Indian Journal of History of Science (2023) 58:171–180
https://doi.org/10.1007/s43539-023-00097-x

ARTICLE

Geometry of *prāṇakalāntara* in the *Lagnaprakaraṇa*

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Received: 21 July 2023 / Accepted: 9 August 2023 / Published online: 18 September 2023

Abstract

The *prāṇakalāntara*, which is the difference between the longitude of a point on the ecliptic and its corresponding right ascension, is an important parameter in the computation of the *lagna* (ascendant). Mādhava, in his *Lagnaprakaraṇa*, proposes six different methods for determining the *prāṇakalāntara*. Kolachana et al. (Indian J Hist Sci 53(1):1–15, 2018) have discussed these techniques and their underlying rationale in an earlier paper. In this paper, we bring out the geometric significance of these computations, which was not fully elaborated upon in the earlier study. We also show how some of the sophisticated relations can be simply derived using similar triangles.

Keywords *Lagnaprakaraṇa* · *Prāṇakalāntara* · *Dyujyā* · Mādhava · Longitude · Right ascension · Radius of diurnal circle

1 Introduction

The *prāṇakalāntara* is the difference between the longitude (λ) of a point on the ecliptic and its corresponding right ascension (α). That is,

$$prāṇakalāntara = \lambda - \alpha.$$

Among other applications, the *prāṇakalāntara* is essential for the precise computation of the *lagna* or the ascendant. In his *Lagnaprakaraṇa*, Mādhava proposes six different methods for determining the *prāṇakalāntara*. Later astronomer, Putumana Somayājī (2018, pp. 249-251), in his *Karaṇapaddhati*, also mentions the first three methods of *prāṇakalāntara* against the six given by Mādhava. These methods and their rationales have been discussed by Kolachana et al. (2018b) in an earlier study. The study also discusses some of the geometry associated with these computations, particularly with respect to the determination of intermediary quantities such as the *dyujyā* or the radius of the diurnal circle, and conceives of epicyclic models to explain the rationales for some methods. However, crucially, the study does not explain how to geometrically visualize the difference $\lambda - \alpha$, and the significance of intermediary quantities such as *bhujaphala*, *koṭīphala*, and *antyaphala* therein. In this paper, we explain how to geometrically visualize the *prāṇakalāntara* (particularly for the last four methods), bring out the interconnected geometry of the different methods, and discuss the significance of the intermediary terms employed. This gives us a clue as to how Mādhava and other Indian astronomers might have approached these sorts of problems in spherical trigonometry and brings out some of the unique aspects of their approach.

It may be noted that this paper is to be read in conjunction with Kolachana et al. (2018b), and we employ the same symbols and terminology employed therein. Further, we have not reproduced the source text but have directly stated the expressions for *prāṇakalāntara* from the earlier paper, which includes the source text and translation. Finally, as many of the given expressions seem to hint at the use of proportions, we have tried to prove them primarily through the use of similar triangles, even when other methods may be possible. With these caveats in mind, we now proceed to discuss the geometric rationales for each of the six methods in the coming sections.

2 Method 1

The first expression given for the *prāṇakalāntara* in the *Lagnaprakaraṇa* (verse 6) is:

$$\lambda - \alpha = \lambda - R\sin^{-1}\left(\frac{R\sin\lambda \times R\cos\epsilon}{R\cos\delta}\right). \quad (1)$$

Kolachana et al. (2018b) derive the above result by spherical triangles. Here, we show how the result can be derived using planar triangles.[1]

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[1] It may be noted that this proof is based on the discussion on the third method of determining the *prāṇakalāntara* by Kolachana et al. (2018b)

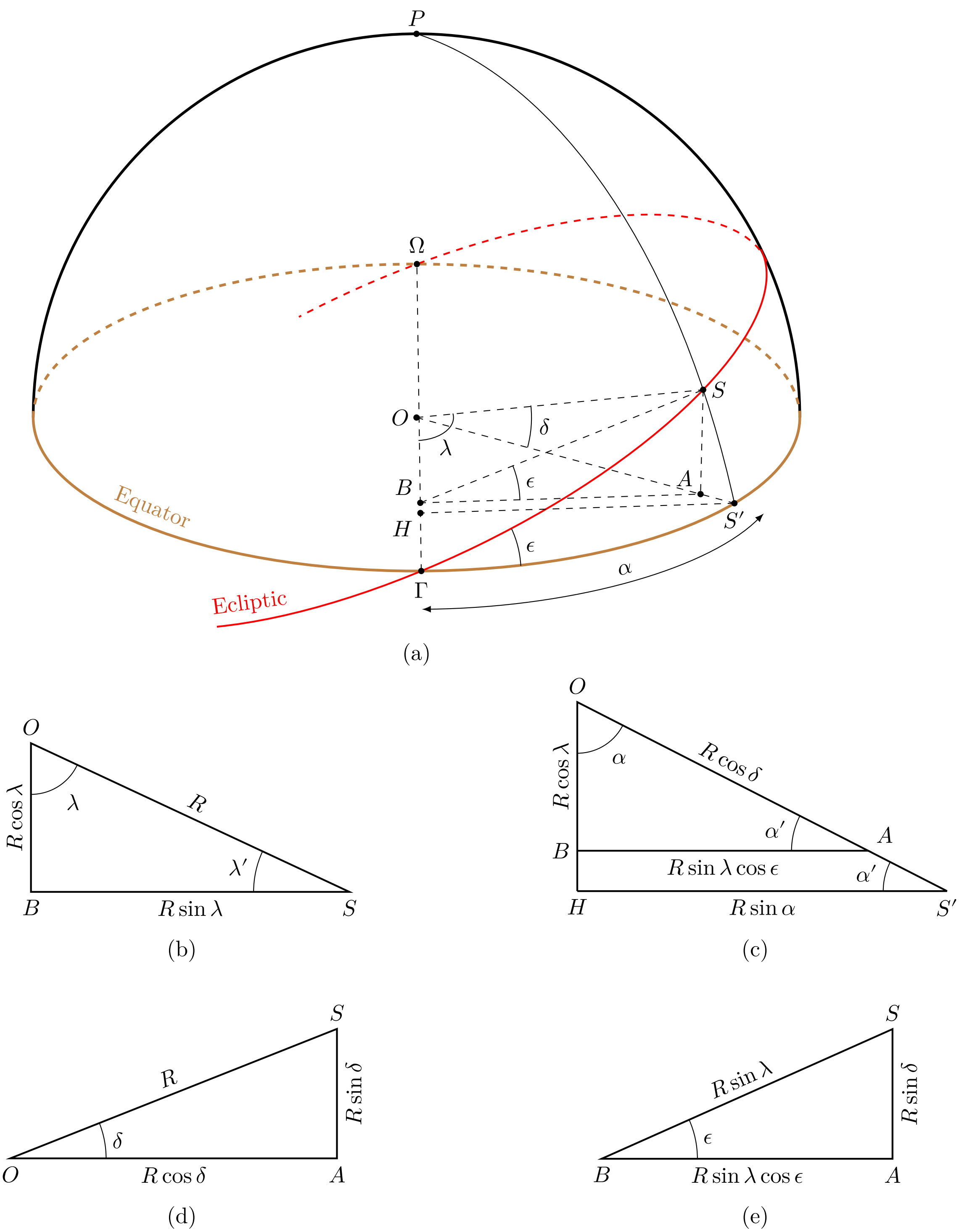

Fig. 1 **a** A diagram showing a part of the celestial sphere depicting different triangles associated with the point S which is on ecliptic, and **b**–**e** are the enlarged views of the triangles therein describing their corresponding sides

Footnote 1 (continued)
and the discussion on the fourth method of determining the ascensional difference by Kolachana et al. (2018a). Also, see Somayājī (2011, p. 78).

2.1 Proof

Figure 1a depicts a portion of the celestial sphere, where the planes of the equator and the ecliptic intersect along the line $\Gamma\Omega$, at an angle of ϵ. Consider a point S on the ecliptic whose longitude (λ) is measured by the angle $\Gamma\hat{O}S$ or arc ΓS, and right ascension (α) is measured by the angle $\Gamma\hat{O}S'$ or arc $\Gamma S'$. From S, drop a perpendicular onto the equatorial plane such that it meets OS' at A. The angle $S\hat{O}S' = S\hat{O}A$ measures the declination (δ) of the point S. From A and S', drop the perpendiculars AB and $S'H$ respectively onto $\Gamma\Omega$.

Now, we obtain five right angled triangles—$\triangle OBS$, $\triangle OHS'$ (and $\triangle OBA$), $\triangle OAS$ and $\triangle BAS$—as depicted in Fig. 1b, c, d and e respectively. $\triangle OBS$ is a triangle in the ecliptic plane, and $\triangle OHS'$ and $\triangle OBA$ are the triangles in the equatorial plane. $\triangle OAS$ lies in the plane of the secondary to the equator passing through S, which is perpendicular to the plane of the equator. $\triangle BAS$ also lies in a plane perpendicular to the plane of the equator.

In $\triangle OAS$, as $S\hat{O}A = \delta$ and $OS = R$ (radius of the celestial sphere),

$$OA = OS \cos\delta = R\cos\delta. \tag{2}$$

In $\triangle OBS$, as $S\hat{O}B = \lambda$ and $OS = R$,

$$BS = OS \sin\lambda = R\sin\lambda. \tag{3}$$

In $\triangle BAS$, as $S\hat{B}A = \epsilon$, employing (3) we obtain[2]

$$BA = BS\cos\epsilon = R\sin\lambda\cos\epsilon. \tag{4}$$

As $\triangle OBA$ and $\triangle OHS'$ are similar, $OS' = R$, and $S'\hat{O}H = A\hat{O}B = \alpha$, employing (2) and (4) we obtain

$$\frac{HS'}{OS'} = \frac{BA}{OA} \implies \frac{R\sin\alpha}{R} = \frac{R\sin\lambda\cos\epsilon}{R\cos\delta}.$$

Thus,

$$\alpha = R\sin^{-1}\left(\frac{R\sin\lambda \times R\cos\epsilon}{R\cos\delta}\right). \tag{5}$$

Hence, we obtain the expression for the *prāṇakalāntara*

$$\lambda - \alpha = \lambda - R\sin^{-1}\left(\frac{R\sin\lambda \times R\cos\epsilon}{R\cos\delta}\right), \tag{6}$$

which is the same as (1).

3 Method 2

The second expression given for the *prāṇakalāntara* in the *Lagnaprakaraṇa* (verse 7) is:

$$\lambda - \alpha = R\sin^{-1}\left(\frac{R\cos\lambda \times R}{R\cos\delta}\right) - R\sin^{-1}(R\cos\lambda). \tag{7}$$

Kolachana et al. (2018b) once again derive the above result using spherical triangles. Here, we show how the result can be derived using planar triangles.

[2] This result can also be derived using similar triangles. See the discussion to the third method by Kolachana et al. (2018b).

3.1 Proof

This expression can be obtained by considering Fig. 1b, c.

In $\triangle OBS$, as $B\hat{S}O = 90 - \lambda = \lambda'$ and $OS = R$, we have

$$OB = OS \times \sin\lambda' = R\cos\lambda, \tag{8}$$

$$\implies R\sin\lambda' = R\cos\lambda, \implies \lambda' = R\sin^{-1}(R\cos\lambda). \tag{9}$$

As $\triangle OBA$ and $\triangle OHS'$ are similar, $OS' = R$ and $H\hat{S'}O = B\hat{A}O = 90 - \alpha = \alpha'$, employing (2) and (8) we have

$$\frac{OH}{OS'} = \frac{OB}{OA} \implies \frac{R\sin\alpha'}{R} = \frac{R\cos\lambda}{R\cos\delta}.$$

Thus,

$$\alpha' = R\sin^{-1}\left(\frac{R\cos\lambda \times R}{R\cos\delta}\right). \tag{10}$$

From (9) and (10), we have the *prāṇakalāntara*

$$\begin{aligned}\lambda - \alpha &= \alpha' - \lambda' = R\sin^{-1}\left(\frac{R\cos\lambda \times R}{R\cos\delta}\right) \\ &\quad - R\sin^{-1}(R\cos\lambda),\end{aligned} \tag{11}$$

which is the same as (7).

4 Method 3

The third method (verse 8) introduces a term known as *antyaphala* and provides an expression for computing it. It is further utilized in the computation of *prāṇakalāntara* as follows:

$$antyaphala\ (A_p) = \frac{R\sin\lambda \times R\,\text{versin}\,\epsilon}{R}, \tag{12}$$

$$R\cos\delta = \sqrt{(R\sin\lambda - A_p)^2 + (R\cos\lambda)^2}, \tag{13}$$

$$\lambda - \alpha = \frac{A_p \times R\cos\lambda}{R\cos\delta}. \tag{14}$$

Kolachana et al. (2018b) discuss the geometry of the first two expressions above, but do not describe how the *prāṇakalāntara* is to be visualized geometrically. We discuss the same here.

4.1 Proof

In deriving the expression for the *prāṇakalāntara*, Mādhava appears to have conceived the idea to superimpose the geometric entities that lie on the equatorial plane

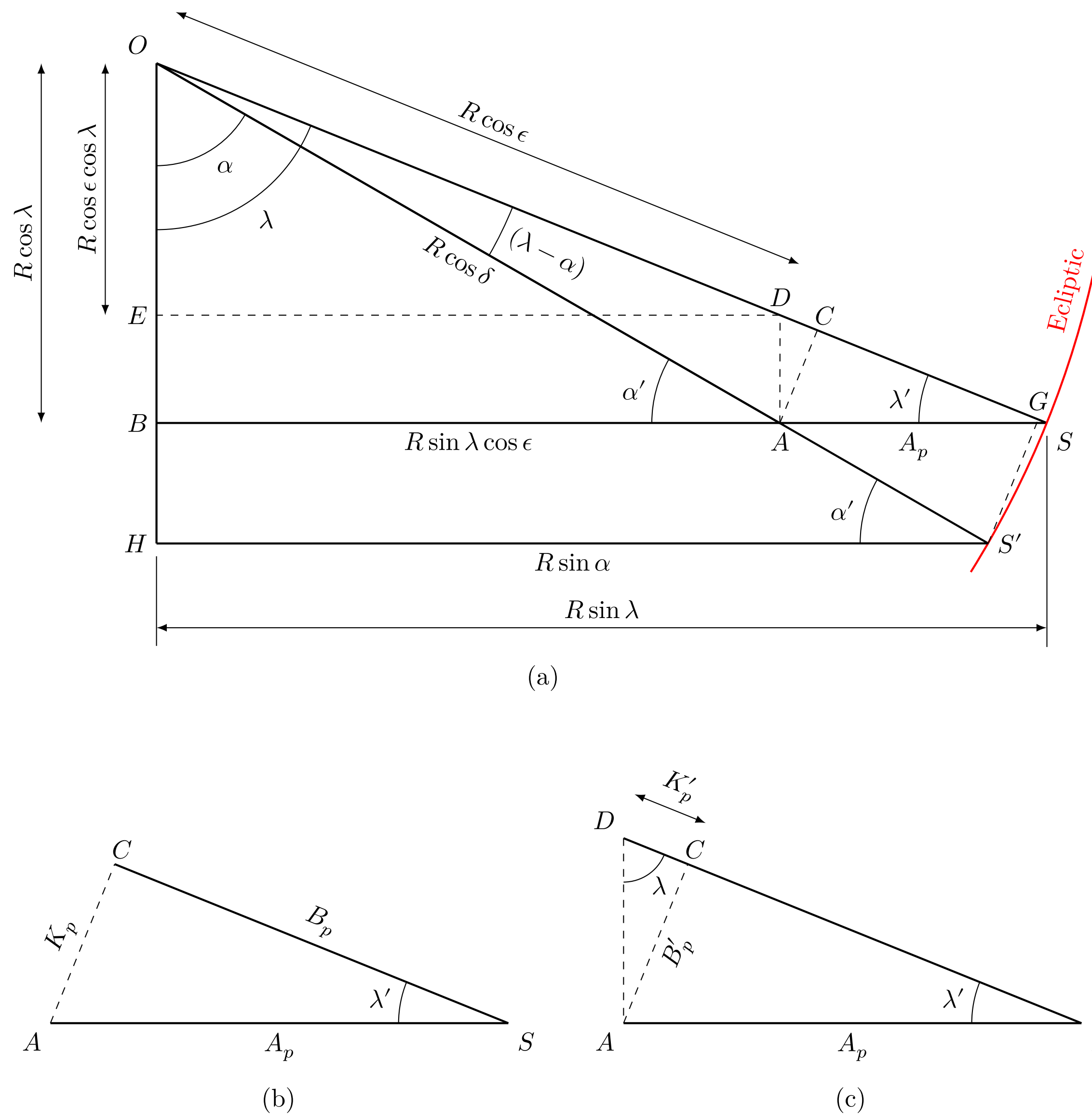

Fig. 2 **a** A diagram showing superimposed geometrical entities of equatorial plane on the ecliptic plane, **b** the enlarged view of $\triangle CAS$ showing *antyaphala* (A_p), *bhujāphala* (B_p) and *koṭīphala* (K_p) used in methods 3 and 4, and **c** the enlarged view of $\triangle DAS$ showing *bhujaphala* (B_p') and *koṭīphala* (K_p') used in method 5

onto the ecliptic plane (or vice-versa). This is achieved by rotating the equatorial plane in Fig. 1a anticlockwise about the ΓΩ axis by an angle ϵ such that the equator aligns with the ecliptic. The resulting geometry is depicted in Fig. 2a. Further, Mādhava introduces different terms like *antyaphala*, *bhujaphala* and *koṭīphala* to refer to different portions of the resulting geometry and assist in the computations.

To derive the given expressions, construct the perpendiculars AC and $S'G$ on OS. Construct a line perpendicular to BS at A such that it meets OS at D. Also, construct DE perpendicular to OB.

Now, from (4)

$$ED = BA = R\sin\lambda\cos\epsilon. \tag{15}$$

As $\triangle OBS$ and $\triangle OED$ are similar, we have

$$\frac{OD}{ED} = \frac{OS}{BS} \implies \frac{OD}{R\sin\lambda\cos\epsilon} = \frac{R}{R\sin\lambda}.$$

Thus,

$$OD = R\cos\epsilon, \tag{16}$$

and

$$DS = OS - OD = R - R\cos\epsilon = R\,\text{versin}\,\epsilon. \tag{17}$$

As $\triangle OBS$ and $\triangle DAS$ are similar, employing (3) and (17), we have

$$\frac{AS}{DS} = \frac{BS}{OS} \implies \frac{AS}{R\,\text{versin}\,\epsilon} = \frac{R\sin\lambda}{R}.$$

AS is the quantity referred to as *antyaphala* (A_p). Thus,

$$A_p = AS = \frac{R\sin\lambda \times R\,\text{versin}\,\epsilon}{R}, \tag{18}$$

which is the same as (12). Alternatively, employing (3) and (4), we get

$$A_p = AS = BS - BA = R\sin\lambda - R\sin\lambda\cos\epsilon = R\sin\lambda\,\text{versin}\,\epsilon.$$

Employing (2), (3), (8) and (18) in $\triangle OBA$,

$$\begin{aligned} R\cos\delta = OA &= \sqrt{BA^2 + OB^2}, \\ &= \sqrt{(BS - AS)^2 + OB^2}, \\ &= \sqrt{(R\sin\lambda - A_p)^2 + (R\cos\lambda)^2}, \end{aligned} \tag{19}$$

which is the same as (13).

As $\triangle ACS$ and $\triangle OBS$ are similar, we have

$$\frac{AC}{AS} = \frac{OB}{OS} \implies \frac{AC}{A_p} = \frac{R\cos\lambda}{R}.$$

Thus,

$$AC = \frac{A_p \times R\cos\lambda}{R}. \tag{20}$$

Now, it may be noted that the arc $SS' = \lambda - \alpha$. Thus,

$$S'G = R\sin(\lambda - \alpha). \tag{21}$$

As $\triangle OAC$ and $\triangle OS'G$ are similar, employing (2), (20) and (21), we have

$$\frac{S'G}{OS'} = \frac{AC}{OA} \implies \frac{R\sin(\lambda - \alpha)}{R} = \frac{A_p \times R\cos\lambda}{R \times R\cos\delta}.$$

Thus,

$$R\sin(\lambda - \alpha) \approx \lambda - \alpha = \frac{A_p \times R\cos\lambda}{R\cos\delta}, \tag{22}$$

which is the same as (14).[3]

5 Method 4

The fourth method (verses 9 and 10) introduces the terms *bhujāphala* and *koṭīphala* for the computation of *prāṇakalāntara*:

$$\textit{bhujāphala}\ (B_p) = \frac{R\sin\lambda \times A_p}{R}, \tag{23}$$

$$\textit{koṭīphala}\ (K_p) = \frac{R\cos\lambda \times A_p}{R}. \tag{24}$$

They are employed in the computation of $R\cos\delta$ and the *prāṇakalāntara* as follows:

$$R\cos\delta = \sqrt{(R - B_p)^2 + (K_p)^2}, \tag{25}$$

$$\lambda - \alpha = \frac{K_p \times R}{R\cos\delta}. \tag{26}$$

Kolachana et al. (2018b) show the mathematical equivalence of (25) and (26) with (13) and (14) respectively, and further propose an epicyclic model to explain the terms *bhujāphala*, *koṭīphala*, and *antyaphala*. Here, we show the geometric significance of the above relations by making use of Fig. 2a, b. Figure 2b is an enlarged view of the $\triangle CAS$ in Fig. 2a.

5.1 Proof

In Fig. 2a, the terms *bhujāphala* (B_p) and *koṭīphala* (K_p) refer to CS and AC respectively. This can be understood as follows.

As $\triangle ACS$ and $\triangle OBS$ are similar, we have

$$\frac{CS}{AS} = \frac{BS}{OS} \implies \frac{B_p}{A_p} = \frac{R\sin\lambda}{R}.$$

Thus,

$$\textit{bhujāphala}\ (B_p) = CS = \frac{R\sin\lambda \times A_p}{R}, \tag{27}$$

which is the same as (23). Similarly, as already derived in (20),

$$\textit{koṭīphala}\ (K_p) = AC = \frac{R\cos\lambda \times A_p}{R}, \tag{28}$$

which is the same as (24).

From $\triangle OAC$,

$$\begin{aligned} R\cos\delta = OA &= \sqrt{OC^2 + AC^2}, \\ &= \sqrt{(OS - CS)^2 + AC^2}, \\ &= \sqrt{(R - B_p)^2 + (K_p)^2}, \end{aligned} \tag{29}$$

which is the same as (25).

As $\triangle OAC$ and $\triangle OS'G$ are similar, employing (21) and (2), we have

[3] As noted by Kolachana et al. (2018b) in their discussion of the third method, $\max(\lambda - \alpha) \approx 2.6°$ at $\lambda = 46°$. Thus, $R\sin(\lambda - \alpha) \approx (\lambda - \alpha)$.

$$\frac{S'G}{OS'} = \frac{AC}{OA} \implies \frac{R\sin(\lambda-\alpha)}{R} = \frac{K_p}{R\cos\delta}.$$

Thus,

$$R\sin(\lambda-\alpha) \approx \lambda-\alpha = \frac{K_p \times R}{R\cos\delta}, \tag{30}$$

which is the same as (26).

6 Method 5

Like the fourth method, the fifth method (verses 11 and 12) also makes use of *bhujaphala* and *koṭīphala* for the computation of *prāṇakalāntara*. However, these terms represent different quantities here:

$$bhujaphala\ (B'_p) = \frac{R\cos\lambda \times R\,\mathrm{versin}\,\epsilon}{R} \times \frac{R\sin\lambda}{R}, \tag{31}$$

$$koṭīphala\ (K'_p) = \frac{R\cos\lambda \times R\,\mathrm{versin}\,\epsilon}{R} \times \frac{R\cos\lambda}{R}. \tag{32}$$

They are used in the computation of $R\cos\delta$ and the *prāṇakalāntara* as follows:

$$R\cos\delta = \sqrt{(R\cos\epsilon + K'_p)^2 + (B'_p)^2}, \tag{33}$$

$$\lambda - \alpha = \frac{B'_p \times R}{R\cos\delta}. \tag{34}$$

Kolachana et al. (2018b) show the mathematical equivalence of (33) and (34) with (13) and (14) respectively, and again propose an epicyclic model to explain the terms *bhujaphala* and *koṭīphala*. Here, we show the geometric significance of the above relations by making use of Fig. 2a, c. Figure 2c is an enlarged view of the $\triangle DAS$ in Fig. 2a.

6.1 Proof

In Fig. 2c, the terms *bhujaphala* (B'_p) and *koṭīphala* (K'_p) refer to AC and DC respectively.[4] This can be understood as follows.

As $\triangle DAS$ and $\triangle OBS$ are similar, employing (8) and (17), we have

$$\frac{DA}{DS} = \frac{OB}{OS} \implies \frac{DA}{R\,\mathrm{versin}\,\epsilon} = \frac{R\cos\lambda}{R}.$$

Thus,[5]

$$DA = \frac{R\cos\lambda \times R\,\mathrm{versin}\,\epsilon}{R}. \tag{35}$$

As $\triangle DCA$ and $\triangle OBS$ are similar, we have

$$\frac{AC}{DA} = \frac{BS}{OS} \implies AC = DA \times \frac{R\sin\lambda}{R}.$$

Thus, employing (35),

$$bhujaphala\ (B'_p) = AC = \frac{R\cos\lambda \times R\,\mathrm{versin}\,\epsilon}{R} \times \frac{R\sin\lambda}{R}, \tag{36}$$

which is the same as (31). Similarly, we have

$$\frac{DC}{DA} = \frac{OB}{OS} \implies DC = DA \times \frac{R\cos\lambda}{R}.$$

Again, employing (35),

$$koṭīphala\ (K'_p) = DC = \frac{R\cos\lambda \times R\,\mathrm{versin}\,\epsilon}{R} \times \frac{R\cos\lambda}{R}, \tag{37}$$

which is the same as (32).

From $\triangle OAC$, employing (16), we have

$$\begin{aligned} R\cos\delta = OA &= \sqrt{OC^2 + AC^2} \\ &= \sqrt{(OD + DC)^2 + AC^2} \\ &= \sqrt{(R\cos\epsilon + K'_p)^2 + (B'_p)^2}, \end{aligned} \tag{38}$$

which is the same as (33).

Finally, as $\triangle OAC$ and $\triangle OS'G$ are similar, employing (21) and (2), we have

$$\frac{S'G}{OS'} = \frac{AC}{OA} \implies \frac{R\sin(\lambda-\alpha)}{R} = \frac{B'_p}{R\cos\delta}.$$

Thus,

$$R\sin(\lambda-\alpha) \approx \lambda-\alpha = \frac{B'_p \times R}{R\cos\delta}, \tag{39}$$

which is the same as (34).

7 Method 6

Like the previous two methods, the sixth method (verses 15–17) also employs the quantities *bhujāphala* and *koṭīphala* for computation of *prāṇakalāntara*. These terms represent the following quantities here:

[4] It may be noted that AC was referred to as the *koṭīphala* (K_p) in the previous method.

[5] It may be noted that DA can be conceived of as the *antyaphala* (A'_p) here, though the text makes no mention of it, as $(A'_p)^2 = (B'_p)^2 + (K'_p)^2$.

$$\textit{bhujāphala}\ (B''_p) = \frac{R\sin 2\lambda \times \frac{1}{2}R\,\text{versin}\,\epsilon}{R}, \quad (40)$$

$$\textit{koṭīphala}\ (K''_p) = \frac{R\cos 2\lambda \times \frac{1}{2}R\,\text{versin}\,\epsilon}{R}. \quad (41)$$

They are used in the computation of $R\cos\delta$ and the *prāṇakalāntara* as follows:

$$R\cos\delta = \sqrt{\left(R - \frac{1}{2}R\,\text{versin}\,\epsilon \pm |K''_p|\right)^2 + (B''_p)^2}, \quad (42)$$

$$\lambda - \alpha = R\sin^{-1}\left(\frac{B''_p \times R}{R\cos\delta}\right). \quad (43)$$

Further, the *Lagnaprakaraṇa* explicitly states the condition for the sign of the *koṭīphala* in (42) through the phrase "*mṛgakarkaṭādyoḥ svarṇam*". That is, the *koṭīphala* is to be added when 2λ is in the range from *mṛga* (Capricorn) to *karkaṭa* (Cancer), and subtracted from Cancer to Capricorn. In other words, the *koṭīphala* is to be added in the range $270° < 2\lambda < 90°$, and subtracted in the range $90° < 2\lambda < 270°$. This is because the *koṭīphala* is a function of the cosine function which is positive or negative in the aforesaid ranges. We discuss more later.

Kolachana et al. (2018b) show the mathematical equivalence of (42) and (43) with (33) and (14) respectively and do not discuss the geometry associated with these expressions. Here, we show the geometric significance of the above expressions by making use of Figs. 3 and 4.

7.1 Proof

The above expressions can be derived by first considering Fig. 3. In Fig. 3a, having superimposed the geometric entities that lie on the equatorial plane onto the ecliptic plane as before, mark points S'' and S''' on the ecliptic such that $S\hat{O}S'' = 2\lambda$ and $S\hat{O}S''' = 180°$. Evidently, $OS'' = OS''' = R$, $S''I = R\sin 2\lambda$ and $IO = R\cos 2\lambda$. Also, $S'''S''S$ is a right-angled triangle, and similar to $\triangle DAS$. Construct a line AF parallel to $S''O$ so that $\triangle CAF$ is similar to $\triangle IS''O$.

In $\triangle S'''S''S$, $S''O$ bisects the side $S'''S$, and also $S''O = OS = \frac{1}{2}S'''S$. Similarly, in $\triangle DAS$, as depicted in the enlarged Fig. 3b, AF will bisect DS, and also[6]

$$AF = FS = \frac{1}{2}DS = \frac{1}{2}R\,\text{versin}\,\epsilon, \quad (44)$$

using (17).

Now, in Fig. 3b, the terms *bhujāphala* (B''_p) and *koṭīphala* (K''_p) refer to AC and FC respectively.[7] This can be understood as follows.

As $\triangle IS''O$ and $\triangle CAF$ are similar, we have

$$\frac{AC}{AF} = \frac{S''I}{S''O} \implies \frac{AC}{\frac{1}{2}R\,\text{versin}\,\epsilon} = \frac{R\sin 2\lambda}{R}.$$

Thus,

$$\textit{bhujāphala}\ (B''_p) = AC = \frac{R\sin 2\lambda \times \frac{1}{2}R\,\text{versin}\,\epsilon}{R}, \quad (45)$$

which is the same as (40). Similarly,

$$\frac{FC}{AF} = \frac{OI}{S''O} \implies \frac{FC}{\frac{1}{2}R\,\text{versin}\,\epsilon} = \frac{R\cos 2\lambda}{R}.$$

Thus, we obtain the magnitude of the

$$\textit{koṭīphala}\ (K''_p) = FC = \left|\frac{R\cos 2\lambda \times \frac{1}{2}R\,\text{versin}\,\epsilon}{R}\right|, \quad (46)$$

which is the same as (41).[8]

Further, from $\triangle OAC$, employing (44),

$$\begin{aligned} R\cos\delta = OA &= \sqrt{OC^2 + AC^2} \\ &= \sqrt{(OS - FS - FC)^2 + AC^2} \\ &= \sqrt{\left(R - \frac{1}{2}R\,\text{versin}\,\epsilon - K''_p\right)^2 + (B''_p)^2}. \end{aligned} \quad (47)$$

The expression of $R\cos\delta$ in (47) is observed to be valid for $90° < 2\lambda < 270°$, when the cosine function is negative. In this case, F lies in between C and S, as depicted in Fig. 3. In case of $270° < 2\lambda < 90°$, when the cosine function is positive, C lies in between F and S, as shown in Fig. 4. Here too, following similar constructions and arguments as earlier, we can again easily obtain (44), (45) and (46) from the similar triangles $IS''O$ and CAF.

However, in the computation of $R\cos\delta$ from $\triangle OAC$, we observe

[6] Alternatively, employing the sine rule in $\triangle FAS$, and substituting (18) and solving, we obtain AF.

[7] It may be noted that AC was referred to as the *koṭīphala* (K_p) in method 4 and *bhujaphala* (B'_p) in method 5.

[8] It may be noted that AF from (44) can be conceived of as the *antyaphala* (A''_p) here, though the text makes no mention of it, as $(A''_p)^2 = (B''_p)^2 + (K''_p)^2$.

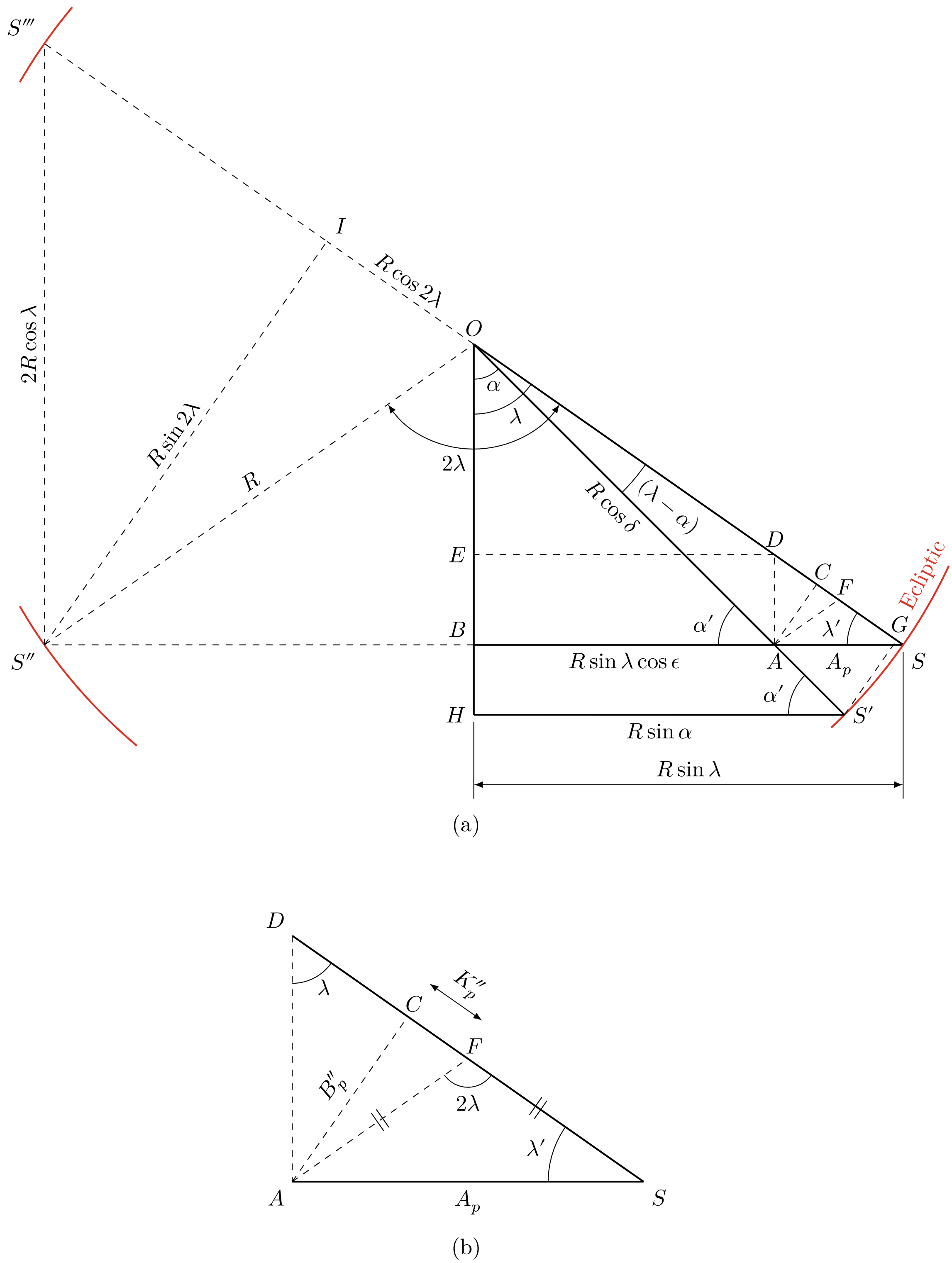

Fig. 3 **a** A diagram showing the superimposed geometrical entities of the equatorial plane on the ecliptic plane and also depicting a situation when 90° < 2λ < 270°, and **b** the enlarged view of the $\triangle DAS$ showing *bhujāphala* (B_p'') and *koṭīphala* (K_p'') pertaining to method 6

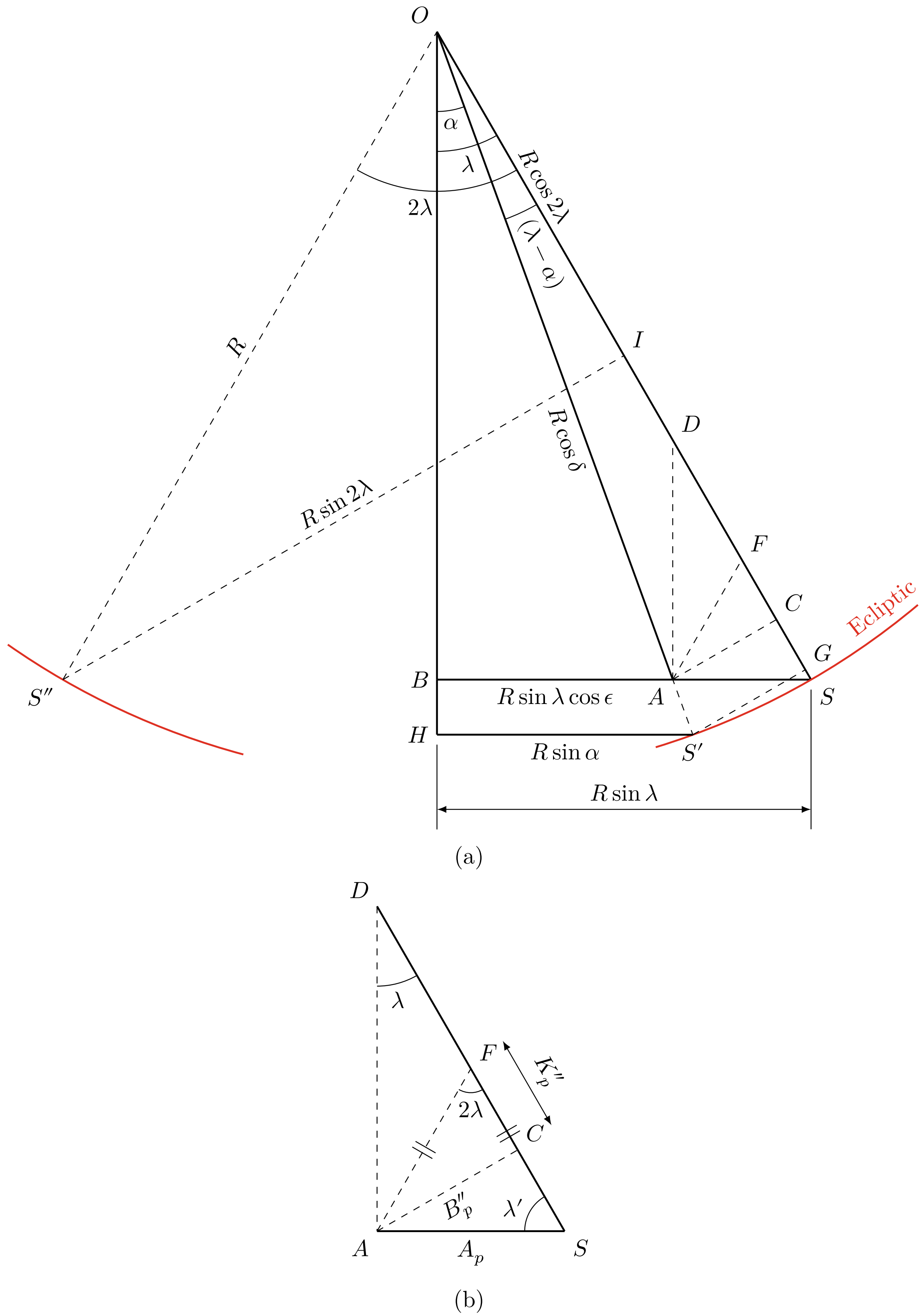

Fig. 4 **a** A diagram showing the superimposed geometrical entities of the equatorial plane on the ecliptic plane and also depicting a situation when 270° < 2λ < 90°, and **b** the enlarged view of the $\triangle DAS$ showing *bhujāphala* (B_p'') and *koṭīphala* (K_p'') pertaining to method 6

$$R\cos\delta = OA = \sqrt{OC^2 + AC^2} = \sqrt{(OS - FS + FC)^2 + AC^2} = \sqrt{\left(R - \frac{1}{2}R\,\text{versin}\,\epsilon + K_p''\right)^2 + (B_p'')^2}. \quad (48)$$

Thus, from the expressions for $R\cos\delta$ in (47) and (48), we obtain (42).

Finally, as $\triangle OAC$ and $\triangle OS'G$ are similar, employing (21) and (2), we have

$$\frac{S'G}{OS'} = \frac{AC}{OA} \implies \frac{R\sin(\lambda - \alpha)}{R} = \frac{B_p''}{R\cos\delta}.$$

Thus,

$$\lambda - \alpha = R\sin^{-1}\left(\frac{B_p'' \times R}{R\cos\delta}\right), \quad (49)$$

which is equivalent to (43).

8 Discussion

In this paper, we have elaborated upon the geometry associated with *prāṇakalāntara* computations in the *Lagnaprakaraṇa*. We observe that the first two expressions for the *prāṇakalāntara* are directly based on results for λ and α. The other four methods introduce intermediary terms such as *antyaphala*, *bhujāphala* and *koṭīphala*, and employ these to determine the radius of the diurnal circle as well as the *prāṇakalāntara*. By superimposing geometric entities that lie in the equatorial plane onto the ecliptic plane, we have shown how to geometrically visualize the *prāṇakalāntara*, the geometric significance of the intermediary terms, and how the former can be expressed in terms of the latter. We have also shown how the given relations can be derived simply through the use of similar triangles.

Our analysis reveals the sophisticated nature of spherical trigonometry employed in the *Lagnaprakaraṇa*. The diversity of approaches employed by Mādhava toward solving a single problem not only showcases his genius, but also reveals him to be a true connoisseur of mathematics and astronomy, and validates the title of '*golavid*' bestowed upon him by later scholars.

Acknowledgements The authors would like to thank the Ministry of Education, Government of India, for the financial support afforded to their research by way of establishing the Centre for Indian Knowledge Systems at IIT Madras. The authors would also like to thank the anonymous referees for their valuable suggestions towards improving the paper.

Data Availability Not applicable.

References

Kolachana, A., Mahesh, K., Montelle, C., & Ramasubramanian, K. (2018a). Determination of ascensional difference in the *Lagnaprakaraṇa*. *Indian Journal of History of Science, 53*(3), 302–316.

Kolachana, A., Mahesh, K., & Ramasubramanian, K. (2018b). Mādhava's multi-pronged approach for obtaining the *prāṇakalāntara*. *Indian Journal of History of Science, 53*(1), 1–15.

Putumana Somayājī (2018). *Karaṇapaddhati*. Trans. and comm. by Venketeswara Pai, K. Ramasubramanian, M. S. Sriram, and M. D. Srinivas. Sources and Studies in the History of Mathematics and Physical Sciences. New Delhi: Hindustan Book Agency and Springer.

Somayājī, N. (2011). *Tantrasaṅgraha*. Trans. and comm. by K. Ramasubramanian and M. S. Sriram. Sources and studies in the history of mathematics and physical sciences. Hindustan Book Agency and Springer.