Theorem 1.7** *Let  $A \subset \mathbb{R}^n$  and  $B \subset \mathbb{R}^m$  be two bounded, measurable sets. Then the product set  $\Omega = A \times B$  can weakly tile its complement in  $\mathbb{R}^n \times \mathbb{R}^m$  by translations if and only if both  $A$  and  $B$  weakly tile their complements in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively.*

Theorem 1.6 is thus obtained by a combination of Theorems 1.2, 1.7 and 1.4.

Similarly, by a combination of Theorems 1.2, 1.7 and 1.3 we obtain that if  $A$  is a (not necessarily convex) polytope in  $\mathbb{R}^n$ , and if  $B$  is any bounded, measurable set in  $\mathbb{R}^m$ , then the spectrality of the product  $\Omega = A \times B$  implies that  $A$  is equidecomposable by translations to a cube. Alternatively, it is possible to obtain this result by a combination of [6, Lemma 5.1] and [19, Theorem 7.1].

## 1.6 Nowhere dense sets

Assume now that  $\Omega \subset \mathbb{R}^d$  is a bounded, nowhere dense set of positive measure. It is not hard to show that such a set  $\Omega$  cannot tile the space by translations. To see this, suppose to the contrary that  $\Omega + \Lambda$  is a tiling, then  $\Lambda$  must be a locally finite set. Since

$\Omega$  is bounded, any ball can thus intersect only finitely many translated copies  $\Omega + \lambda$ ,  $\lambda \in \Lambda$ . Since  $\Omega$  is a nowhere dense set, then also any finite union of translates of  $\Omega$  is a nowhere dense set. Hence the union of all the translated copies  $\Omega + \lambda$  intersecting a given ball is a nowhere dense set, so this union does not cover a set of full measure in the ball, a contradiction.

In [21], the following question was posed: can a bounded, nowhere dense set be spectral? The answer is expected to be negative, but so far this has been proved only in dimension one. In fact, it was proved in [8, Corollary 1.6] that if a bounded, measurable set  $\Omega \subset \mathbb{R}$  is spectral, then it can *multi-til*e the real line by translations. This excludes the possibility that  $\Omega$  is a nowhere dense set, by the same argument as above.

In this paper we use the weak tiling method as a different approach to the spectrality problem for bounded, nowhere dense sets. We will prove the following result:

**Theorem 1.8** *Let  $E \subset \mathbb{R}$  be a symmetric Cantor set of positive measure. Then  $E$  cannot weakly tile its complement by translations. As a consequence,  $E$  is not spectral.*

The definition of a symmetric Cantor set will be given in Sect. 5.

If we combine Theorem 1.8 with Theorem 1.7 then we obtain the following conclusion for multi-dimensional nowhere dense sets with a cartesian product structure:

**Theorem 1.9** *Let  $\Omega = E \times B$  be the product of a symmetric Cantor set  $E \subset \mathbb{R}$  of positive measure, and an arbitrary bounded, measurable set  $B \subset \mathbb{R}^m$ . Then  $\Omega$  is not a spectral set in  $\mathbb{R} \times \mathbb{R}^m$ .*

We give two proofs of Theorem 1.8. In Sect. 5 we prove it by closely examining the *essential difference set* of a symmetric Cantor set in  $\mathbb{R}$  and proving that the Cantor set admits a *packing region* which is strictly larger than the set itself (see Sect. 4). In Sect. 6 we prove directly that the Cantor set cannot weakly tile its complement by translations, by establishing in a quantitative way the same property for the  $n$ -th generation set that approximates the Cantor set.

## 2 Weak tiling and equidecomposability

In this section we prove Theorem 1.3, that is, we show that if a (not necessarily convex) polytope  $A \subset \mathbb{R}^d$  can weakly tile its complement by translations, then  $A$  is equidecomposable by translations to a cube. As a consequence, any *convex* polytope that can weakly tile its complement by translations must be centrally symmetric and have centrally symmetric facets. In turn, this implies Theorem 1.4 (see Sect. 1.4).

2.1. The Fourier transform of a function  $f \in L^1(\mathbb{R}^d)$  is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx, \quad \xi \in \mathbb{R}^d.$$

We will need a result from [19] concerning the zeros of the Fourier transform  $\widehat{\mathbb{1}_A}$  of the indicator function  $\mathbb{1}_A$  of a (not necessarily convex) polytope  $A \subset \mathbb{R}^d$ .

Given  $\delta > 0$  and real numbers  $\tau_1, \dots, \tau_k$  we consider the set

$$T = T(\delta; \tau_1, \dots, \tau_k) = \{n \in \mathbb{Z} : |e^{2\pi i n \tau_j} - 1| < \delta, 1 \leq j \leq k\}. \tag{2.1}$$

If we are also given  $R > 0$  then we let

$$J = J(R) = \{n \in \mathbb{Z} : 0 < |n| < R\}. \tag{2.2}$$

Finally, given also  $\varepsilon > 0$  and a vector  $v \in \mathbb{R}^d$  we define

$$S = S(T, J, v, \varepsilon) = \{nv + w : n \in T \setminus J, w \in \mathbb{R}^d, |w| < \varepsilon\}. \tag{2.3}$$

The following result was proved in [19] although it was not explicitly stated there in this form.

**Theorem 2.1** ([19]) *Let  $A$  be a polytope in  $\mathbb{R}^d$ . If  $A$  is not equidecomposable by translations to a cube, then there exist  $\delta > 0$ , real numbers  $\tau_1, \dots, \tau_k$ , a nonzero vector  $v \in \mathbb{R}^d$ ,  $\varepsilon > 0$  and  $R > 0$  such that the Fourier transform  $\widehat{\mathbb{1}}_A$  has no zeros in the set  $S$ , where  $T, J$  and  $S$  are the three sets defined by (2.1), (2.2) and (2.3).*

We briefly indicate how this can be inferred from [19]. If  $A$  is not equidecomposable by translations to any cube, then there exists a Hadwiger functional  $H_\Phi$  such that  $H_\Phi(A) \neq 0$ . Let  $p(u)$  be the trigonometric polynomial constructed in [19, Section 6.1], and expand it in the form  $p(u) = \sum_{j=1}^k c_j e^{2\pi i u \tau_j}$ . Let  $\delta = \delta(p, \eta) > 0$  be small enough such that  $|p(n) - p(0)| < \eta$  for every  $n$  in the set (2.1). We then continue the proof as in [19, Section 6] to conclude that there exist a nonzero vector  $v \in \mathbb{R}^d$ ,  $\varepsilon > 0$  and  $R > 0$  such that  $\widehat{\mathbb{1}}_A$  has no zeros in the set (2.3).

The results in [19] are based on an intricate analysis of the asymptotic behavior of the Fourier transform  $\widehat{\mathbb{1}}_A$ , see [19, Theorem 4.1]. The analysis becomes considerably simpler if the polytope  $A$  is assumed to be convex, see [5, Sections 3 and 4].

2.2. A measure  $\mu$  on  $\mathbb{R}^d$  is said to be *translation-bounded* if for every (or equivalently, for some) open ball  $B$  we have

$$\sup_{x \in \mathbb{R}^d} |\mu|(B + x) < +\infty.$$

If  $\mu$  is a translation-bounded measure on  $\mathbb{R}^d$ , then  $\mu$  is a tempered distribution.

If  $f$  is a function in  $L^1(\mathbb{R}^d)$  and  $\mu$  is a translation-bounded measure on  $\mathbb{R}^d$ , then the convolution  $f * \mu$  is a locally integrable function on  $\mathbb{R}^d$  defined uniquely (up to equality a.e.) by the condition that  $(f * \mu) * \varphi = f * (\mu * \varphi)$  for every continuous, compactly supported function  $\varphi$  on  $\mathbb{R}^d$ .

**Theorem 2.2** *Let  $f \in L^1(\mathbb{R}^d)$ ,  $\int f \neq 0$ , and let  $\mu$  be a translation-bounded measure on  $\mathbb{R}^d$ . If we have  $f * \mu = 1$  a.e., then  $\widehat{\mu} = (\int f)^{-1} \cdot \delta_0$  in the open set  $Z(\widehat{f})^c$ .*

Here we let  $Z(\widehat{f}) := \{\xi \in \mathbb{R}^d : \widehat{f}(\xi) = 0\}$  denote the (closed) set of zeros of  $\widehat{f}$ .

Theorem 2.2 can be proved in a similar way to [13, Theorem 4.1].

2.3. We now turn to the proof of the main result of this section.

**Proof of Theorem 1.3** Let  $A$  be a polytope in  $\mathbb{R}^d$ , and assume that  $A$  can tile its complement weakly by translations, that is, there exists a positive, locally finite measure  $\nu$  on  $\mathbb{R}^d$  such that  $\mathbb{1}_A * \nu = \mathbb{1}_{A^c}$  a.e. By [20, Lemma 2.4] the measure  $\nu$  is not only locally finite, but in fact  $\nu$  must be translation-bounded. It follows that the measure  $\mu := \delta_0 + \nu$  is translation-bounded and satisfies  $\mathbb{1}_A * \mu = 1$  a.e. In turn, Theorem 2.2 implies that  $\widehat{\mu} = m(A)^{-1} \cdot \delta_0$  in the open set  $Z(\widehat{\mathbb{1}}_A)^c$ .

We must prove that  $A$  is equidecomposable by translations to a cube of the same volume. Suppose to the contrary that this is not the case. Then by Theorem 2.1 there exist  $\delta > 0$ , real numbers  $\tau_1, \dots, \tau_k$ , a nonzero vector  $v \in \mathbb{R}^d$ ,  $\varepsilon > 0$  and  $R > 0$  such that  $\widehat{\mathbb{1}}_A$  has no zeros in the set  $S$ , where  $T, J$  and  $S$  are the three sets defined by (2.1), (2.2) and (2.3). It follows that we have  $\widehat{\mu} = m(A)^{-1} \cdot \delta_0$  in the open set  $S$ .

Now suppose that we are given a real-valued Schwartz function  $g$  on  $\mathbb{R}^d$  satisfying

$$\text{supp}(g) \subset S, \quad \widehat{g} \geq 0. \tag{2.4}$$

Then we have

$$\int_{\mathbb{R}^d} g(x) dx = \widehat{g}(0) \leq \widehat{g}(0) + \int \widehat{g}(\xi) d\nu(\xi) = \int \widehat{g}(\xi) d\mu(\xi), \tag{2.5}$$

where the inequality in (2.5) is due to  $\widehat{g}$  being a nonnegative function and  $\nu$  being a positive measure. On the other hand, we have

$$\int \widehat{g}(\xi) d\mu(\xi) = \mu(\widehat{g}) = \widehat{\mu}(g) = m(A)^{-1} g(0), \tag{2.6}$$

where the last equality holds since we have  $\text{supp}(g) \subset S$  and  $\widehat{\mu} = m(A)^{-1} \cdot \delta_0$  in the open set  $S$ . We conclude that

$$\int_{\mathbb{R}^d} g(x) dx \leq m(A)^{-1} g(0) \tag{2.7}$$

for every real-valued Schwartz function  $g$  satisfying (2.4). We will show that this leads to a contradiction, by constructing an example of a real-valued Schwartz function  $g$  satisfying (2.4), but such that (2.7) does not hold.

We choose a nonnegative Schwartz function  $\varphi$  such that  $\int \varphi = 1$ ,  $\varphi$  is supported in the open ball of radius  $\varepsilon$  centered at the origin, and  $\widehat{\varphi} \geq 0$ . Next, let  $\psi$  be a nonnegative, smooth function on the  $k$ -dimensional torus  $\mathbb{T}^k = (\mathbb{R}/\mathbb{Z})^k$ , such that  $\int \psi = 1$ ,  $\psi$  is supported in the set

$$\{(t_1, \dots, t_k) : |e^{2\pi i t_j} - 1| < \delta, 1 \leq j \leq k\}, \tag{2.8}$$

and the Fourier coefficients  $\widehat{\psi}(m), m \in \mathbb{Z}^k$ , are nonnegative. We may assume that  $R$  is an integer (by enlarging it if necessary) and define also the trigonometric polynomial

$p_N(t) := K_N(Rt)$ , where  $K_N$  is the classical Fejér kernel,

$$K_N(t) = \sum_{|n| < N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n t}, \quad t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}. \tag{2.9}$$

Then  $p_N$  is nonnegative,  $p_N(0) = N$ , the Fourier coefficients  $\widehat{p}_N(n)$ ,  $n \in \mathbb{Z}$ , are also nonnegative,  $\widehat{p}_N(0) = 1$ , and we have  $\widehat{p}_N(n) = 0$  for every  $n \in J$ .

Finally, we define the function

$$g_N(x) := \sum_{n \in \mathbb{Z}} \psi(n\tau) \widehat{p}_N(n) \varphi(x - nv), \quad x \in \mathbb{R}^d, \tag{2.10}$$

where we denote  $\tau = (\tau_1, \dots, \tau_k)$ . Notice that there are only finitely many nonzero terms in the sum (2.10), and that the nonzero terms correspond to integers  $n$  belonging to the set  $T \setminus J$ . Hence  $g_N$  is a real-valued (in fact, nonnegative) Schwartz function such that  $\text{supp}(g_N)$  is contained in the set  $S$ . The Fourier transform of  $g$  is given by

$$\widehat{g}_N(\xi) = \widehat{\varphi}(\xi) \sum_{n \in \mathbb{Z}} \psi(n\tau) \widehat{p}_N(n) e^{-2\pi i n \langle v, \xi \rangle}. \tag{2.11}$$

Using the (absolutely convergent) Fourier expansion of the function  $\psi$ , we obtain

$$\psi(n\tau) = \sum_{m \in \mathbb{Z}^k} \widehat{\psi}(m) e^{2\pi i n \langle m, \tau \rangle}. \tag{2.12}$$

If we plug (2.12) into (2.11) and exchange the order of summation (which is justified as  $n$  goes through a finite set of values) we obtain

$$\widehat{g}_N(\xi) = \widehat{\varphi}(\xi) \sum_{m \in \mathbb{Z}^k} \widehat{\psi}(m) p_N(\langle m, \tau \rangle - \langle v, \xi \rangle). \tag{2.13}$$

Since all the terms on the right hand side of (2.13) are nonnegative, we obtain that  $\widehat{g}_N$  is a nonnegative function. We conclude that  $g_N$  satisfies the conditions (2.4).

To complete the proof we will show that if  $N$  is sufficiently large, then  $g_N$  does not satisfy (2.7). Indeed, we may assume that  $\varepsilon < \frac{1}{2}|v|$  which ensures that the terms in the sum (2.10) have pairwise disjoint supports. This implies that

$$g_N(0) = \varphi(0)\psi(0), \tag{2.14}$$

so that the value  $g_N(0)$  does not depend on  $N$ . On the other hand, using (2.13) we have

$$\int_{\mathbb{R}^d} g_N(x) dx = \widehat{g}_N(0) \geq \widehat{\varphi}(0)\widehat{\psi}(0)p_N(0) = N, \tag{2.15}$$

which can be arbitrarily large, contradicting (2.7). We have thus arrived at the desired contradiction and so Theorem 1.3 is proved.  $\square$



### 3 Weak tiling and product domains

In this section we prove Theorem 1.7. That is, we show that if  $A \subset \mathbb{R}^n$  and  $B \subset \mathbb{R}^m$  are two bounded, measurable sets, then the product set  $\Omega = A \times B$  weakly tiles its complement in  $\mathbb{R}^n \times \mathbb{R}^m$  by translations if and only if both  $A$  and  $B$  can weakly tile their complements in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively.

**Proof of Theorem 1.7** Suppose first that both  $A$  and  $B$  weakly tile their complements. Then there exist two positive, locally finite measures  $\nu_1$  on  $\mathbb{R}^n$  and  $\nu_2$  on  $\mathbb{R}^m$ , such that

$$\mathbb{1}_A * (\delta_0 + \nu_1) = 1 \text{ a.e. in } \mathbb{R}^n, \quad \mathbb{1}_B * (\delta_0 + \nu_2) = 1 \text{ a.e. in } \mathbb{R}^m. \quad (3.1)$$

It follows that the (also locally finite) product measure

$$\mu := (\delta_0 + \nu_1) \times (\delta_0 + \nu_2) \quad (3.2)$$

satisfies  $\mathbb{1}_\Omega * \mu = 1$  a.e. in  $\mathbb{R}^n \times \mathbb{R}^m$ . Let  $\nu$  be the positive measure

$$\nu := \nu_1 \times \delta_0 + \delta_0 \times \nu_2 + \nu_1 \times \nu_2, \quad (3.3)$$

and observe that we have  $\mu = \delta_0 + \nu$ . Hence  $\mathbb{1}_\Omega * \nu = \mathbb{1}_{\Omega^c}$  a.e. and  $\Omega$  weakly tiles its complement in  $\mathbb{R}^n \times \mathbb{R}^m$  by translations.

Conversely, suppose that  $\nu$  is a positive, locally finite measure in  $\mathbb{R}^n \times \mathbb{R}^m$  satisfying

$$\mathbb{1}_\Omega * \nu = \mathbb{1}_{\Omega^c} \text{ a.e.} \quad (3.4)$$

For each  $y \in \mathbb{R}^m$  we define a measure  $\nu_y$  on  $\mathbb{R}^n$  by the condition

$$\int_{\mathbb{R}^n} \varphi(u) d\nu_y(u) = \iint_{\mathbb{R}^n \times \mathbb{R}^m} \varphi(u) \mathbb{1}_B(y - v) d\nu(u, v) \quad (3.5)$$

for any bounded, compactly supported Borel function  $\varphi$  on  $\mathbb{R}^n$ . It follows that

$$(\mathbb{1}_A * \nu_y)(x) = \int_{\mathbb{R}^n} \mathbb{1}_A(x - u) d\nu_y(u) \quad (3.6)$$

$$= \iint_{\mathbb{R}^n \times \mathbb{R}^m} \mathbb{1}_A(x - u) \mathbb{1}_B(y - v) d\nu(u, v) = (\mathbb{1}_\Omega * \nu)(x, y). \quad (3.7)$$

We use this together with (3.4) to conclude by an application of Fubini's theorem that there is a set  $Y \subset \mathbb{R}^m$  of full measure, such that for any  $y \in Y$  we have

$$(\mathbb{1}_A * \nu_y)(x) = \mathbb{1}_{\Omega^c}(x, y), \quad x \in X(y), \quad (3.8)$$

where  $X(y)$  is a set of full measure in  $\mathbb{R}^n$ . In particular, for  $y \in Y \cap B$  this yields  $\mathbb{1}_A * \nu_y = \mathbb{1}_{A^c}$  a.e. in  $\mathbb{R}^n$ , which shows that  $A$  weakly tiles its complement in  $\mathbb{R}^n$ . In the same way one can show that also  $B$  weakly tiles its complement in  $\mathbb{R}^m$ .  $\square$

## 4 Packing regions

In this section we introduce the notion of a *packing region*, and use it to establish a geometric condition (different from the weak tiling condition) that is necessary for the spectrality of a set  $A \subset \mathbb{R}^d$ . To prove our result (Theorem 4.3) it will not suffice though to invoke Theorem 1.2, since the proof requires additional properties of the weak tiling measure that are established in [20] but not stated in Theorem 1.2.

4.1. In order to define the notion of a packing region, we first need to introduce the following definition:

**Definition 4.1** If  $A \subset \mathbb{R}^d$  is a bounded, measurable set, then the set

$$\Delta(A) := \{t \in \mathbb{R}^d : \text{mes}(A \cap (A + t)) > 0\} \tag{4.1}$$

will be called the *essential difference set* of  $A$ .

The set  $\Delta(A)$  is a bounded open set, symmetric with respect to the origin.

The essential difference set  $\Delta(A)$  can be viewed as the measure-theoretic analog of the algebraic difference set  $A - A$ . In particular, if  $A$  is an open set then  $\Delta(A) = A - A$ . In general we have  $\Delta(A) \subset A - A$ , but this inclusion can be strict.

**Definition 4.2** We say that a bounded, measurable set  $D \subset \mathbb{R}^d$  is a *packing region* for  $A$  if we have  $\Delta(D) \subset \Delta(A)$ .

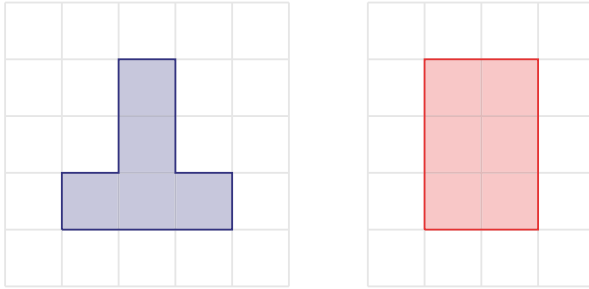
Let us explain the reason for the name “packing region”. If  $\Lambda$  is a finite or countable set in  $\mathbb{R}^d$ , then we say that  $A + \Lambda$  is a *packing* if the translated copies  $\{A + \lambda\}$ ,  $\lambda \in \Lambda$ , are disjoint up to measure zero. Then a bounded, measurable set  $D$  is a packing region for  $A$  if and only if whenever we have that  $A + \Lambda$  is a packing, then also  $D + \Lambda$  is a packing.

4.2. The following theorem is the main result of this section.

**Theorem 4.3** *Let  $A$  be a bounded, measurable set in  $\mathbb{R}^d$ . Suppose that  $A$  admits a packing region  $D$  such that  $m(D) > m(A)$ . Then  $A$  is neither spectral nor can it tile the space by translations.*

**Proof** We first show that  $A$  cannot tile by translations. Indeed, if  $A + \Lambda$  is a tiling then  $D + \Lambda$  must be a packing, and it follows e.g. from [6, Lemma 3.2] that  $m(D) \leq m(A)$ . But we assumed that  $m(D) > m(A)$ , so we arrive at a contradiction.

Next we show that  $A$  can neither be spectral. Suppose to the contrary that  $A$  is spectral, and let  $\gamma$  be the measure given by [20, Theorem 3.1]. Define the function  $f := |\widehat{\mathbb{1}}_D|^2$ , then the Fourier transform of  $f$  is the function  $\widehat{f} = \mathbb{1}_D * \mathbb{1}_{-D}$ , that is, we have  $\widehat{f}(x) = m(D \cap (D + x))$  for every  $x \in \mathbb{R}^d$ . In particular,  $\widehat{f}(0) = m(D)$ , and  $\widehat{f}$  vanishes on the set  $\Delta(D)^c$ . On the other hand, we have  $\widehat{\gamma} = m(A) \delta_0$  in the open set  $\Delta(A)$ , so due to the assumption that  $\Delta(D) \subset \Delta(A)$  this implies that  $\widehat{\gamma} \cdot \widehat{f} = m(A)m(D) \delta_0$ . The measure  $\widehat{\gamma} \cdot \widehat{f}$  is the Fourier transform of the function  $\gamma * f$  (see [14, Section 2.3]), and it follows that  $\gamma * f = m(A)m(D)$  a.e.



**Fig. 1** The planar domain  $A$  shown on the left is not spectral, since it has a packing region  $D$  (shown on the right) satisfying  $m(D) > m(A)$

Recall now that  $\gamma$  is a positive measure, and that  $\gamma = \delta_0$  in some open ball centered at the origin. Since  $f$  is a nonnegative function, this implies that

$$f = \delta_0 * f \leq \gamma * f = m(A)m(D) \quad \text{a.e.}$$

But since  $f$  is a continuous function, it follows that the inequality  $f(t) \leq m(A)m(D)$  must in fact hold for every  $t \in \mathbb{R}^d$ . However  $f(0) = m(D)^2$ , so this is possible only if  $m(D) \leq m(A)$ . Since we assumed that  $m(D) > m(A)$ , we arrive at a contradiction.  $\square$

**Remark 4.4** Let us comment that the proofs of Theorems 1.3 and 4.3 share a common idea: in order to arrive at a contradiction we construct a positive definite function with prescribed support, whose integral is “too large” compared to its value at the origin.

**Remark 4.5** We note that there is a notion of an *orthogonal packing region*, that was introduced in [18] and extended to general bounded, measurable sets in [6, Section 3.6]. By definition, a bounded measurable set  $D \subset \mathbb{R}^d$  is an orthogonal packing region for  $A$  if we have  $\Delta(D) \subset Z(\widehat{\mathbb{1}}_A)^c$ . This notion was used in [18, Lemma 2.3] and [11, Theorem 7] to establish a result analogous to Theorem 4.3, namely, if  $A$  admits an orthogonal packing region  $D$  such that  $m(D) > m(A)^{-1}$ , then  $A$  is neither spectral nor can it tile by translations. Notice however that the notion of an orthogonal packing region is not purely geometric, as it involves the set  $Z(\widehat{\mathbb{1}}_A)$  of the zeros of the Fourier transform  $\widehat{\mathbb{1}}_A$ .

4.3. We give two examples demonstrating how to use Theorem 4.3.

**Example 4.6** Let  $A$  and  $D$  be the two planar domains illustrated respectively on the left and right hand sides of Fig. 1. It is straightforward to verify that  $\Delta(D)$  is the open rectangle  $(-2, 2) \times (-3, 3)$ , and that  $\Delta(A)$  contains this rectangle. Hence  $D$  serves as a packing region for  $A$ . But notice that  $m(A) = 5$ ,  $m(D) = 6$ , so that  $m(D) > m(A)$ . It thus follows from Theorem 4.3 that  $A$  is not spectral.

**Example 4.7** Let  $A$  be a convex body in  $\mathbb{R}^d$ , and assume that  $A$  is not centrally symmetric. Then the convex body  $D = \frac{1}{2}(A - A)$  is a packing region for  $A$ , and we have

$m(D) > m(A)$  by the Brunn-Minkowski inequality. Using Theorem 4.3 this yields that a spectral convex body must be centrally symmetric, a result first proved in [10] based on the same consideration.

**Remark 4.8** Due to [20] we know that a convex body  $A \subset \mathbb{R}^d$  is spectral if and only if it satisfies the following four conditions: (i)  $A$  is a convex polytope; (ii)  $A$  is centrally symmetric; (iii) all the facets of  $A$  are centrally symmetric; and (iv) each belt of  $A$  has either 4 or 6 facets. Example 4.7 shows that by an application of Theorem 4.3 one can prove the necessity of condition (ii) for the spectrality of  $A$ . We observe however that the same is not true for the other three conditions (i), (iii) and (iv). Indeed, let a convex body  $A \subset \mathbb{R}^d$  be centrally symmetric, say,  $-A = A$ . Then  $\Delta(A) = 2 \text{int}(A)$ , where  $\text{int}(A)$  is the set of interior points of  $A$ . If  $D$  is a packing region for  $A$ , then  $\Delta(D) \subset \Delta(A)$  and hence

$$m(\Delta(D)) \leq m(\Delta(A)) = 2^d m(A). \tag{4.2}$$

On the other hand, we can invoke a version of the Brunn-Minkowski inequality for the essential difference set  $\Delta(D)$ , see the inequality (2.6) in [1], which implies that

$$m(\Delta(D)) \geq 2^d m(D). \tag{4.3}$$

It thus follows from (4.2) and (4.3) that  $m(D) \leq m(A)$ , whenever  $D$  is a packing region for  $A$ . Hence, assuming that one of the conditions (i), (iii) or (iv) fails to hold does not imply the existence of a packing region  $D$  such that  $m(D) > m(A)$ , and therefore one cannot establish the non-spectrality of  $A$  by an application of Theorem 4.3.

4.4. The next result connects the notions of a packing region and weak tiling.

**Theorem 4.9** *Let  $A$  be a bounded, measurable set in  $\mathbb{R}^d$ . Suppose that  $A$  admits a packing region  $D$  such that not only  $m(D) > m(A)$ , but moreover  $D \supset A$ . Then  $A$  cannot weakly tile its complement by translations.*

**Proof** We use the same argument as in [20, Theorem 3.5]. Suppose to the contrary that  $\mathbb{1}_A * \nu = \mathbb{1}_{A^c}$  a.e., where  $\nu$  is a positive, locally finite measure on  $\mathbb{R}^d$ . Then

$$m(D \cap A^c) = \int_D \mathbb{1}_{A^c} dm = \int_D (\mathbb{1}_A * \nu) dm = \int \varphi d\nu, \tag{4.4}$$

where  $\varphi(t) := m((A+t) \cap D)$ . The assumptions that  $D$  is a packing region for  $A$  and  $D \supset A$  imply that  $\varphi$  must vanish on the set  $\Delta(A)^c$ . On the other hand, the measure  $\nu$  is supported on the set  $\Delta(A)^c$  due to [20, Corollary 2.6]. Hence the integral on the right hand side of (4.4) vanishes. We conclude that  $m(D \cap A^c) = 0$ , or equivalently,  $m(D) = m(D \cap A)$ . But this contradicts our assumption that  $m(D) > m(A)$ .  $\square$

## 5 Cantor sets of positive measure: via the essential difference set

In this section we prove Theorem 1.8, which asserts that a symmetric Cantor set of positive measure in  $\mathbb{R}$  cannot weakly tile its complement by translations. The approach is based on the construction of a packing region  $D$  for the set  $E$  with  $m(D) > m(E)$  and moreover  $D \supset E$ . The conclusion then follows from Theorem 4.9.

5.1. We start by recalling the definition of a symmetric Cantor set in  $\mathbb{R}$ . Let  $\{\xi_k\}$ ,  $k = 1, 2, \dots$ , be a sequence of real numbers such that  $0 < \xi_k < \frac{1}{2}$  for all  $k$ . The *symmetric Cantor set associated to the ratio sequence*  $\{\xi_k\}$  is the closed set  $E(\xi_1, \xi_2, \dots) \subset \mathbb{R}$  obtained from the interval  $[0, 1]$  by a Cantor type construction, where at the  $k$ 'th step we remove from each one of the intervals of the previous step a central open interval of relative length  $1 - 2\xi_k$ , so that at the  $k$ 'th step we obtain  $2^k$  closed intervals of common length  $\xi_1 \cdots \xi_k$ .

The Lebesgue measure of the set  $E(\xi_1, \xi_2, \dots)$  is

$$m(E(\xi_1, \xi_2, \dots)) = \lim_{k \rightarrow \infty} 2^k \xi_1 \cdots \xi_k. \quad (5.1)$$

We are interested in the case where  $E(\xi_1, \xi_2, \dots)$  is a set of positive measure.

5.2. Our proof of Theorem 1.8 is based on the following lemma:

**Lemma 5.1** *Let  $E$  be a symmetric Cantor set of positive measure in  $\mathbb{R}$ . Then  $E$  admits a packing region  $D$  satisfying the two conditions  $D \supset E$  and  $m(D) > m(E)$ .*

Theorem 1.8 follows as an immediate consequence of Theorem 4.9 and Lemma 5.1. So it remains to prove the lemma.

We note that there exist results in the literature which concern the set of differences  $E - E$  of a symmetric Cantor set  $E$ , see e.g. [2] and the references therein. For example, it is well known that if  $E$  is the classical ternary Cantor set (associated to the constant ratio sequence  $\xi_k = \frac{1}{3}$ ) then  $E - E = [-1, 1]$ . However, such results do not suffice for our present purpose, as in order to prove Lemma 5.1 we must establish that the *essential difference set*  $\Delta(E)$  (and not only  $E - E$ ) is quite large.

We now turn to the proof of Lemma 5.1. This will be done in several steps.

5.3. We will need the following proposition.

**Proposition 5.2** *Let  $A$  be a bounded, measurable set in  $\mathbb{R}^d$ , and let  $B = \bigcup_{j=1}^n (A + t_j)$  where  $t_1, \dots, t_n \in \mathbb{R}^d$ . Then*

$$\Delta(B) = \bigcup_{i,j} (\Delta(A) + t_i - t_j). \quad (5.2)$$

The proof is straightforward and we omit the details.

5.4. Let  $E(\xi_1, \xi_2, \dots)$  be the symmetric Cantor set associated to a given ratio sequence  $\{\xi_k\}$  (that is, a sequence of real numbers such that  $0 < \xi_k < \frac{1}{2}$  for all  $k$ ). Then the set

$E(\xi_1, \xi_2, \dots)$  consists of all points  $x$  of the form

$$x = \sum_{k=1}^{\infty} r_k \varepsilon_k, \tag{5.3}$$

where

$$r_k := \xi_1 \cdots \xi_{k-1} (1 - \xi_k) \tag{5.4}$$

and each  $\varepsilon_k$  is either 0 or 1. If we let  $T_k$  denote the finite set of points  $x$  of the form

$$x = \sum_{j=1}^k r_j \varepsilon_j \tag{5.5}$$

where  $\varepsilon_j = 0$  or 1, then we have

$$E(\xi_1, \xi_2, \dots) = T_k + \xi_1 \cdots \xi_k E(\xi_{k+1}, \xi_{k+2}, \dots), \tag{5.6}$$

so that  $E(\xi_1, \xi_2, \dots)$  is the union of  $2^k$  disjoint homothetic copies of  $E(\xi_{k+1}, \xi_{k+2}, \dots)$ .

It follows from Proposition 5.2 that

$$\Delta(E(\xi_1, \xi_2, \dots)) = T_k - T_k + \xi_1 \cdots \xi_k \Delta(E(\xi_{k+1}, \xi_{k+2}, \dots)). \tag{5.7}$$

5.5. The following result establishes that Lemma 5.1 holds in the special case where the set  $E(\xi_1, \xi_2, \dots)$  has measure sufficiently close to 1.

**Lemma 5.3** *Suppose that  $E(\xi_1, \xi_2, \dots)$  has measure at least  $\frac{4}{5}$ . Then*

$$\Delta(E(\xi_1, \xi_2, \dots)) = \Delta(I), \tag{5.8}$$

where  $I := [0, 1]$ .

It follows from (5.8) that the set  $D := I$  serves as a packing region for  $E := E(\xi_1, \xi_2, \dots)$ , and we have  $D \supset E$  and  $m(D) > m(E)$ . We note that  $\Delta(I)$  is simply the interval  $(-1, 1)$ .

**Proof of Lemma 5.3** Write  $m(E(\xi_1, \xi_2, \dots)) = 1 - \varepsilon$ . From (5.1) it follows that

$$2\xi_1 \geq 1 - \varepsilon, \tag{5.9}$$

and

$$\lim_{n \rightarrow \infty} 2^n \xi_{k+1} \cdots \xi_{k+n} \geq 1 - \varepsilon \tag{5.10}$$

for every  $k$ . Note that  $0 < \varepsilon \leq \frac{1}{5}$  by assumption.

The set  $\Delta(E(\xi_1, \xi_2, \dots))$  is symmetric with respect to the origin and it is a subset of  $(-1, 1)$ , so to establish (5.8) it would be enough to show that  $\Delta(E(\xi_1, \xi_2, \dots))$  contains the interval  $[0, 1]$ . This will be done in two steps.

*Step 1:* We first show that  $\Delta(E(\xi_1, \xi_2, \dots))$  contains the interval  $[0, 1 - \xi_1]$ .  
 Indeed, let  $t \in [0, 1 - \xi_1]$ . For any two bounded, measurable sets  $E$  and  $F$  we have

$$m(E \cap (E + t)) \geq m(F \cap (F + t)) - 2m(E \Delta F), \tag{5.11}$$

where  $E \Delta F$  is the symmetric difference of  $E$  and  $F$ . We apply this estimate to the two sets  $E = E(\xi_1, \xi_2, \dots)$  and  $F = [0, 1]$ . On one hand, we have  $F \cap (F + t) = [t, 1]$  and thus

$$m(F \cap (F + t)) = 1 - t > \xi_1 \geq \frac{1}{2} - \frac{1}{2}\varepsilon, \tag{5.12}$$

where the last inequality follows from (5.9). On the other hand, notice that

$$m(F \Delta E) = 1 - m(E) = \varepsilon. \tag{5.13}$$

So (5.11), (5.12) and (5.13) imply that

$$m(E \cap (E + t)) > \frac{1}{2} - \frac{5}{2}\varepsilon \geq 0, \tag{5.14}$$

hence  $E \cap (E + t)$  is a set of positive measure and  $t \in \Delta(E)$ .

*Step 2:* Next we show that  $\Delta(E(\xi_1, \xi_2, \dots))$  contains also the interval  $[1 - \xi_1, 1]$ .

Let therefore  $t \in [1 - \xi_1, 1]$ , then we have  $r_1 \leq t < 1$ . Since  $\sum_{j=1}^{\infty} r_j = 1$ , there is  $k$  such that

$$r_1 + \dots + r_k \leq t < r_1 + \dots + r_k + r_{k+1}. \tag{5.15}$$

Then we can write

$$t = r_1 + \dots + r_k + \xi_1 \dots \xi_k s, \quad 0 \leq s < 1 - \xi_{k+1}. \tag{5.16}$$

The set  $E(\xi_{k+1}, \xi_{k+2}, \dots)$  also has measure at least  $\frac{4}{5}$ , due to (5.10), so by what we have already shown in Step 1 we have  $s \in \Delta(E(\xi_{k+1}, \xi_{k+2}, \dots))$ . Moreover, the number  $r_1 + \dots + r_k$  belongs to the set  $T_k - T_k$ . Hence using (5.7) we conclude that

$$t \in T_k - T_k + \xi_1 \dots \xi_k \Delta(E(\xi_{k+1}, \xi_{k+2}, \dots)) = \Delta(E(\xi_1, \xi_2, \dots)), \tag{5.17}$$

which completes the proof. □

5.6. Let  $E_k(\xi_1, \dots, \xi_k)$  be the set consisting of the  $2^k$  disjoint closed intervals of common length  $\xi_1 \dots \xi_k$  obtained after the  $k$ 'th step of the Cantor type construction. Then

$$E_k(\xi_1, \dots, \xi_k) = T_k + \xi_1 \dots \xi_k I \tag{5.18}$$

where (as before)  $I = [0, 1]$ , and we have

$$E(\xi_1, \xi_2, \dots) = \bigcap_{k=1}^{\infty} E_k(\xi_1, \dots, \xi_k). \tag{5.19}$$

It follows from Proposition 5.2 that

$$\Delta(E_k(\xi_1, \dots, \xi_k)) = T_k - T_k + \xi_1 \cdots \xi_k \Delta(I). \tag{5.20}$$

The next result concludes the proof of Theorem 1.8 by establishing that Lemma 5.1 holds in the general case.

**Lemma 5.4** *Let  $E(\xi_1, \xi_2, \dots)$  have positive measure. Then there is  $k$  such that*

$$\Delta(E(\xi_1, \xi_2, \dots)) = \Delta(E_k(\xi_1, \dots, \xi_k)). \tag{5.21}$$

It follows from (5.21) that the set  $D := E_k(\xi_1, \dots, \xi_k)$  serves as a packing region for  $E := E(\xi_1, \xi_2, \dots)$ . Moreover, the two conditions  $D \supset E$  and  $m(D) > m(E)$  are satisfied, so Lemma 5.1 is indeed established.

**Proof of Lemma 5.4** We can choose  $k$  sufficiently large such that

$$\lim_{n \rightarrow \infty} 2^n \xi_{k+1} \xi_{k+2} \cdots \xi_{k+n} \geq \frac{4}{5} \tag{5.22}$$

(for otherwise, there would exist a sequence  $k_1 < k_2 < k_3 < \dots$  such that

$$2^{k_{j+1}-k_j} \xi_{k_j+1} \xi_{k_j+2} \cdots \xi_{k_{j+1}} < \frac{4}{5} \tag{5.23}$$

for every  $j$ , which implies that the limit in (5.1) is zero contrary to our assumption).

Then by Lemma 5.3 we have

$$\Delta(E(\xi_{k+1}, \xi_{k+2}, \dots)) = \Delta(I). \tag{5.24}$$

Together with (5.7) and (5.20) this implies that

$$\Delta(E(\xi_1, \xi_2, \dots)) = T_k - T_k + \xi_1 \cdots \xi_k \Delta(I) = \Delta(E_k(\xi_1, \dots, \xi_k)), \tag{5.25}$$

and thus (5.21) is proved. □

**Remark 5.5** It follows from (5.25) that if  $E(\xi_1, \xi_2, \dots)$  has positive measure, then the essential difference set  $\Delta(E(\xi_1, \xi_2, \dots))$  is the union of finitely many open intervals.

## 6 Cantor sets of positive measure: via weak tiling

In this section we give another proof of Theorem 1.8, which states that a symmetric Cantor set of positive measure cannot weakly tile its complement by translations. In this proof the approach is direct and does not go through the essential difference set.

6.1. Suppose that we have a Cantor set  $E \subset \mathbb{R}$  which is the intersection of the compact sets  $E_n$ , each of which is a finite collection of closed intervals of length  $\ell_n$ . The set  $E_n$  is constructed from  $E_{n-1}$  by throwing away an open middle interval of length  $d_n$



from each of the intervals of  $E_{n-1}$ . For  $E$  to have positive measure it is *necessary* that  $d_n/\ell_n \rightarrow 0$ .

The idea of this proof is to use the fact that the sets  $E_n$  obviously cannot weakly tile their complement (the holes are too small compared to the intervals making up  $E_n$ ). Making this observation quantitative is the key.

Suppose  $\mu = \delta_0 + \nu$ , with  $\nu$  being a nonnegative Borel measure and

$$\mathbb{1}_E * \mu = 1 \text{ on } \mathbb{R} \text{ so also } \mathbb{1}_E * \nu = 0 \text{ on } E. \tag{6.1}$$

Write  $A_n$  for the union of intervals (each of length  $d_n$ ) that we threw away from the intervals of  $E_{n-1}$  in order to obtain the intervals of  $E_n$  (each of length  $\ell_n$ ).

Observe that

$$m(A_n) = \frac{d_n}{2\ell_n} m(E_n). \tag{6.2}$$

Since  $E_{n+1} \subset E_n$  we have by the monotone convergence theorem that

$$\int_{E_n} \mathbb{1}_{E_n} * \nu \rightarrow \int_E \mathbb{1}_E * \nu = 0, \text{ as } n \rightarrow \infty. \tag{6.3}$$

The last equality is due to (6.1).

We also have the crucial inequality (for all  $n$  for which  $d_n/\ell_n < 1$ )

$$\frac{\ell_n - d_n}{d_n} \int_{A_n} \mathbb{1}_{E_n} * \nu \leq \int_{E_n} \mathbb{1}_{E_n} * \nu. \tag{6.4}$$

To see this note that every time we “load” an interval of  $A_n$  with a fractional copy (via the measure  $\nu$ ) of some interval of  $E_n$  we also “load” one or both the intervals of  $E_n$  on each side of  $A_n$  by at least a multiple  $(\ell_n - d_n)/d_n$  of the load that goes onto the  $A_n$ -interval.

More formally, we first observe that for each interval  $J$  of  $E_{n-1}$  and for all  $t \in \mathbb{R}$  we have

$$\frac{\ell_n - d_n}{d_n} m((E_n + t) \cap A_n \cap J) \leq m((E_n + t) \cap E_n \cap J). \tag{6.5}$$

Summing over all intervals  $J$  comprising  $E_{n-1}$  we obtain

$$\frac{\ell_n - d_n}{d_n} m((E_n + t) \cap A_n) \leq m((E_n + t) \cap E_n) \tag{6.6}$$

or, equivalently,

$$\frac{\ell_n - d_n}{d_n} \int_{A_n} \mathbb{1}_{E_n}(x - t) dx \leq \int_{E_n} \mathbb{1}_{E_n}(x - t) dx, \tag{6.7}$$

and integration  $d\nu(t)$  followed by Fubini’s theorem gives (6.4).

We continue from (6.4) using (6.2):

$$\begin{aligned}
 \int_{E_n} \mathbb{1}_{E_n} * \nu &\geq \frac{\ell_n - d_n}{d_n} \int_{A_n} \mathbb{1}_{E_n} * \nu \\
 &\geq \frac{\ell_n - d_n}{d_n} \int_{A_n} \mathbb{1}_E * \nu \text{ (since } E \subset E_n) \\
 &= \frac{\ell_n - d_n}{d_n} m(A_n) \text{ (due to (6.1) since } A_n \subset E^{\complement}) \\
 &= \frac{\ell_n - d_n}{d_n} \frac{d_n}{2\ell_n} m(E_n) \text{ (from (6.2))} \\
 &= \left( \frac{1}{2} - \frac{d_n}{2\ell_n} \right) m(E_n) \rightarrow \frac{1}{2} m(E) \text{ as } n \rightarrow \infty.
 \end{aligned}$$

This positive lower bound contradicts the limit (6.3) and finishes the proof.

6.2. It should be apparent that the proof above is quite flexible and does not impose much rigidity on the Cantor sets of positive measure to which it applies. Instead of trying to state the most general result possible let us indicate this flexibility by giving an example in two dimensions, to which the method applies. This Cantor set is not a cartesian product.

Define a Cantor set  $E \subset [0, 1]^2$  of positive measure as follows (refer to Fig. 2).

The set  $E$  will be the intersection of the decreasing sequence of compact sets  $E_n$ , with  $E_0 = [0, 1]^2$ . The  $n$ -th stage set  $E_n$  will be a union of non-overlapping cubes of the same side-length  $s_n$ , all of them aligned at multiples of  $s_n$ . To obtain the set  $E_n$  from  $E_{n-1}$  we visit each of the cubes of side-length  $s_{n-1}$  comprising  $E_{n-1}$ , we subdivide such a cube  $Q$  into cubes of side-length  $s_n = s_{n-1}/M_n$  (here  $M_n > 0$  is a fast increasing integer sequence) and we throw away any one of these cubes situated in the middle third of  $Q$ .

If the integer sequence  $M_n$  grows sufficiently fast then the resulting Cantor set has positive measure. The proof of the previous section then applies with no essential changes.

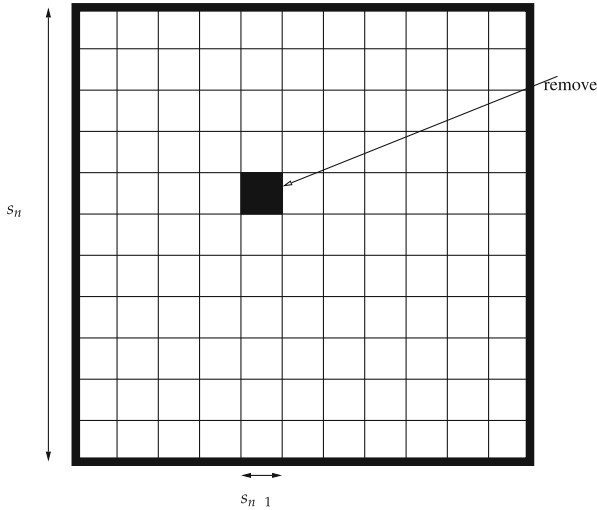
6.3. Finally let us mention that there do exist spectral *unbounded* nowhere dense sets of positive and finite measure. One way to obtain such a set is to construct a set that tiles by  $\mathbb{Z}^d$  translates, and which therefore admits  $\mathbb{Z}^d$  also as a spectrum. We describe the construction in dimension one, but a similar idea works also in several dimensions.

Assume first that we have

$$\mathbb{1}_{[0,1]}(x) = \sum_{n \in \mathbb{Z}} \mathbb{1}_{E_n}(x) \text{ a.e.}$$

where each  $E_n$  is a nowhere dense subset of  $[0, 1]$ . It follows then that the union

$$E = \bigcup_{n \in \mathbb{Z}} (E_n + n)$$



**Fig. 2** To go from  $E_{n-1}$  to  $E_n$  we remove one of the small cubes near the center of the big cube

is a nowhere dense (unbounded) set in  $\mathbb{R}$  which tiles by  $\mathbb{Z}$  translates, and is therefore spectral.

To construct this partition  $E_n$  we first pick a fat Cantor set in  $[0, 1]$  (a Cantor set of positive measure – we can construct such sets of arbitrarily large measure in any given interval) as our first set. At any stage in the construction the complement of the (finitely many) closed sets we have selected so far is an open subset of  $(0, 1)$ , which is therefore a disjoint countable union of intervals. In each of these intervals we select another fat Cantor set, taking care for the total measure of the complement to go to zero. This process exhausts the measure and it is clear that the resulting set coincides a.e. with a fundamental domain of the lattice  $\mathbb{Z}$  in  $\mathbb{R}$ . In other words  $E + \mathbb{Z}$  is a tiling.

## 7 Open problems

We conclude the paper by posing some open problems.

7.1. Let  $\Omega \subset \mathbb{R}^d$  be a bounded, nowhere dense set of positive measure. Can  $\Omega$  be a spectral set? As we have mentioned in Sect. 1.6, the answer is known to be negative in dimension one, but in dimensions two and higher the problem is open.

7.2. Let  $\Omega = A \times B$  where  $A$  is a convex body in  $\mathbb{R}^n$ , and  $B$  is a bounded, measurable set in  $\mathbb{R}^m$ . If  $\Omega$  is a spectral set, then  $A$  must be spectral according to Theorem 1.6. Is it true that also  $B$  must be spectral?

At present this is proved only for dimensions  $n = 1$  and 2 [4, 6].

7.3. Let  $K$  be a convex body in  $\mathbb{R}^d$ , and assume that  $K$  can weakly tile its complement by translations. Let  $W(K)$  be the (nonempty, convex) set of all positive, locally finite measures  $\nu$  such that  $\mathbb{1}_K * \nu = \mathbb{1}_{K^c}$  a.e. If we endow  $W(K)$  with the topology of vague

convergence, then  $W(K)$  is also compact, so by the Krein–Milman theorem  $W(K)$  is the closed convex hull of its extremal points. In particular,  $W(K)$  has extremal points.

It is not difficult to verify that any proper tiling (that is, any measure  $\nu \in W(K)$  which is the sum of unit masses) is an extremal point of  $W(K)$ . Is the converse true?

7.4. Let  $\Omega$  be a bounded, measurable set in  $\mathbb{R}^d$ . Consider the following properties:

- (i)  $\Omega$  can tile the space (properly) by translations;
- (ii)  $\Omega$  is spectral;
- (iii)  $\Omega$  can weakly tile its complement by translations.

It is obvious that (i) implies (iii), and by Theorem 1.2 we know that also (ii) implies (iii). On the other hand, (iii) does not imply (i) (as an example, take a spectral set that cannot tile), and also (iii) does not imply (ii) (take a tile that is not spectral).

Does there exist a set  $\Omega$  satisfying (iii), but such that both (i) and (ii) do not hold?

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