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ORIGINAL RESEARCH

Single Server Batch Arrival Bernoulli Feedback Queueing System with Waiting Server, K-Variant Vacations and Impatient Customers

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Abstract
We consider an infinite capacity batch arrival single server Markovian Bernoulli feedback queueing system with waiting server, *K*-variant vacations, impatient customers and retention of reneged customers. The model is analyzed using probability generating function (PGF) technique. Various queueing system characteristics are derived. Then, by setting the appropriate parameters, some special cases are discussed. Moreover, a cost model for the queueing system is developed. The parameter optimization is numerically illustrated using particle swarm optimization (PSO). Finally, numerical results are provided to explore the impact of system parameters on performance measures and costs of the queueing system.

Keywords Variant multiple vacations · Impatient customers · Bernoulli feedback · Probability generating function · Optimization · PSO algorithm

1 Introduction

Queueing systems with vacation policies have been greatly studied due to their broad applications in different real-life queueing system situations, such as manufacturing/production systems, distribution/service systems, transportation systems, telecommunication industry, and computer and communication systems. For various results on different vacation models, the readers may refer to the survey paper of Doshi [1], monographs of Takagi [2], and Tian and Zhang [3].

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The queueing models discussed in the above literature suppose that the customers arrive one at a time. There are various situations wherein customers arrive in groups. Such queues are called batch arrival queueing models; digital communication system is a perfect practical example of such models. Eminent research papers on the subject can be found in Lee et al. [4], Madan and AI-Rawwash [5], Wang et al. [6], Haridass and Arrumuganathan [7], Chang and Ke [8], Aissani [9], Haridass and Arumuganathan [10], and Inoue et al. [11].

Vacation queueing systems with impatience play important roles in the analysis of many telephone switching systems, communication/telecommunication networks, and computer systems and manufacturing systems. These queues have extensively examined (cf. Zhang et al.[12], Altman and Yechiali [13], Yue et al. [14], Altman and Yechiali [15], Adan et al.[16], Ammar [17], and Bouchentouf et al. [18, 19]).

In real-life situations, the server does not go on vacation just as the system is empty. When we consider the human behavior as a server, we observe that it waits a certain amount of time before taking a vacation, even if there are no customers in the system. Such queueing models have attracted the attention of many researchers (cf. Padmavathy et al.[20], Ammar [21], Deepa and Kalidass [22], Bouchentouf et al.[23], and Bouchentouf and Guendouzi [24]).

Variant of multiple vacation scheme is relatively recent where it is permitted to the server to take a certain fixed number of consecutive vacations if the system remains empty at the end of a vacation. Such vacation policy was treated by different researchers including Zhang and Tian [25], Ke [26], Wang et al. [27], Yue et al. [28], Laxmi and Rajesh [29], and Laxmi and Rajesh [30].

The optimization of manufacturing/production, telecommunication and computer systems using queueing theory has been the subject of many studies in recent decades. Interesting papers in this area include the research works of Whitt [31] in open and closed queueing networks, Dallery and Gershwin [32], which describe the main queueing models and the results of the literature on the production lines, Cruz et al. [33], which present the optimization of the performance of general finite single-server acyclic queueing networks, and Martins et al. [34], which present performance analysis and optimization of buffers and servers in finite queueing networks.

In the earlier literature, as it was mentioned, very few authors dealt with queueing models with variant vacations and impatience at which the server may take a sequence of finite vacations at its idle time. But as far as the best of our knowledge, there is no research work on Bernoulli feedback queueing system with batch arrival, waiting server, variant vacations, impatient customers, and retention of reneged customers. This motivates us to develop such a model and carry out its cost model.

The remainder of the paper is arranged as follows. The mathematical model is constructed in Section 2. The probability generating function of the steady state of the system is obtained in Section 3. In Section 4, various performance measures are derived. In Section 5, we give some special cases of our model. In Section 6, a cost model for the queueing system is developed. Then, we adopt PSO algorithm to implement the optimization tasks. Section 7 is consecrated to numerical illustrations. Finally, we conclude the paper in Section 8.

2 Model Description

We consider an $M^X/M/1$ queueing system at which customers arrive in batches according to a Poisson process with rate λ. Let X denote the batch size random variable of the arrival with probability mass function $P(X = l) = b_l, l = 1, 2, ...$

The service time is assumed to be exponentially distributed with parameter μ. The customers are served on FCFS discipline. When the busy period is ended, the server waits a random period before taking a vacation, this waiting time is assumed to be exponentially distributed with parameter η. When duration of the waiting server expires, the server leaves for vacation. Then, at a vacation period termination, if it finds a customer at the vacation completion instant, it comes back to the busy period; otherwise, it takes a finite number, say K, of successive vacations. When the K consecutive vacations are complete, the server returns to busy period and depending on the arrival batch of customers, it stays idle or busy. The period of a vacation follows an exponential distribution with parameter ϕ.

During vacation period, each incoming customer starts up an impatience timer independently of the other customers in the system, assumed to be exponentially distributed with parameter ξ. The impatient customers may leave the system with probability α. Via certain mechanism, they can be retained in the system with probability $\alpha' = 1 - \alpha$.

After completion of each service, the latter may be incomplete or unsatisfactory, at this situation, the customer may either decide to leave the system with probability β or to join the tail of the queue with probability β', $(\beta + \beta' = 1)$.

The inter-arrival times, batch sizes, waiting server times, vacation times, service time, and impatience times are independent of each other.

2.1 Practical Applications of the Model

The proposed queueing model has prominent applications in diverse practical systems dealing with human behavior including private healthcare and private business firms at which customers may arrive in batches. At the end of busy periods, the server waits for a while before proceeding for a vacation. Once the vacation period is over, the server switches to the busy period if there are customers in the queue; otherwise it may take a fixed consecutive vacations, at the end of the successive vacations, the server switches to busy period and stays idle or busy depending on the availability of the customers in the system. During the vacation period, a customer may quit the system whenever his waiting time is longer than his patience time. Further, customers may be dissatisfied with the quality of the service. In this case, they can rejoin the system as feedback customers to complete their service. Such systems can be modeled by our model developed in this paper.

Another practical application of the proposed model arises in communication systems: It is broadly recognized that the impatience phenomenon is one of the determining factors for the performance of call centers. From a business point of view, a call center is an entity that combines voice and data communication technologies, enabling a company to implement critical business strategies in order to reduce costs and increase revenues. It is typically set up for sales, marketing, technical support, and customer service purposes. Once the calls (arrival stream of customers in batches) are connected to the system, they can be filtered and forwarded through a proactive support service. The filter may be a software or a live representative who assesses the customer's problem and then transfers the calls to a designated representative. Once the calls are forwarded to the suitable representatives, the customer service representatives will work on resolving the customer's problems (service). In addition, in call center, arrival streams of customers in batches called outbound calls, in the form of e-mails sent to the call center with a request to be called back will be processed in the center in the order of their arrival when there is no incoming call. Once the customers are serviced and no call are connected (empty system), the agents stay active and look for new calls (waiting server) for a certain period of time. After that, they go on vacation. At the end of the vacation period, they come back to the busy period, if there are a new calls, they work on them; otherwise, they may take a fixed consecutive vacations. When the number of fixed vacation in taken, the agents return to the busy period, and start working if they find customers waiting in the queue; otherwise, they stay idle. When the system is on vacation, the flow of new requests (customers) continues, but each customer activates its own impatience timer, such that, if the system is still on vacation when the time expires, the customers leave the system. The reneging has a very bad effect on the system. To avert this serious problem, the agents, using certain retention strategies, may convince the impatient customer to remain in the system. This could be either by increasing the rate of the service, introducing an extra service channel or presenting more advantageous offers to customers.

3 The Equilibrium State Distribution

Let $L(t)$ be the number of customers in the system, and $S(t)$ denote the status of the server at time t, such that

$$S(t) = \begin{cases} j, & \text{when the server is taking the (j+1)th vacation at time } t, \\ & j = \overline{0, K-1}; \\ K, & \text{the server is in busy period at time } t. \end{cases}$$

The bi-variate $\{(L(t);S(t));t \geq 0\}$ represents two-dimensional infinite state continuous-time Markov chain with state space $\Omega = \{(n,j) : n \geq 0, j = \overline{0, K}\}$.

Let $P_{n,j} = \lim_{t\to\infty} P\{L(t) = n, S(t) = j\}$, $n \geq 0, j = \overline{0, K}$ denote the system state probabilities of the process $\{(L(t), S(t)), t \geq 0\}$.

The steady-state balance equations that govern our model are deduced as:

$$(\lambda + \phi)P_{0,0} = \alpha\xi P_{1,0} + \eta P_{0,K}, \tag{1}$$

$$(\lambda+\phi+\alpha\xi)P_{1,0} = \lambda b_1 P_{0,0} + 2\alpha\xi P_{2,0},\ \ n=1, \tag{2}$$

$$(\lambda+\phi+n\alpha\xi)P_{n,0} = \lambda\sum_{m=1}^{n} b_m P_{n-m,0} + (n+1)\alpha\xi P_{n+1,0},\ \ n\geq 2, \tag{3}$$

$$(\lambda+\phi)P_{0,j} = \alpha\xi P_{1,j} + \phi P_{0,j-1},\ \ 1\leq j\leq K-1, \tag{4}$$

$$(\lambda+\phi+n\alpha\xi)P_{n,j} = \lambda\sum_{m=1}^{n} b_m P_{n-m,j} + (n+1)\alpha\xi P_{n+1,j},\ n\geq 1,\ \ 1\leq j\leq K-1, \tag{5}$$

$$(\lambda+\eta)P_{0,K} = \phi P_{0,K-1} + \beta\mu P_{1,K}, \tag{6}$$

$$(\lambda+\beta\mu)P_{n,K} = \lambda\sum_{m=1}^{n} b_m P_{n-m,K} + \beta\mu P_{n+1,K} + \phi\sum_{j=0}^{K-1} P_{n,j},\ \ n\geq 1. \tag{7}$$

Theorem 1 *If $\lambda E(X) < \beta\mu$, then the steady-state-probabilities $P_{n,j}$ are given as*

$$P_{.,j} = \sum_{n=0}^{\infty} P_{n,j} = A^{j-1}P_{0,0}, \qquad j=\overline{0,K-1}, \tag{8}$$

and

$$P_{.,K} = \sum_{n=0}^{\infty} P_{n,K} = \frac{1}{\beta\mu-\lambda B'(1)}\left\{\frac{\phi\lambda B'(1)}{\alpha\xi+\phi}\frac{1-A^K}{A(1-A)} + \frac{\beta\mu\alpha\xi}{\eta C}\right\}P_{0,0}, \tag{9}$$

where

$$P_{0,0} = \left\{\frac{\beta\mu\alpha\xi}{\eta C(\beta\mu-\lambda B'(1))} + \frac{1-A^K}{A(1-A)}\left(\frac{\phi\lambda B'(1)}{(\beta\mu-\lambda B'(1))(\alpha\xi+\phi)}+1\right)\right\}^{-1},$$

such that

$$A = \frac{\phi C}{\alpha\xi},$$

with

$$C = \int_0^1 e^{\frac{\lambda}{\alpha\xi}H(x)}(1-x)^{\frac{\phi}{\alpha\xi}-1}dx, \quad \text{and} \quad H(z) = \int_0^z \frac{B(x)-1}{1-x}dx,$$

where $B(x)$ is the probability generating function of the batch arrival size X, and $B'(1)=E(X)$ is the first moment of random variable X.

Proof The state probabilities are obtained by solving equations (1)-(7), using probability generating functions (PGFs).

Let us define the PGFs of $P_{n,j}$ as

$$G_j(z) = \sum_{n=0}^{\infty} P_{n,j} z^n, \ \ |z| \le 1, \ \ j = \overline{0, K},$$

and the PGF of the batch arrival size X as

$$B(z) = \sum_{n=1}^{\infty} b_n z^n, \ \ |z| \le 1.$$

Multiplying equations (1)-(3) by z^n and summing all possible values of n, then re-arranging all the terms, we obtain

$$(1-z)\alpha\xi G_0'(z) - [\lambda(B(z)-1) - \phi]G_0(z) = -\eta P_{0,K}, \tag{10}$$

In the same way, using equations (4)-(5) and (6)-(7), we, respectively, get

$$(1-z)\alpha\xi G_j'(z) - [\lambda(B(z)-1) - \phi]G_j(z) = -\phi P_{0,j-1}, \ \ j = \overline{1, K-1}, \tag{11}$$

and

$$\begin{aligned} &[\lambda z(B(z)-1) + \beta\mu(1-z)]G_K(z) + z\phi \sum_{j=0}^{K-1} G_j(z) = z\phi \sum_{j=0}^{K-2} P_{0,j} \\ &+[\beta\mu(1-z) + z\eta]P_{0,K}. \end{aligned} \tag{12}$$

By taking $z = 1$ in equations (10) and (11), we, respectively, find

$$\phi G_0(1) = \eta P_{0,K}, \tag{13}$$

and

$$G_j(1) = P_{0,j-1}, \ \ j = \overline{1, K-1}. \tag{14}$$

Now, we can write equations (10) and (11) for $z \neq 1$ as

$$G_0'(z) + \left[\frac{\lambda}{\alpha\xi}H'(z) - \frac{\phi}{\alpha\xi(1-z)}\right]G_0(z) = -\frac{\eta}{\alpha\xi(1-z)}P_{0,K}, \tag{15}$$

and

$$G_j'(z) + \left[\frac{\lambda}{\alpha\xi}H'(z) - \frac{\phi}{\alpha\xi(1-z)}\right]G_j(z) = -\frac{\phi}{\alpha\xi(1-z)}P_{0,j-1}, \ \ j = \overline{1, K-1}, \tag{16}$$

where

$$H'(z) = \frac{B(z) - 1}{1 - z}.$$

Next, by multiplying (15) and (16) by $e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}}$, we obtain

$$\frac{d}{dz}\left(e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}}G_0(z)\right) = -\frac{\eta}{\alpha\xi}e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}-1}P_{0,K}, \tag{17}$$

and

$$\frac{d}{dz}\left(e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}}G_j(z)\right) = -\frac{\phi}{\alpha\xi}e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}-1}P_{0,j-1},\ j = \overline{1, K-1}. \tag{18}$$

Then, integrating form 0 to z, we get

$$G_0(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{G_0(0) - \frac{\eta}{\alpha\xi}C(z)P_{0,K}\right\}, \tag{19}$$

and

$$G_j(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{G_j(0) - \frac{\phi}{\alpha\xi}C(z)P_{0,j-1}\right\},\ j = \overline{1, K-1}, \tag{20}$$

where

$$C(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi}H(x)}(1-x)^{\frac{\phi}{\alpha\xi}-1}dx.$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{n,0} > 0$ and $z = 1$ is the root of the denominator of the right hand side of equation (19), so $z = 1$ must be the root of the numerator of the right hand side of equation (19). Thus, we get

$$P_{0,0} = G_0(0) = \frac{\eta}{\alpha\xi}CP_{0,K}, \tag{21}$$

where

$$C =: C(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi}H(x)}(1-x)^{\frac{\phi}{\alpha\xi}-1}dx.$$

This implies

$$P_{0,K} = \frac{\alpha\xi}{\eta C}P_{0,0}. \tag{22}$$

Similarly, as $G_j(1) = \sum_{n=0}^{\infty} P_{n,j} > 0$, $j = \overline{1, K-1}$, and $z = 1$ is the root of the denominator of the right hand side of equation (20), so $z = 1$ must be the root of the numerator of the right hand side of equation (20). Thus,

$$P_{0,j} = G_j(0) = \frac{\phi}{\alpha\xi} C P_{0,j-1}, \quad j = \overline{1, K-1}. \tag{23}$$

Using repeatedly equation (23) , we obtain

$$P_{0,j} = A^j P_{0,0}, \ j = \overline{1, K-1}, \ \text{where } A = \frac{\phi C}{\alpha\xi}. \tag{24}$$

Substituting equations (22) and (24) into (19) and (20), respectively, we get

$$G_0(z) = e^{-\frac{\lambda}{\alpha\xi} H(z)} (1-z)^{-\frac{\phi}{\alpha\xi}} \left\{ 1 - \frac{C(z)}{C} \right\} P_{0,0}, \tag{25}$$

and

$$G_j(z) = e^{-\frac{\lambda}{\alpha\xi} H(z)} (1-z)^{-\frac{\phi}{\alpha\xi}} \left\{ 1 - \frac{C(z)}{C} \right\} A^j P_{0,0}, \ j = \overline{1, K-1}. \tag{26}$$

From equations (25)-(26), we find the expression of the probability generating function $G_j(z)$, for $j = \overline{0, K-1}$ in terms of $P_{0,0}$, and from equation (22), we have $P_{0,K}$ in terms of $P_{0,0}$. While $P_{0,j}$ in terms of $P_{0,0}$ is given in equation (24). Thus, substituting equations (22) and (24)-(26) in equation (12), we obtain the expression of the probability generating function $G_K(z)$ in terms of $P_{0,0}$.

Substituting equations (13)-(14) into (12), we get

$$G_K(z) = \frac{\beta\mu(1-z)P_{0,K} - z\phi \sum_{j=0}^{K-1} (G_j(z) - G_j(1))}{\lambda z(B(z) - 1) + \beta\mu(1-z)}. \tag{27}$$

Applying L'Hospital's rule, we find

$$G_K(1) = \frac{\beta\mu P_{0,K} + \phi \sum_{j=0}^{K-1} G_j'(1)}{\beta\mu - \lambda B'(1)}. \tag{28}$$

Next, from equations (10) and (13), using L'Hospital's rule, we get

$$G_0'(1) = \frac{\lambda B'(1)}{\alpha\xi + \phi} G_0(1). \tag{29}$$

Similarly, from equations (11) and (14), we have

$$G_j'(1) = \frac{\lambda B'(1)}{\alpha\xi + \phi} G_j(1), \ j = \overline{1, K-1}. \tag{30}$$

Equations (29) and (30) imply

$$G_j'(1) = \frac{\lambda B'(1)}{\alpha\xi + \phi} G_j(1),\ \ j = \overline{0, K-1}. \tag{31}$$

Via equations (25) and (26), using L'Hospital's rule, we obtain

$$G_j(1) = A^{j-1} P_{0,0}, j = \overline{0, K-1}. \tag{32}$$

This implies

$$\sum_{j=0}^{K-1} G_j'(1) = \frac{\lambda B'(1)}{\alpha\xi + \phi} \frac{1-A^K}{A(1-A)} P_{0,0}. \tag{33}$$

Substituting equations (22) and (33) into (28), we get

$$G_K(1) = \frac{1}{\beta\mu - \lambda B'(1)} \left\{ \frac{\phi\lambda B'(1)}{\alpha\xi + \phi} \frac{1-A^K}{A(1-A)} + \frac{\beta\mu\alpha\xi}{\eta C} \right\} P_{0,0}. \tag{34}$$

Finally, in order to obtain $P_{0,0}$, we use the normalization condition given as

$$\sum_{n=0}^{\infty} \sum_{j=0}^{K-1} P_{n,j} + \sum_{n=0}^{\infty} P_{n,K} = 1 \Longleftrightarrow \sum_{j=0}^{K-1} G_j(1) + G_K(1) = 1. \tag{35}$$

Substituting equations (32) and (34) into (35), we get

$$P_{0,0} = \left\{ \frac{\beta\mu\alpha\xi}{\eta C(\beta\mu - \lambda B'(1))} + \frac{1-A^K}{A(1-A)} \left(\frac{\phi\lambda B'(1)}{(\beta\mu - \lambda B'(1))(\alpha\xi + \phi)} + 1 \right) \right\}^{-1}.$$

4 System Performance Measures

The indices that are of general interest for the evaluation of the performances of our system include:

– The probability that the server is idle during busy period.

$$P_{0,K} = \frac{\alpha\xi}{\eta C} P_{0,0}.$$

– The probability that the server is in vacation period.

$$P_v = \sum_{j=0}^{K-1} A^{j-1} P_{0,0} = \frac{1-A^K}{A(1-A)} P_{0,0}.$$

– The probability that the server is serving customers during busy period.

$$P_b = 1 - P_v - P_{0,K}.$$

– The mean system size when the server is on vacation.

$$E[L_V] = \sum_{j=0}^{K-1} \lim_{z\to 1} G_j'(z).$$

From equation (33), we have

$$E[L_V] = \frac{\lambda B'(1)}{\alpha\xi + \phi} \frac{1 - A^K}{A(1 - A)} P_{0,0}.$$

– The mean system size when the server is in busy period.

$$E[L_K] = \lim_{z\to 1} G_K'(z).$$

Differentiating equation (27) and using L'Hospital's rule, we obtain

$$E[L_K] = \frac{\phi}{2(\beta\mu - \lambda B'(1))} \sum_{j=0}^{K-1} G_j''(1) + \frac{\phi(2\beta\mu + \lambda B''(1))}{2(\beta\mu - \lambda B'(1))^2} \sum_{j=0}^{K-1} G_j'(1)$$
$$+\frac{\beta\mu\lambda(2B'(1) + B''(1))}{2(\beta\mu - \lambda B'(1))^2} P_{0,K}, \tag{36}$$

where $G_j''(1)$ is obtained by differentiating twice $G_j(z)$ at $z = 1$ for $j = \overline{0, K}$. So, differentiating twice equations (10) and (11) and taking $z = 1$, we find

$$\sum_{j=0}^{K-1} G_j''(1) = \frac{2\lambda B'(1)}{2\alpha\xi + \phi} E[L_V]. \tag{37}$$

Next, substituting equation (37) into (36), we find

$$E[L_K] = \left[\frac{\phi\lambda B'(1)}{(2\alpha\xi + \phi)(\beta\mu - \lambda B'(1))} + \frac{\phi(2\beta\mu + \lambda B''(1))}{2(\beta\mu - \lambda B'(1))^2}\right] E[L_V]$$
$$+\frac{\beta\mu\lambda(2B'(1) + B''(1))}{2(\beta\mu - \lambda B'(1))^2} P_{0,K}.$$

– The mean system size.

$$E[L] = E[L_V] + E[L_K].$$

– The mean queue length.

$$E[L_q] = \sum_{j=0}^{K} \sum_{n=1}^{\infty} (n - 1) P_{n,j} = E[L] - \left[1 - \sum_{j=0}^{K} P_{0,j}\right].$$

– The mean number of customers served per unit time.

$$N_s = \beta\mu \sum_{n=1}^{\infty} P_{n,K} = \beta\mu P_b.$$

– The average rate of reneging.

$$R_a = \alpha\xi \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} nP_{n,j} = \alpha\xi E[L_V].$$

– The average rate of retention of impatient customers.

$$R_e = (1-\alpha)\xi \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} nP_{n,j} = (1-\alpha)\xi E[L_V].$$

5 Particular Cases

In this section, we present some special cases of our queueing model which are consistent with the existing literature.

5.1 Case 1: No Variant Vacations, No Batch Arrival, No Retention, and No Feedback

When $K = 1, b_1 = 1, \alpha = 1$, and $\beta = 1$, that is, if the server comes back from vacation to an empty system, it remains idle waiting for new arrivals, then it starts a busy period. Customers arrive to the system one by one, they are persistent and never return to the system as a feedback customers. In this case, the equations (1)–(7) can be abstracted as follow:

$$(\lambda + \phi)P_{0,0} = \xi P_{1,0} + \eta P_{0,1},$$

$$(\lambda + \phi + n\xi)P_{n,0} = \lambda P_{n-1,0} + (n+1)\xi P_{n+1,0}, \;\; n \geq 1,$$

$$(\lambda + \eta)P_{0,1} = \phi P_{0,0} + \mu P_{1,1},$$

$$(\lambda + \mu)P_{n,1} = \lambda P_{n-1,1} + \mu P_{n+1,1} + \phi P_{n,0}, \;\; n \geq 1.$$

Then, the steady-state probabilities $P_{.,0}$ and $P_{.,1}$ are as follows:

$$P_{.,0} = \frac{(\phi + \xi)(\mu(\eta C - \xi P_{0,0}) - \lambda\eta C)}{\phi C(\phi\mu + \xi(\mu - \lambda))},$$

and

$$P_{.,1} = \frac{\lambda\phi\eta C + \xi\mu(\phi+\xi)P_{0,0}}{\eta C(\mu\phi + \xi(\mu-\lambda))},$$

where

$$P_{0,0} = \frac{\phi\eta C(\phi+\xi)(\mu-\lambda)}{(\mu\phi\eta + \mu\eta\xi - \lambda\eta\xi + \mu\phi^2 + \mu\phi\xi)\xi},$$

and

$$C = \int_0^1 e^{\frac{-\lambda}{\xi}x}(1-x)^{\frac{\phi}{\xi}-1}dx.$$

These results coincide with that of an $M/M/1$ queueing model with single vacation, waiting server, and impatient customers, given in Padmavathy et al. [20].

5.2 Case 2: No Waiting Server, No Batch Arrival, No Retention, and No Feedback

When $\eta \to +\infty$, $b_1 = 1$, $\alpha = 1$, and $\beta = 1$, that is, Whenever a system becomes empty, the server goes on vacation. Customers arrive to the system one by one, they are persistent and never return to the system as a feedback customer. In this case, the equations (1)–(7) can be abstracted as follow:

$$(\lambda+\phi)P_{0,0} = \xi P_{1,0} + \mu P_{1,K},$$

$$(\lambda+\phi+n\xi)P_{n,0} = \lambda P_{n-1,0} + (n+1)\xi P_{n+1,0}, \ \ n \geq 1,$$

$$(\lambda+\phi)P_{0,j} = \xi P_{1,j} + \phi P_{0,j-1}, \ \ 1 \leq j \leq K-1,$$

$$(\lambda+\phi+n\xi)P_{n,j} = \lambda P_{n-1,j} + (n+1)\xi P_{n+1,j}, \ \ n \geq 1, \ \ 1 \leq j \leq K-1,$$

$$\lambda P_{0,K} = \phi P_{0,K-1},$$

$$(\lambda+\mu)P_{n,K} = \lambda P_{n-1,K} + \mu P_{n+1,K} + \phi \sum_{j=0}^{K-1} P_{n,j}, \ \ n \geq 1.$$

From the latest equations, the steady-state-probabilities of the number of customers in the system have the following from:

$$P_{.,j} = A^{j-1}P_{0,0}, \qquad j = \overline{0, K-1},$$

and

$$P_{.,K} = \frac{\phi}{\mu - \lambda}\left(\frac{\lambda(1 - A^K)}{(\phi + \xi)A(1 - A)} + \frac{\mu}{\lambda}A^{K-1}\right)P_{0,0},$$

where

$$P_{0,0} = \left\{\frac{(\mu\phi + (\mu - \lambda)\xi)(1 - A^K)}{(\mu - \lambda)(\phi + \xi)A(1 - A)} + \frac{\mu\phi A^{K-1}}{\lambda(\mu - \lambda)}\right\}^{-1},$$

with

$$A = \frac{\phi C}{\xi},$$

such that

$$C = \int_0^1 e^{\frac{-\lambda}{\xi}x}(1 - x)^{\frac{\phi}{\xi}-1}dx.$$

The obtained results match with that of an $M/M/1$ queueing model with impatient customers and a variant of multiple vacation policy presented by Yue et al. [28].

6 Cost Model

Let

- C_1 be a cost per unit time when the server is working during busy period.
- C_2 be a cost unit time when the server is idle during busy period.
- C_3 be a cost per unit time when the server is on vacation.
- C_4 be a cost per unit time when customers join the queue and wait for service.
- C_5 be a cost per service per unit time.
- C_6 be a cost per unit time of serving a feedback customer.
- C_7 be a cost per unit time when a customer reneges.
- C_8 be a cost per unit time when a customer is retained.

Using the cost parameters listed above, the total expected cost per unit time of the system is presented as

$$\mathcal{T}_c = C_1P_b + C_2P_{0,K} + C_3P_v + C_4E[L_q] + \mu(C_5 + \beta' C_6) + C_7R_a + C_8R_e.$$

The total expected profit per unit time of the system is given by

$$\mathcal{T}_p = \mathcal{T}_r - \mathcal{T}_c,$$

where $\mathcal{T}_r$ is the total expected revenue per unit time of the system,

$$\mathcal{T}_r = Rev \times \mu \times P_b,$$

where Rev denotes the revenue earned by providing service to a customer.

We consider in this paper the cost optimization problem under a particle swarm optimization (PSO). We employ this method to solve the optimization problem.

We focus on the optimization of the service rate μ in different cases in order to minimize the cost function F. Therefore, a total expected cost function must be developed in order to determine an optimum regular service rate μ^* and the optimum expected cost $F(\mu^*)$. Consequently, the optimization problem can be illustrated mathematically as:

$$\text{Minimize: } F(\mu) = \mathcal{T}_c.$$

7 Numerical Results

To study the behavior of system characteristics with respect to the changes of its parameters, we execute a numerical experiment by coding computer program in R software. For computational convenience, we arbitrarily choose the values of different system parameters and costs. We suppose that the arrival batch size X follows a geometric distribution with parameter p, that is

$$P(X = l) = b_l = (1-p)^{l-1}p,\ 0 < p < 1\ (l = 1, 2, ...).$$

Then, we easily have

$$B(z) = \frac{pz}{1-(1-p)z}, \quad E(X) = B'(1) = \frac{1}{p}, \quad \text{and} \quad E(X^2) = B''(1) = \frac{2(1-p)}{p^2}.$$

Next, for the whole analysis, we fix $C_1 = 20, C_2 = 10, C_3 = 8, C_4 = 20, C_5 = 30, C_6 = 15, C_7 = 20, C_8 = 25$, and $Rev = 200$.

7.1 Optimization of Service Rate μ

The main goal is to determine the optimal value of the service rate μ for different cases in order to minimize the cost function F. Swarm size, maximum number of iterations and learning factors are taken as 20, 100, and $c_1 = c_2 = 2$.

The total expected cost incurred on the system $\mathcal{T}_c$ can be minimized with respect to the decision parameter service μ. The total cost function is presented in Tables 1-4 by varying values of λ, K, η, and ϕ,, respectively.

Table 1 μ^* and $F(\mu^*)$ for various values of λ

λ	μ^*	$F(\mu^*)$
0.55	1.818215	102.873936
0.65	2.082011	115.988351
0.75	2.340349	128.746239
0.85	2.594235	141.207636
0.95	2.844401	153.416642

- Table 1 presents the minimum values of μ along with $F(\mu^*)$ for various λ. The other parameters are chosen as $K = 7, p = 0.75, \beta = 0.7, \eta = 3, \phi = 1.1, \alpha = 0.6$, and $\xi = 1.5$.
- Table 2 illustrates the optimum values of μ along with $F(\mu^*)$ for various values of K. The other parameters are taken as $\lambda = 1, p = 0.75, \beta = 0.7, \eta = 3, \phi = 1.1$, $\alpha = 0.6$, and $\xi = 1.5$.
- Table 3 displays the optimal values of μ along with $F(\mu^*)$ for different values of η. The other parameters are fixed as $K = 7$, $p = 0.75$, $\beta = 0.7$, $\lambda = 1$, $\phi = 1.1$, $\alpha = 0.6$, and $\xi = 1.5$.
- Table 4 illustrates the minimum values of μ along with $F(\mu^*)$ for various values of ϕ. The other parameters are taken as $K = 7$, $p = 0.75$, $\beta = 0.7$, $\lambda = 1, \eta = 3$, $\alpha = 0.6$, and $\xi = 1.5$.

It is worth noting that we have to choose the values for the two parameters in such a way that the stability condition $\lambda E(X) < \beta\mu$ is verified.

Tables 1-4 present the optimum values of μ along with the minimum expected cost for various values of λ, K, η, and ϕ,, respectively. A decreasing (resp. increasing) trend is seen in μ^* with the increase in η and K (resp. λ and ϕ). Further, the

Table 2 μ^* and $F(\mu^*)$ for various values of λ

K	μ^*	$F(\mu^*)$
1	2.989518	158.255259
3	2.972585	159.195156
5	2.969294	159.379618
7	2.968266	159.437363
9	2.967902	159.457829

Table 3 μ^* and $F(\mu^*)$ for various values of η

η	μ^*	$F(\mu^*)$
0.7	3.006193	157.345151
1.4	2.985068	158.50078
2.1	2.975202	159.048926
2.8	2.969484	159.368998
3.5	2.965751	159.578848

Table 4 μ^* and $F(\mu^*)$ for various values of ϕ

ϕ	μ^*	$F(\mu^*)$
0.6	2.904913	162.926958
1.2	2.975776	158.931596
1.8	3.005696	156.699431
2.4	3.022255	155.312738
3.0	3.032777	154.382524

optimal expected cost $F(\mu^*)$ increases with the increases of λ, η and K. This is because the mean number of the customers in the system as well as the average rate of lost customers increase with the increases of λ, η and K which results in the increasing of the optimal expected cost. In addition, $F(\mu^*)$ decreases with ϕ which is quite reasonable; the mean queue length and the average rate of reneging decrease which leads to the decreasing in the optimal expected cost.

The results obtained are quite interesting and can be applied to many real-time machining systems for upgrading the system by suitable choice of service rate μ.

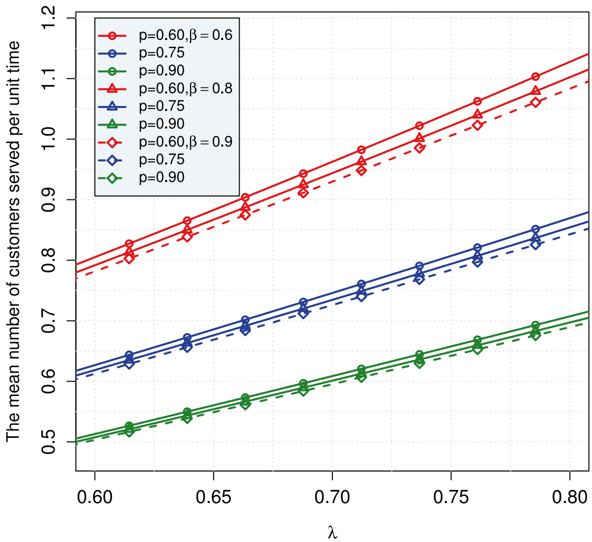
7.2 Impact of λ, p, and β

We check the effect of λ, p, and β on different performance measures and costs, the values of these parameters are presented in Table 5 and Figs. 1 and 2. The

Table 5 Impact of $\lambda, p,$ and β

λ	p	β	$E[L_V]$	$E[L_K]$	R_a	P_b	P_v	$P_{0,K}$	$\mathcal{T}_c$	$\mathcal{T}_r$
		0.7	0.3612	0.8683	0.1950	0.3286	0.5923	0.0791	154.4013	229.9915
	0.60	0.8	0.3858	0.6840	0.2083	0.2827	0.6327	0.0846	146.5769	197.9150
		0.9	0.4044	0.5629	0.2184	0.2481	0.6633	0.0886	139.7439	173.6907
		0.7	0.3209	0.5137	0.1733	0.2558	0.6578	0.0864	146.0160	179.0679
0.6	0.75	0.8	0.3359	0.4186	0.1814	0.2209	0.6886	0.0905	139.5682	154.6544
		0.9	0.3473	0.3530	0.1876	0.1944	0.7120	0.0936	133.5388	136.0991
		0.7	0.2846	0.3538	0.1537	0.2094	0.7000	0.0906	141.6145	146.5738
	0.9	0.8	0.2947	0.2939	0.1591	0.1813	0.7249	0.0938	135.6575	126.8854
		0.9	0.3024	0.2513	0.1633	0.1598	0.7439	0.0963	129.9293	111.8600
		0.7	0.3779	1.2075	0.2041	0.3929	0.5312	0.0759	161.3242	275.0287
	0.60	0.8	0.4126	0.9266	0.2228	0.3371	0.5802	0.0829	151.9645	235.9600
		0.9	0.4387	0.7493	0.2369	0.2952	0.6167	0.0881	144.3036	206.6104
		0.7	0.3471	0.6796	0.1874	0.3045	0.6099	0.0856	149.8402	213.1156
0.7	0.75	0.8	0.3681	0.5452	0.1988	0.2624	0.6468	0.0908	142.8413	183.6394
		0.9	0.3840	0.4547	0.2074	0.2304	0.6748	0.0948	136.4914	161.3262
		0.7	0.3131	0.4550	0.1691	0.2485	0.6602	0.0913	144.2319	173.9113
	0.9	0.8	0.3272	0.3740	0.1767	0.2147	0.6899	0.0954	138.0070	150.2730
		0.9	0.3379	0.3173	0.1825	0.1890	0.7125	0.0985	132.1152	132.2917
		0.7	0.3810	1.6623	0.2058	0.4602	0.4687	0.0711	170.1157	322.1654
	0.60	0.8	0.4280	1.2348	0.2311	0.3936	0.5265	0.0799	158.3052	275.5482
		0.9	0.4632	0.9783	0.2501	0.3439	0.5697	0.0864	149.4117	240.7166
		0.7	0.3651	0.8834	0.1972	0.3549	0.5614	0.0837	154.1822	248.4298
0.8	0.75	0.8	0.3933	0.6961	0.2124	0.3051	0.6047	0.0902	146.4034	213.5712
		0.9	0.4145	0.5735	0.2239	0.2676	0.6374	0.0950	139.6203	187.2912
		0.7	0.3362	0.5737	0.1815	0.2887	0.6202	0.0911	147.0579	202.1041
	0.9	0.8	0.3549	0.4660	0.1917	0.2490	0.6548	0.0962	140.4808	174.3076
		0.9	0.3692	0.3921	0.1993	0.2189	0.6810	0.1001	134.3817	153.2326

Fig. 1 Effect of $\lambda, p,$ and β on N_s



other parameters of the model are taken as $K = 5, \eta = 3, \phi = 1.1, \alpha = 0.6, \xi = 0.9,$ and $\mu = 3.5$.

– For fixed p and β, along the increases of λ, an increasing trend is observed in P_b, $E[L_V]$, $E[L_K]$, and R_a, while a decreasing trend is seen in $P_{0,K}$ and P_v with λ. This implies an increasing in $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_p$. The obtained results are reasonable; the mean number of the customers in the system increases with the increasing of λ. Thus, the larger the mean number of customers in the system, the higher the mean number of customers served.

Fig. 2 Effect of $\lambda, p,$ and β on $\mathcal{T}_p$

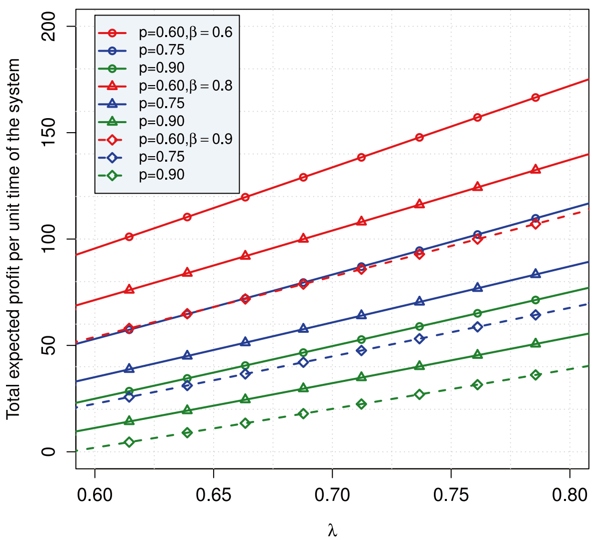


Table 6 Impact of K, ξ, and α

K	ξ	α	$E[L_K]$	$E[L_V]$	N_s	R_a	R_e	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$
		0.3	2.4102	0.2899	1.2638	0.0696	0.1624	163.7101	451.3396	287.6295
	0.8	0.6	2.2180	0.2625	1.2073	0.1260	0.0840	158.6905	431.1953	272.5049
		0.9	2.0845	0.2398	1.1607	0.1726	0.0192	155.0977	414.5361	259.4384
		0.3	2.2180	0.2625	1.2073	0.1260	0.2940	163.9399	431.1953	267.2554
1	1.6	0.6	1.9847	0.2207	1.1215	0.2119	0.1412	156.7723	400.5296	243.7573
		0.9	1.8426	0.1904	1.0592	0.2741	0.0305	152.2130	378.2885	226.0755
		0.3	2.0845	0.2398	1.1607	0.1726	0.4028	164.6884	414.5361	249.8477
	2.4	0.6	1.8426	0.1904	1.0592	0.2741	0.1828	156.0203	378.2885	222.2682
		0.9	1.7061	0.1578	0.9924	0.3409	0.0379	150.8832	354.4259	203.5427
		0.3	2.4259	0.3399	1.2518	0.0816	0.1903	165.4641	447.0590	281.5949
	0.8	0.6	2.1984	0.3134	1.1829	0.1504	0.1003	159.6924	422.4637	262.7714
		0.9	2.0342	0.2913	1.1236	0.2097	0.0233	155.4353	401.2820	245.8467
		0.3	2.1984	0.3134	1.1829	0.1504	0.3510	165.9605	422.4637	256.5032
4	1.6	0.6	1.9062	0.2725	1.0717	0.2616	0.1744	157.5341	382.7672	225.2331
		0.9	1.7138	0.2418	0.9851	0.3482	0.0387	151.8575	351.8160	199.9585
		0.3	2.0342	0.2913	1.1236	0.2097	0.4894	167.0877	401.2820	234.1943
	2.4	0.6	1.7138	0.2418	0.9851	0.3482	0.2322	156.6943	351.8160	195.1217
		0.9	1.5136	0.2074	0.8853	0.4481	0.0498	150.0523	316.1646	166.1123
		0.3	2.4273	0.3443	1.2507	0.0826	0.1928	165.6206	446.6771	281.0565
	0.8	0.6	2.1964	0.3184	1.1805	0.1529	0.1019	159.7915	421.5995	261.8079
		0.9	2.0287	0.2969	1.1195	0.2138	0.0238	155.4721	399.8358	244.3637
		0.3	2.1964	0.3184	1.1805	0.1529	0.3567	166.1605	421.5995	255.4390
7	1.6	0.6	1.8969	0.2787	1.0658	0.2675	0.1783	157.6248	380.6526	223.0278
		0.9	1.6961	0.2489	0.9749	0.3584	0.0398	151.8088	348.1880	196.3792
		0.3	2.0287	0.2969	1.1195	0.2138	0.4989	167.3495	399.8358	232.4863
	2.4	0.6	1.6961	0.2489	0.9749	0.3584	0.2389	156.7867	348.1880	191.4013
		0.9	1.4831	0.2153	0.8683	0.4650	0.0517	149.9207	310.1067	160.1860

- For fixed λ and β, with the increases of p, an increasing trend is remarked in P_v and $P_{0,K}$. Further, a decreasing trend is observed in P_b, $E[L_V]$, $E[L_K]$, and R_a with p. This implies a diminution in $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_p$. This is because the mean number of customers in the system decreases with p. Thus, the mean number of customers served is reduced.
- For fixed λ and p, along the increases of β, we observe an increasing trend in $P_{0,K}$, P_v, $E[L_V]$, and R_a. In addition, a decreasing trend is seen in P_b and $E[L_K]$ with β. This leads to a decrease in $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_p$.
- For fixed p and β, along the increase in λ, N_s monotonically increases , as it should be. Moreover, one may also see that for higher values of β and smaller values p, N_s is reduced. Therefore, $\mathcal{T}_p$ decreases.

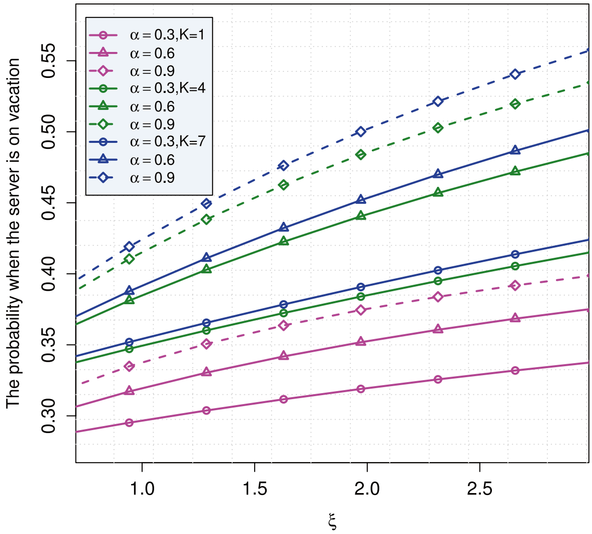


Fig. 3 Effect of K, ξ, and α on P_v

7.3 Impact of K, ξ, and α

We vary K, ξ, and α, their values are given in the respective Table 6 and Figs. 3 and 4. The other default parameters are chosen as $p = 0.75$, $\beta = 0.7$, $\lambda = 1$, $\eta = 3$, $\phi = 1.1$, and $\mu = 3$.

– For fixed α, increases in K and ξ implies an increase in R_a, R_e, and $\mathcal{T}_c$, while $E[L_V]$ increases with K and decreases with ξ. The other performance measures and costs, $E[L_K]$, N_s, $\mathcal{T}_r$, and $\mathcal{T}_p$ decrease with the increasing of K and ξ.

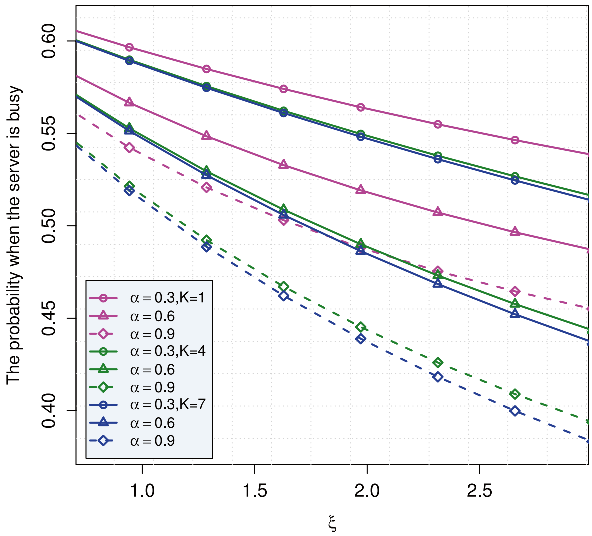


Fig. 4 Effect of K, ξ, and α on P_b

Table 7 Impact of μ, ϕ, and η

μ	ϕ	η	P_b	P_v	$P_{0,K}$	$E[L]$	$E[L_q]$	$\mathcal{T}_p$
		2.0	0.9368	0.0385	0.0247	19.6869	18.7270	377.7967
	2.0	2.5	0.9368	0.0418	0.0214	19.7672	18.8054	376.1619
		3.0	0.9367	0.0444	0.0189	19.8288	18.8656	374.9070
		2.0	0.9370	0.0360	0.0270	19.4320	18.4755	383.0391
5.0	2.5	2.5	0.9369	0.0394	0.0237	19.4966	18.5383	381.7286
		3.0	0.9369	0.0420	0.0211	19.5470	18.5872	380.7077
		2.0	0.9371	0.0339	0.0290	19.2690	18.3150	386.3820
	3.0	2.5	0.9370	0.0374	0.0256	19.3221	18.3665	385.3081
		3.0	0.9370	0.0401	0.0229	19.3640	18.4070	384.4610
		2.0	0.8667	0.0813	0.0520	9.1744	8.2590	573.7768
	2.0	2.5	0.8665	0.0883	0.0452	9.2558	8.3364	572.0877
		3.0	0.8665	0.0936	0.0399	9.3182	8.3957	570.7913
		2.0	0.8670	0.0759	0.0571	8.9177	8.0094	579.2146
5.4	2.5	2.5	0.8669	0.0831	0.0500	8.9834	8.0713	577.8618
		3.0	0.8668	0.0887	0.0445	9.0346	8.1195	576.8080
		2.0	0.8672	0.0716	0.0612	8.7536	7.8506	582.6724
	3.0	2.5	0.8671	0.0789	0.0540	8.8078	7.8506	581.5640
		3.0	0.8671	0.0846	0.0483	8.8505	7.9412	580.6899
		2.0	0.8063	0.1181	0.0756	6.2294	5.3524	618.5547
	2.0	2.5	0.8062	0.1282	0.0656	6.3117	5.4288	616.8190
		3.0	0.8060	0.1360	0.0580	6.3749	5.4875	615.4870
		2.0	0.8068	0.1103	0.0829	5.9713	5.1044	624.1603
5.8	2.5	2.5	0.8067	0.1207	0.0726	6.0379	5.1655	622.7712
		3.0	0.8065	0.1289	0.0646	6.0898	5.2131	621.6892
		2.0	0.8070	0.1040	0.0890	5.8061	4.9472	627.7167
	3.0	2.5	0.8070	0.1146	0.0784	5.8612	4.9971	626.5789
		3.0	0.8069	0.1230	0.0701	5.9047	5.0365	625.6816

– For fixed K and ξ, except R_a, all the other performance measures and costs decrease with the increase in α. Moreover, as intuitively expected, P_v (resp. P_b) increases (resp. decreases) with ξ. In addition, the higher α and K, the smaller the probability that the server is working during the busy period P_b and the bigger the probability that the server is on vacation P_v. This is due to the fact that, the number of customers in the system decreases with α and ξ, and increase with K. Thus, the probability of busy period decreases which implies a decrease in the mean number of customers served. Further, one can conclude that the retention probability α' has a nice impact on the economy of the queueing system.

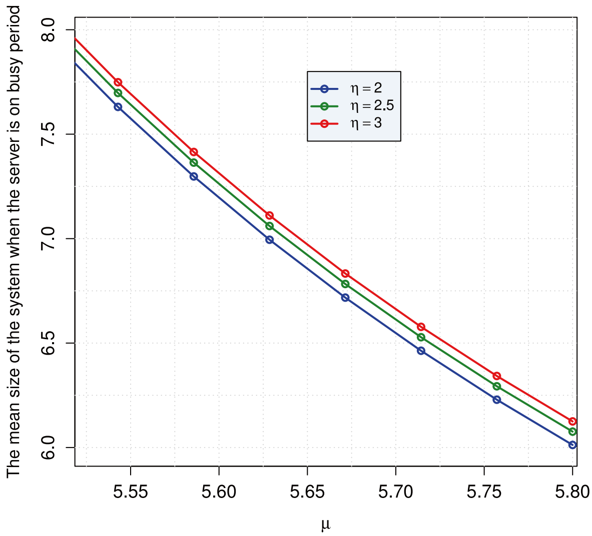


Fig. 5 $E[L_K]$ vs. μ and η

7.4 Impact of μ, ϕ, and η.

We examine the impact of μ, ϕ, and η, their values are mentioned in Table 7 and Fig. 5. The other model parameters are arbitrarily selected as $\lambda = 3$, $\beta = 0.8$, $p = 0.8$, $K = 3$, $\xi = 0.1$, $\alpha = 0.4$, $\eta = 3$, and $\phi = 2$.

- As we expect, for fixed ϕ and η, the performance measures P_v, and $P_{0,K}$ increase with the increase in μ, while P_b, $E[L_K]$, $E[L]$ and $E[L_q]$ monotonically decrease. Therefore, $\mathcal{T}_p$ increases because of the number of customers served.
- For fixed μ and η, it is depicted that P_b and $P_{0,K}$ increase with ϕ, while P_v, $E[L]$, and $E[L_q]$ decrease with the increasing values of ϕ. Thus, $\mathcal{T}_p$ increases significantly. This trend matches with the realistic situation.
- For fixed μ and ϕ, with the increasing of η, an increasing trend is observed in P_v, $E[L]$, and $E[L_q]$ and a decreasing trend is seen in P_b, $E[L_K]$, $P_{0,K}$, and $\mathcal{T}_p$. This is because the number of customers during the vacation period increases with η. Hence, the number of customers reneged augments which results in the increasing of the total expected profit.

8 Conclusion

This study focused on the analysis of an infinite capacity batch arrival single server Markovian Bernoulli feedback queueing system subject to functioning K-variant vacations by including the assumption of waiting server, reneging, and retention of reneged customers. The steady-state study of the system was presented, using the PGFs method, to evaluate various system metrics in terms of steady-state probabilities. We also considered a cost optimization problem using particle swarm optimization (PSO). Further, we investigated the effect of different parameters on the performance measures and the cost functions of the system through numerical experiments. Our queueing system may be considered as a generalized version of some existing queueing models given by Padmavathy et al. [20] and Ye et al. [28]. The model studied can be further extended to a more general case with general type service times and lead times. Furthermore, the realistic feature of bulk failure can be included.

Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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