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2-Rotund norms for unconditional and symmetric sequence spaces

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Abstract
A reflexive Banach space with an unconditional basis admits an equivalent 1-unconditional $2R$ norm and embeds into a reflexive space with a 1-symmetric $2R$ norm. Partial results on 1-symmetric $2R$ renormings of spaces with a symmetric basis are obtained.

Keywords 2-Rotundity · Renorming · $2R$ norm · Unconditional basis · Symmetric basis · Reflexivity

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1 Introduction

The notions of 2-rotund and weakly 2-rotund norms were introduced by Milman [22] and are defined as follows.

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Definition 1.1 Let X be a Banach space. We say that a norm $\|\cdot\|$ on X is 2-rotund $(2R)$ (resp. weakly 2-rotund $(W2R)$) if for every $(x_n) \subset X$ such that $\|x_n\| \leq 1$ $(n \geq 1)$ and

$$\lim_{m,n\to\infty} \|x_m + x_n\| = 2,$$

there exists $x \in X$ such that $x = \lim_{n\to\infty} x_n$ strongly (resp. weakly).

Note that a $W2R$ norm is strictly convex. It follows from a characterization of reflexivity due to James [17] that if X admits an equivalent $W2R$ norm then X is reflexive. Hájek and Johanis [15] proved the converse: every reflexive Banach space admits an equivalent $W2R$ norm. Odell and Schlumprecht [23] proved that every separable reflexive Banach space X admits an equivalent $2R$ norm (cf. [13]). However, it is an open question whether every reflexive Banach space admits an equivalent $2R$ norm (cf. [15, p. 72]).

One motivation for the present article is the following result of Figiel and Johnson which combines Theorem 3.1 and Remark 3.2 of [12]. Only the unconditional case is stated explicitly in [12], but the argument for the unconditional case also proves the symmetric case.

Theorem A *Let X be a superreflexive Banach space with an unconditional (respectively, symmetric) basis $(e_n)_{n=1}^{\infty}$. Then X admits an equivalent uniformly convex norm for which $(e_n)_{n=1}^{\infty}$ is 1-unconditional (respectively, 1-symmetric).*

Enflo [11] showed that a space is superreflexive if and only if it admits an equivalent uniformly convex norm. By the theorem of Odell and Schlumprecht above a separable space is reflexive if and only if it admits an equivalent $2R$ norm. Therefore it is natural to ask whether the analogue of Theorem A holds for $2R$ renormings of separable reflexive spaces.

In Sect. 3 we prove the analogous result in the unconditional case: a reflexive space with an unconditional basis admits a 1-unconditional $2R$ norm. For the symmetric case, however, we have only partial results. In particular, the following question is open.

Question 1.2 *Let X be a reflexive Banach space with a symmetric basis $(e_n)_{n=1}^{\infty}$. Does X admit an equivalent $2R$ norm for which $(e_n)_{n=1}^{\infty}$ is 1-symmetric?*

We show that the answer is positive if the lower Boyd index p_X of X satisfies $p_X > 1$. We also prove that if X is a reflexive space with an unconditional basis then X is isomorphic to a 1-complemented subspace of a space with a $2R$ norm and a 1-symmetric basis. This is a refinement of a theorem of Szankowski [24]. A similar argument proves that the non-superreflexive space with a symmetric basis which does not contain c_0 or ℓ_p constructed in [12] and the space with a unique symmetric basic sequence (not equivalent to the unit vector basis of c_0 or ℓ_p) constructed by Altshuler [1] both admit an equivalent $2R$ norm for which the basis is 1-symmetric.

A second motivation is the open question whether nonseparable reflexive spaces admit a $2R$ norm. To attack this problem it is natural to examine specific classes of nonseparable reflexive spaces for potential counterexamples or positive results. In [9] partial positive results were obtained for nonseparable generalized Baernstein spaces. Another natural class to examine is the class of spaces with an uncountable symmetric basis.

Question 1.3 *Suppose X is a reflexive space with an uncountable symmetric basis $(e_\gamma)_{\gamma\in\Gamma}$. Does X admit an equivalent (not necessarily 1-symmetric) $2R$ norm?*

A positive answer to Quesion 1.2 would imply a positive answer to Question 1.3. In fact, it would imply that X admits a 1-symmetric $2R$ norm.

In the final section we show that $L_\infty[0,1]$ admits an equivalent rearrangement-invariant norm which restricts to a $W2R$ norm on every reflexive subspace.

Finally, let us mention that related results are proved in [14, 16]. In [14] it is proved that a uniformly smooth (resp. uniformly convex) space with a Schauder basis admits a uniformly smooth (resp. uniformly convex) renorming for which the basis is monotone, while in [16] the spaces with a symmetric basis which admit equivalent symmetric norms that are Gâteaux differentiable or uniformly rotund in every direction are characterized.

2 Preliminary results

We shall use the following characterization of 2-rotundity (see e.g., [8, II.6.4] or [15]): $\|\cdot\|$ is a $2R$ norm on X if for all $(x_n)_{n=1}^\infty \subset X$ such that

$$\lim_{m,n\to\infty}[\|x_m+x_n\|^2-2(\|x_m\|^2+\|x_n\|^2)]=0, \tag{2.1}$$

there exists $x\in X$ such that $x=\lim_{n\to\infty}x_n$ strongly.

Day [7] introduced the norm $\|\cdot\|_{\mathrm{Day}}$ on c_0 defined by

$$\|(a_n)_{n=1}^\infty\|_{\mathrm{Day}}=\left(\sum_{n=1}^\infty 4^{-n}a_n^{*2}\right)^{1/2},$$

where $(a_n^*)_{n=1}^\infty$ is the non-increasing rearrangement of $(|a_n|)_{n=1}^\infty$. Let $(Y,\|\cdot\|)$ be a reflexive Banach space with normalized basis $(e_n)_{n=1}^\infty$. We define an equivalent norm on Y thus:

$$\left|\left|\left|\sum_{n=1}^\infty a_ne_n\right|\right|\right|=\left(\left\|\sum_{n=1}^\infty a_ne_n\right\|^2+\|(a_n)_{n=1}^\infty\|_{\mathrm{Day}}^2\right)^{1/2}. \tag{2.2}$$

We will use the following result of Hájek and Johanis. It is a consequence of Theorem 3 and Corollary 4 of [15] and the reflexivity of Y. (Here $\|\sum_{n=1}^\infty a_ne_n\|_\infty=\sup_{n\ge1}|a_n|$ as usual.)

Theorem B *Suppose* $(y_n)_{n=1}^{\infty} \subset Y$ *satisfies*

$$\lim_{m,n\to\infty} [\||y_n + y_m\||^2 - 2(\||y_n\||^2 + \||y_m\||^2)] = 0. \tag{2.3}$$

Then there exists $y \in Y$ *such that*

$$y_n \to y \quad \textit{weakly as } n \to \infty$$

and

$$\lim_{n\to\infty} \|y_n - y\|_\infty = 0.$$

For $K \geq 1$, a basis $(e_n)_{n=1}^{\infty}$ is K-unconditional if

$$\left\| \sum_{n=1}^{\infty} \pm a_n e_n \right\| \leq K \left\| \sum_{n=1}^{\infty} a_n e_n \right\|$$

for all scalars $(a_n)_{n=1}^{\infty}$ and all choices of signs. The basis is K-symmetric if

$$\left\| \sum_{n=1}^{\infty} \pm a_{\sigma(n)} e_n \right\| \leq K \left\| \sum_{n=1}^{\infty} a_n e_n \right\|$$

for all scalars $(a_n)_{n=1}^{\infty}$, all choices of signs, and all permutations $\sigma : \mathbb{N} \to \mathbb{N}$.

We refer the reader to [20] for other unexplained Banach space notation and terminology.

3 1-Unconditional bases

Theorem 3.1 *Suppose that X has an unconditional basis. Then X admits an equivalent* 1*-unconditional norm* $\||\cdot\||$ *such that if* $(x_n)_{n=1}^{\infty} \subset X$ *is relatively weakly compact and satisifies* (2.1)*, then* (x_n) *converges strongly. In particular, if X is reflexive, then* $\||\cdot\||$ *is* $2R$ *and* 1*-unconditional.*

Proof The proof closely follows [23, Main Theorem]. So it suffices to indicate how to adapt [23, Main Theorem] to produce a 1-unconditional basis as well as a $2R$ norm.

Let $(e_n)_{n=1}^{\infty}$ be a semi-normalized unconditional basis for X and let $\|\cdot\|$ denote any equivalent norm on X which is strictly convex and for which $(e_n)_{n=1}^{\infty}$ is 1-unconditional. To see that such a norm exists, let $|\cdot|$ be any equivalent norm on X. Let

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\| := \sup \left| \sum_{n=1}^{\infty} \pm a_n e_n \right| + \left(\sum_{n=1}^{\infty} 2^{-4n} a_n^2 \right)^{1/2},$$

where the supremum is taken over all choices of signs. Then $\|\cdot\|$ is strictly convex and $(e_n)_{n=1}^{\infty}$ is a 1-unconditional basis for $(X, \|\cdot\|)$. For $x \in X$, following [23, p. 148], define an equivalent norm $\|\cdot\|_x$ thus:

$$\|y\|_x := \|\|y\|x + y\| + \|\|y\|x - y\| \quad (y \in X).$$

Then [23, Lemma 2.1]

$$2\|y\| \leq \|y\|_x \leq (2 + 2\|x\|)\|y\|. \tag{3.1}$$

Let C be the countable vector space over $\mathbb{Q}$ defined by

$$C := \left\{\sum a_n e_n : (a_n) \in c_{00}, a_n \in \mathbb{Q}, n \geq 1\right\}.$$

Let us say that $c = \sum a_n e_n \in C$ and $d = \sum b_n e_n \in C$ are absolutely equivalent if $|a_n| = |b_n|$ for all $n \geq 1$. Note that absolute equivalence is an equivalence relation on C and that the equivalence classes are finite. Let $\mathcal{A}$ be the collection of equivalence classes. For all $A \in \mathcal{A}$ and for all absolutely equivalent $y, z \in C$, by 1 unconditionality of $\|\cdot\|$ and absolute equivalence of y and z, we have

$$\begin{aligned}\sum_{c\in A} \|y\|_c &= \sum_{c\in A}(\|\|y\|c + y\| + \|\|y\|c - y\|)\\ &= \sum_{c\in A}(\|\|z\|c + z\| + \|\|z\|c - z\|)\\ &= \sum_{c\in A} \|z\|_c.\end{aligned} \tag{3.2}$$

Choose $p_A > 0$ $(A \in \mathcal{A})$ such that $\sum_{A\in\mathcal{A}} p_A(1 + \sum_{c\in A} \|c\|) < \infty$. Define a norm $|||\cdot|||$ on X as follows:

$$|||x||| = \sum_{A\in\mathcal{A}} p_A \sum_{c\in A} \|x\|_c.$$

It follows from (3.1) that $|||\cdot|||$ is an equivalent norm. Note that $|||\cdot|||$ is strictly convex since $\|\cdot\|_0 = 2\|\cdot\|$ which is strictly convex. Suppose $y, z \in C$ are absolutely equivalent. Then (3.2) implies that

$$|||y||| = \sum_{A\in\mathcal{A}} p_A \sum_{c\in A} \|y\|_c = \sum_{A\in\mathcal{A}} p_A \sum_{c\in A} \|z\|_c = |||z|||.$$

Since C is dense in X, it follows that (e_n) is a 1-unconditional basis for $(X, |||\cdot|||)$. The proof of [23, Main Theorem], especially Lemmas 2.2(a), 2.3(a), and 2.4, now shows that $\|\cdot\|_M$ is $2R$. □

Remark 3.2 We are unable to adapt the proof of Theorem 3.1 to the case of a symmetric basis. The natural approach would be to say that vectors from C are equivalent if their coefficient sequences are permuted. However, in this case the equivalence classes are infinite, so the proof does not go through.

Lindenstrauss [19] proved that every space X with an unconditional basis is isomorphic to a complemented subspace of a space Y with a symmetric basis. Subsequently, Szankowski [24] proved that if X is reflexive then Y can be chosen to be reflexive and Davis [5] proved that if X is superreflexive then Y can be chosen to be superreflexive. Davis's method also proves the reflexive case. As an application of Theorem 3.1 we use Davis's method to prove the following refinement of Szankowski's result.

Theorem 3.3 *Suppose that X is reflexive and has an unconditional basis. Then X is isomorphic to a 1-complemented subspace of a space with a 1-symmetric basis and a $2R$ norm. Moreover, that subspace has a 1-unconditional basis.*

We recall the presentation of Davis's approach in [20, p. 125]. Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be two Banach sequence spaces such that $(e_n)_{n=1}^{\infty}$ is a 1-symmetric basis for both E and F. We assume also that $\|x\|_E \leq \|x\|_F$ for all $x \in F$ and that

$$\lim_{n\to\infty} \frac{\|\sum_{i=1}^{n} e_i\|_E}{\|\sum_{i=1}^{n} e_i\|_F} = 0.$$

For each $m \geq 1$, define a 1-symmetric norm $\|\cdot\|_m$ on E as follows:

$$\|x\|_m = \inf\left\{(\|y\|_E^2 + \|z\|_F^2)^{1/2} \colon x = \frac{1}{m}y + mz,\ y \in E, z \in F\right\}.$$

Then [20, p. 125]

$$\frac{1}{m}\|x\|_m \leq \|x\|_E \leq 2m\|x\|_m \quad (x \in E). \tag{3.3}$$

Hence $\|\cdot\|_m$ is equivalent to $\|\cdot\|_E$.

Now suppose that X is a Banach space with a normalized 1-unconditional basis $(f_n)_{n=1}^{\infty}$. For every strictly increasing sequence $(m_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} 1/m_n < \infty$, we define the space $Y := Y(E, F, X, (m_n)_{n=1}^{\infty})$ to be the collection of all $x \in E$ for which

$$\|x\|_Y := \left\|\sum_{n=1}^{\infty} \|x\|_{m_n} f_n\right\|_X < \infty. \tag{3.4}$$

Then (e_n) is a 1-symmetric basis for Y. (The condition $\sum_{n=1}^{\infty} 1/m_n < \infty$ guarantees that F embeds continuously into Y and, in particular, that $(e_n)_{n=1}^{\infty} \subset Y$ [20, p. 126].)

Theorem C [20, Prop. 3.b.4] *For every E, F and X as above there exists an increasing sequence of numbers $(m_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} 1/m_n < \infty$ such that $Y = Y(E, F, X, (m_n)_{n=1}^{\infty})$ contains a complemented subspace isomorphic to X.*

The following lemma generalizes [20, Lemma 3.b.11].

Lemma 3.4 *Suppose that $E = c_0$ and $\sum_{n=1}^{\infty} 1/m_n < \infty$. Let*

$$u_n = \sum_{i=q_n+1}^{q_{n+1}} c_i e_i \quad (m \geq 1)$$

be a normalized block basis (with respect to $(e_n)_{n=1}^{\infty}$) in $Y(c_0, F, X, (m_n)_{n=1}^{\infty})$ such that $\lim_{i\to\infty} c_i = 0$. Then some subsequence of $(u_n)_{n=1}^{\infty}$ is equivalent to a block basis of $(f_n)_{n=1}^{\infty}$ in X.

Proof Fix $N \geq 1$ and $n \geq 1$. It follows from (3.3) that

$$\sum_{i=1}^{N} \|u_n\|_{m_i} \leq \left(\sum_{i=1}^{N} m_i\right) \|u_n\|_E = \left(\sum_{i=1}^{N} m_i\right) \max\{|c_i|\colon q_n + 1 \leq i \leq q_{n+1}\}.$$

Since $\lim_{i\to\infty} c_i = 0$, we can inductively define an increasing sequence of natural numbers $1 = N_1 < N_2 < \cdots$ and a subsequence $(u_{n_k})_{k=1}^{\infty}$ such that for all $k \geq 1$

$$\left\| \sum_{i=1}^{N_k} \|u_{n_k}\right\|_{m_i} f_i \|_X + \left\| \sum_{N_{k+1}+1}^{\infty} \right\| u_{n_k} \|_{m_i} f_i \|_X < 2^{-k-1}.$$

It follows that the block basis $(u_{n_k})_{k=1}^{\infty} \subset Y$ is equivalent to the block basis

$$\left(\sum_{i=N_k+1}^{N_{k+1}} \|u_{n_k}\right\|_{m_i} f_i)_{k=1}^{\infty} \subset X.$$

□

The next lemma is more general than is needed for the proof of Theorem 3.3, but we believe that the additional generality may be of independent interest.

Lemma 3.5 *Suppose that $E = c_0$ and X is reflexive. If $\sum_{n=1}^{\infty} 1/m_n < \infty$ then $Y(c_0, F, X, (m_n)_{n=1}^{\infty})$ is reflexive.*

Proof Since Y has a symmetric (hence unconditional) basis it follows from a result of James [18] that Y is reflexive unless Y contains a subspace isomorphic to c_0 or ℓ_1. We use the fact that every (infinite-dimensional) subspace of Y contains a further subspace isomorphic to a subspace of X or to a subspace of E (see [20, p. 127]). Since every subspace of ℓ_1 contains a further subspace isomorphic to ℓ_1, and since neither c_0 nor X contain a subspace isomorphic to ℓ_1, it follows that Y does not contain a subspace isomorphic to ℓ_1. Suppose, to obtain a contradiction, that Y contains a sequence equivalent to the unit vector basis of c_0. Since the unit vector basis of c_0

is weakly null, a standard gliding hump argument shows that $(e_n)_{n=1}^{\infty}$ admits a block basis

$$u_n = \sum_{i=q_n+1}^{q_{n+1}} c_i e_i$$

equivalent to the unit vector basis of c_0. By the proof of [20, Lemma 3.b.3],

$$\max_{n\geq 1} \left\| \sum_{i=1}^{n} e_i \right\|_m \to \infty \quad \text{as } m \to \infty.$$

So, using (3.4), $\| \sum_{i=1}^{n} e_i \|_Y \to \infty$ as $n \to \infty$. It follows from the unconditionality of $(e_n)_{n=1}^{\infty}$ and the fact that $\sup_{n\geq 1} \| \sum_{i=1}^{n} u_i \|_Y < \infty$ that $\lim_{i\to\infty} c_i = 0$. By Lemma 3.4, some subsequence of $(u_n)_{n=1}^{\infty}$ is equivalent to a block basis of $(f_n)_{n=1}^{\infty} \subset X$. Hence c_0 is isomorphic to a subspace of X, which contradicts the reflexivity of X. □

Remark 3.6 Reflexivity of $Y(c_0, \ell_1, X, (2^n)_{n=1}^{\infty})$ was proved in [12] using results from [6].

Proof (Proof of Theorem 3.3) By Theorem 3.1, X has an equivalent $2R$ norm $\| \cdot \|_X$ for which $(f_n)_{n=1}^{\infty}$ is a 1-unconditional basis. Let $Y := Y(c_0, F, X, (m_n)_{n=1}^{\infty})$ be as in Theorem C. We equip Y with the equivalent norm defined by (2.2).

Suppose $(y_n)_{n=1}^{\infty}$ satisfies (2.3). Since Y is reflexive, by Theorem B there exists $y \in Y$ such that $y_n \to y$ weakly and $\|y_n - y\|_{\infty} \to 0$ as $n \to \infty$. Moreover, (2.3) implies that

$$\lim_{m,n\to\infty} [\|y_n + y_m\|_Y^2 - 2(\|y_n\|_Y^2 + \|y_m\|_Y^2)] = 0. \tag{3.5}$$

Let $x_n = \sum_{i=1}^{\infty} \|y_n\|_{m_i} f_i$ $(n \geq 1)$. By definition of $\| \cdot \|_Y$, $\|x_n\|_X = \|y_n\|_Y$. Moreover, by 1-unconditionality of the basis $(f_n)_{n=1}^{\infty}$ of X,

$$\begin{aligned} \|x_n + x_k\|_X &= \left\| \sum_{i=1}^{\infty} (\|y_n\|_{m_i} + \|y_k\|_{m_i}) f_i \right\|_X \\ &\geq \left\| \sum_{i=1}^{\infty} \|y_n + y_k\|_{m_i} f_i \right\|_X \\ &= \|y_n + y_k\|_Y. \end{aligned}$$

Hence (3.5) implies that

$$\lim_{m,n\to\infty} [\|x_n + x_m\|_X^2 - 2(\|x_n\|_X^2 + \|x_m\|_X^2)] = 0.$$

Since $\|\cdot\|_X$ is $2R$, it follows that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X. Hence, given $\varepsilon > 0$, there exists $N_1 \geq 1$ such that

$$\left\| \sum_{i=N_1+1}^{\infty} \|y_n\|_{m_i} f_i \right\|_X < \frac{\varepsilon}{4} \quad (n \geq 1).$$

Hence

$$\left\| \sum_{i=N_1+1}^{\infty} \|y_n - y_k\|_{m_i} f_i \right\|_X < \frac{\varepsilon}{2} \quad (n, k \geq 1). \tag{3.6}$$

Recall that, by (3.3), $\|\cdot\|_m$ is equivalent to $\|\cdot\|_\infty$ for all $m \geq 1$. Since $\lim_{n\to\infty} \|y_n - y\|_\infty = 0$, it follows that $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\|\cdot\|_m$ for all $m \geq 1$. Hence there exists $N_2 \geq 1$ such that

$$\left\| \sum_{i=1}^{N_1} \|y_n - y_k\|_{m_i} f_i \right\|_X < \frac{\varepsilon}{2} \quad (n, k \geq N_2). \tag{3.7}$$

Combining (3.6) and (3.7), we have $\|y_n - y_k\|_Y < \varepsilon$ for all $n, k \geq N_2$. So $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in Y and hence $\lim_{n\to\infty} \|y_n - y\|_Y = 0$.

The proof of Theorem C shows that X is isomorphic to the closed linear span Z of disjointly supported constant coefficient vectors in Y. Hence Z has a 1-unconditional basis and is the range of an averaging projection on Y. So Z is 1-complemented in Y. □

Let T be the space introduced in [12] (the dual of the space that does not contain c_0 or ℓ_p constructed by Tsirelson [4]). It was proved in [12] that $Y(c_0, \ell_1, T, (2^n)_{n=1}^{\infty})$ does not contain a subspace isomorphic to c_0 or ℓ_p.

Let $d_{w,1}$ be the Lorentz sequence space corresponding to the weight sequence $w = (1/n)$. The norm in $d_{w,1}$ is given by

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\| = \sum_{n=1}^{\infty} \frac{a_n^*}{n}.$$

It was proved by Altshuler [1] that $Y(c_0, d_{w,1}, T, (2^n)_{n=1}^{\infty})$ has a unique symmetric basic sequence which, moreover, is not equivalent to the unit vector basis of c_0 or ℓ_p.

The proof of Theorem 3.3 also establishes the following result.

Theorem 3.7 *The spaces $Y(c_0, \ell_1, T, (2^n)_{n=1}^{\infty})$ of [12] and $Y(c_0, d_{w,1}, T, (2^n)_{n=1}^{\infty})$ of [1] both have equivalent $2R$ norms with a 1-symmetric basis.*

4 1-Symmetric bases

In this section $(X, \|\cdot\|)$ denotes a reflexive Banach space with a symmetric basis $(e_n)_{n=1}^{\infty}$.

Let us recall the definition of the lower Boyd index [3] p_X of X (cf. [21, p. 130]). For $m \in \mathbb{N}$, the linear operator $D_m\colon X \to X$ is defined by

$$D_m\left(\sum_{n=1}^{\infty} a_n e_n\right) = \sum_{n=1}^{\infty} a_n \left(\sum_{j=(n-1)m+1}^{nm} e_j\right).$$

The lower Boyd index p_X is defined by

$$p_X = \sup_{m\geq 2} \frac{\log m}{\log \|D_m\|} = \lim_{m\to\infty} \frac{\log m}{\log \|D_m\|}. \tag{4.1}$$

The following main result of this section is an immediate consequence of Theorem 4.7 proved below.

Theorem 4.1 *Suppose that X is a reflexive Banach space with a symmetric basis such that $p_X > 1$. Then X admits a 1-symmetric $2R$ norm.*

[21, Prop. 2.b.7], which characterizes when $p_X > 1$, yields a geometrical formulation of Theorem 4.1.

Corollary 4.2 *Suppose that X is a reflexive Banach space with a symmetric basis which does not admit uniformly isomorphic copies of ℓ_1^n spanned by disjointly supported vectors with the same distribution. Then X admits a 1-symmetric $2R$ norm.*

For $x = \sum_{i=1}^{\infty} x(i)e_i \in X$, define the formal series

$$\widehat{x} := \sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{i=1}^{n} x^*(i)\right) e_n.$$

We prove the following lemma for the sake of completeness. More general results in the setting of rearrangement-invariant function spaces rather than symmetric sequence spaces are proved in [2, Theorem 5.15].

Lemma 4.3 *Suppose that $p_X > 1$. Then there exists a constant $c > 0$ such that*

$$\|\widehat{x}\| \leq c\|x\| \quad (x \in X).$$

Proof We may assume that $(e_n)_{n=1}^{\infty}$ is a 1-symmetric basis of X. Let $1 < p < p_X$. It follows from (4.1) that there exists $A > 0$ such that $\|D_m\| \leq Am^{1/p}$ for all $m \geq 1$. Consider $x = \sum_{n=1}^{\infty} x(n)e_n \in X$, where $(x(n))_{n=1}^{\infty}$ is a nonnegative decreasing sequence. Define $f\colon (0,\infty) \to (0,\infty)$ by $f(t) = x(n)$ for $n \geq 1$ and $n-1 < t \leq n$.

Then

$$\widehat{x} = \sum_{n=1}^{\infty} \left(\int_0^1 f(tn)\, dt \right) e_n = \int_0^1 \left(\sum_{n=1}^{\infty} f(tn) e_n \right) dt$$

Hence

$$\begin{aligned} \|\widehat{x}\| &\le \int_0^1 \left\| \sum_{n=1}^{\infty} f(tn) e_n \right\| dt \\ &\le \sum_{m=1}^{\infty} 2^{-m} \left\| \sum_{n=1}^{\infty} f(2^{-m} n) e_n \right\| \end{aligned}$$

(by 1-unconditionality)

$$\begin{aligned} &= \sum_{m=1}^{\infty} 2^{-m} \| D_{2^m}(x) \| \\ &\le \sum_{m=1}^{\infty} 2^{-m} (A 2^{m/p}) \|x\| \\ &= \frac{A}{2^{1-1/p} - 1} \|x\|. \end{aligned}$$

□

Henceforth, we suppose that $p_X > 1$, and, using Theorem 3.1, that $\|\cdot\|$ is $2R$, and that $(e_n)_{n=1}^{\infty}$ is a symmetric 1-unconditional basis for $\|\cdot\|$. Suppose that $(e_n)_{n=1}^{\infty}$ is K-symmetric for $\|\cdot\|$. Define a quasi-norm $|||\cdot|||$ as follows:

$$|||x||| = (\|\widehat{x}\|^2 + \|(x(n))_{n=1}^{\infty}\|_{\mathrm{Day}}^2)^{1/2} \quad \left(x = \sum_{n=1}^{\infty} x(n) e_n \in X \right). \tag{4.2}$$

Lemma 4.4 *$|||\cdot|||$ is a 1-symmetric equivalent norm on X.*

Proof Clearly, $|||\cdot|||$ is a 1-symmetric quasi-norm since $\widehat{x}$ depends only on $(x^*(n))_{n=1}^{\infty}$ and $\|\cdot\|_{\mathrm{Day}}$ is 1-symmetric. For $x \in X$,

$$\begin{aligned} \frac{1}{K} \|x\| &= \frac{1}{K} \left\| \sum_{n=1}^{\infty} x(n) e_n \right\| \\ &\le \left\| \sum_{n=1}^{\infty} x^*(n) e_n \right\| \end{aligned}$$

(since $(e_n)_{n=1}^{\infty}$ is a K-symmetric basis)

$$\leq \|\hat{x}\|$$

(since $x^*(n) \leq \frac{1}{n}\sum_{i=1}^{n} x^*(i)$ and $(e_n)_{n=1}^{\infty}$ is a 1-unconditional basis)

$$\leq c\|x\|.$$

Since $\|\cdot\|_{\mathrm{Day}}$ is equivalent to $\|\cdot\|_{\infty}$, it follows that $\|\cdot\|$ and $|||\cdot|||$ are equivalent quasi-norms. It remains to prove that $|||\cdot|||$ is in fact a norm, i.e., that $|||\cdot|||$ satisfies the triangle inequality. It suffices to show that $x \mapsto \|\widehat{x}\|$ satisfies the triangle inequality. Let $x, y \in X$. Note that

$$\widehat{(x+y)}(n) \leq \widehat{x}(n) + \widehat{y}(n) \quad (n \in \mathbb{N}). \tag{4.3}$$

Since $(e_n)_{n=1}^{\infty}$ is a 1-unconditional basis, it follows that

$$\|\widehat{x+y}\| \leq \|\widehat{x} + \widehat{y}\| \leq \|\widehat{x}\| + \|\widehat{y}\|. \tag{4.4}$$

Hence $x \mapsto \|\widehat{x}\|$ and $\|\cdot\|$ are equivalent norms on X.

□

For $x \in X$ and $N, M \in \mathbb{N}$, define

$$x \cdot 1_{[N,M]} := \sum_{n=N}^{M} x(n)e_n.$$

Lemma 4.5 *For $x, y \in X$ and $N \in \mathbb{N}$,*

$$|\|\widehat{x} \cdot 1_{[N,\infty)}\| - \|\widehat{y} \cdot 1_{[N,\infty)}\|| \leq c\|x - y\|.$$

Proof (4.3) yields

$$\|\widehat{x} \cdot 1_{[N,\infty)}\| \leq \|\widehat{y} \cdot 1_{[N,\infty)}\| + \|\widehat{(x-y)} \cdot 1_{[N,\infty)}\|.$$

Hence

$$\|\widehat{x} \cdot 1_{[N,\infty)}\| - \|\widehat{y} \cdot 1_{[N,\infty)}\| \leq \|\widehat{(x-y)} \cdot 1_{[N,\infty)}\| \leq \|\widehat{(x-y)}\| \leq c\|x-y\|.$$

Interchanging x and y gives the result. □

Lemma 4.6 *Suppose that $x \in X$, $y_n \in X$ $(n \geq 1)$, $\|y_n\| \geq \delta > 0$, $\lim_{n\to\infty}\|y_n\|_{\infty} = 0$ and $\min(\mathrm{supp}(y_n)) \to \infty$. Then, for all $N \geq 1$,*

$$\liminf_{n\to\infty} \|\widehat{(x+y_n)} \cdot 1_{[N,\infty)}\| \geq \frac{\delta}{K}.$$

Proof Let $\alpha > 0$. Choose $N_1 \in \mathbb{N}$ so that $x' := x1_{[1,N_1]}$ satisfies $\|x - x'\| < \alpha$. Then, for all sufficiently large n, we have

$$\min(\operatorname{supp}(y_n)) > N_1 \geq \max(\operatorname{supp}(x')).$$

Hence, for all sufficiently large n and for all $i \geq 1$,

$$(\widehat{x' + y_n})(i) \geq y_n^*(i).$$

So for all $N \geq 1$ and for all sufficiently large n,

$$\begin{aligned}\|(\widehat{x' + y_n})1_{[N,\infty)}\| &\geq \left\| \sum_{i=N+1}^{\infty} y_n^*(i)e_i \right\| \\ &\geq \left\| \sum_{i=1}^{\infty} y_n^*(i)e_i \right\| - N\|y_n\|_\infty \\ &\geq \frac{1}{K}\|y_n\| - N\|y_n\|_\infty \\ &\geq \frac{\delta}{K} - N\|y_n\|_\infty.\end{aligned}$$

Hence, by Lemma 4.5, for all $N \geq 1$ and for all sufficiently large n,

$$\begin{aligned}\|(\widehat{x + y_n}) \cdot 1_{[N,\infty)}\| &\geq \|(\widehat{x' + y_n}) \cdot 1_{[N,\infty)}\| - c\|x - x'\| \\ &\geq \frac{\delta}{K} - N\|y_n\|_\infty - c\alpha.\end{aligned}$$

Since $\lim_{n\to\infty} \|y_n\|_\infty = 0$ and $\alpha > 0$ is arbitrary the result follows. □

Theorem 4.7 $|||\cdot|||$ *is a* 1*-symmetric* $2R$ *equivalent norm on* X.

Proof Let us summarize the relevant progress we have made so far in this section. We used Theorem 3.1 to equip X with an equivalent 2R norm $\|\cdot\|$ that is 1-unconditional but not necessarily 1-symmetric (see the paragraph before Lemma 4.4). We then defined the equivalent norm $|||\cdot|||$, which we have shown in Lemma 4.4 to be 1-symmetric. It remains to prove that it is 2R. Suppose $(x_n)_{n=1}^\infty \subset X$ satisfies

$$\lim_{m,n\to\infty} [|||x_n + x_m|||^2 - 2(|||x_n|||^2 + |||x_m|||^2)] = 0. \tag{4.5}$$

By Theorem B, $(x_n)_{n=1}^\infty$ converges weakly to some $x \in X$ and $\lim_{n\to\infty} \|x_n - x\|_\infty = 0$. Let $x_n = x + y_n$ and suppose that $(y_n)_{n=1}^\infty$ does not converge to zero in norm. Since $\lim_{n\to\infty} \|y_n\|_\infty = 0$, a gliding hump and an approximation argument show, after passing to a subsequence and relabelling, that without loss of generality each y_n has finite support, that $(y_n)_{n=1}^\infty$ is a block basis with respect to $(e_n)_{n=1}^\infty$, and hence $\min(\operatorname{supp}(y_n)) \to \infty$ as $n \to \infty$, and that $\|y_n\| > \delta > 0$ $(n \geq 1)$.

It follows from (4.5) and the definition of $|||\cdot|||$ in (4.2) that

$$\lim_{m,n\to\infty} \|\widehat{x_n + x_m}\|^2 - 2(\|\widehat{x_n}\|^2 + \|\widehat{x_m}\|^2) = 0.$$

Note that $\|(\widehat{x_n + x_m})\| \le \|\widehat{x_n} + \widehat{x_m}\|$ by (4.4). Hence

$$\lim_{m,n\to\infty} \|\widehat{x_n} + \widehat{x_m}\|^2 - 2(\|\widehat{x_n}\|^2 + \|\widehat{x_m}\|^2) = 0.$$

Since $\|\cdot\|$ is a $2R$ equivalent norm on X, it follows that $(\widehat{x_n})_{n=1}^{\infty}$ converges strongly in X. By Lemma 4.6, for all $N \ge 1$,

$$\liminf_{m\to\infty} \|\widehat{x_m} \cdot 1_{[N,\infty)}\| = \liminf_{m\to\infty} \|(\widehat{x + y_m}) \cdot 1_{[N,\infty)}\| \ge \frac{\delta}{K},$$

which contradicts the fact that $(\widehat{x_n})_{n=1}^{\infty}$ is a Cauchy sequence in X. □

5 Symmetric renormings of ℓ_∞ and L_∞

A symmetric renorming of ℓ_∞ [16] is an equivalent norm $\|\cdot\|$ on ℓ_∞ such that

$$\|(a_n)_{n=1}^{\infty}\| = \|(a_{\sigma(n)})_{n=1}^{\infty}\| \quad ((a_n)_{n=1}^{\infty} \in \ell_\infty)$$

for all permutations σ of $\mathbb{N}$. It was proved in [16, Proposition 5] that for every symmetric renorming $\|\cdot\|$, $(\ell_\infty, \|\cdot\|)$ contains a subspace isometric to $(\ell_\infty, \|\cdot\|_\infty)$. For the sake of completeness we include an elementary proof that avoids uncountable cardinals.

Theorem 5.1 *Let $\|\cdot\|$ be a 1-symmetric norm on ℓ_∞. Then there exists a subspace Y of $(\ell_\infty, \|\cdot\|)$ that is isometrically isomorphic to $(\ell_\infty, \|\cdot\|_\infty)$. In fact, there exists $\alpha > 0$ such that, for all $y \in Y$, $\|y\| = \alpha\|y\|_\infty$.*

Proof Let $\|\cdot\|$ be a 1-symmetric norm on ℓ_∞. We let $2\mathbb{N}$ denote the set of even positive integers and $\ell_\infty(2\mathbb{N})$ the subspace of ℓ_∞ comprising all x with $\mathrm{supp}(x) = \{i \in \mathbb{N} : x(i) \ne 0\} \subset 2\mathbb{N}$. We will first show that $\|\cdot\|$ restricted on $\ell_\infty(2\mathbb{N})$ is 1-suppression unconditional, i.e., for every $x, y \in \ell_\infty(2\mathbb{N})$ such that, for all $i \in \mathbb{N}$, $x(i) = y(i)$ or $y(i) = 0$, we have $\|x\| \ge \|y\|$. We verify this on the dense linear subspace of $\ell_\infty(2\mathbb{N})$ consisting of all x that have the form

$$x = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where $n \in \mathbb{N}$, $a_1, \dots, a_n$ are (not necessarily different) scalars, and $A_1, \dots, A_n$ are disjoint (finite or infinite) subsets of $2\mathbb{N}$. By symmetry, it suffices to show that, letting

$$y = \sum_{i=1}^{n-1} a_i \chi_{A_i},$$

$\|x\| \geq \|y\|$. Fix an infinite $S \subset \mathbb{N} \setminus 2\mathbb{N}$. For each $N \in \mathbb{N}$, choose disjoint subsets $A_n^1, \dots, A_n^N$ of S that are equinumerous to A_n, and, for $1 \leq j \leq N$, let $x_N^j = y + a_n \chi_{A_n^j}$. By symmetry, $\|x_N^j\| = \|x\|$, and thus letting

$$y_N = \frac{1}{N} \sum_{j=1}^{N} x_N^j$$

we have $\|x\| \geq \|y_N\|$. At the same time, $\|y_N - y\|_\infty \to 0$, and thus by equivalence $\|y_N - y\| \to 0$, which yields $\|x\| \geq \|y\|$.

By scaling and symmetry, we may assume that, for all $n \in \mathbb{N}$, $\|e_n\| = 1$ and thus, by 1-suppression unconditionality, for all $x \in \ell_\infty(2\mathbb{N})$, $\|x\| \geq \|x\|_\infty$. Put

$$\alpha = \sup\{\|x\| \colon x \in \ell_\infty(2\mathbb{N}), \|x\|_\infty = 1\}.$$

Then there exists $x_0 \in \ell_\infty(2\mathbb{N})$ such that $\|x_0\|_\infty = 1$ and $\|x_0\| = \alpha$. Indeed, by symmetry we can pick disjointly supported vectors $(x_n)_{n=1}^\infty$ in $\ell_\infty(2\mathbb{N})$ such that, for all $n \in \mathbb{N}$, $\|x_n\|_\infty = 1$ and $\|x_n\| \geq \alpha - 1/n$. This follows from the symmetry of $\|\cdot\|$ and the fact that any bijection between infinite subsets of $2\mathbb{N}$ extends to a permutation of $\mathbb{N}$. Define $x_0 = \sum_{n=1}^\infty x_n$ pointwise. Then, $\|x_0\|_\infty = 1$ and thus $\|x_0\| \leq \alpha$. By 1-suppression unconditionality, for all $n \in \mathbb{N}$, $\|x_0\| \geq \|x_n\| \geq \alpha - 1/n$. So $\|x_0\| = \alpha$. Pick a disjointly supported sequence $(y_n)_{n=1}^\infty$ in $\ell_\infty(2\mathbb{N})$ so that each y_n has the same distribution as x_0. Then, for all $(a_n) \in \ell_\infty$, $\sum_{n=1}^\infty a_n y_n$ (defined pointwise) satisfies $\|\sum_{n=1}^\infty a_n y_n\| = \alpha \|(a_n)\|_\infty$. Hence $Y = \{\sum_{n=1}^\infty a_n y_n \colon (a_n) \in \ell_\infty\}$ has the required property. ◻

On the other hand, by [15, Corollary 4] ℓ_∞ admits an equivalent norm which restricts to a $W2R$ norm on reflexive subspaces. Clearly, every such norm is strictly convex and hence cannot be symmetric by Theorem 5.1.

Next we consider rearrangement-invariant renormings of $L_\infty[0, 1]$. Curiously, we reach a rather different conclusion from the case of ℓ_∞.

We will apply the following result from [10].

Theorem D [10] *There is an equivalent rearrangement-invariant (Orlicz) norm* $|||\cdot|||$ *on* $L_1[0, 1]$ *satisfying the following restricted uniform convexity condition. Let* $K \subset \{x \in L_1[0, 1] \colon |||x||| \leq 1\}$ *be weakly compact. Then, given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for all* $x, y \in K$*,*

$$|||x + y||| > 2 - \delta \Rightarrow |||x - y||| < \varepsilon.$$

Corollary 5.2 *Suppose* $(y_n)_{n=1}^{\infty}$ *is relatively weakly compact in* $L_1[0,1]$ *and satisfies*

$$\lim_{m,n\to\infty}[\vert\vert\vert y_n + y_m\vert\vert\vert^2 - 2(\vert\vert\vert y_n\vert\vert\vert^2 + \vert\vert\vert y_m\vert\vert\vert^2)] = 0. \tag{5.1}$$

Then $(y_n)_{n=1}^{\infty}$ *converges in* $L_1[0,1]$.

Proof The proof is omitted as it is essentially the same as the proof that a uniformly convex norm is $2R$. □

Theorem 5.3 *Let* $(Y, |\cdot|)$ *be a rearrangement-invariant space on* $[0,1]$. *Then* Y *admits an equivalent rearrangement-invariant norm* $\|\cdot\|$ *such that if* $(y_n)_{n=1}^{\infty}$ *is relatively weakly compact in* Y *and satisfies*

$$\lim_{m,n\to\infty}[\|y_m + y_n\|^2 - 2(\|y_m\|^2 + \|y_n\|^2)] = 0, \tag{5.2}$$

then $(y_n)_{n-1}^{\infty}$ *converges weakly in* Y. *In particular,* $\|\cdot\|$ *restricts to a* $W2R$ *norm on every reflexive subspace of* Y.

Proof Note that Y embeds continuously into $L_1[0,1]$. Define $\|\cdot\|$ as follows:

$$\|y\| = (|y|^2 + \vert\vert\vert y\vert\vert\vert^2)^{1/2} \quad (y \in Y).$$

Suppose that $(y_n)_{n=1}^{\infty}$ satisfies (5.2). Then $(y_n)_{n=1}^{\infty}$ also satisfies (5.1) and $(y_n)_{n=1}^{\infty}$ is relatively weakly compact in $L_1[0,1]$. It follows from Theorem D that $(y_n)_{n=1}^{\infty}$ converges in $L_1[0,1]$, which implies that $(y_n)_{n=1}^{\infty}$ has a unique weak cluster point in Y, i.e. that $(y_n)_{n=1}^{\infty}$ converges weakly in Y. □

Corollary 5.4 $L_\infty[0,1]$ *admits an equivalent rearrangement-invariant norm which restricts to a* $W2R$ *norm on every reflexive subspace.*

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