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On hyperplane sections and projections in l_p^n

Hermann König[1]

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Abstract

For $2 < p < p_0 \simeq 26.265$, the hyperplane section of the l_p^n-unit ball B_p^n perpendicular to $a^{(n)} = \frac{1}{\sqrt{n}}(1, \ldots, 1)$ for large n has larger volume than the one orthogonal to $a^{(2)} = \frac{1}{\sqrt{2}}(1, 1, 0, \ldots, 0)$, as shown by Oleszkiewicz. This is different from the case of l_∞^n considered by Ball. We give a quantitative estimate for which dimensions n this happens, namely for $n > c(\frac{1}{p_0 - p} + \frac{1}{p-2})$ for some absolute constant $c > 0$. Correspondingly for projections of B_q^n onto hyperplanes, Barthe and Naor showed that projections onto hyperplanes perpendicular to $a^{(n)}$ have smaller volume for large n than onto the one orthogonal to $a^{(2)}$, if $\frac{4}{3} < q < 2$, different from the case $q = 1$. We show that this happens for all $n > 5(\frac{1}{q - \frac{4}{3}} + \frac{1}{2-q})$.

Keywords Volume · Hyperplane sections · Projections · l_p^n-ball · Random variables

Mathematics Subject Classification 52A38 · 52A40 · 46 B07 · 60F05

1 Introduction and main results

In a well-known paper Ball [2] proved that the hyperplane section of the n-cube $B_\infty^n := [-1, 1]^n$ perpendicular to $a^{(2)} = \frac{1}{\sqrt{2}}(1, 1, 0, \ldots, 0) \in S^{n-1} \subset \mathbb{R}^n$ has maximal volume among all hyperplane sections. Earlier Hadwiger [6] and Hensley [7] had shown independently of one another that coordinate hyperplanes, e.g. orthogonal to $a^{(1)} = (1, 0, \ldots, 0) \in S^{n-1}$, yield the minimal $(n-1)$-dimensional cubic sections.

For $0 < p < \infty$, let $B_p^n := \{x = (x_j)_{j=1}^n \in \mathbb{R}^n \mid \|x\|_p := (\sum_{j=1}^n |x_j|^p)^{\frac{1}{p}} \le 1 \}$ be the unit ball of $l_p^n = (\mathbb{R}^n, \| \cdot \|_p)$. Meyer and Pajor [13] found extremal sections of the l_p^n balls B_p^n: They proved that the normalized volume of sections of B_p^n by a fixed

Dedicated to Bill Johnson on the occasion of his 80th birthday.

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✉ Hermann König
hkoenig@math.uni-kiel.de

1 Mathematisches Seminar, Universität Kiel, 24098 Kiel, Germany

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hyperplane is monotone increasing in p. This implies that coordinate planes provide the minimal sections for $2 \le p < \infty$, as for $p = \infty$, and the maximal sections for $1 \le p \le 2$. The minimal hyperplane sections of B_1^n are those orthogonal to a main diagonal, e.g. $a^{(n)} = \frac{1}{\sqrt{n}}(1, \dots, 1) \in S^{n-1}$, see also [13]. Koldobsky [10] extended this to the full range $1 \le p \le 2$.

This left open the case of the maximal hyperplane section of B_p^n for $2 < p < \infty$. The situation there is more complicated, since the maximal hyperplane may depend as well on p as on the dimension n: Oleszkiewicz [14] proved that Ball's result does not transfer to the balls B_p^n if $2 < p < p_0 \simeq 26.265$: then the intersection of the hyperplane perpendicular to $a^{(n)}$ has larger volume than the one orthogonal to $a^{(2)}$, for sufficiently large dimensions n. The value p_0 is the unique value in $(2, \infty)$ such that $\lim_{n\to\infty} \frac{\mathrm{vol}_{n-1}(B_p^n \cap (a^{(n)})^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap (a^{(2)})^\perp)} > 1$ for all $2 < p < p_0$. Oleszkiewicz's result is an asymptotic one, not determining dimensions n for which this happens. We derive a quantitative estimate for dimensions n such that this holds, namely for $n > c\left(\frac{1}{p_0 - p} + \frac{1}{p-2}\right)$. On the other hand, recently Eskenazis, Nayar and Tkocz [5] proved that Ball's result is stable for l_p^n and very large p: $(a^{(2)})^\perp \cap B_p^n$ is the maximal hyperplane section of B_p^n for all dimensions, provided that $p_1 := 10^{15} \le p < \infty$. They call it "resilience of cubic sections".

Dual to hyperplane sections of convex bodies are projections of convex bodies onto hyperplanes. The known results for l_p^n-balls show a duality between sections and projections, when maximal and minimal directions a and p and the conjugate index $q = \frac{p}{p-1}$ are interchanged. Nevertheless the proofs in both situations are different, since volume does not behave well under duality. Barthe and Naor [3] determined the extremal hyperplane projections of l_q^n-balls except for the minimal hyperplane projections when $1 < q < 2$, corresponding to the dual maximal section case mentioned above when $2 < p < \infty$. For $q = 1$, the projection of B_1^n onto the hyperplane perpendicular to $a^{(1)} = (1, 0, \dots, 0)$ is maximal, the projection onto the hyperplane orthogonal to $a^{(2)}$ is minimal, which essentially is a consequence of Szarek's result [16] on the best constants in the Khintchine inequality for $q = 1$. Barthe and Naor [3] proved that this does no transfer to $\frac{4}{3} < q < 2$, namely that the projection onto $a^{(n)\perp}$ has smaller volume than the one onto $a^{(2)\perp}$ for large dimensions n. Here $q_0 := \frac{4}{3}$ is the unique value in $(1, 2)$ such that $\lim_{n\to\infty} \frac{\mathrm{vol}_{n-1}(P_{(a^{(n)})^\perp}(B_q^n))}{\mathrm{vol}_{n-1}(P_{(a^{(2)})^\perp}(B_q^n))} < 1$ for all $q_0 < q < 2$, where $P_{a^\perp}$ denotes the orthogonal projection onto the hyperplane $a^\perp$. In this case, we also give a quantitative estimate for dimensions n when this happens, namely when $n > 5\left(\frac{1}{p-\frac{4}{3}} + \frac{1}{2-p}\right)$. Note that there is no complete duality here, since $\frac{4}{3}$ is not the dual index of $p_0 \simeq 26.265$.

Let $a \in S^{n-1} := \{\, x \in \mathbb{R}^n \mid \|x\|_2 = 1 \,\}$ be a direction vector. We introduce the normalized section function

$$A_{n,p}(a) := \frac{\mathrm{vol}_{n-1}(a^\perp \cap B_p^n)}{\mathrm{vol}_{n-1}(B_p^{n-1})}$$

and the normalized projection function

$$P_{n,p}(a) := \frac{\mathrm{vol}_{n-1}(P_{a^\perp}(B_p^n))}{\mathrm{vol}_{n-1}(B_p^{n-1})}.$$

In terms of this notation, Ball's result states $A_{n,\infty}(a) \le A_{n,\infty}(a^{(2)}) = \sqrt{2}$ for all $a \in S^{n-1}$ and Eskenazis, Nayar and Tkocz's result reads $A_{n,p}(a) \le A_{n,p}(a^{(2)}) = 2^{\frac{1}{2}-\frac{1}{p}}$ for all $a \in S^{n-1}$ and $10^{15} \le p < \infty$. We note that $A_{n,p}(a^{(2)}) = 2^{\frac{1}{2}-\frac{1}{p}}$ is independent of $n \ge 2$. But as shown by Oleszkiewicz, $\lim_{n\to\infty} A_{n,p}(a^{(n)}) > A_{n,p}(a^{(2)}) = 2^{\frac{1}{2}-\frac{1}{p}}$ for $2 < p < p_0$. In the projection case, $P_{n,1}(a^{(2)}) \le P_{n,1}(a)$ for all $a \in S^{n-1}$, which Eskenazis, Nayar and Tkocz [5] extended to $P_{n,q}(a^{(2)}) \le P_{n,q}(a)$ for all $a \in S^{n-1}$ and $1 < q \le 1+10^{-12}$. Again $P_{n,q}(a^{(2)}) = 2^{\frac{1}{2}-\frac{1}{q}}$ is independent of $n \ge 2$. However, by Barthe and Naor [3], $P_{n,q}(a^{(2)}) > \lim_{n\to\infty} P_{n,q}(a^{(n)})$ for $\frac{4}{3} < q < 2$.

Our two main results study these limits in more detail.

Theorem 1.1 *Let* $2 < p < \infty$ *and* $n \in \mathbb{N}$. *Then for all* $2 < p < p_0 \simeq 26.265$

$$\lim_{n\to\infty} \frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} = \sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1+\frac{1}{p}\right)^3}{\Gamma\left(1+\frac{3}{p}\right)}} > 1.$$

We have the following quantitative estimate: $A_{n,p}(a^{(n)}) > A_{n,p}(a^{(2)})$ *holds if*

(a) *either* $5 \le p < p_0$ *and* $n \ge \frac{650}{p_0-p}$ *or*

(b) $2 < p < 5$ *and* $n > \frac{65}{p-2}$ *is satisfied.*

Remarks. (a) The constant 650 in the statement for $5 \le p < p_0$ is not optimal, but by necessity fairly large since the p-derivative of $f(p) := \sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1+\frac{1}{p}\right)^3}{\Gamma\left(1+\frac{3}{p}\right)}}$ at p_0 with $f(p_0) = 1$ is small, $f'(p_0) \simeq -\frac{1}{1316}$. The derivative at 2 is positive and larger in modulus, namely $f'(2) = \frac{1}{4}(1-\ln 2) \simeq \frac{1}{13}$.

(b) The case of complex hyperplane sections of $l_p^n(\mathbb{C})$ is considered in [8].

Theorem 1.2 *Let* $1 < q < 2$ *and* $n \in \mathbb{N}$. *Then for all* $\frac{4}{3} < q < 2$

$$\lim_{n\to\infty} \frac{P_{n,q}(a^{(n)})}{P_{n,q}(a^{(2)})} = \sqrt{\frac{1}{\pi}} \sqrt{2^{\frac{2}{q}} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2-\frac{1}{q}\right)} < 1.$$

We have the following quantitative estimate: $P_{n,q}(a^{(n)}) < P_{n,q}(a^{(2)})$ *holds if*

$$n > \frac{\frac{32}{15}}{q-\frac{4}{3}} + \frac{\frac{24}{5}}{2-q}.$$

Remark. For the derivative of $g(q) := \sqrt{\frac{1}{\pi}}\sqrt{2^{\frac{2}{q}}\Gamma\left(\frac{1}{q}\right)\Gamma\left(2-\frac{1}{q}\right)}$ we have $g'(\frac{4}{3}) = \frac{9}{32}(4-\pi-2\ln 2) \simeq -\frac{1}{6.73}$ and $g'(2) = \frac{1}{4}(1-\ln 2) \simeq \frac{1}{13}$.

The limits in Theorems 1.1 and 1.2 were already determined by Oleszkiewicz [14] and Barthe, Naor [3]. Meyer and Pajor [13] showed that $A_{n,p}(a)$ is monotone increasing in p for any fixed n and a. Barthe and Naor proved that $P_{n,q}(a)$ is monotone increasing in q for any fixed n and a.

2 Formulas

Eskenazis, Nayar and Tkocz [5], Proposition 6, proved the following formula for the normalized volume of hyperplane sections.

Proposition 2.1 *Let* $1 \le p < \infty$, $n \in \mathbb{N}$ *and* $a = (a_j)_{j=1}^n \in S^{n-1} \subset \mathbb{R}^n$. *Then*

$$A_{n,p}(a) = \Gamma\left(1+\frac{1}{p}\right) \mathbb{E}_{\xi,R} \frac{1}{\|\sum_{j=1}^n a_j R_j \xi_j\|_2} \tag{2.1}$$

where $(\xi_j)_{j=1}^n$ *are i.i.d. random vectors uniformly distributed on the sphere* $S^2 \subset \mathbb{R}^3$ *and* $(R_j)_{j=1}^n$ *are i.i.d. random variables with density* $c_p^{-1} t^p \exp(-t^p)$ *on* $[0,\infty)$, $c_p := \frac{1}{p}\Gamma\left(1+\frac{1}{p}\right)$, *independent of the* $(\xi_j)_{j=1}^n$.

For $p = \infty$ with $R_j = 1$ one has $A_{n,\infty}(a) = \mathbb{E}_\xi \frac{1}{\|\sum_{j=1}^n a_j \xi_j\|_2}$, see König, Koldobsky [11]. We will use another formula for $A_{n,p}(a)$ derived from (2.1).

Proposition 2.2 *Let* $1 \le p < \infty$, $n \in \mathbb{N}$ *and* $a = (a_j)_{j=1}^n \in S^{n-1} \subset \mathbb{R}^n$. *Then*

$$A_{n,p}(a) = \Gamma\left(1+\frac{1}{p}\right) \frac{2}{\pi} \int_0^\infty \prod_{j=1}^n \gamma_p(a_j s)\, ds\ ,$$

$$\gamma_p(s) := \frac{1}{\Gamma\left(1+\frac{1}{p}\right)} \int_0^\infty \cos(sr)\exp(-r^p)\, dr\ . \tag{2.2}$$

Proof Define $\mathrm{sinc}(x) := \frac{\sin x}{x}$ for $x \neq 0$, $\mathrm{sinc}(0) := 1$. Let $t > 0$, $e \in S^2$ be fixed and m denote the normalized Haar surface measure on S^2. Then

$$\mathrm{sinc}(t) = \int_{S^2} \exp(it\langle e, u\rangle)\, dm(u). \tag{2.3}$$

Here $\langle\ \cdot, \cdot\ \rangle$ denotes the standard inner product on $\mathbb{R}^n$. This implies for $(b_j)_{j=1}^n \in \mathbb{R}^n$

$$\prod_{j=1}^{n} \operatorname{sinc}(b_j s) = \int_{(S^2)^n} \exp\left(is \left\langle e, \sum_{j=1}^{n} b_j u_j \right\rangle \right) \prod_{j=1}^{n} dm(u_j)$$
$$= \int_{(S^2)^n} \operatorname{sinc}\left(\left\| \sum_{j=1}^{n} b_j u_j \right\|_2 s \right) \prod_{j=1}^{n} dm(u_j) = \mathbb{E}_\xi \operatorname{sinc}\left(\left\| \sum_{j=1}^{n} b_j \xi_j \right\|_2 s \right), \tag{2.4}$$

where the second equality follows from (2.3) by integration over $dm(e)$. Note that the first equality holds for all $e \in S^2$.
For all $t > 0$ we have $\frac{2}{\pi} \int_0^\infty \operatorname{sinc}(ts)ds = \frac{1}{t}$ and (2.1) may be rewritten

$$A_{n,p}(a) = \Gamma\left(1 + \frac{1}{p}\right) \frac{2}{\pi} \mathbb{E}_{\xi,R} \int_0^\infty \operatorname{sinc}\left(\left\| \sum_{j=1}^{n} a_j R_j \xi_j \right\|_2 s \right) ds$$
$$= \Gamma\left(1 + \frac{1}{p}\right) \frac{2}{\pi} \int_0^\infty \mathbb{E}_{\xi,R} \operatorname{sinc}\left(\left\| \sum_{j=1}^{n} a_j R_j \xi_j \right\|_2 s \right) ds .$$

The sinc-integral is only a conditionally convergent Riemann integral. The verification that $\mathbb{E}_{\xi,R}$ and $\int_0^\infty$ may be interchanged is the same as in the proof of Proposition 3.2 (a) of König, Rudelson [12]. Using (2.4) and the independence of the $(R_j)_{j=1}^n$, we get

$$A_{n,p}(a) = \Gamma\left(1 + \frac{1}{p}\right) \frac{2}{\pi} \int_0^\infty \mathbb{E}_R \left(\prod_{j=1}^{n} \operatorname{sinc}(a_j R_j s) \right) ds$$
$$= \Gamma\left(1 + \frac{1}{p}\right) \frac{2}{\pi} \int_0^\infty \prod_{j=1}^{n} \mathbb{E}_{R_j} \operatorname{sinc}(a_j R_j s)\, ds .$$

Denoting $\gamma_p(s) := \mathbb{E}_{R_1} \operatorname{sinc}(R_1 s)$, integration by parts gives

$$\gamma_p(s) = c_p^{-1} \int_0^\infty \operatorname{sinc}(sr) r^p \exp(-r^p) dr$$
$$= c_p^{-1} \frac{1}{p} \int_0^\infty \cos(sr) \exp(-r^p) dr = \frac{1}{\Gamma\left(1 + \frac{1}{p}\right)} \int_0^\infty \cos(sr) \exp(-r^p) dr .$$

□

Equation (2.1) yields $A_{n,p}(a^{(2)}) = 2^{\frac{1}{2} - \frac{1}{p}}$, see [5, Section 3.2].

Remarks. (a) Proposition 2.2 is also found in Koldobsky [9, Theorem 3.2], with a different proof.
(b) For $1 \leq p \leq 2$ the γ_p are just the densities of the (positive) p-stable random variables. In the case interesting for us, namely $2 < p < \infty$, the functions γ_p take

positive and negative values. For $p \notin 2\mathbb{N}$, γ_p has only finitely many real zeros, see Pólya [15], whereas for $p \in 2\mathbb{N}$, γ_p has infinitely many real zeros, see Boyd [4].

Barthe and Naor [3] proved the following formula for the volume of the orthogonal projection of B_q^n onto hyperplanes.

Proposition 2.3 *Let $1 \leq q < \infty$, $p := \frac{q}{q-1}$ be the conjugate index, $n \in \mathbb{N}$ and $a = (a_j)_{j=1}^n \in S^{n-1} \subset \mathbb{R}^n$. Then*

$$P_{n,q}(a) = \Gamma\left(\frac{1}{q}\right) \mathbb{E}\left|\sum_{j=1}^n a_j X_j\right| \tag{2.5}$$

where the X_j are i.i.d. symmetric random variables with density function $d_q^{-1}|t|^{p-2}\exp(-|t|^p)$, $t \in \mathbb{R}$, $d_q = \frac{2}{p}\Gamma\left(\frac{1}{q}\right)$. A second formula for $P_{n,q}(a)$ is

$$\begin{aligned} P_{n,q}(a) &= \Gamma\left(\frac{1}{q}\right) \frac{2}{\pi} \int_0^\infty \frac{1 - \prod_{j=1}^n \delta_q(a_j s)}{s^2} ds \ , \\ \delta_q(s) &:= \frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^\infty \cos(sr)\, r^{p-2} \exp(-r^p) dr \ . \end{aligned} \tag{2.6}$$

Note that $\mathbb{E}|X_1| = \frac{1}{\Gamma\left(\frac{1}{q}\right)}$. To deduce (2.6) from (2.5), apply the usual formula $|x| = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1-Re(\exp(ixs))}{s^2} ds$ to find

$$\begin{aligned} \mathbb{E}\left|\sum_{j=1}^n a_j X_j\right| &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{1-\mathbb{E}\exp(i(\sum_{j=1}^n a_j X_j)s)}{s^2} ds = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1-\prod_{j=1}^n \mathbb{E}\exp(i a_j X_j s))}{s^2} ds \\ &= \frac{2}{\pi}\int_0^\infty \frac{1-\prod_{j=1}^n \mathbb{E}\cos(a_j X_j s))}{s^2} ds = \frac{2}{\pi}\int_0^\infty \frac{1-\prod_{j=1}^n \delta_q(a_j s)}{s^2} ds \ , \\ \delta_q(s) &= \frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^\infty \cos(sr)\, r^{p-2} \exp(-r^p) dr \ . \end{aligned}$$

Differentiation and integration by parts yields a relation between the functions δ_q and γ_p in (2.6) and (2.2):

$$\begin{aligned} \delta_q'(s) &= -\frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^\infty \sin(sr)\, r^{p-1} \exp(-r^p) dr \\ &= -\frac{s}{\Gamma\left(\frac{1}{q}\right)} \int_0^\infty \cos(sr) \exp(-r^p) dr = -\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)}\, s\, \gamma_p(s) \ . \end{aligned} \tag{2.7}$$

Since $\gamma_4''(s) = -\frac{\Gamma(\frac{3}{4})}{4\Gamma(\frac{5}{4})}\,\delta_{\frac{4}{3}}(s)$, we have $\gamma_4'''(s) = \frac{1}{4}\,s\gamma_4(s)$. Similarly, for all $k \in \mathbb{N}$, $\gamma_{2k}^{(2k-1)}(s) = (-1)^k \frac{1}{2k} s\gamma_{2k}(s)$. Therefore the functions γ_{2k} studied by Boyd [4] satisfy a linear differential equation.

For $q \searrow 1$, the variables X_j tend to the Rademacher variables with $\delta_q(s) \to \delta_1(s) = \cos(s)$, and the best constants in the Khintchine inequality, which were determined by Szarek [16], yield the extrema of $P_{n,1}$: $a^{(2)}$ for the minimum and $a^{(1)}$ for the maximum.

3 Prerequisites for the proof of Theorem 1.1

For the proof of Theorem 1.1 we need two lemmas on Γ-functions.

Lemma 3.1 (a) *Let* $f(p) := \frac{\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}$. *Then* $f(p) \geq 0.9429$ *for all* $3 \leq p < \infty$.

(b) *Let* $g(p) := \left(\frac{3}{\pi}\frac{2^{\frac{2}{p}}\Gamma\left(1+\frac{1}{p}\right)^3}{\Gamma\left(1+\frac{3}{p}\right)}\right)^{\frac{1}{2}}$. *Then there is exactly one solution* $p_0 \in (2,\infty)$ *of* $g(p) = 1$, $p_0 \simeq 26.265$. *For all* $2 < p < p_0$ *we have* $g(p) > 1$. *The function* g' *has exactly one zero* $p_1 \in [2,\infty)$, $p_1 \simeq 4.192$. *For* $2 \leq p < p_1$, g *is strictly increasing, for* $p_1 < p < \infty$, g *is strictly decreasing. The following lower estimates hold:*

$$g(p) \geq 1 + \frac{p_0 - p}{1317} \quad p \in [5, p_0]\ ,\ g(p) > \frac{25}{24} \quad p \in [4,5]\ ,\ g(p) \geq 1 + \frac{p-2}{44} \quad p \in [2,4].$$

Proof (a) In terms of the Digamma function $\Psi := (\ln\Gamma)'$ we have

$$f'(p) = \frac{f(p)}{p^2}\left(\Psi\left(1+\frac{1}{p}\right) - 3\Psi\left(1+\frac{3}{p}\right)\right).$$

For $F(p) := \Psi\left(1+\frac{1}{p}\right) - 3\Psi\left(1+\frac{3}{p}\right)$ one has $F'(p) = \frac{1}{p^2}\left(9\Psi'\left(1+\frac{3}{p}\right) - \Psi'\left(1+\frac{1}{p}\right)\right)$. By Abramowitz, Stegun [1, 6.3.16 and 6.4.10] for all $x > 0$

$$\Psi(1+x) = -\gamma + \sum_{n=1}^{\infty}\frac{x}{n(n+x)}\ ,\ \Psi'(1+x) = \sum_{n=1}^{\infty}\frac{1}{(n+x)^2} \tag{3.1}$$

where $\gamma \simeq 0.5772$ denotes the Euler–Mascheroni constant. Therefore Ψ' is decreasing, and we conclude for all $0 \leq x \leq 1$ that $\frac{\pi^2}{6} - 1 = \Psi'(2) \leq \Psi'(1+x) \leq \Psi'(1) = \frac{\pi^2}{6}$. Hence $F'(p) \geq \frac{1}{p^2}\left(\frac{4\pi^2}{3} - 9\right) > 0$ for all $p \geq 3$. Thus F is increasing. Since $F(9) \simeq -0.012$, $F(10) \simeq 0.084$, F has exactly one zero $p_1 \in (3,\infty)$,

$p_1 \simeq 9.115$. Hence f is decreasing in $(3, p_1)$ and increasing in (p_1, ∞). For all $p \geq 3$, $f(p) \geq f(p_1) > 0.9429$.

(b) Let $h(p) := \frac{2^{\frac{2}{p}}\Gamma\left(1+\frac{1}{p}\right)^3}{\Gamma\left(1+\frac{3}{p}\right)}$. Then

$$h'(p) = \frac{h(p)}{p^2}\left(3\Psi\left(1+\frac{3}{p}\right) - 3\Psi\left(1+\frac{1}{p}\right) - 2\ln 2\right).$$

By (3.1) and the geometric series we find for $p > 3$

$$\begin{aligned}(\ln h)'(p) = \frac{h'(p)}{h(p)} &= \frac{1}{p^2}\left(3\sum_{n=1}^{\infty}\left(\frac{\frac{3}{p}}{n(n+\frac{3}{p})} - \frac{\frac{1}{p}}{n(n+\frac{1}{p})}\right) - 2\ln 2\right) \\ &= \frac{1}{p^2}\left(3\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{n^{k+2}}\frac{3^{k+1}-1}{p^{k+1}} - 2\ln 2\right) \\ &= \frac{1}{p^2}\left(3\sum_{k=0}^{\infty}(-1)^k\zeta(k+2)\frac{3^{k+1}-1}{p^{k+1}} - 2\ln 2\right).\end{aligned}$$

The sum is an alternating series with decreasing coefficients. We find that

$$(\ln h)'(p) \leq -\frac{2\ln 2}{p^2} + \frac{\pi^2}{p^3} - \frac{24\zeta(3)}{p^4} + \frac{13}{15}\frac{\pi^4}{p^5} - \frac{240\zeta(5)}{p^6} + \frac{242}{315}\frac{\pi^6}{p^7} < 0 \tag{3.2}$$

holds for all $5 \leq p < \infty$. Therefore $\ln h, h$ and $g(p) = \sqrt{\frac{3}{\pi}h(p)}$ are strictly decreasing in $[5, \infty)$. We have $\lim_{p\to\infty} g(p) = \sqrt{\frac{3}{\pi}} < 1$, $g(26) \simeq 1.00020$, $g(27) \simeq 0.99945$: There is exactly one $p_0 \in [5, \infty)$ with $g(p_0) = 1$, $p_0 \simeq 26.265$, and for $5 \leq p < p_0$ we have $g(p) > 1$. Inequality (3.2) yields for $5 \leq p \leq p_0$ that $(\ln h)'(p) \leq -\frac{1.04768}{p^2}$. Hence for these p

$$g'(p) = g(p)(\ln g)'(p) = \frac{1}{2}g(p)(\ln h)'(p) \leq -\frac{1}{2}\frac{1.04768}{p^2} \leq -\frac{1}{2}\frac{1.04768}{p_0^2} < -\frac{1}{1317}.$$

This implies $g(p) \geq 1 + \frac{1}{1317}(p_0 - p)$ for all $5 \leq p \leq p_0$.

To show $g(p) > 1$ also for $2 < p < 5$, note that $g(2) = 1$ and

$$g'(p) = \frac{1}{2}g(p)(\ln h)'(p) = \frac{3}{2}\frac{g(p)}{p^2}\left(\Psi\left(1+\frac{3}{p}\right) - \Psi\left(1+\frac{1}{p}\right) - \frac{2}{3}\ln 2\right).$$

Again by (3.1)

$$\Psi\left(1+\frac{3}{p}\right)-\Psi\left(1+\frac{1}{p}\right)=\sum_{n=1}^{\infty}\left(\frac{\frac{3}{p}}{n(n+\frac{3}{p})}-\frac{\frac{1}{p}}{n(n+\frac{1}{p})}\right)=\sum_{n=1}^{\infty}\frac{2p}{(np+1)(np+3)}.$$

All summands are decreasing in p. Thus $k(p) := \Psi(1+\frac{3}{p}) - \Psi(1+\frac{1}{p}) - \frac{2}{3}\ln 2$ is strictly decreasing in p, with $k(4) = \pi - \frac{8}{3} - \frac{2}{3}\ln 2 \simeq 0.0128 > 0$ and $k(5) \simeq -0.0470 < 0$. Thus g' has exactly one zero $p_1 \in (2, \infty)$, $p_1 \simeq 4.193$, and g is strictly increasing in $(2, p_1)$ and strictly decreasing in (p_1, ∞). We know already that $g(5) > 1$ and hence $g(p) > 1$ for all $2 < p \le 5$. For $p \in [4, 5]$, $g(p) \ge \min(g(4), g(5)) = g(5) > \frac{25}{24}$. Further

$$\left(\frac{g(p)}{p^2}\right)'=\frac{3}{2}\frac{g(p)}{p^4}\left(k(p)-\frac{4}{3}p\right)<0$$

since $k(p) - \frac{4}{3}p \le k(2) - \frac{8}{3} = -\left(2 + \frac{2}{3}\ln(2)\right) < 0$. Therefore $\frac{g(p)}{p^2}$ and $k(p)$ are both strictly decreasing and positive for $p \in [2, p_1)$, and with $g'(p) = \frac{g(p)}{p^2}k(p)$, g' is decreasing and hence g is concave in $[2, p_1]$. Therefore for $2 \le p \le 4$

$$g(p)\ge 1+\frac{g(4)-1}{2}(p-2)>1+\frac{p-2}{44}$$

which proves all lower estimates stated in Lemma 3.1. □

For $p \to \infty$, the functions γ_p in (2.2) tend to γ_∞, $\gamma_\infty(s) = \mathrm{sinc}(s)$. We estimate their difference for $p \ge 2$.

Lemma 3.2 *Let* $2 \le p < \infty$. *Then for all* $s > 0$

$$\left|\mathrm{sinc}(s)-\int_0^\infty \cos(sr)\exp(-r^p)dr\right|\le 0.3926.$$

This implies $\gamma_p(s) > 0$ *for all* $0 \le s \le \frac{2}{3}\pi$.

Proof We have $\int_0^\infty \exp(-s^p)ds = \Gamma\left(1+\frac{1}{p}\right) < 1$. Since $\mathrm{sinc}(s) = \int_0^1 \cos(sr)dr$, we find

$$\begin{aligned}&\left|\mathrm{sinc}(s)-\int_0^\infty \cos(sr)\exp(-r^p)dr\right|\\&=\left|\int_0^1 \cos(sr)(1-\exp(-r^p))dr-\int_1^\infty \cos(sr)\exp(-r^p)dr\right|\\&\le\left(1-\Gamma\left(1+\frac{1}{p}\right)\right)+2\int_1^\infty \exp(-r^p)dr\\&=\left(1-\Gamma\left(1+\frac{1}{p}\right)\right)+\frac{2}{p}\int_1^\infty u^{\frac{1}{p}-1}\exp(-u)du=:\phi(p)\,.\end{aligned}$$

Then $\phi' < 0$, since for $p \geq 2$

$$\phi'(p) = -\frac{1}{p^2}\left(2\int_1^\infty u^{\frac{1}{p}-1}\left(1+\frac{\ln(u)}{p}\right)\exp(-u)du - \Gamma'\left(1+\frac{1}{p}\right)\right).$$

Since $\Gamma(x)$ has a (positive) minimum at $\tilde{x} \simeq 1.4616$, $\Gamma'(1+\frac{1}{\tilde{p}}) = 0$ for $\tilde{p} \simeq 2.1662$. For $p > \tilde{p}, \Gamma'(1+\frac{1}{p}) < 0$ and for $2 \leq p < \tilde{p}, 0 \leq \Gamma'(1+\frac{1}{p}) < \frac{3}{100}$. The last inequality holds since Γ is log-convex and positive in $(1,2)$, thus convex, and for $2 \leq p < \tilde{p}$, $\Gamma'(1+\frac{1}{p})$ is decreasing, with $\Gamma'\left(\frac{3}{2}\right) = \Gamma\left(\frac{3}{2}\right)\Psi\left(\frac{3}{2}\right) = \sqrt{\pi}(1-\ln 2 - \frac{\gamma}{2}) < \frac{3}{100}$, see Abramowitz, Stegun [1, 6.3.4]. Therefore for all $p \geq 2$

$$\phi'(p) \leq -\frac{1}{p^2}\left(2\int_1^\infty u^{-1}\exp(-u)du - \frac{3}{100}\right) < -\frac{2}{5}\frac{1}{p^2} < 0$$

using that $\int_1^\infty u^{-1}\exp(-u)du \simeq 0.219$. We conclude $\phi(p) \leq \phi(2) < 0.3926$.

This yields for all $0 \leq s \leq \frac{2}{3}\pi$ and $p \geq 2$

$$\begin{aligned}\Gamma\left(1+\frac{1}{p}\right)\gamma_p(s) &= \int_0^\infty \cos(sr)\exp(-r^p)dr \geq \operatorname{sinc}(s) \\ &\quad -0.3926 \geq \frac{3\sqrt{3}}{4\pi} - 0.3926 > \frac{1}{50} > 0.\end{aligned}$$

□

Remark. The derivative of $p\phi(p)$ is increasing with

$$\lim_{p\to\infty}(p\phi(p))' = \gamma + 2\int_1^\infty \frac{1}{u}\exp(-u)du \leq 1.016.$$

Thus for all $p \geq 1$, $\left|\operatorname{sinc}(s) - \int_0^\infty \cos(sr)\exp(-r^p)dr\right| \leq \frac{1.016}{p}$. Actually, $\gamma_p(s) > 0$ for all $p \geq 1$ and $s \in [0,\pi]$. However, we do not need this.

4 Proof of Theorem 1.1

Proof of Theorem 1.1.
(i) To estimate $A_{n,p}(a^{(n)})$ from below, we first find a lower bound for $\gamma_p\left(\frac{s}{\sqrt{n}}\right)$ for all $s \leq \frac{3}{2}\sqrt{n}$. By the series representation for $\cos x$ we have for $0 \leq x \leq \frac{3}{2}$, by pairing up terms in the alternating series,

$$\cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{26}\right) = \frac{x^4}{312} - \frac{x^6}{720} + \sum_{k=4}^\infty (-1)^m \frac{x^{2m}}{(2m)!} > 0$$

$1 - \frac{x^2}{2} + \frac{x^4}{26} > 0$. Therefore for $s \le \frac{3}{2}\sqrt{n}$

$$\gamma_p\left(\frac{s}{\sqrt{n}}\right) \ge \frac{1}{\Gamma\left(1+\frac{1}{p}\right)}\Big[\int_0^{\frac{3}{2}\frac{\sqrt{n}}{s}}\left(1 - \frac{s^2 r^2}{2n} + \frac{s^4 r^4}{26 n^2}\right)\exp(-r^p)dr + \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty}\cos\left(\frac{sr}{\sqrt{n}}\right)\exp(-r^p)dr\Big].$$

To estimate this from below, write $\int_0^{\frac{3}{2}\frac{\sqrt{n}}{s}} = \int_0^{\infty} - \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty}$ and use that $|\cos(x)| \le 1$,

$$\begin{aligned}\gamma_p\left(\frac{s}{\sqrt{n}}\right) &\ge \frac{1}{\Gamma\left(1+\frac{1}{p}\right)}\left[\int_0^{\infty}\left(1 - \frac{s^2 r^2}{2n} + \frac{s^4 r^4}{26 n^2}\right)\exp(-r^p)dr - R\right] \\ &= \frac{1}{\Gamma\left(1+\frac{1}{p}\right)}\left[\Gamma\left(1+\frac{1}{p}\right) - \frac{\Gamma\left(1+\frac{3}{p}\right)s^2}{6n} + \frac{\Gamma\left(1+\frac{5}{p}\right)s^4}{130 n^2} - R\right],\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}R &:= \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty}\left(2 - \frac{s^2 r^2}{2n} + \frac{s^4 r^4}{26 n^2}\right)\exp(-r^p)dr \\ &= \frac{1}{p}\int_{\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p}^{\infty}\left(2u^{\frac{1}{p}-1} - \frac{s^2}{2n}u^{\frac{3}{p}-1} + \frac{s^4}{26n^2}u^{\frac{5}{p}-1}\right)\exp(-u)du\ ;\ u \ge 1\ .\end{aligned} \tag{4.2}$$

(ii) We first consider the case $p \ge 5$. Since $u \ge 1$ in the above integral, $u^{\frac{5}{p}-1} \le 1$, $2u^{\frac{1}{p}-1} \le 2\left(\frac{2}{3}\frac{s}{\sqrt{n}}\right)^{p-1}$ and $\int_{\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p}\exp(-u)du = \exp\left(-\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p\right)$. The remainder term R will be smaller than the fourth order term $\frac{\Gamma\left(1+\frac{5}{p}\right)s^4}{130n^2}$ provided that

$$\frac{1}{p\Gamma\left(1+\frac{5}{p}\right)}\left(2\left(\frac{2}{3}\frac{s}{\sqrt{n}}\right)^{p-1} + \frac{s^4}{26n^2}\right)\exp\left(-\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p\right) < \frac{s^4}{130n^2}.$$

For $s \le \frac{3}{2}\sqrt{n}$ the left side is decreasing in p, and therefore this condition is the strongest for $p = 5$. Writing $y := \frac{s}{\sqrt{n}}$, it means

$$\frac{1}{5}\left(\frac{32}{81}y^4 + \frac{1}{26}y^4\right)\exp\left(-\left(\frac{3}{2}\frac{1}{y}\right)^5\right) < \frac{1}{130}y^4$$

or $\frac{913}{81} < \exp\left(\left(\frac{3}{2}\frac{1}{y}\right)^5\right)$, $y < \frac{3}{2}\frac{1}{\ln\left(\frac{913}{81}\right)^{\frac{1}{5}}} \simeq 1.2567$. Choosing $s \le \frac{7}{6}\sqrt{n}$, $R \le \frac{\Gamma\left(1+\frac{5}{p}\right)s^4}{130n^2}$ is satisfied and therefore

$$\gamma_p\left(\frac{s}{\sqrt{n}}\right) \ge 1 - c\frac{s^2}{n}, \quad c := \frac{1}{6}\frac{\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}.$$

By the proof of Lemma 3.1 (a), $\frac{\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}$ is decreasing for $5 \le p \le p_1 \simeq 9.115$ and increasing for $p > p_1$. Its value at p_0 is less than the one at 5, so that $c \le 0.1622 < \frac{8}{49}$, for its value at $p = 5$. Then for $s \le \frac{7}{6}\sqrt{n}$, $x := c\frac{s^2}{n} \le \frac{8}{49}\frac{49}{36} = \frac{2}{9}$. We have

$$\ln(1-x) = -\sum_{j=1}^{\infty}\frac{x^j}{j} \ge -x - \frac{1}{2}x^2\sum_{k=0}^{\infty}x^k = -x - \frac{1}{2}\frac{x^2}{1-x} \ge -x - \frac{9}{14}x^2$$

and hence

$$\gamma_p\left(\frac{s}{\sqrt{n}}\right)^n \ge \left(1 - c\frac{s^2}{n}\right)^n \ge \exp\left(-cs^2 - \frac{9}{14}c^2\frac{s^4}{n}\right) \ge \exp(-cs^2)\left(1 - \frac{9}{14}c^2\frac{s^4}{n}\right).$$

By Lemma 3.1 (a), $\frac{\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)} \ge 0.9429$ and hence $c \ge 0.1571$ and for $s \le \frac{7}{6}\sqrt{n}$

$$\begin{aligned}\int_0^{\frac{7}{6}\sqrt{n}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds &\ge \int_0^{\frac{7}{6}\sqrt{n}} \exp(-cs^2)\left(1 - \frac{9}{14}c^2\frac{s^4}{n}\right) ds \\ &= \int_0^{\infty} \exp(-cs^2)\left(1 - \frac{9}{14}c^2\frac{s^4}{n}\right) ds - \int_{\frac{7}{6}\sqrt{n}}^{\infty} \exp(-cs^2)\left(1 - \frac{9}{14}c^2\frac{s^4}{n}\right) ds\end{aligned}$$

For $s \ge \frac{7}{6}\sqrt{n}$ and $c \ge 0.1571$ we have $1 - \frac{9}{14}c^2\frac{s^4}{n} < 0$ for all $n \ge 35$. Actually, evaluating the last integral in terms of the error function shows that the integral is already negative for $n \ge 24$. Thus for $n \ge 24$

$$\begin{aligned}\int_0^{\frac{7}{6}\sqrt{n}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds &\ge \int_0^{\infty} \exp(-cs^2)\left(1 - \frac{9}{14}c^2\frac{s^4}{n}\right) ds \\ &= \frac{1}{2}\sqrt{\frac{\pi}{c}}\left(1 - \frac{27}{56}\frac{1}{n}\right) = \sqrt{\frac{3\pi}{2}}\sqrt{\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)}}\left(1 - \frac{27}{56}\frac{1}{n}\right),\end{aligned}$$

where we used that $\int_0^\infty \exp(-cs^2)ds = \frac{1}{2}\sqrt{\frac{\pi}{c}}$ and $\int_0^\infty \exp(-cs^2)c^2 s^4 ds = \frac{3}{8}\sqrt{\frac{\pi}{c}}$. By Lemma 3.2 we have $\gamma_p(s) > 0$ for all $0 \le s \le 2$. Hence

$$0 < \sqrt{n} \int_{\frac{7}{6}}^{2} \gamma_p(s)^n du = \int_{\frac{7}{6}\sqrt{n}}^{2\sqrt{n}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds.$$

By the proof of Proposition 2.2

$$\begin{aligned} |\gamma_p(s)| &= \left| \frac{p}{\Gamma\left(1+\frac{1}{p}\right)} \int_0^\infty \operatorname{sinc}(sr) r^p \exp(-r^p) dr \right| \\ &= \left| \frac{1}{s\Gamma\left(1+\frac{1}{p}\right)} \int_0^\infty \sin(sr)\ pr^{p-1} \exp(-r^p) dr \right|, \ |\sin(sr)| \le 1 \\ &\le \frac{1}{s\Gamma\left(1+\frac{1}{p}\right)} \int_0^\infty \exp(-u) du = \frac{1}{s\Gamma(1+\frac{1}{p})}. \end{aligned}$$

This yields the tail estimate for $p \ge 5$

$$\begin{aligned} \int_{2\sqrt{n}}^\infty \left|\gamma_p\left(\frac{s}{\sqrt{n}}\right)\right|^n ds &= \sqrt{n} \int_2^\infty |\gamma_p(s)|^n ds \le \frac{\sqrt{n}}{\Gamma\left(1+\frac{1}{p}\right)^n} \int_2^\infty s^{-n} ds \\ &= \frac{2\sqrt{n}}{n-1} \left(\frac{1}{2\Gamma\left(1+\frac{1}{p}\right)} \right)^n < \frac{2\sqrt{n}}{n-1} 0.5446^n. \end{aligned}$$

We conclude for $p \ge 5$ and $n \ge 24$ that

$$\begin{aligned} \int_0^\infty \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds &\ge \int_0^{\frac{7}{6}\sqrt{n}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds - \int_{2\sqrt{n}}^\infty \left|\gamma_p\left(\frac{s}{\sqrt{n}}\right)\right|^n ds \\ &\ge \sqrt{\frac{3\pi}{2}} \sqrt{\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)}} \left(1 - \frac{27}{56}\frac{1}{n}\right) - \frac{2\sqrt{n}}{n-1} 0.5446^n \\ &\ge \sqrt{\frac{3\pi}{2}} \sqrt{\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)}} \left(1 - \frac{27}{56}\frac{1}{n} - \frac{\sqrt{n}}{n-1} 0.5446^n\right), \end{aligned}$$

using $\sqrt{\frac{3\pi}{2}}\sqrt{\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)}} > 2$. For $n \ge 24$, $\frac{\sqrt{n}}{n-1}0.5446^n < \frac{10^{-5}}{n}$ and $\frac{27}{56} + 10^{-5} < \frac{193}{400}$, so that

$$A_{n,p}(a^{(n)}) = \Gamma\left(1+\frac{1}{p}\right)\frac{2}{\pi}\int_0^\infty \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n \, ds$$
$$\ge \sqrt{\frac{6}{\pi}}\sqrt{\frac{\Gamma\left(1+\frac{1}{p}\right)^3}{\Gamma\left(1+\frac{1}{p}\right)}}\left(1-\frac{193}{400}\frac{1}{n}\right).$$

This is $> A_{n,p}(a^{(2)}) = 2^{\frac{1}{2}-\frac{1}{p}}$, provided that

$$\frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} \ge \sqrt{\frac{3}{\pi}}\sqrt{\frac{2^{\frac{2}{p}}\Gamma\left(1+\frac{1}{p}\right)^3}{\Gamma\left(1+\frac{3}{p}\right)}}\left(1-\frac{193}{400}\frac{1}{n}\right) > 1.$$

By Lemma 3.1 (b), the quotient $g(p) := \sqrt{\frac{3}{\pi}}\sqrt{\frac{2^{\frac{2}{p}}\Gamma\left(1+\frac{1}{p}\right)^3}{\Gamma\left(1+\frac{3}{p}\right)}}$ is > 1 for all $2 < p < p_0 \simeq 26.265$, with $g(p) \ge 1 + \frac{1}{1317}(p_0 - p)$ for all $5 \le p \le p_0$. We find for $p \ge 5$ and $n \ge 24$,

$$\frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} \ge \left(1+\frac{1}{1317}(p_0-p)\right)\left(1-\frac{193}{400}\frac{1}{n}\right).$$

This is > 1 provided that $5 \le p \le p_0$ and $n \ge \frac{650}{p_0-p}$; $n \ge 24$ being automatically satisfied.

(iii) Secondly we consider the case $2 < p \le 5$. To estimate the remainder R in (4.2), we use that in this case $u^{\frac{5}{p}-1} \le u^{\frac{3}{2}}$. For $x > 0$

$$\int_x^\infty u^{\frac{3}{2}}\exp(-u)\,du \le \left(\left(\int_x^\infty u\exp(-u)\,du\right)\left(\int_x^\infty u^2\exp(-u)\,du\right)\right)^{\frac{1}{2}}$$
$$= \left((1+x)(2+2x+x^2)\right)^{\frac{1}{2}}\exp(-x),$$

Since the function $\phi(x) := \frac{(1+x)(2+2x+x^2)}{x^4} = \left(\frac{1}{x}+\frac{1}{x^2}\right)\left(2+\frac{2}{x}+\frac{1}{x^2}\right)$ is decreasing in x with $\phi\left(\frac{9}{2}\right) = \left(\frac{5\sqrt{110}}{81}\right)^2 < \left(\frac{13}{20}\right)^2$, we conclude that $\int_x^\infty u^{\frac{3}{2}}\exp(-u)\,du \le \frac{13}{20}x^2\exp(-x)$ for all $x \ge \frac{9}{2}$. Now choose $s \le \sqrt{\frac{n}{2}}$ for $2 \le p \le 5$. Then $x :=$

$\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p \geq \left(\frac{3}{\sqrt{2}}\right)^p \geq \frac{9}{2}$ and

$$R \leq \frac{1}{p}\left(2\left(\frac{2}{3}\frac{s}{\sqrt{n}}\right)^{p-1} + \frac{s^4}{26n^2}\frac{13}{20}\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^{2p}\right)\exp\left(-\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p\right).$$

Again we want this to be smaller than the fourth order term $\Gamma(1+\frac{5}{p})\frac{s^4}{130n^2}$, a condition which is strongest for $p=2$. We then require for $y := \frac{s}{\sqrt{n}}$

$$\frac{1}{2}\left(\frac{4}{3}y + \frac{81}{640}\right)\exp\left(-\left(\frac{3}{2}\frac{1}{y}\right)^2\right) < \Gamma\left(\frac{7}{2}\right)\frac{y^4}{130}.$$

This is equivalent to $g(y) := \frac{1}{\sqrt{\pi}}\left(26\frac{16}{9}\frac{1}{y^3} + 13\frac{27}{80}\frac{1}{y^4}\right)\exp\left(-\left(\frac{3}{2}\frac{1}{y}\right)^2\right) < 1$. The function g is increasing and positive in $y > 0$, with $g(0.7161) < 1$. Hence the condition is satisfied for all $0 < y \leq 0.7161$, and in particular for our choice $y = \frac{s}{\sqrt{n}} \leq \frac{1}{\sqrt{2}}$. Therefore for $s \leq \sqrt{\frac{n}{2}}$, as in part (ii),

$$\gamma_p\left(\frac{s}{\sqrt{n}}\right) \geq 1 - c\frac{s^2}{n}, \quad c := \frac{1}{6}\frac{\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}.$$

We have $c \leq \frac{1}{4}$ for $2 \leq p \leq 5$ and $x := c\frac{s^2}{n} \leq \frac{1}{8}$. Similarly as in (ii), $\ln(1-x) \geq -x - \frac{x^2}{1-x} \geq -x - \frac{4}{7}x^2$ and

$$\begin{aligned}\gamma_p\left(\frac{s}{\sqrt{n}}\right) &\geq \left(1 - c\frac{s^2}{n}\right)^n = \exp\left(n\ln\left(1 - c\frac{s^2}{n}\right)\right) \\ &\geq \exp\left(-cs^2 - \frac{4}{7}c^2\frac{s^4}{n}\right) \geq \exp(-cs^2)\left(1 - \frac{4}{7}c^2\frac{s^4}{n}\right)\end{aligned}$$

and

$$\begin{aligned}\int_0^{\sqrt{\frac{n}{2}}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds &\geq \int_0^{\sqrt{\frac{n}{2}}} \exp(-cs^2)\left(1 - \frac{4}{7}c^2\frac{s^4}{n}\right) ds \\ &\geq \int_0^{\infty} \exp(-cs^2)\left(1 - \frac{4}{7}c^2\frac{s^4}{n}\right) ds - \int_{\sqrt{\frac{n}{2}}}^{\infty} \exp(-cs^2)\, ds \\ &= \frac{1}{2}\sqrt{\frac{\pi}{c}}\left(1 - \frac{3}{7}\frac{1}{n}\right) - \int_{\sqrt{\frac{n}{2}}}^{\infty} \exp(-cs^2)\, ds\ .\end{aligned}$$

To estimate the error term, note that $c \geq .16219$, its approximate value for $p = 5$, and

$$\int_{\sqrt{\frac{n}{2}}}^{\infty} \exp(-cs^2)ds = \frac{1}{2\sqrt{c}} \int_{\frac{c}{2}n}^{\infty} \frac{1}{\sqrt{u}} \exp(-u)du$$
$$\leq \frac{1}{c\sqrt{2n}} \int_{\frac{c}{2}n}^{\infty} \exp(-u)du = \frac{1}{c\sqrt{2n}} \exp(-\frac{c}{2}n) \leq \frac{4.36}{\sqrt{n}} \, 0.9222^n \, .$$

Again $\int_{\sqrt{\frac{n}{2}}}^{2\sqrt{n}} \gamma_p \left(\frac{s}{\sqrt{n}}\right)^n ds \geq 0$, since by Lemma 3.2 $\gamma_p(x) > 0$ for all $0 \leq x \leq 2$ and, as in (ii), $\int_{2\sqrt{n}}^{\infty} \left|\gamma_p \left(\frac{s}{\sqrt{n}}\right)\right|^n ds \leq \frac{2\sqrt{n}}{n-1} \, 0.5446^n$, so that for $2 \leq p \leq 5$

$$\int_0^{\infty} \gamma_p \left(\frac{s}{\sqrt{n}}\right)^n ds \geq \frac{1}{2}\sqrt{\frac{\pi}{c}} \left(1 - \frac{3}{7}\frac{1}{n}\right) - \left(\frac{4.36}{\sqrt{n}} \, 0.9222^n + \frac{2\sqrt{n}}{n-1} \, 0.5446^n\right).$$

Since $c \leq \frac{1}{4}$, $\frac{1}{2}\sqrt{\frac{\pi}{c}} \geq \sqrt{\pi}$. Further for $n \geq 33$

$$\frac{3}{7} + \frac{1}{\sqrt{\pi}} \left(4.36\sqrt{n} \, 0.9222^n + \frac{2n\sqrt{n}}{n-1} \, 0.5446^n\right) \leq 1.405$$

so that $\int_0^{\infty} \gamma_p \left(\frac{s}{\sqrt{n}}\right)^n ds \geq \frac{1}{2}\sqrt{\frac{\pi}{c}} \left(1 - \frac{1.405}{n}\right)$. For $p \in [2, 4]$, using Lemma 3.1 (b),

$$\frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} \geq \sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1 + \frac{1}{p}\right)^3}{\Gamma\left(1 + \frac{3}{p}\right)}} \left(1 - \frac{1.405}{n}\right)$$
$$\geq \left(1 + \frac{p-2}{44}\right)\left(1 - \frac{1.405}{n}\right) \, .$$

For $n \geq \frac{65}{p-2}$ this is > 1, with $n \geq 33$ being automatically satisfied. For $p \in [4, 5]$, again by Lemma 3.1 (b), $\sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1+\frac{1}{p}\right)^3}{\Gamma\left(1+\frac{3}{p}\right)}} > \frac{25}{24}$ and $\frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} > 1$ is satisfied for $n \geq \frac{65}{p-2}$, too. □

5 Prerequisites for the proof of Theorem 1.2

We need two lemmas for the proof of Theorem 1.2.

Lemma 5.1 (a) *For $q \in [1, 2]$, let $f(q) := \frac{\Gamma\left(2-\frac{1}{q}\right)}{\Gamma\left(\frac{1}{q}\right)}$. Then f is decreasing, with $f(1) = 1$, $f(2) = \frac{1}{2}$ and $f\left(\frac{4}{3}\right) \leq 0.7397$.*

(b) *For* $q \in [1,2]$, *let* $g(q) := \sqrt{\frac{2^{\frac{2}{q}}}{\pi}\Gamma\left(\frac{1}{q}\right)\Gamma\left(2-\frac{1}{q}\right)}$. *Then* g' *has exactly one zero in* $q_1 \in (1,2)$, $q_1 \simeq 1.612$, *and* g *is strictly decreasing in* $[1, q_1)$ *and strictly increasing in* $(q_1, 2]$, *with* $g\left(\frac{4}{3}\right) = g(2) = 1$. *For* $q \in (\frac{4}{3}, 2)$, *we have*

$$g(q) \leq 1 - M\left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right), \quad M = 0.86326.$$

Proof (a) Differentiation gives $f'(q) = \frac{f(q)}{q^2}\left(\Psi\left(\frac{1}{q}\right)+\Psi\left(2-\frac{1}{q}\right)\right)$. Since Γ is logarithmic convex,Ψ is increasing. Hence $\Psi\left(\frac{1}{q}\right) \leq \Psi(1) = -\gamma$ and $\Psi\left(2-\frac{1}{q}\right) \leq \Psi\left(\frac{3}{2}\right) = 2(1-\ln(2)) - \gamma$, $\Psi\left(\frac{1}{q}\right)+\Psi\left(2-\frac{1}{q}\right) \leq -2(\gamma+\ln(2)-1) < 0$. Therefore f is decreasing in $[1,2]$. Moreover, $f\left(\frac{4}{3}\right) \leq 0.7397$.

(b) For g we find $(\ln(g))'(q) = \frac{1}{2q^2}\left(\Psi\left(2-\frac{1}{q}\right)-\Psi\left(\frac{1}{q}\right)-2\ln(2)\right)$. The function $h(q) := \Psi\left(2-\frac{1}{q}\right)-\Psi\left(\frac{1}{q}\right)-2\ln(2)$ is strictly increasing, since with $\frac{1}{q}+\frac{1}{p}=1$ we have, using (3.1),

$$h'(q) = \frac{1}{q^2}\left(\Psi'\left(2-\frac{1}{q}\right)+\Psi'\left(\frac{1}{q}\right)\right) = \frac{1}{q^2}\sum_{n=1}^{\infty}\left(\frac{1}{(n+\frac{1}{p})^2}+\frac{1}{(n-\frac{1}{p})^2}\right) > 0.$$

Moreover $h(1) = -2\ \ln(2) < 0$, $h(2) = 2(1-\ln(2)) > 0$. Thus h has exactly one zero $q_1 \in (1,2)$, $q_1 \simeq 1.612$. We get that g is decreasing in $[1, q_1)$ and increasing in $(q_1, 2]$. We have $g(1) = \frac{2}{\sqrt{\pi}} > 1$, $g\left(\frac{4}{3}\right) = g(2) = 1$ and $g(q) < 1$ for $q \in (\frac{4}{3}, 2)$.

For $\frac{4}{3} < q < 2$, choose $\theta \in (0,1)$ with $\frac{1}{q} = (1-\theta)\frac{1}{2}+\theta\frac{3}{4}$, $\theta = \frac{4}{q}-2$, $1-\theta = 3-\frac{4}{q}$. Since Γ is logarithmic convex, $F := \ln\Gamma$ satisfies $F'' = \Psi' > 0$ and by Taylor's formula with second degree remainder we have for some $\eta \in (\frac{1}{2}, \frac{3}{4})$

$$F\left(\frac{1}{q}\right) \leq (1-\theta)F\left(\frac{1}{2}\right)+\theta F\left(\frac{3}{4}\right)-\frac{\Psi'(\eta)}{2}\left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right).$$

Since by (3.1) Ψ' is decreasing, $\min_{\eta\in[\frac{1}{2},\frac{3}{4}]}\Psi'(\eta) = \Psi'\left(\frac{3}{4}\right)$ and

$$F\left(\frac{1}{q}\right) \leq \left(3-\frac{4}{q}\right)F\left(\frac{1}{2}\right)+\left(\frac{4}{q}-2\right)F\left(\frac{3}{4}\right)-\frac{\Psi'\left(\frac{3}{4}\right)}{2}\left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right).$$

Similarly, for $\frac{5}{4} < 2-\frac{1}{q} < \frac{3}{2}$, choose $\theta \in (0,1)$ with $2-\frac{1}{q} = (1-\theta)\frac{5}{4}+\theta\frac{3}{2}$, $\theta = 3-\frac{4}{q}$, $1-\theta = \frac{4}{q}-2$, such that with $\min_{\eta\in[\frac{5}{4},\frac{3}{2}]}\Psi'(\eta) = \Psi'\left(\frac{3}{2}\right)$

$$F\left(2-\frac{1}{q}\right) \leq \left(\frac{4}{q}-2\right)F\left(\frac{5}{4}\right)+\left(3-\frac{4}{q}\right)F\left(\frac{3}{2}\right)-\frac{\Psi'\left(\frac{3}{2}\right)}{2}\left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right).$$

This yields after exponentiation with $c := \frac{\Psi'\left(\frac{3}{4}\right)+\Psi'\left(\frac{3}{2}\right)}{4}$

$$\Gamma\left(\frac{1}{q}\right)\Gamma\left(2-\frac{1}{q}\right) \le \left(\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\right)^{3-\frac{4}{q}} \times \left(\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)^{\frac{4}{q}-2} \exp\left(-2c\left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right)\right).$$

Clearly $\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right) = \frac{\pi}{2}$, and by the complement formula for the Γ-function, see Abramowitz, Stegun [1], $\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right) = \frac{\pi}{4\sin\left(\frac{\pi}{4}\right)} = \frac{\pi}{2\sqrt{2}}$, so that

$$g(q) = \sqrt{\frac{2^{\frac{2}{q}}}{\pi}\Gamma\left(\frac{1}{q}\right)\Gamma\left(2-\frac{1}{q}\right)} \le \exp\left(-c\left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right)\right) =: k(q).$$

Let $\varepsilon := \left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right)$. Then $\varepsilon \le \frac{1}{64}$. Numerical evaluation yields $c \ge 0.86917$. By Taylor expansion $k(q) \le 1 - c\varepsilon + \frac{c^2\varepsilon^2}{2} = 1 - c\left(1-\frac{c\varepsilon}{2}\right)\varepsilon \le 1 - d\varepsilon$, $d := c - \frac{c^2}{128} \ge 0.86326 =:$ M. Therefore $g(q) \le 1 - M\left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right)$. □

Lemma 5.2 *For all* $\frac{4}{3} \le q \le 2$ *and* $0 \le s \le \frac{16}{5}$, *the functions* δ_q *of* (2.6) *satisfy* $\delta_{\frac{4}{3}}(s) \le \delta_q(s) \le \delta_2(s) = \exp\left(-\frac{s^2}{4}\right)$. *Further* $\delta_{\frac{4}{3}}\left(\frac{48}{25}\right) > 0$ *and* $\delta_{\frac{4}{3}}\left(\frac{16}{5}\right) > -0.588$.

Proof Let $2 \le p = \frac{q}{q-1} \le 4$ be the conjugate index of q and $s \in [\frac{48}{25}, \frac{16}{5}]$. We will show that $\frac{d}{dq}\delta_q(s) > 0$, or equivalently $\frac{d}{dp}\delta_q(s) < 0$. For $m > -1$ we have

$$\int_0^\infty r^m \exp(-r^p)dr = \frac{1}{p}\int_0^\infty u^{\frac{m+1}{p}-1}\exp(-u)du = \frac{1}{p}\Gamma\left(\frac{m+1}{p}\right). \quad (5.1)$$

Expanding $\cos(sr)$ into its Taylor series at zero, we find using (5.1)

$$\delta_q(s) = \sum_{n=0}^{\infty}\left(f_{2n}(p)\frac{s^{4n}}{(4n)!} - f_{2n+1}(p)\frac{s^{4n+2}}{(4n+2)!}\right) =: \sum_{n=0}^{\infty} F_n(p,s) \quad (5.2)$$

where $f_{2n}(p) := \frac{\Gamma\left(1+\frac{4n-1}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)}$, $f_{2n+1}(p) := \frac{\Gamma\left(1+\frac{4n+1}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)}$. Since Γ is logarithmic convex, we have for $x > 0$ and $0 \le \theta \le 1$ that $\Gamma(x+\theta) \le \Gamma(x)^{1-\theta}\Gamma(x+1)^\theta = x^\theta\Gamma(x)$. Let $n \ge 2$, $x := 1+\frac{4n-1}{p} \ge 1$ and $\theta := \frac{2}{p}$. We claim that $x^\theta = \left(1+\frac{4n-1}{p}\right)^\theta \le \frac{4n+1}{p}$. This is equivalent to $(4n+p-1)^{\frac{2}{p}}p^{1-\frac{2}{p}} \le 4n+1$. Applying the inequality $ab \le \frac{a^r}{r} + \frac{b^{r'}}{r'}$ with $r := \frac{p}{2}$ and $r' = \frac{p}{p-2}$, we get $(4n+p-1)^{\frac{2}{p}}p^{1-\frac{2}{p}} \le \frac{2}{p}(4n+p-1)+p-2 =$

$\frac{2}{p}(4n-1)+p$ which is $\leq 4n+1$ if and only if $n \geq \frac{p+1}{4}$, which is satisfied, since $p \leq 4$ and $n \geq 2$. Therefore

$$\frac{f_{2n+1}(p)}{f_{2n}(p)} = \frac{\Gamma\left(1+\frac{4n+1}{p}\right)}{\Gamma\left(1+\frac{4n-1}{p}\right)} \leq \frac{4n+1}{p} \tag{5.3}$$

and

$$\begin{aligned} F_n(p,s) &= f_{2n}(p)\frac{s^{4n}}{(4n)!}\left(1-\frac{f_{2n+1}(p)}{f_{2n}(p)}\frac{s^2}{(4n+1)(4n+2)}\right) \\ &\geq f_{2n}(p)\frac{s^{4n}}{(4n)!}\left(1-\frac{s^2}{2p(2n+1)}\right) > 0 \end{aligned}$$

for $n \geq 2$ and $s \leq \frac{16}{5} < \sqrt{20}$. Hence for all $m \geq 1$, $\delta_q(s) \geq \sum_{n=0}^{m} F_n(p,s)$. In particular, for $q = \frac{4}{3}$, we find by numerical evaluation $\delta_{\frac{4}{3}}\left(\frac{48}{25}\right) > 0.0026 > 0$, choosing $m=2$, and $\delta_{\frac{4}{3}}\left(\frac{16}{5}\right) > -0.588$, choosing $m=4$. Formula (5.2) implies

$$\begin{aligned} \frac{d}{dp}\delta_q(s) &= \sum_{n=0}^{\infty}\frac{d}{dp}F_n(p,s) =: -\frac{1}{p^2}\sum_{n=0}^{\infty}G_n(p,s) \\ &=: -\frac{1}{p^2}\sum_{n=0}^{\infty}\left(f_{2n}(p)g_{2n}(p)\frac{s^{4n}}{(4n)!} - f_{2n+1}(p)g_{2n+1}(p)\frac{s^{4n+2}}{(4n+2)!}\right), \end{aligned} \tag{5.4}$$

where in terms of the Digamma function Ψ, $g_{2n}(p) = (4n-1)\Psi\left(1+\frac{4n-1}{p}\right) + \Psi\left(1-\frac{1}{p}\right)$, $g_{2n+1}(p) = (4n+1)\Psi\left(1+\frac{4n+1}{p}\right) + \Psi\left(1-\frac{1}{p}\right)$. By Abramowitz, Stegun [1], 6.3, Ψ' is positive, decreasing and $\Psi'(1+x) \leq \frac{1}{x+\frac{1}{2}}$. Therefore

$$\begin{aligned} \Psi\left(1+\frac{4n+1}{p}\right) &\leq \Psi\left(1+\frac{4n-1}{p}\right) + \frac{2}{p}\Psi'\left(1+\frac{4n-1}{p}\right) \\ &\leq \Psi\left(1+\frac{4n-1}{p}\right) + \frac{1}{2n}(4n+1)\Psi\left(1+\frac{4n+1}{p}\right) \\ &\leq (4n-1)\Psi\left(1+\frac{4n-1}{p}\right) + 2\Psi\left(1+\frac{4n-1}{p}\right) + \frac{4n+1}{2n} \end{aligned}$$

so that

$$\frac{g_{2n+1}(p)}{g_{2n}(p)} \leq 1 + \frac{2\Psi\left(1+\frac{4n-1}{p}\right)+2+\frac{1}{2n}}{g_{2n}(p)} = 1 + \frac{2}{4n-1} + C_n(p) \tag{5.5}$$

where $C_n(p) := \frac{2-\frac{2\Psi\left(1-\frac{1}{p}\right)}{4n-1}+\frac{1}{2n}}{g_{2n}(p)}$. The function $k(x) := \Psi(1+x) - \ln(1+x)$ is increasing, since using (3.1) we find

$$k'(x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} - \frac{1}{1+x} > \int_1^{\infty} \frac{dy}{(y+x)^2} - \frac{1}{1+x} = 0.$$

Further, by Abramowitz, Stegun [1, 6.3.18], $\lim_{x\to\infty} \Psi(1+x) - \ln(1+x) = 0$. Therefore for any $x > 0$, $k(x) < 0$. For $x \geq \frac{7}{4}$, $k(x) \geq k\left(\frac{7}{4}\right) \simeq -0.193 > -\frac{1}{5}$. Hence for $n \geq 2$, $\Psi\left(1+\frac{4n-1}{p}\right) \geq \ln\left(1+\frac{4n-1}{p}\right) - \frac{1}{5}$ and

$$C_n(p) \leq \frac{2-\frac{2\Psi\left(1-\frac{1}{p}\right)}{4n-1}+\frac{1}{2n}}{(4n-1)\ln\left(1+\frac{4n-1}{p}\right) - \frac{4n-1}{5} + \Psi\left(1-\frac{1}{p}\right)} =: R.$$

We have that $\Psi\left(1-\frac{1}{p}\right) \in [-1.964, -1.085]$. The right side is maximal for $p = 4$, with $\Psi\left(\frac{3}{4}\right) \simeq -1.086$, the logarithmic term in the denominator being minimal then,

$$C_n(p) \leq \frac{2-\frac{2\Psi\left(\frac{3}{4}\right)}{4n-1}+\frac{1}{2n}}{(4n-1)\ln\left(n+\frac{3}{4}\right) - \frac{4n-1}{5} + \Psi\left(\frac{3}{4}\right)}.$$

A tedious calculation yields that $C_n(p) \leq c_n(p) := \frac{2}{3n\ln(n)}$ and $\frac{2}{4n-1} + c_n(2) \leq \frac{1}{n+\frac{1}{4}}$ for all $n \geq 5$. For $n = 2, 3, 4$, keeping a dependence on p, we may estimate $C_n(p) \leq R \leq c_n(p)$, where

$$c_2(p) = \frac{1}{6} + \frac{p}{10}, c_3(p) = \frac{1}{11} + \frac{p}{36}, c_4(p) = \frac{1}{18} + \frac{p}{72}.$$

Using this and Eqs. (5.5) and (5.3) we find for $n \geq 2$

$$\begin{aligned} \frac{f_{2n+1}(p)g_{2n+1}(p)}{f_{2n}(p)g_{2n}(p)} &\leq \frac{4n+1}{p}\left(1+\frac{2}{4n-1}+c_n(p)\right) \leq \frac{4n+1}{2}\left(1+\frac{2}{4n-1}+c_n(2)\right) \\ &=: q_n \leq \frac{4n+1}{2}\left(1+\frac{1}{n+\frac{1}{4}}\right) = \frac{4n+5}{2}, \end{aligned}$$

the last estimate for q_n holding for $n \geq 5$, whereas $q_2 \leq \frac{15}{2}$, $q_3 \leq \frac{25}{3}$ and $q_4 \leq \frac{21}{2}$. This implies

$$G_n(p,s) \geq f_{2n}(p)g_{2n}(p)\frac{s^{4n}}{(4n)!}\left(1-\frac{q_n s^2}{(4n+1)(4n+2)}\right) =: \tilde{G}_n(p,s) > 0$$

for all $n \geq 2$ and $s \leq \frac{16}{5} < \sqrt{12}$. Hence by (5.4) for all $m \geq 1$

$$\frac{d}{dp}\delta_q(s) \leq -\frac{1}{p^2}\sum_{n=0}^{m} G_n(p,s).$$

For $m = 1$, with $g_0(s) = 0$, we have

$$\frac{d}{dp}\delta_q(s) \leq +\frac{s^2}{2p^2}\left[a(p) - b(p)\frac{s^2}{12} + c(p)\frac{s^4}{360}\right] =: \phi(p,s)$$

with $-0.972 \leq a(p) := f_1(p)g_1(p) \leq -0.954$ varying very little, $-0.255 \leq b(p) := f_2(p)g_2(p) \leq 0.114$, $0 < b(p)$ for $p \leq 2.83$ and $c(p) := f_3(p)g_3(p)$ decreasing in $p \in [2,4]$, with value 6.66 at $p = 2$ and 1.64 at $p = 4$. Therefore $360|a(p)| \geq 343.5$ and $\phi(p,s) < 0$ will be satisfied if

$$s^2 < 15\frac{b(p)}{c(p)} + \sqrt{\left(15\frac{b(p)}{c(p)}\right)^2 + \frac{343.5}{c(p)}}.$$

This holds for all $0 \leq s \leq \frac{16}{5}$, if $c(p) \leq 3.275$, i.e. $p \geq 2.81$. For $0 \leq p \leq 2$, the right side is minimal for $p = 2$ and we require $s \leq 2.72$. If $p < 2.81$ and $s > 2.72$ one needs two more terms, $m = 3$, to show $\frac{d}{dp}\delta_q(s) < 0$,

$$\frac{d}{dp}\delta_q(s) \leq -\frac{1}{p^2}\left[G_0(p,s) + G_1(p,s) + \tilde{G}_2(p,s) + \tilde{G}_3(p,s)\right] < 0.$$

$\square$

Corollary 5.3 *For all* $\frac{4}{3} \leq q \leq 2$ *and* $\frac{48}{25} \leq s \leq \frac{16}{5}$, $\left|\delta_q(s)\right| \leq 0.588$.

Proof By Lemma 5.2, $\delta_{\frac{4}{3}}(s) \leq \delta_q(s) \leq \delta_2(s) = \exp\left(-\frac{s^2}{4}\right) \leq \exp\left(-\left(\frac{24}{25}\right)^2\right) < \frac{2}{5}$ for all $s \in [\frac{48}{25}, \frac{16}{5}]$. By (2.7), $\delta'_{\frac{4}{3}}(s) = -\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} s\gamma_4(s)$. According to Boyd [4], page 83, $\gamma_4(s) > 0$ for all $0 \leq s \leq 3.45$, the first positive zero of γ_4 being at $s_1 \simeq 3.4535$. Therefore $\delta_{\frac{4}{3}}$ is strictly decreasing in $[\frac{48}{25}, \frac{16}{5}]$, with $\frac{2}{5} > \delta_{\frac{4}{3}}(s) \geq \delta_{\frac{4}{3}}\left(\frac{16}{5}\right) > -0.588$ by Lemma 5.2. We conclude that

$$\max\left\{\left|\delta_q(s)\right| \mid q \in \left[\frac{4}{3}, 2\right] \; s \in \left[\frac{48}{25}, \frac{16}{5}\right]\right\} \leq 0.588.$$

$\square$

Remark. For $q \searrow 1$, $\delta_q(s) \to \cos(s)$, so that $|\delta_q(\pi)| \to 1$. Corollary 5.3 does not extend to the range $1 \leq q < \frac{4}{3}$.

6 Proof of Theorem 1.2

Proof of Theorem 1.1.

Barthe and Naor [3] showed for $1 \le q < \infty$ that

$$\lim_{n\to\infty} \frac{P_{n,q}(a^{(n)})}{P_{n,q}(a^{(2)})} = \sqrt{\frac{2^{\frac{2}{q}}}{\pi}\Gamma\left(\frac{1}{q}\right)\Gamma\left(2-\frac{1}{q}\right)}$$

and this is < 1 if and only if $\frac{4}{3} < q < 2$, see Lemma 5.1 (b). Now consider $\frac{4}{3} < q < 2$ and let $p = \frac{q}{q-1}$ be the conjugate index, $2 < p < 4$. As in the proof of Theorem 1.1, we use $\cos(x) \ge 1 - \frac{x^2}{2} + \frac{x^4}{26} > 0$ for all $0 \le x \le \frac{3}{2}$, so that by (2.6) for all $s \le \frac{3}{2}\sqrt{n}$

$$\begin{aligned}
\delta_q\left(\frac{s}{\sqrt{n}}\right) &\ge \frac{p}{\Gamma\left(\frac{1}{q}\right)}\Big(\int_0^{\frac{3}{2}\frac{\sqrt{n}}{s}}\left(1-\frac{s^2r^2}{2n}+\frac{s^4r^4}{26n^2}\right) r^{p-2}\exp(-r^p)\,dr \\
&\quad + \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty}\cos\left(\frac{sr}{\sqrt{n}}\right) r^{p-2}\exp(-r^p)\,dr\Big) \\
&\ge \frac{p}{\Gamma\left(\frac{1}{q}\right)}\left(\int_0^{\infty}\left(1-\frac{s^2r^2}{2n}+\frac{s^4r^4}{26n^2}\right) r^{p-2}\exp(-r^p)\,dr - R\right) \\
&= \frac{1}{\Gamma\left(1-\frac{1}{p}\right)}\left(\Gamma\left(1-\frac{1}{p}\right)-\Gamma\left(1+\frac{1}{p}\right)\frac{s^2}{2n}+\Gamma\left(1+\frac{3}{p}\right)\frac{s^4}{26n^2}-R\right), \\
R &:= \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty}\left(2-\frac{s^2}{2n}r^2+\frac{s^4}{26n^2}r^4\right) r^{p-2}\exp(-r^p)\,dr \\
&= \frac{1}{p}\int_{\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p}^{\infty}\left(2u^{-\frac{1}{p}}-\frac{s^2}{2n}u^{\frac{1}{p}}+\frac{s^4}{26n^2}u^{\frac{3}{p}}\right)\exp(-u)\,du\,.
\end{aligned}$$

Then $u^{-\frac{1}{p}} \le \frac{2}{3}\frac{s}{\sqrt{n}}$ and $u^{\frac{3}{p}} \le u^{\frac{3}{2}}$. As in the proof of Theorem 1 (iii), for $x \ge \frac{9}{2}$

$$\int_x^{\infty} u^{\frac{3}{2}}\exp(-u)du \le \left((1+x)(2+2x+x^2)\right)^{\frac{1}{2}}\exp(-x) \le \frac{13}{20}x^2\exp(-x).$$

Choose again $s \le \sqrt{\frac{n}{2}}$. Then with $x := \left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p \ge \left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2}$ and $y := \frac{s}{\sqrt{n}}$

$$R \le \frac{1}{p}\left(2\frac{2}{3}y+\frac{y^4}{26}\frac{13}{20}\left(\frac{3}{2}\frac{1}{y}\right)^{2p}\right)\exp\left(-\left(\frac{3}{2}\frac{1}{y}\right)^p\right).$$

We want $R \le \Gamma\left(1+\frac{3}{p}\right)\frac{y^4}{26}$, a condition which is strongest for $p=2$ when it means

$$\frac{1}{2}\left(\frac{4}{3}y+\frac{81}{640}\right) < \frac{\Gamma\left(\frac{5}{2}\right)}{26}y^4 \ \exp\left(\frac{9}{4}\frac{1}{y^2}\right)$$

which is the same requirement as in (iii) of the proof of Theorem 1.1, being valid for $0 \le y \le 0.7161$. Thus the choice of $s \le \sqrt{\frac{n}{2}}$ is allowed and then

$$\delta_q\left(\frac{s}{\sqrt{n}}\right) \ge 1-c\frac{s^2}{n}, \quad c := \frac{1}{2}\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)} = \frac{1}{2}\frac{\Gamma\left(2-\frac{1}{q}\right)}{\Gamma\left(\frac{1}{q}\right)}.$$

We have by Lemma 5.1 (a) $\frac{1}{4} \le c \le 0.3699 < \frac{37}{100}$, the lower estimate attained for $p=q=2$, the maximum for c attained for $p=4$, $q=\frac{4}{3}$. Therefore for $x := c\frac{s^2}{n} \le \frac{37}{200}$, $\ln(1-x) \ge -x-\frac{1}{2}\frac{x^2}{1-x} \ge -x-\frac{100}{163}x^2$ and $\left(1-c\frac{s^2}{n}\right)^n \ge \exp(-cs^2)\left(1-\frac{100}{163}c^2\frac{s^4}{n}\right)$. This yields the estimate

$$\begin{aligned}
\int_0^{\sqrt{\frac{n}{2}}} \frac{1-\delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2}\,ds &\le \int_0^{\sqrt{\frac{n}{2}}} \frac{1-\exp(-cs^2)\left(1-\frac{100}{163}c^2\frac{s^4}{n}\right)}{s^2}\,ds \\
&\le \int_0^{\infty} \frac{1-\exp(-cs^2)}{s^2}ds - \int_{\sqrt{\frac{n}{2}}}^{\infty} \frac{1-\exp(-cs^2)}{s^2}ds + \frac{100}{163}\frac{c^2}{n}\int_0^{\infty} s^2\exp(-cs^2)\,ds \\
&= \sqrt{\pi c}\left(1+\frac{25}{163}\frac{1}{n}\right) - \int_{\sqrt{\frac{n}{2}}}^{\infty} \frac{1-\exp(-cs^2)}{s^2}\,ds\,.
\end{aligned}$$

Therefore, using (2.6),

$$\begin{aligned}
P_{n,q}(a^{(n)}) &= \Gamma\left(\frac{1}{q}\right)\frac{2}{\pi}\int_0^{\infty} \frac{1-\delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2}\,ds \\
&= \Gamma\left(\frac{1}{q}\right)\frac{2}{\pi}\left(\int_0^{\sqrt{\frac{n}{2}}} \frac{1-\delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2}\,ds + \int_{\sqrt{\frac{n}{2}}}^{\infty} \frac{1-\delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2}\,ds\right) \\
&\le \Gamma\left(\frac{1}{q}\right)\frac{2}{\pi}\left(\sqrt{\pi c}\left(1+\frac{25}{163}\frac{1}{n}\right) + \int_{\sqrt{\frac{n}{2}}}^{\infty} \frac{\exp(-cs^2)-\delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2}\,ds\right) \\
&= \sqrt{\frac{2}{\pi}\Gamma\left(\frac{1}{q}\right)\Gamma\left(2-\frac{1}{q}\right)}\left(1+\frac{25}{163}\frac{1}{n}\right) + \Gamma\left(\frac{1}{q}\right)\frac{2}{\pi}S\,,
\end{aligned}$$

$S := \frac{1}{\sqrt{n}} \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{\exp(-cu^2)^n - \delta_q(u)^n}{u^2} du$. Since $c \geq \frac{1}{4}$ and $\delta_q(s) > 0$ for all $0 \leq u \leq \frac{48}{25}$ by Lemma 5.2, we find

$$S \leq \frac{1}{\sqrt{n}} \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{\exp\left(-\frac{u^2}{4} n\right)}{u^2} \, du + \frac{1}{\sqrt{n}} \int_{\frac{48}{25}}^{\infty} \frac{|\delta_q(u)|^n}{u^2} \, du.$$

For $n \geq 8$, $v = \frac{u^2}{4} n \geq 1$ and

$$\begin{aligned} \frac{1}{\sqrt{n}} \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{\exp\left(-\frac{u^2}{4} n\right)}{u^2} \, du &= \frac{1}{4} \int_{\frac{n}{8}}^{\infty} \frac{\exp(-v)}{v^{\frac{3}{2}}} \, dv \\ &\leq \frac{1}{4} \left(\frac{8}{n}\right)^{\frac{3}{2}} \int_{\frac{n}{8}}^{\infty} \exp(-v) \, dv = \frac{4\sqrt{2}}{n^{\frac{3}{2}}} \exp\left(-\frac{n}{8}\right) \leq \frac{4\sqrt{2}}{n^{\frac{3}{2}}} \, 0.8825^n \, . \end{aligned}$$

By Corollary 5.3 we have $|\delta_q(s)| \leq 0.588$ for all $\frac{48}{25} \leq u \leq \frac{16}{5}$. Therefore

$$\frac{1}{\sqrt{n}} \int_{\frac{48}{25}}^{\frac{16}{5}} \frac{|\delta_q(u)|^n}{u^2} \, du \leq \frac{1}{\sqrt{n}} \, 0.588^n \int_{\frac{48}{25}}^{\frac{16}{5}} \frac{du}{u^2} = \frac{5}{24} \frac{1}{\sqrt{n}} \, 0.588^n.$$

Integration by parts shows for $\frac{4}{3} \leq q < 2$, $2 < p \leq 4$ that

$$\begin{aligned} \left|\delta_q(u)\right| &= \left| \frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^{\infty} \frac{\sin(ur)}{u} (p - 2 - pr^p) \, r^{p-3} \exp(-r^p) \, dr \right| \\ &\leq \frac{1}{u} \frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^{\infty} |p - 2 - pr^p| \, r^{p-3} \, \exp(-r^p) \, dr \\ &= \frac{1}{u} \frac{2p}{\Gamma\left(1 - \frac{1}{p}\right)} \left(\frac{1 - \frac{2}{p}}{e}\right)^{1-\frac{2}{p}} =: \phi(p) \frac{1}{u}, \end{aligned}$$

using that $\int (p - 2 - pr^p) \, r^{p-3} \, \exp(-r^p) \, dr = r^{p-2} \exp(-r^p)$. For ϕ, we have that $(\ln \phi)'(p) = \frac{1}{p^2} \left(2 \ln\left(1 - \frac{2}{p}\right) + p - \Psi\left(1 - \frac{1}{p}\right)\right)$ and $\lambda(p) := 2 \ln\left(1 - \frac{2}{p}\right) + p - \Psi\left(1 - \frac{1}{p}\right)$ is increasing, since $\lambda'(p) = \frac{p(3+(p-1)^2) - (p-2)\Psi'\left(1-\frac{1}{p}\right)}{p^2(p-2)}$ and $\Psi'\left(1 - \frac{1}{p}\right) \leq \Psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2}$, so that $\lambda'(p) \geq \frac{p(3+(p-1)^2) - (p-2)\frac{\pi^2}{2}}{p^2(p-2)} > 0$. We have $\lambda(2) < 0$, $\lambda(4) > 0$, so that ϕ is first decreasing and then increasing in $[2, 4]$. Since $\phi(2) < \phi(4) < \frac{14}{5}$, we conclude that $\left|\delta_q(u)\right| < \frac{14}{5}\frac{1}{u}$. Then

$$\begin{aligned}\frac{1}{\sqrt{n}}\int_{\frac{16}{5}}^{\infty}\frac{|\delta_q(u)|^n}{u^2}\,du &\le \frac{1}{\sqrt{n}}\left(\frac{14}{5}\right)^n\int_{\frac{16}{5}}^{\infty}\frac{du}{u^{n+2}} \\ &= \frac{5}{16}\frac{1}{\sqrt{n}(n+1)}\left(\frac{7}{8}\right)^n \le \frac{5}{16}\frac{1}{n^{\frac{3}{2}}}\,0.875^n\,.\end{aligned}$$

We finally get that

$$S \le \frac{4\sqrt{2}}{n^{\frac{3}{2}}}\,0.8825^n + \frac{5}{24}\frac{1}{\sqrt{n}}\,0.588^n + \frac{5}{16}\frac{1}{n^{\frac{3}{2}}}\,0.875^n.$$

By Lemma 5.1 (a), $\sqrt{\frac{2}{\pi}\frac{\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(2-\frac{1}{q}\right)}} \le \frac{2}{\sqrt{\pi}}$, with equality for $p = q = 2$, and $\frac{2}{\sqrt{\pi}}S \le \frac{0.10559}{n}$ for all $n > 20$. We conclude with $\frac{25}{163} + 0.10559 < 0.25896$ that

$$P_{n,q}(a^{(n)}) \le \sqrt{\frac{2}{\pi}\Gamma\left(\frac{1}{q}\right)\Gamma\left(2-\frac{1}{q}\right)}\left(1+\frac{0.25896}{n}\right).$$

By Barthe, Naor [3] $P_{n,q}(a^{(2)}) = 2^{\frac{1}{2}-\frac{1}{q}}$, so that together with Lemma 5.2 (b)

$$\begin{aligned}\frac{P_{n,q}(a^{(n)})}{P_{n,q}(a^{(2)})} &\le \sqrt{\frac{2^{\frac{2}{q}}}{\pi}\Gamma\left(\frac{1}{q}\right)\Gamma\left(2-\frac{1}{q}\right)}\left(1+\frac{0.25896}{n}\right) \\ &\le \left(1 - M\left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right)\right)\left(1+\frac{0.25896}{n}\right),\ \ M = 0.86326.\end{aligned}$$

Suppose that n satisfies $n \ge \frac{4}{5}\frac{q^2}{(q-\frac{4}{3})(2-q)}$. Then the last product is < 1, since $\frac{0.25896}{n} \le M\left(\frac{1}{q}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{1}{q}\right)$ suffices and this requires $n > \frac{\frac{8}{3}\frac{0.25896}{M}q^2}{(q-\frac{4}{3})(2-q)}$, with $\frac{8}{3}\frac{0.25896}{M} < \frac{4}{5}$. The condition $n \ge \frac{4}{5}\frac{q^2}{(q-\frac{4}{3})(2-q)}$ will be satisfied if $n > \frac{\frac{32}{15}}{q-\frac{4}{3}} + \frac{\frac{24}{5}}{2-q} = \frac{\frac{8}{3}q-\frac{32}{15}}{(q-\frac{4}{3})(2-q)} \ge \frac{\frac{4}{5}q^2}{(q-\frac{4}{3})(2-q)}$, where the last inequality is an equality for $q = \frac{4}{3}$ and $q = 2$. Hence for $n > \frac{\frac{32}{15}}{q-\frac{4}{3}} + \frac{\frac{24}{5}}{2-q} =: \phi(q)$ we have $P_{n,q}(a^{(n)}) < P_{n,q}(a^{(2)})$. The restriction $n > 20$ is automatically satisfied since the minimum of ϕ is $\phi\left(\frac{8}{5}\right) = 20$. We have the equality $\lim_{n\to\infty}\frac{P_{n,q}(a^{(n)})}{P_{n,q}(a^{(2)})} = \sqrt{\frac{2^{\frac{2}{q}}}{\pi}\Gamma\left(\frac{1}{q}\right)\Gamma\left(2-\frac{1}{q}\right)}$. □

Remark. Barthe and Naor [3] show that $P_{n,q}(a) \le P_{n,q}(a^{(n)})$ for all $a \in S^{n-1}$ and $q \ge 2$. Since the quantities are differentiable and coincide for $q = 2$, they conclude that $\frac{d}{dq}P_{n,q}(a)|_{q=2} \le \frac{d}{dq}P_{n,q}(a^{(n)})|_{q=2}$ holds. If this inequality were strict,

$P_{n,q}(a) \geq P_{n,q}(a^{(n)})$ might hold for some $2 - \varepsilon_n < q < 2$. In this case, for q close to 2, $a^{(n)}$ might always yield the minimum, and no singularity $O(\frac{1}{2-q})$ would occur in Theorem 1.2.

Similarly, in the case of sections, Koldobsky [10] showed that $A_{n,p}(a^{(n)}) \leq A_{n,p}(a)$ holds for all $1 < p < 2$ and $a \in S^{n-1}$, which together with $A_{n,2}(a^{(n)}) = A_{n,2}(a)$ implies $\frac{d}{dp} A_{n,p}(a)|_{p=2} \leq \frac{d}{dp} A_{n,p}(a^{(n)})|_{p=2}$. If this inequality were strict, it might be that $A_{n,p}(a) \leq A_{n,p}(a^{(n)})$ for p close to 2, $2 < p < 2 + \varepsilon_n$.

However, numerical evaluation of (2.3) yields for $p = 6$, $p = 8$ and $n = 3, 4$ that $a^{(n)}$ is not maximal, since

$$A_{3,6}(a^{(2)}) = 2^{\frac{1}{3}} \simeq 1.260 > A_{3,6}(a^{(3)}) \simeq 1.250$$
$$A_{4,8}(a^{(2)}) = 2^{\frac{3}{8}} \simeq 1.297 > A_{4,8}(a^{(4)}) \simeq 1.295 > A_{4,8}(a^{(3)}) \simeq 1.270.$$

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