Proceedings of the Indian National Science Academy
https://doi.org/10.1007/s43538-024-00295-w

RESEARCH PAPER

Boros integral involving the class of polynomials and incomplete ℵ-functions

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Received: 18 January 2024 / Accepted: 27 March 2024

Abstract

In this paper, we explore the Boros integral with three parameters containing the incomplete ℵ- functions and Srivastava polynomial (genaral class of polynomial). We build the Boros integral for the product of Srivastava polynomials and the incomplete ℵ-function. Also, We mentioned numerous specific instances of our main finding. For extending our given result one can generalize these formulas by using the classes of multivariable polynomials.

Keywords Incomplete Gamma function · Mellin-Barnes integrals contour · Incomplete ℵ-function · Class of polynomials · Boros integral

Mathematics Subject Classification 33C05 · 33C60

Introduction

A wide range of Mathematical functions have been introduced and investigated by numerous writers (see, for instance, the gamma function, beta function, ML function, Bessel function, Fox's H-function and hypergeometric function). They have also conducted a comprehensive research of these functions and provided applicability of these functions in engineering, applied sciences, and so on. Equations on a sphere have been the subject of recent research and have numerous significant applications in physics, phenomena, and oceanography. Phong and Long (2022) investigated the Caputo-Fabrizio derivative in relation to the parabolic issue with non-local conditions. Li et al. (2023) used fractional calculus and the Adomian decomposition method to analyze and provide approximate and analytical solutions to the Schrodinger problem. For the conformable space-time nonlinear Schrodinger equation (CSTNLSE) with Kerr law nonlinearity, Asjad et al. (2022) found several types of solitons solutions. We used two suggested approaches-the novel extended hyperbolic function method and the Sardar-subequation method-to look for such answers. Using Banach's fixed point theorem and Babenko's method, Using Banach's fixed point theorem and Babenko's method, Li et al. (2023) constructed a necessary condition for the uniqueness of solutions to a novel boundary value issue of the fractional nonlinear

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Published online: 26 April 2024

partial integro-differential equation. According to Abdullah et al. (2022), the Atangana-Baleanu derivative in the sense of Caputo (ABC) with ϕ_p-Laplacian operator may solve a system of fractional differential equations (FDEs) and ensure its uniqueness (EU) and stability.

The Aleph function was first developed and explored in the 19th century by Südland et al. (1998, 2001), subsequent to that, a various authors established fascinating outcomes and outlined feasible applications in Physics, Applied Mathematics, and Engineering., for example, see Purohit et al. (2020), Südland et al. (2019), etc.

Srivastava (1972) has advocated the use of class of polynomials $S_N^M(x)$ given by

$$S_N^M[y] = \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A[N,K], \tag{1}$$

where $A[N, K]$ is an arbitrary constant (real or complex) and M is any positive integer. We shall note that

$$A_{N,K} = \frac{(-N)_{MK}}{K!} A[N,K]. \tag{1}$$

For some particular values of the coefficients $A[N, K]$, $S_N^M[.]$ reduces to a variety of well-known polynomials, including the Hermite, Jacobi, Laguerre, bessel, and other polynomials; see Srivastava and Singh (1983).

The incomplete hypergeometric function and incomplete Gamma-function were recently introduced by Srivastava et al. (2012). The incomplete H and $\bar{H}$-function, which is modified version of Fox's H-function (see Baleanu et al. (2016), Daiya et al. (2016) and Gupta et al. (2016)), were first introduced and investigated by Srivastava et al. (2018) in more recent years. Several authors, Bansal et al. (2019, 2020), and Bansal and Kumar (2020) presented and investigated the incomplete I and $\aleph$-function and integrals calculations for the incomplete H-function respectively. Recently, Kumar et al. (2022) and Bhatter et al. (2023a) have researched and examined the Boros integral with three parameters involving generalized multi-index Mittag-Leffler function and the incomplete I-functions. Also, for further study reader can refer recent work (Bhatter et al. 2023b; c; Parmar and Saxena 2017; Purohit et al. 2024; Srivastava 2013; Srivastava and Cho 2012). In this paper, we introduce and investigate the incomplete $\aleph$-function and incomplete Gamma-functions. We calculated the Boros integral concerning the product of the incomplete $\aleph$-function. The $\aleph$-function is extended by the incomplete $\aleph$-function.

The study and development of the incomplete Gamma type functions, such as $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$ presented in (3) and (4), respectively, has recently attracted a lot of research attention. Incomplete Gamma functions are important special functions and their closely related functions are extensively used in physics and engineering; therefore, they are of interest to physicists, engineers, statisticians, and mathematicians. The theory of the incomplete Gamma functions, as a part of the theory of confluent hypergeometric functions, has received its first efficient exposition by Tricomi (1950).

The familiar incomplete Gamma functions $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$ are defined, respectively, by

$$\gamma(\alpha,x) = \int_0^x u^{\alpha-1} e^{-u} \mathrm{d}u, \quad x > 0, \ (\Re(\alpha) > 0, x \geq 0), \tag{1}$$

and

$$\Gamma(\alpha,x) = \int_x^\infty u^{\alpha-1} e^{-u} \mathrm{d}u, \quad (x \geqslant 0, \ \Re(\alpha) > 0 \text{ when } x = 0). \tag{1}$$

The decomposition formula shown below is valid:

$$\gamma(\alpha,x) + \Gamma(\alpha,x) = \Gamma(\alpha), \quad (\Re(\alpha) > 0). \tag{1}$$

Recently, The subsequent two classes of extended incomplete hypergeometric functions were recently introduced and investigated in well manner by Srivastava et al. (2012):

$${}_p\gamma_q\left[\begin{matrix}(\alpha_1,s),\alpha_2,\dots,\alpha_p; \\ \beta_1,\dots,\beta_q;\end{matrix}\ z\right] = \sum_{n=0}^\infty \frac{(\alpha_1;s)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \tag{1}$$

and

$${}_p\Gamma_q\left[\begin{matrix}(\alpha_1,s),\alpha_2,\dots,\alpha_p; \\ \beta_1,\dots,\beta_q;\end{matrix}\ z\right] = \sum_{n=0}^\infty \frac{[\alpha_1;s]_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \tag{1}$$

where the incomplete Pochhammer symbols $(\alpha;s)_\nu$ and $[\alpha;s]_\nu$ $(\alpha;\nu \in \mathbb{C}; s \geqq 0)$ are defined as follows:

$$(\alpha;s)_\nu = \frac{\gamma(\alpha+\nu,s)}{\Gamma(s)} \quad (\alpha;\nu \in \mathbb{C}; s \geqq 0), \tag{1}$$

and

$$[\alpha;s]_\nu = \frac{\Gamma(\alpha+\nu,s)}{\Gamma(s)} \quad (\alpha;\nu \in \mathbb{C}; s \geqq 0), \tag{1}$$

Incomplete Pochhammer symbols $(\alpha;s)_\nu$ & $[\alpha;s]_\nu$ fulfill the aforementioned decomposition connection, which is obviously true.

$$(\alpha;s)_\nu + [\alpha;s]_\nu = (\alpha)_\nu \quad (\alpha;\nu \in \mathbb{C}; s \geqq 0). \tag{10}$$

The shifted factorial, often known as the Pochhammer symbol, is defined by

$$(\chi)_\nu = \frac{\Gamma(\chi+\nu)}{\Gamma(\chi)} = \begin{cases} 1 & (\nu=0;\ \chi\in\mathbb{C}\backslash\{0\}), \\ \chi(\chi+1)\cdots(\chi+n-1) & (\nu=n\in\mathbb{N};\ \chi\in\mathbb{C}), \end{cases} \tag{11}$$

The definitions of (6) and (7) easily produce an alternate decomposition relation for the generalized hypergeometric function ${}_pF_q$ (Slater 1966), according to Srivastava et al. (2012):

$${}_p\gamma_q\left[\begin{matrix}(\alpha_1,s),\alpha_2,\dots,\alpha_p;\\ \beta_1,\dots,\beta_q;\end{matrix}\ z\right] + {}_p\Gamma_q\left[\begin{matrix}(\alpha_1,s),\alpha_2,\dots,\alpha_p;\\ \beta_1,\dots,\beta_q;\end{matrix}\ z\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\dots,\alpha_p;\\ \beta_1,\dots,\beta_q;\end{matrix}\ z\right] \tag{12}$$

The incomplete $\aleph$-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,r}(z)$ defined by Bansal et al. (2020) are expressed as follows:

$${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z) = {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}\left(z\left|\begin{matrix}(a_1,A_1,x),\ (a_j,A_j)_{2,n},\ [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},\ [\tau_i(g_{ji},G_{ji})]_{m+1,q_i}\end{matrix}\right.\right)$$
$$= \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]} z^{-s}\mathrm{d}s, \tag{13}$$

and

$${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z) = {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}\left(z\left|\begin{matrix}(a_1,A_1,x),\ (a_j,A_j)_{2,n},\ [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},\ [\tau_i(g_{ji},G_{ji})]_{m+1,q_i}\end{matrix}\right.\right)$$
$$= \frac{1}{2\pi\omega}\int_L \frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]} z^{-s}\mathrm{d}s. \tag{14}$$

The incomplete $\aleph$-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,r}(z)$ in (13) and (14) exists for $x \geq 0$ based on a specific set of requirements.

The contour L in the s-plane extends from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number constructing a loop. It is essential to guarantee that the poles of $\Gamma(1-a_j-A_js),\ j=2,\dots,n$ lie to the right of the contour L and the poles of $\Gamma(g_j+G_js),\ j=1,\dots,m$ lie to the left of the contour L. The parameters τ_i, m, n, p_i, q_i are positive numbers satisfying $0 \leq n \leq p_i, 0 \leq m \leq q_i$ and a_j, g_j, a_{ji}, g_{ji} are complex numbers. It is presumed that these poles are simple. We have subsequent conditions:

$$\Omega_i > 0,\ |\arg(z)| < \frac{\pi}{2}\Omega_i,\quad i=1,\dots,r \tag{15}$$

$$\Omega_j \geq 0,\ |\arg(z)| < \frac{\pi}{2}\Omega_i \ \text{ and }\ \Re(\zeta_i)+1<0, \tag{16}$$

where

$$\Omega_i = \sum_{j=1}^n A_j + \sum_{j=1}^m G_j - \tau_i\left(\sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=m+1}^{q_i} G_{ji}\right), \tag{17}$$

and

$$\zeta_i = \sum_{j=1}^m g_j - \sum_{j=1}^n a_j + \tau_i\left(\sum_{j=m+1}^{q_i} g_{ji} - \sum_{j=n+1}^{p_i} a_{ji}\right) + \frac{p_i-q_i}{2},\quad i=1,\dots,r. \tag{18}$$

As a result, the relationship is as follows:

$${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z) + {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z) = \aleph^{m,n}_{p_i,q_i,\tau_i,r}(z), \tag{19}$$

$\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ is the $\aleph$-function introduced by Südland et al. (1998) which is the generalization of I-function (Saxena 2008).

For $\tau_i = 1$, the incomplete $\aleph$-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,r}(z)$ reduce to incomplete I-functions ${}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)$ respectively (see, Kumar et al. 2022), we have

$${}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z) = {}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}\left(z\left|\begin{matrix}(a_1,A_1,x),\ (a_j,A_j)_{2,n},\ (a_{ji},A_{ji})_{n+1,p_i}\\ (g_j,G_j)_{1,m},\ (g_{ji},G_{ji})_{m+1,q_i}\end{matrix}\right.\right)$$
$$= \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\sum_{i=1}^r\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]} z^{-s}\mathrm{d}s, \tag{20}$$

and

$${}^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z) = {}^{(\gamma)}I^{m,n}_{p_i,q_i,r}\left(z\left|\begin{matrix}(a_1,A_1,x),\ (a_j,A_j)_{2,n},\ (a_{ji},A_{ji})_{n+1,p_i}\\ (g_j,G_j)_{1,m},\ (g_{ji},G_{ji})_{m+1,q_i}\end{matrix}\right.\right)$$
$$= \frac{1}{2\pi\omega}\int_L \frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\sum_{i=1}^r\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]} z^{-s}\mathrm{d}s. \tag{21}$$

Now, if we set $r = 1$, then the incomplete I-functions ${}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)$ reduce to incomplete H-functions ${}^{(\Gamma)}H^{m,n}_{p,q}(z)$ and ${}^{(\gamma)}H^{m,n}_{p,q}(z)$ respectively (see, Bansal et al. 2019),

$${}^{(\Gamma)}H^{m,n}_{p,q}(z) = {}^{(\Gamma)}H^{m,n}_{p,q}\left(z\left|\begin{matrix}(a_1,A_1,x),\ (a_j,A_j)_{2,p}\\ (g_j,G_j)_{1,q}\end{matrix}\right.\right)$$
$$= \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\left[\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)\right]} z^{-s}\mathrm{d}s, \tag{22}$$

and

$${}^{(\gamma)}H^{m,n}_{p,q}(z) = {}^{(\Gamma)}H^{m,n}_{p,q}\left(z\left|\begin{matrix}(a_1,A_1,x),\ (a_j,A_j)_{2,p}\\ (g_j,G_j)_{1,q}\end{matrix}\right.\right)$$
$$= \frac{1}{2\pi\omega}\int_L \frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\left[\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)\right]} z^{-s}\mathrm{d}s, \tag{23}$$

under the same conditions verified by the incomplete I-functions with $r = 1$.

By using the relation (5), we have

$${}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z) + {}^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z) = I^{m,n}_{p_i,q_i,r}(z), \tag{24}$$

the function $I^{m,n}_{p_i,q_i,r}(z)$ being the function introduced by Saxena (2008) and

$$^{(\Gamma)}H^{m,n}_{p,q}(z) + {}^{(\gamma)}H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}(z). \tag{25}$$

Boros integral

Well-known Boros integral representation is given by

$$\int_0^\infty \left(\frac{y^2}{y^4+2ay^2+1}\right)^p \frac{y^2+1}{y^b+1}\frac{\mathrm{d}y}{y^2} = 2^{-\frac{1}{2}-p}(1+a)^{\frac{1}{2}-p} B\Big(p-\frac{1}{2},\frac{1}{2}\Big), \tag{26}$$

where $b>0, a>-1, \Re(p)>-\frac{1}{2}, B(.,.)$ is the beta function.

Proof Concerning the proof, see Qureshi et al. (2013) and Boros and Moll (1998). Let

$$X = \left(\frac{y^2}{y^4+2ay^2+1}\right)^p. \tag{27}$$

□

Main integrals

In this section, the Boros integral with three parameters including the incomplete ℵ function is evaluated.

Theorem 1 *We have the proceeding integral*

$$\begin{aligned}\int_0^\infty &\left(\frac{y^2}{y^4+2ay^2+1}\right)^p \frac{y^2+1}{y^b+1} S^M_N(uX)\, {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(X^e z)\frac{\mathrm{d}y}{y^2}\\
&= \frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{[N/M]} A_{N,K}\, u^K \frac{1}{[2(1+a)]^K}\\
&\times {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left(z[2(a+1)]^{-e}\,\middle|\, \begin{matrix}(a_1,A_1,x),\ \left(\frac{3}{2}-p-K,e\right),\\ (g_j,G_j)_{1,m},\end{matrix}\right.\\
&\left.\begin{matrix}(a_j,A_j)_{2,n},\ [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ [\tau_i(g_{ji},G_{ji})]_{m+1,q_i},(1-p-K,e)\end{matrix}\right),\end{aligned} \tag{28}$$

under the following conditions:

$$x\geq 0,\ e,b>0,\ a>-1,\ \Re(p+K) - e\min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) > -\frac{1}{2},$$

and

$$\Omega_i>0,\ |\arg(z)|<\frac{\pi}{2}\Omega_i, i=1,\dots,r \ \text{ or }\ \Omega_i\geq 0,\ |\arg(z)| < \frac{\pi}{2}\Omega_i \ \text{ and }\ \Re(\zeta_i)+1<0.$$

Proof Replacing the class of polynomials and the incomplete ℵ-function $^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ by (1) and (13) respectively, we get

$$\begin{aligned}G = &\int_0^\infty \left(\frac{y^2}{y^4+2ay^2+1}\right)^p \frac{y^2+1}{y^b+1}\sum_{K=0}^{[N/M]} A_{N,K}(Xu)^K\\
&\times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1 s,x)\prod_{j=2}^n\Gamma(1-a_j-A_j s)\prod_{j=1}^m\Gamma(g_j+G_j s)}{\sum_{i=1}^r \tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}\\
&(zX^e)^{-s}\,\mathrm{d}s\,\frac{\mathrm{d}y}{y^2},\end{aligned} \tag{29}$$

interchanging the order of finite sum and integral (feasible due to the absolute convergence of integrals), we arrive at

$$\begin{aligned}G = &\sum_{K=0}^{[N/M]} A_{N,K}\, u^K\\
&\times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1 s,x)\prod_{j=2}^n\Gamma(1-a_j-A_j s)\prod_{j=1}^m\Gamma(g_j+G_j s)}{\sum_{i=1}^r \tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]} z^{-s}\\
&\times \int_0^\infty \left(\frac{y^2}{y^4+2ay^2+1}\right)^{p+K-es}\frac{y^2+1}{y^b+1}\frac{\mathrm{d}y}{y^2}\,\mathrm{d}s,\end{aligned} \tag{30}$$

evaluating the integral with the help of (26), we get

$$\begin{aligned}&\int_0^\infty \left(\frac{y^2}{y^4+2ay^2+1}\right)^{p+K-es}\frac{y^2+1}{y^b+1}\frac{\mathrm{d}y}{y^2}\\
&= 2^{-\frac{1}{2}-p-K+es}(1+a)^{\frac{1}{2}-p-K+es} B\Big(p+K-es-\frac{1}{2},\frac{1}{2}\Big),\end{aligned} \tag{31}$$

substituting (31) in (30), after algebraic manipulations we get

$$\begin{aligned}G = &\frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{[N/M]} A_{N,K}\, u^K \frac{1}{[2(1+a)]^K}\\
&\times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1 s,x)\prod_{j=2}^n\Gamma(1-a_j-A_j s)\prod_{j=1}^m\Gamma(g_j+G_j s)}{\sum_{i=1}^r \tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}\\
&\times \frac{\Gamma\left(p+K-es-\frac{1}{2}\right)}{\Gamma(p+K-es)}\frac{z^{-s}}{[2(1+a)]^{-se}}\,\mathrm{d}s.\end{aligned} \tag{32}$$

Illustrating (32) by means of Mellin-Barnes integral contour to get (28). □

Theorem 2 *We have the proceeding integral*

$$\int_0^\infty \left(\frac{y^2}{y^4+2ay^2+1}\right)^p \frac{y^2+1}{y^b+1} S_N^M(uX)\, {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(X^e z)\frac{\mathrm{d}y}{y^2} = \frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{[N/M]} A_{N,K}\, u^K \frac{1}{[2(1+a)]^K} \times {}^{(\gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left(z[2(a+1)]^{-e}\left|\begin{matrix}(a_1,A_1,x),\left(\frac{3}{2}-p-K,e\right),(a_j,A_j)_{2,n},\ [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},\ [\tau_i(g_{ji},G_{ji})]_{m+1,q_i},(1-p-K,e)\end{matrix}\right.\right), \tag{33}$$

under the conditions as that in Theorem 1.

Proof On similar steps of Theorems 1 and 2 will follow. □

Special cases

For $\tau_i = 1$, the incomplete $\aleph$-function reduce to incomplete I-functions, we have

Corollary 1 *The following integral hold true*

$$\int_0^\infty \left(\frac{y^2}{y^4+2ay^2+1}\right)^p \frac{y^2+1}{y^b+1} S_N^M(uX)\, {}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(X^e z)\frac{\mathrm{d}y}{y^2} = \frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{[N/M]} A_{N,K}\, u^K \frac{1}{[2(1+a)]^K} \times {}^{(\Gamma)}I^{m,n+1}_{p_i+1,q_i+1,r}\left(z[2(a+1)]^{-e}\left|\begin{matrix}(a_1,A_1,x),\left(\frac{3}{2}-p-K,e\right),(a_j,A_j)_{2,n},\ (a_{ji},A_{ji})_{n+1,p_i}\\ (g_j,G_j)_{1,m},\ (g_{ji},G_{ji})_{m+1,q_i},\ (1-p-K,e)\end{matrix}\right.\right), \tag{34}$$

considering the identical circumstances as those in Theorem 1 for $\tau_i = 1$.

Corollary 2 *The following integral hold true*

$$\int_0^\infty \left(\frac{y^2}{y^4+2ay^2+1}\right)^p \frac{y^2+1}{y^b+1} S_N^M(uX)\, {}^{(\gamma)}I^{m,n}_{p_i,q_i,r}(X^e z)\frac{\mathrm{d}y}{y^2} = \frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{[N/M]} A_{N,K}\, u^K \frac{1}{[2(1+a)]^K} \times {}^{(\gamma)}I^{m,n+1}_{p_i+1,q_i+1,r} \tag{35}$$

$$\left(z[2(a+1)]^{-e}\left|\begin{matrix}(a_1,A_1,x),\left(\frac{3}{2}-p-K,e\right),(a_j,A_j)_{2,n},(a_{ji},A_{ji})_{n+1,p_i}\\ (g_j,G_j)_{1,m},(g_{ji},G_{ji})_{m+1,q_i},(1-p-K,e)\end{matrix}\right.\right),$$

considering the identical circumstances as those in Theorem 2 for $\tau_i = 1$.

For $r = 1$, the incomplete I-function reduces to incomplete H-function, we have

Corollary 3 *The following integral hold true*

$$\int_0^\infty \left(\frac{y^2}{y^4+2ay^2+1}\right)^p \frac{y^2+1}{y^b+1} S_N^M(uX)\, {}^{(\Gamma)}H^{m,n}_{p,q}(X^e z)\frac{\mathrm{d}y}{y^2} = \frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{[N/M]} A_{N,K}\, u^K \frac{1}{[2(1+a)]^K} \times {}^{(\Gamma)}H^{m,n+1}_{p+1,q+1}\left(z[2(a+1)]^{-e}\left|\begin{matrix}(a_1,A_1,x),\left(\frac{3}{2}-p-K,e\right),(a_j,A_j)_{2,p}\\ (g_j,G_j)_{1,q},(1-p-K,e)\end{matrix}\right.\right), \tag{36}$$

considering the identical circumstances as those in Theorem 1 for $r = \tau_i = 1$.

Corollary 4 *The following integral hold true*

$$\int_0^\infty \left(\frac{y^2}{y^4+2ay^2+1}\right)^p \frac{y^2+1}{y^b+1} S_N^M(uX)\, {}^{(\gamma)}H^{m,n}_{p,q}(X^e z)\frac{\mathrm{d}y}{y^2} = \frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{[N/M]} A_{N,K}\, u^K \frac{1}{[2(1+a)]^K} \times {}^{(\gamma)}H^{m,n+1}_{p+1,q+1}\left(z[2(a+1)]^{-e}\left|\begin{matrix}(a_1,A_1,x),\left(\frac{3}{2}-p-K,e\right),(a_j,A_j)_{2,p}\\ (g_j,G_j)_{1,q},(1-p-K,e)\end{matrix}\right.\right), \tag{37}$$

considering the identical circumstances as those in Theorem 2 for $r = \tau_i = 1$.

Now, we consider several special cases concerning the class of polynomials.

Definition 4.1 Hermite polynomials: In 1810, Pierre-Simon Laplace defined Hermite polynomials and these polynomials were studied in the expanded form by Pafnuty Chebyshev in 1859.Chebyshev's research was disregarded, and the polynomials were eventually called after Charles Hermite, who wrote about them in 1864 and claimed they were novel. A traditional sequence of orthogonal polynomials is the Hermite polynomials. The Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} = e^{\frac{x^2}{2}}\left(x-\frac{d}{dx}\right)^n e^{\frac{-x^2}{2}}, \tag{38}$$

they satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\mathrm{d}x = \sqrt{\pi}2^n n!\delta_{n,m}. \tag{39}$$

Definition 4.2 Jacobi polynomials: A category of conventional orthogonal polynomials called Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ are also referred to as hypergeometric polynomials.

The Jacobi polynomials are provided by the formula for $\alpha, \beta > -1$.

$$P_n^{(\xi,\tau)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\xi}(1+x)^{-\tau}\frac{d^n}{dx^n}\left\{(1-x)^{\xi}(1+x)^{\tau}\left(1-x^2\right)^n\right\}. \tag{40}$$

They are normalized (standardized) by

$$P_n^{(\xi,\tau)}(1) = \binom{n+\xi}{n}, \tag{41}$$

and satisfy the orthogonality condition

$$\begin{aligned}\int_{-1}^{1}&(1-x)^{\xi}(1+x)^{\tau}P_m^{(\xi,\tau)}(x)P_n^{(\xi,\tau)}(x)\mathrm{d}x\\ &= \frac{2^{\xi+\tau+1}}{2n+\xi+\tau+1}\frac{\Gamma(n+\xi+1)\Gamma(n+\tau+1)}{\Gamma(n+\xi+\tau+1)\,n!}\delta_{nm}.\end{aligned} \tag{42}$$

Definition 4.3 Laguerre polynomials: The solutions to Laguerre's equation are known as Laguerre polynomials, after Edmond Laguerre (1834–1866):

$$xy'' + (1-x)y' + ny = 0 \tag{43}$$

which is a linear differential equation of second order. Only when n is an integer that is not negative does this equation have nonsingular solutions. The definition of the modified Laguerre polynomials is

$$L_n^{(\xi)}(x) = \frac{x^{-\xi}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-}xx^{n+\xi}\right)$$
(the classical Laguerre polynomials correspond to $\xi = 0$). (44)

They satisfy the orthogonality relation

$$\int_0^{\infty} x^{\xi}e^{-x}L_n^{(\xi)}(x)L_m^{(\xi)}(x)\mathrm{d}x = \frac{\Gamma(n+\xi+1)}{n!}\delta_{n,m}. \tag{45}$$

Next, by setting $M = 2, A[N,K] = (-1)^K$, we have $S_N^2(y) \rightarrow y^{\frac{N}{2}}H_N\left(\frac{1}{2\sqrt{y}}\right)$ (Hermite polynomials), we get

Corollary 5 *The following integral hold true*

$$\begin{aligned}&\int_0^{\infty}\left(\frac{y^2}{y^4+2ay^2+1}\right)^p\frac{y^2+1}{y^b+1}(uX)^{\frac{N}{2}}H_N\left(\frac{1}{2\sqrt{uX}}\right){}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(X^e z)\,\frac{\mathrm{d}y}{y^2}\\ &= \frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{[N/2]}\frac{(-N)_{2K}}{K!}(-1)^K u^K\frac{1}{[2(1+a)]^K}\\ &\times {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left(z[2(a+1)]^{-e}\left|\begin{matrix}(a_1,A_1,x),\left(\frac{3}{2}-p-K,e\right),\\ (g_j,G_j)_{1,m},\end{matrix}\right.\right.\\ &\left.\begin{matrix}(a_j,A_j)_{2,n},\ \left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left[\tau_i\left(g_{ji},G_{ji}\right)\right]_{m+1,q_i},\ (1-p-K,e)\end{matrix}\right),\end{aligned} \tag{46}$$

considering the identical circumstances as in Theorem 1.

Setting $M = 1, A[N,K] = \binom{N+\alpha}{N}\frac{(\alpha+\beta+N+1)_K}{(\alpha+1)_K}$, we have $S_N^1(y) \rightarrow P_N^{(\alpha,\beta)}(1-2y)$ (Jacobi polynomials), we have

Corollary 6 *The following integral hold true*

$$\begin{aligned}&\int_0^{\infty}\left(\frac{y^2}{y^4+2ay^2+1}\right)^p\frac{y^2+1}{y^b+1}P_N^{(\alpha,\beta)}(1-2uX)\,{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(X^e z)\,\frac{\mathrm{d}y}{y^2}\\ &= \frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{N}\frac{(-N)_K}{K!}\binom{N+\alpha}{N}\frac{(\alpha+\beta+N+1)_K}{(\alpha+1)_K}\frac{u^K}{[2(1+a)]^K}\\ &\times {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left(z[2(a+1)]^{-e}\left|\begin{matrix}(a_1,A_1,x),\left(\frac{3}{2}-p-K,e\right),\\ (g_j,G_j)_{1,m},\end{matrix}\right.\right.\\ &\left.\begin{matrix}(a_j,A_j)_{2,n},\ \left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left[\tau_i\left(g_{ji},G_{ji}\right)\right]_{m+1,q_i},\ (1-p-K,e)\end{matrix}\right),\end{aligned} \tag{47}$$

considering the identical circumstances as Theorem 1.

Further, by setting $M = 1, A[N,K] = \binom{N+\alpha}{N}\frac{1}{(\alpha+1)_K}$, we have $S_N^1(y) \rightarrow L_N^{(\alpha)}(y)$ (Laguerre polynomials), we get

Corollary 7 *The following integral hold true*

$$\begin{aligned}&\int_0^{\infty}\left(\frac{y^2}{y^4+2ay^2+1}\right)^p\frac{y^2+1}{y^b+1}L_N(uX)\,{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(X^e z)\frac{\mathrm{d}y}{y^2}\\ &= \frac{\Gamma\left(\frac{1}{2}\right)}{2[2(1+a)]^{p-\frac{1}{2}}}\sum_{K=0}^{N}\frac{(-N)_K}{K!}\binom{N+\alpha}{N}\frac{1}{(\alpha+1)_K}\frac{u^K}{[2(1+a)]^K}\\ &\times {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left(z[2(a+1)]^{-e}\left|\begin{matrix}(a_1,A_1,x),\left(\frac{3}{2}-p-K,e\right),\\ (g_j,G_j)_{1,m},\end{matrix}\right.\right.\\ &\left.\begin{matrix}(a_j,A_j)_{2,n},\ \left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left[\tau_i\left(g_{ji},G_{ji}\right)\right]_{m+1,q_i},\ (1-p-K,e)\end{matrix}\right),\end{aligned} \tag{48}$$

considering the identical circumstances as Theorem 1.

Conclusions and future work

In the current study, we have investigated the Boros integral with three parameters, which contains the Srivastava polynomials and incomplete ℵ-functions.Numerous other known results come as special cases of our main conclusions because the functions in the provided outcomes are unified and general in nature. We have also given some particular cases in terms of incomplete I and H-function. We further explored Boros integral involving the incomplete ℵ -functions and pointed out many leading cases concerning the class of polynomials (Hermite, Jacobi, Laguerree polynomials).

For extending this work one can generalize these formulas by using the classes of multivariable polynomials defined by Srivastava (1985) and Srivastava and Garg (1987). We can generalize these formulas by using the class of multivariable polynomials described through Srivastava (1985) and Srivastava and Garg (1987).

Acknowledgements Not applicable

Author contributions All authors contributed equally to the present investigation. All authors read and approved the final manuscript.

Funding This study did not receive any funding in any form.

Data availibility No data were used to support this study.

Declarations

Conflict of interest There is no conflict of interest regarding the publication of this article.

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