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# Carleson measures and Berezin-type operators on Fock spaces

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## Abstract

We characterize (vanishing) Fock–Carleson measures by products of functions in Fock spaces. We also study the boundedness of Berezin-type operators from a weighted Fock space to a Lebesgue space. Due to the special properties of Fock–Carleson measures, the boundedness of Berezin-type operators on Fock spaces is different from the corresponding results on Bergman spaces.

**Keywords** Carleson measure · Fock space · Product of functions · Integral operator · Berezin transform

**Mathematics Subject Classification** 32A37 · 47G10

## 1 Introduction

Carleson measures are important research objects and play powerful role in function spaces and the theory of operators. The concept of it was first introduced by Carleson in [5, 6], where it is used to study interpolating sequences and the corona problem of

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all bounded analytic functions on the unit disc. After that, there are a large amount of works on characterizations of Carleson measures in or between spaces, including Hardy spaces and Bergman spaces, on various domains, such as Hardy–Carleson measures on the unit disc [9, 22, 31] and on the unit ball [11, 24, 27], Bergman–Carleson measures on the unit ball [19–21, 23] and on the polydisc [10] and so on. Then it has been further investigated with the development of the theory of Toeplitz operators, such as harmonic Bergman–Carleson measures on the upper half space [7], holomorphic Bergman-Carleson measures on the strongly pseudoconvex domain [1–4, 15] and on the Siegel upper half space [17, 30], Besov–Carleson measures on the unit ball [16, 26, 29], and Fock–Carleson measures in $\mathbb{C}^n$ [12–14, 33].

In 2010, Zhao [34] gave a characterization of Carleson measures using products of functions in Hardy spaces on the unit disc. In 2015, Pau and Zhao [25] overcame the lack of Riesz factorization theorem on weighted Bergman spaces and gave a corresponding result on the unit ball. Recently, Abate, Mongodi, and Raissy [1, 4] extended Pau and Zhao's results to strongly pseudoconvex domains and gave a characterization of skew Carleson measures through products of functions in weighted Bergman spaces. They also obtained the boundedness and compactness of a larger class of Toeplitz operators from one weighted Bergman space to another. Wang and Zhou [32] investigated tent Carleson measures in terms of products of functions in Hardy-type tent spaces and also obtained the boundedness and compactness of Toeplitz operators between distinct Hardy-type tent spaces. Motivated by these results, we aim to give some characterizations of Fock–Carleson measures involving products of functions in Fock spaces.

Let $\mathbb{C}^n$ be the $n$-dimensional complex Euclidean space. For $0 < p < \infty$ and $0 < \alpha < \infty$, the function space $L^p_\alpha$ consists of all measurable functions $f$ on $\mathbb{C}^n$ for which

$$\|f\|_{p,\alpha} = \left( \int_{\mathbb{C}^n} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p \mathrm{d}v(z) \right)^{\frac{1}{p}} < \infty,$$

where $\mathrm{d}v$ denotes the Lebesgue volume measure. When $\alpha = 0$, we write $L^p_0 = L^p$. Let $H(\mathbb{C}^n)$ be the family of all holomorphic functions on $\mathbb{C}^n$. Fock space $F^p_\alpha$ is defined by

$$F^p_\alpha := L^p_\alpha \cap H\left(\mathbb{C}^n\right).$$

It is clear that $F^p_\alpha$ is a Banach space under the norm $\|\cdot\|_{p,\alpha}$ for $1 \le p < \infty$ and is an Fréchet space under $d(f, g) = \|f - g\|^p_{p,\alpha}$ for $0 < p < 1$.

For $0 < p, q < \infty$, a positive Borel measure $\mu$ on $\mathbb{C}^n$ is called a $(p, q)$-Fock–Carleson measure if there is a constant $C > 0$ such that

$$\int_{\mathbb{C}^n} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^q \mathrm{d}\mu(z) \le C \|f\|^q_{p,\alpha}$$

for any $f \in F^p_\alpha$. We say $\mu$ is a vanishing $(p, q)$-Fock–Carleson measure if

$$\lim_{k\to\infty} \int_{\mathbb{C}^n} \left| f_k(z) e^{-\frac{\alpha}{2}|z|^2} \right|^q \mathrm{d}\mu(z) = 0$$

whenever $\{f_k\}$ is a bounded sequence in $F^p_\alpha$ that converges to 0 uniformly on any compact subset of $\mathbb{C}^n$ as $k \to \infty$. Since $(p, q)$-Fock–Carleson measures only depend on the ratio $\lambda := q/p$, we simply say (vanishing) $(\lambda, \alpha)$-Fock–Carleson measures for (vanishing) $(p, q)$-Fock–Carleson measures. We refer the reader to [12] for details. Our main results are the following three theorems.

**Theorem 1.1** *Let $\mu$ be a positive Borel measure on $\mathbb{C}^n$, $0 < p_i, q_i < \infty$ and $0 < \alpha_i < \infty$, where $i = 1, 2, \ldots, k$ and $k \geq 1$. Set*

$$\lambda = \sum_{i=1}^{k} \frac{q_i}{p_i}, \quad \gamma = \sum_{i=1}^{k} \alpha_i. \tag{1.1}$$

*Then $\mu$ is a $(\lambda, \gamma)$-Fock–Carleson measure if and only if there is a constant $C > 0$ such that for any $f_i \in F^{p_i}_{\alpha_i}$,*

$$\int_{\mathbb{C}^n} \prod_{i=1}^{k} |f_i(z)|^{q_i}\, e^{-\frac{\alpha_i}{2}|z|^2 q_i} d\mu(z) \leq C \prod_{i=1}^{k} \|f_i\|^{q_i}_{p_i,\alpha_i}. \tag{1.2}$$

**Theorem 1.2** *Let $\mu$ be a positive Borel measure on $\mathbb{C}^n$, $0 < p_i, q_i < \infty$ and $0 < \alpha_i < \infty$, where $i = 1, 2, \ldots, k$ and $k \geq 1$. Let*

$$\lambda = \sum_{i=1}^{k} \frac{q_i}{p_i}, \quad \gamma = \sum_{i=1}^{k} \alpha_i.$$

*Then the following statements are equivalent:*

(i) *$\mu$ is a vanishing $(\lambda, \gamma)$-Fock–Carleson measure;*
(ii) *For any sequence $\{f_{1,l}\}$ in $F^{p_1}_{\alpha_1}$ that is convergent to 0 uniformly on any compact subset of $\mathbb{C}^n$,*

$$\lim_{l\to\infty} F(l) = 0,$$

*where*

$$F(l) = \sup_{\substack{\|f_i\|_{p_i,\alpha_i} \leq 1 \\ i = 2,\ldots,k}} \left\{ \int_{\mathbb{C}^n} \left| f_{1,l}(z) e^{-\frac{\alpha_1}{2}|z|^2} \right|^{q_1} \prod_{i=2}^{k} \left| f_i(z) e^{-\frac{\alpha_i}{2}|z|^2} \right|^{q_i} d\mu(z) \right\};$$

(iii) *For any sequences* $\{f_{1,l}\}, \{f_{2,l}\}, \ldots, \{f_{k,l}\}$ *in* $F^{p_1}_{\alpha_1}, F^{p_2}_{\alpha_2}, \ldots, F^{p_k}_{\alpha_k}$, *respectively, that are all convergent to 0 uniformly on any compact subset of* $\mathbb{C}^n$,

$$\lim_{l\to\infty} \int_{\mathbb{C}^n} \prod_{i=1}^{k} \left| f_{i,l}(z) e^{-\frac{\alpha_i}{2}|z|^2} \right|^{q_i} d\mu(z) = 0.$$

The proof of Theorem 1.1, especially the necessity for the case of $0 < \lambda < 1$, is related to the boundedness of a class of integral operators defined by

$$S^{r,t}_{\mu} f(z) = \int_{\mathbb{C}^n} e^{-\frac{\alpha_2 t}{2}|z-w|^2} |f(w)|^r e^{-\frac{\alpha_1}{2}|w|^2 r} \mathrm{d}\mu(w)$$

for $0 < r, t < \infty$. When $r = 0$, it is that

$$S^{0,t}_{\mu}(z) := \widetilde{\mu}_t(z) = \int_{\mathbb{C}^n} e^{-\frac{\alpha t}{2}|z-w|^2} \mathrm{d}\mu(w). \tag{1.3}$$

It is just the $t$-Berezin transform of $\mu$ defined in [12]. And we see (2.1) in detail. Therefore, we call $S^{r,t}_{\mu}$ a Berezin-type operator of $\mu$ on Fock spaces.

Here and follows, we say that the Berezin-type operator $S^{r,t}_{\mu}$ is bounded from a weighted Fock space $F^{p_1}_{\alpha_1}$ to a Lebesgue space $L^{p_2}$ if there is a constant $C > 0$ such that $\|S^{r,t}_{\mu} f\|_{p_2} \le C\|f\|^r_{p_1,\alpha_1}$ for any $f \in F^{p_1}_{\alpha_1}$.

**Theorem 1.3** *Let* $\mu$ *be a positive Borel measure on* $\mathbb{C}^n$, $0 < p_1, p_2 < \infty$ *and* $0 < \alpha_1, \alpha_2 < \infty$. *Let*

$$\lambda = 1 + \frac{r}{p_1} - \frac{1}{p_2}, \quad \gamma = \alpha_1 + \alpha_2.$$

*Suppose* $\lambda > 0$. *If* $S^{r,t}_{\mu}$ *is bounded from* $F^{p_1}_{\alpha_1}$ *to* $L^{p_2}$, *then* $\mu$ *is a* $(\lambda, \gamma)$*-Fock–Carleson measure.*

A similar class of integral operators on the unit ball of $\mathbb{C}^n$ was studied in [25, Lemma 3.3], and then its generalizations were studied in [18, 28]. Since the characterization of the Fock–Carleson measure is different from that of the Bergman–Carleson measure on the unit ball, we only give the necessity of the boundedness of $S^{r,t}_{\mu}$ from $F^{p_1}_{\alpha_1}$ to $L^{p_2}$. This is enough for proving Theorem 1.1, but we do not know whether the sufficiency of it holds or not. We list it as an open problem.

**Open Problem** Let $\mu$ be a positive Borel measure on $\mathbb{C}^n$, $0 < p_1, p_2 < \infty$ and $0 < \lambda, \gamma < \infty$. If $\mu$ is a $(\lambda, \gamma)$-Fock–Carleson measure, then how is the boundedness of $S^{r,t}_{\mu}$ from $F^{p_1}_{\alpha_1}$ to $L^{p_2}$?

The paper is organized as follows. We collect some notations and preliminary results in Sect. 2. We are devoted to proving Theorem 1.3 in Sect. 3 and proving Theorems 1.1 and 1.2 in Sect. 4.

In what follows, the positive constant $C$ may change from line to line but does not depend on the functions. The notation $A \lesssim B$ means that there is a constant $C$ such that $A \le CB$, and $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

## 2 Preliminaries

We list some lemmas in this section. Given $r > 0$ and $z \in \mathbb{C}^n$, the Euclidean ball centered at $z$ with radius $r$ is denoted by

$$B(z,r) = \left\{ w \in \mathbb{C}^n : |w - z| < r \right\}.$$

**Lemma 2.1** *There exists a positive integer $N$ such that for any $r > 0$, we can find a sequence $\{a_k\}$ in $\mathbb{C}^n$ with the following properties:*

(i) $\bigcup_{k=1}^{\infty} B(a_k, r) = \mathbb{C}^n$;
(ii) $\left\{B\left(a_k, \frac{r}{2}\right)\right\}_{k=1}^{\infty}$ *are pairwise disjoint;*
(iii) *Each point $z \in \mathbb{C}^n$ belongs to at most $N$ of the sets $B(a_k, 2r)$.*

The sequence $\{a_k\}$ satisfying the conditions of Lemma 2.1 is called an $r$-lattice. We write $B_k = B(a_k, r)$, $\widetilde{B}_k = B(a_k, 2r)$ for convenience throughout the paper.

Let $\mu$ be a positive Borel measure on $\mathbb{C}^n$. Notice that $v(B(z,r))$ is a constant independent of $z$, the average function of $\mu$ is defined by

$$\widehat{\mu}_r(z) := \mu(B(z,r))$$

for $z \in \mathbb{C}^n$. Let $K_\alpha(z,w) = e^{\alpha\langle z,w\rangle}$ denote the reproducing kernel of the Fock space $F^2_\alpha$. For $t > 0$, the $t$-Berezin transform of $\mu$ is defined by

$$\begin{aligned} \widetilde{\mu}_t(z) &= \int_{\mathbb{C}^n} \left| \frac{K_\alpha(z,w)}{\sqrt{K_\alpha(z,z)K_\alpha(w,w)}} \right|^t \mathrm{d}\mu(w) \\ &= \int_{\mathbb{C}^n} e^{-\frac{\alpha t}{2}|z-w|^2} \mathrm{d}\mu(w). \end{aligned} \tag{2.1}$$

**Lemma 2.2** [12, Theorem 3.1] *Let $\mu$ be a positive Borel measure on $\mathbb{C}^n$. Set $\lambda = q/p$ and $1 \le \lambda < \infty$. Then the following statements are equivalent:*

(i) *$\mu$ is a $(\lambda, \alpha)$-Fock–Carleson measure;*
(ii) *$\widetilde{\mu}_t$ is bounded on $\mathbb{C}^n$ for some (or any) $t > 0$;*
(iii) *$\mu(B(\cdot, \delta))$ is bounded on $\mathbb{C}^n$ for some (or any) $\delta > 0$;*
(iv) *For some (or any) $r > 0$, the sequence $\{\mu(B(a_k, r))\}_{k=1}^{\infty}$ is bounded.*

*Furthermore,*

$$\|\mu\| \simeq \|\widetilde{\mu}_t\|_\infty \simeq \|\mu(B(\cdot,\delta))\|_\infty \simeq \|\{\mu(B(a_k,r))\}\|_{\ell^\infty},$$

*where*

$$\|\mu\| = \sup_{f \in F^p_\alpha, \|f\|_{p,\alpha} \le 1} \int_{\mathbb{C}^n} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^q d\mu(z).$$

**Lemma 2.3** [12, Theorem 3.2] *Let $\mu$ be a positive Borel measure on $\mathbb{C}^n$. Set $\lambda = q/p$ and $1 \le \lambda < \infty$. Then the following statements are equivalent:*

(i) *$\mu$ is a vanishing $(\lambda,\alpha)$-Fock–Carleson measure;*
(ii) *$\widetilde{\mu}_t(z)\to 0$ as $z\to\infty$ for some (or any) $t>0$;*
(iii) *$\mu(B(z,\delta))\to 0$ as $z\to\infty$ for some (or any) $\delta>0$;*
(iv) *$\mu\left(B\left(a_k,r\right)\right)\to 0$ as $k\to\infty$ for some (or any) $r>0$.*

**Lemma 2.4** [12, Theorem 3.3] *Let $\mu$ be a positive Borel measure on $\mathbb{C}^n$. Set $\lambda=q/p$ and $0<\lambda<1$. Then the following statements are equivalent:*

(i) *$\mu$ is a $(\lambda,\alpha)$-Fock–Carleson measure;*
(ii) *$\mu$ is a vanishing $(\lambda,\alpha)$-Fock–Carleson measure;*
(iii) *$\widetilde{\mu}_t\in L^{1/(1-\lambda)}$ for some (or any) $t>0$;*
(iv) *$\mu(B(\cdot,\delta))\in L^{1/(1-\lambda)}$ for some (or any) $\delta>0$;*
(v) *$\sum_{k=1}^{\infty}\mu\left(B\left(a_k,r\right)\right)^{1/(1-\lambda)}<\infty$ for some (or any) $r>0$.*

*Furthermore,*

$$\|\mu\|\simeq\|\widetilde{\mu}_t\|_{1/(1-\lambda)}\simeq\|\mu(B(\cdot,\delta))\|_{1/(1-\lambda)}\simeq\|\{\mu\left(B\left(a_k,r\right)\right)\}\|_{\ell^{1/(1-\lambda)}}.$$

**Lemma 2.5** [8, Lemma 3] *Suppose $0<p<\infty$. Then*

$$\|K_\alpha(\cdot,z)\|_{p,\alpha}\simeq e^{\frac{\alpha}{2}|z|^2}. \tag{2.2}$$

**Lemma 2.6** *Let $\nu$ be a positive Borel measure on $\mathbb{C}^n$ and $0<p,t<\infty$. If $\{f_k\}$ is a sequence of Lebesgue measurable functions such that*

$$\int_{\mathbb{C}^n}\left(\sum_{k=1}^{\infty}|f_k(z)|^t\right)^{p/t}d\nu(z)<\infty,$$

*then*

$$\sum_{k=1}^{\infty}\int_{\widetilde{B}_k}|f_k(z)|^p d\nu(z)\le\max\{1,N^{(t-p)/t}\}\int_{\mathbb{C}^n}\left(\sum_{k=1}^{\infty}|f_k(z)|^t\right)^{p/t}d\nu(z).$$

***Proof*** The proof is divided into two cases. If $p/t\ge 1$, then $\ell^t$ injects continuously into $\ell^p$. Thus, we obtain

$$\begin{aligned}\int_{\mathbb{C}^n}\sum_{k=1}^{\infty}|f_k(z)|^p\chi_{\widetilde{B}_k}(z)\mathrm{d}\nu(z)\le&\int_{\mathbb{C}^n}\left(\sum_{k=1}^{\infty}|f_k(z)|^t\chi_{\widetilde{B}_k}(z)\right)^{p/t}\mathrm{d}\nu(z)\\ \le&\int_{\mathbb{C}^n}\left(\sum_{k=1}^{\infty}|f_k(z)|^t\right)^{p/t}\mathrm{d}\nu(z).\end{aligned}$$

If $0 < p/t < 1$, then $t/p > 1$. Using Hölder's inequality, we have

$$\int_{\mathbb{C}^n} \sum_{k=1}^{\infty} |f_k(z)|^p \chi_{\widetilde{B}_k}(z) \mathrm{d}\nu(z) \le \int_{\mathbb{C}^n} \left( \sum_{k=1}^{\infty} |f_k(z)|^t \right)^{p/t} \left( \sum_{k=1}^{\infty} \chi_{\widetilde{B}_k}(z) \right)^{(t-p)/t} \mathrm{d}\nu(z)$$
$$\le N^{(t-p)/t} \int_{\mathbb{C}^n} \left( \sum_{k=1}^{\infty} |f_k(z)|^t \right)^{p/t} \mathrm{d}\nu(z).$$

The proof is complete. □

**Lemma 2.7** *Let $\{a_k\}$ be an $r$-lattice, $0 < p \le \infty$ and $\{\lambda_k\} \in \ell^p$. If*

$$f(z) = \sum_{k=1}^{\infty} \lambda_k e^{\alpha \langle z, a_k \rangle - \frac{\alpha}{2}|a_k|^2},$$

*then $f \in F^p_\alpha$ and $\|f\|_{p,\alpha} \lesssim \| \{\lambda_k\} \|_{\ell^p}$.*

***Proof*** Since the result has been proven in [12, Lemma 2.4] for $1 \le p \le \infty$, it suffices to prove the case $0 < p < 1$. If $0 < p < 1$, then we get

$$\|f\|^p_{p,\alpha} \le \sum_{k=1}^{\infty} |\lambda_k|^p \int_{\mathbb{C}^n} \left| e^{\alpha \langle z, a_k \rangle - \frac{\alpha}{2}|a_k|^2} e^{-\frac{\alpha}{2}|z|^2} \right|^p \mathrm{d}v(z)$$
$$\le C \sum_{k=1}^{\infty} |\lambda_k|^p \int_{\mathbb{C}^n} e^{-\frac{\alpha p}{2}|z-a_k|^2} \mathrm{d}v(z)$$
$$\lesssim \sum_{k=1}^{\infty} |\lambda_k|^p.$$

The proof is complete. □

## 3 Proof of Theorem 1.3

We are devoted to proving Theorem 1.3 in this section. Two inequalities related to Rademacher functions will be used. We use the notation $r_k$ to denote Rademacher functions as follows

$$r_0(s) = \begin{cases} 1, & 0 \le s - [s] < 1/2, \\ -1, & 1/2 \le s - [s] < 1, \end{cases}$$
$$r_k(s) = r_0(2^k s) \text{ for } k = 1, 2, \dots .$$

Here $[s]$ denotes the largest integer not greater than $s$. One is *Khinchine's inequality*. For $0 < p < \infty$, there exist constants $0 < A_p \le B_p < \infty$ such that

$$A_p\left(\sum_{k=1}^{m}|c_k|^2\right)^{p/2} \le \int_0^1\left|\sum_{k=1}^{m}c_k r_k(s)\right|^p \mathrm{d}s \le B_p\left(\sum_{k=1}^{m}|c_k|^2\right)^{p/2}$$

for all nonnegative integers $m$ and all complex numbers $c_1, c_2, \ldots, c_m$. The other one is Khinchine-*Kahane-Kalton inequality*. If $0 < p, q < \infty$, then

$$\left(\int_0^1\left\|\sum_{k=1}^{\infty}x_k r_k(s)\right\|_X^q \mathrm{d}s\right)^{\frac{1}{q}} \simeq \left(\int_0^1\left\|\sum_{k=1}^{\infty}x_k r_k(s)\right\|_X^p \mathrm{d}s\right)^{\frac{1}{p}}$$

for any sequence $\{x_k\} \subset X$, where $X$ is a quasi-Banach space with quasi-norm $\|\cdot\|_X$. See [23] in detail.

***Proof of Theorem 1.3*** We divide the proof into two cases: $\lambda \ge 1$ and $0 < \lambda < 1$.

Case I: $\lambda \ge 1$. For fixed $a \in \mathbb{C}^n$, set

$$f_a(z) = \frac{e^{\alpha_1\langle z,a\rangle}}{e^{\frac{\alpha_1}{2}|a|^2}}, \quad z \in \mathbb{C}^n.$$

It follows from Lemma 2.5 that $f_a \in F^{p_1}_{\alpha_1}$ and $\|f_a\|_{p_1,\alpha_1} \lesssim 1$. Since for any $z, w \in B(a, \delta)$ with some given $\delta > 0$, we have

$$|z - w| \le |z - a| + |w - a| < 2\delta.$$

Therefore, for any $z \in B(a, \delta)$, we get

$$\begin{aligned}\mu(B(a,\delta)) &= \int_{B(a,\delta)} \mathrm{d}\mu(w)\\ &\le e^{\frac{\alpha_1 r}{2}\delta^2}\cdot e^{\frac{\alpha_2 t}{2}(2\delta)^2}\cdot\int_{B(a,\delta)} e^{-\frac{\alpha_2 t}{2}|z-w|^2}\cdot e^{-\frac{\alpha_1 r}{2}|w-a|^2}\mathrm{d}\mu(w)\\ &= e^{\frac{\alpha_1 r}{2}\delta^2+\frac{\alpha_2 t}{2}(2\delta)^2}\int_{B(a,\delta)} e^{-\frac{\alpha_2 t}{2}|z-w|^2}|f_a(w)|^r e^{-\frac{\alpha_1}{2}|w|^2 r}\mathrm{d}\mu(w)\\ &\le e^{\frac{\alpha_1 r}{2}\delta^2+\frac{\alpha_2 t}{2}(2\delta)^2} S^{r,t}_\mu f_a(z). \end{aligned} \tag{3.1}$$

Thus,

$$\begin{aligned}\mu(B(a,\delta))^{p_2} &\lesssim \int_{B(a,\delta)} \mu(B(a,\delta))^{p_2}\mathrm{d}v(z) \lesssim \int_{B(a,\delta)} S^{r,t}_\mu f_a(z)^{p_2}\mathrm{d}v(z)\\ &\le \int_{\mathbb{C}^n}\left|S^{r,t}_\mu f_a(z)\right|^{p_2}\mathrm{d}v(z) \le C\|f_a\|^{rp_2}_{p_1,\alpha_1} \lesssim C.\end{aligned}$$

This shows that $\mu$ is a $(\lambda, \gamma)$-Fock–Carleson measure by Lemma 2.2.

Case II: $0 < \lambda < 1$. Let $\{a_k\}$ be an $r$-lattice on $\mathbb{C}^n$. For any sequence of real numbers $\{\lambda_k\} \in \ell^{p_1}$, set

$$f_s(z) = \sum_{k=1}^{\infty} \lambda_k r_k(s) g_k(z),$$

where

$$g_k(z) = e^{\alpha_1 \langle z, a_k \rangle - \frac{\alpha_1}{2} |a_k|^2}.$$

Then Lemma 2.7 implies $f_s(z) \in F^{p_1}_{\alpha_1}$ and $\|f_s\|_{p_1,\alpha_1} \lesssim \|\{\lambda_k\}\|_{\ell^{p_1}}$ for almost every $s \in (0,1)$. Hence, the condition shows that

$$\left\| S^{r,t}_{\mu} f_s \right\|^{p_2}_{p_2} = \int_{\mathbb{C}^n} \left| S^{r,t}_{\mu} f_s(z) \right|^{p_2} \mathrm{d}v(z) \le C^{p_2} \|f_s\|^{rp_2}_{p_1,\alpha_1} \lesssim C^{p_2} \, \|\{\lambda_k\}\|^{rp_2}_{\ell^{p_1}}$$

for almost every $s \in (0,1)$. Integrating both sides with respect to $s$ from 0 to 1, we have

$$\int_0^1 \left\| S^{r,t}_{\mu} f_s \right\|^{p_2}_{p_2} \mathrm{d}s \lesssim C^{p_2} \, \|\{\lambda_k\}\|^{rp_2}_{\ell^{p_1}}. \tag{3.2}$$

Applying Fubini's theorem and Khinchine-Kahane-Kalton inequality, we obtain

$$\begin{aligned}
\int_0^1 \left\| S^{r,t}_{\mu} f_s \right\|^{p_2}_{p_2} \mathrm{d}s &= \int_{\mathbb{C}^n} \int_0^1 \left| S^{r,t}_{\mu} f_s(z) \right|^{p_2} \mathrm{d}s \mathrm{d}v(z) \\
&= \int_{\mathbb{C}^n} \int_0^1 \left( \int_{\mathbb{C}^n} e^{-\frac{\alpha_2 t}{2}|z-w|^2} \Big| \sum_{k=1}^{\infty} \lambda_k r_k(s) g_k(w) \Big|^r e^{-\frac{\alpha_1}{2}|w|^2 r} \mathrm{d}\mu(w) \right)^{p_2} \mathrm{d}s \mathrm{d}v(z) \\
&\gtrsim \int_{\mathbb{C}^n} \left( \int_0^1 \int_{\mathbb{C}^n} e^{-\frac{\alpha_2 t}{2}|z-w|^2} \Big| \sum_{k=1}^{\infty} \lambda_k r_k(s) g_k(w) \Big|^r e^{-\frac{\alpha_1}{2}|w|^2 r} \mathrm{d}\mu(w) \mathrm{d}s \right)^{p_2} \mathrm{d}v(z).
\end{aligned}$$

Therefore, using Fubini's theorem and Khinchine's inequality, we have

$$\int_0^1 \left\| S^{r,t}_{\mu} f_s \right\|^{p_2}_{p_2} \mathrm{d}s \gtrsim \int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} \left( \sum_{k=1}^{\infty} |\lambda_k|^2 |g_k(w)|^2 \right)^{r/2} e^{-\frac{\alpha_2 t}{2}|z-w|^2} e^{-\frac{\alpha_1}{2}|w|^2 r} \mathrm{d}\mu(w) \right)^{p_2} \mathrm{d}v(z). \tag{3.3}$$

Remember that $B_k := B(a_k, \delta)$ and $\widetilde{B}_k := B(a_k, 2\delta)$ with $\delta > 0$. A similar discussion to (3.1) implies that

$$\mu(B_k) \lesssim \int_{B_k} e^{-\frac{\alpha_2 t}{2}|z-w|^2} |g_k(w)|^r e^{-\frac{\alpha_1}{2}|w|^2 r} \mathrm{d}\mu(w)$$

for $z \in B_k$. Thus,

$$\begin{aligned}\mu(B_k)^{p_2} &\lesssim \left(\int_{\widetilde{B}_k} e^{-\frac{\alpha_2 t}{2}|z-w|^2}|g_k(w)|^r e^{-\frac{\alpha_1}{2}|w|^2 r}\mathrm{d}\mu(w)\right)^{p_2}\\ &\lesssim \int_{B_k}\left(\int_{\widetilde{B}_k} e^{-\frac{\alpha_2 t}{2}|z-w|^2}|g_k(w)|^r e^{-\frac{\alpha_1}{2}|w|^2 r}\mathrm{d}\mu(w)\right)^{p_2}\mathrm{d}v(z)\\ &\lesssim \int_{\widetilde{B}_k}\left(\int_{\widetilde{B}_k} e^{-\frac{\alpha_2 t}{2}|z-w|^2}|g_k(w)|^r e^{-\frac{\alpha_1}{2}|w|^2 r}\mathrm{d}\mu(w)\right)^{p_2}\mathrm{d}v(z).\end{aligned}$$

It follows that

$$\sum_{k=1}^{\infty}|\lambda_k|^{rp_2}\mu(B_k)^{p_2} \lesssim \sum_{k=1}^{\infty}\int_{\widetilde{B}_k}|\lambda_k|^{rp_2}\left(\int_{\widetilde{B}_k} e^{-\frac{\alpha_2 t}{2}|z-w|^2}|g_k(w)|^r e^{-\frac{\alpha_1}{2}|w|^2 r}\mathrm{d}\mu(w)\right)^{p_2}\mathrm{d}v(z).$$

Using Lemma 2.6 twice, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}|\lambda_k|^{rp_2}\mu(B_k)^{p_2} &\lesssim \int_{\mathbb{C}^n}\left(\sum_{k=1}^{\infty}|\lambda_k|^r\int_{\widetilde{B}_k} e^{-\frac{\alpha_2 t}{2}|z-w|^2}|g_k(w)|^r e^{-\frac{\alpha_1}{2}|w|^2 r}\mathrm{d}\mu(w)\right)^{p_2}\mathrm{d}v(z)\\ &\lesssim \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\left(\sum_{k=1}^{\infty}|\lambda_k|^2|g_k(w)|^2\right)^{r/2} e^{-\frac{\alpha_2 t}{2}|z-w|^2}e^{-\frac{\alpha_1}{2}|w|^2 r}\mathrm{d}\mu(w)\right)^{p_2}\mathrm{d}v(z).\end{aligned}$$

This, together with (3.2) and (3.3), yields

$$\sum_{k=1}^{\infty}|\lambda_k|^{rp_2}\mu(B_k)^{p_2} \lesssim \|\{\lambda_k\}\|_{\ell^{p_1}}^{rp_2}.$$

The duality of $l^{p_1/(rp_2)}$ and $l^{1/((1-\lambda)p_2)}$ gives

$$\{\mu(B_k)\} \in \ell^{1/(1-\lambda)}.$$

This shows $\mu$ is a $(\lambda, \gamma)$-Fock–Carleson measure in view of Lemma 2.4. The proof is complete. □

## 4 Proofs of Theorems 1.1 and 1.2

**Lemma 4.1** *Let* $0 < p_i, q_i < \infty$, $0 < \alpha_i < \infty$, $f_i \in F_{\alpha_i}^{p_i/q_i}$, *where* $i = 1, 2, \ldots, k$. *If*

$$\lambda = \sum_{i=1}^{k}\frac{q_i}{p_i}, \quad \gamma = \sum_{i=1}^{k}\alpha_i,$$

*then* $\prod_{i=1}^{k} f_i \in F_{\gamma}^{1/\lambda}$ *and*

$$\left\| \prod_{i=1}^{k} f_i \right\|_{1/\lambda,\gamma} \lesssim \prod_{i=1}^{k} \| f_i \|_{p_i/q_i,\alpha_i} . \tag{4.1}$$

***Proof*** Let $f_i \in F_{\alpha_i}^{p_i/q_i}$, where $i = 1, 2, \ldots, k$. Since $\sum_{i=1}^{k} q_i/(p_i\lambda) = 1$, it follows from Hölder's inequality that

$$\begin{aligned}
\left\| \prod_{i=1}^{k} f_i \right\|_{1/\lambda,\gamma} &= \left( \int_{\mathbb{C}^n} \left| \prod_{i=1}^{k} |f_i(z)| e^{-\frac{\gamma}{2}|z|^2} \right|^{1/\lambda} \mathrm{d}v(z) \right)^{\lambda} \\
&= \left( \int_{\mathbb{C}^n} \prod_{i=1}^{k} |f_i(z)|^{1/\lambda} e^{-\frac{\alpha_i}{2}|z|^2/\lambda} \mathrm{d}v(z) \right)^{\lambda} \\
&\lesssim \prod_{i=1}^{k} \left( \int_{\mathbb{C}^n} |f_i(z)|^{(1/\lambda)(p_i\lambda/q_i)} e^{(-\frac{\alpha_i}{2}|z|^2/\lambda)(p_i\lambda/q_i)} \mathrm{d}v(z) \right)^{q_i/p_i} \\
&= \prod_{i=1}^{k} \left( \int_{\mathbb{C}^n} |f_i(z)|^{p_i/q_i} e^{-\frac{\alpha_i}{2}|z|^2(p_i/q_i)} \mathrm{d}v(z) \right)^{q_i/p_i} \\
&\lesssim \prod_{i=1}^{k} \| f_i \|_{p_i/q_i,\alpha_i} .
\end{aligned}$$

The proof is complete. □

***Proof of Theorem 1.1*** First, we prove the necessity. Assume that $\mu$ is a $(\lambda, \gamma)$-Fock–Carleson measure. It suffices to prove $k \geq 2$, since the result is just the definition when $k = 1$. It follows from Lemma 4.1 that if $h_i \in F_{\alpha_i}^{p_i/q_i}$ for any $i = 1, 2, \ldots, k$, then $\prod_{i=1}^{k} h_i \in F_{\gamma}^{1/\lambda}$. Because $\mu$ is a $(\lambda, \gamma)$-Fock–Carleson measure, we have

$$\int_{\mathbb{C}^n} \left| \prod_{i=1}^{k} h_i(z) \right| e^{-\frac{\gamma}{2}|z|^2} \mathrm{d}\mu(z) \leq C \| \prod_{i=1}^{k} h_i \|_{1/\lambda,\gamma} .$$

This, together with (4.1), gives

$$\int_{\mathbb{C}^n} \prod_{i=1}^{k} \left( |h_i(z)| e^{-\frac{\alpha_i}{2}|z|^2} \right) \mathrm{d}\mu(z) \leq C \prod_{i=1}^{k} \|h_i\|_{p_i/q_i,\alpha_i} . \tag{4.2}$$

Let

$$d\mu_1 = \left(\prod_{i=2}^{k} |h_i| e^{-\frac{\alpha_i}{2}|z|^2} d\mu\right) \Big/ \left(\prod_{i=2}^{k} \|h_i\|_{p_i/q_i,\alpha_i}\right).$$

Then (4.2) is equivalent to

$$\int_{\mathbb{C}^n} |h_1(z)|\, e^{-\frac{\alpha_1}{2}|z|^2} d\mu_1(z) \le C\|h_1\|_{p_1/q_1,\alpha_1}.$$

Thus, $\mu_1$ is a $(q_1/p_1, \alpha_1)$-Fock–Carleson measure. Therefore, for any $f_1 \in F^{p_1}_{\alpha_1}$,

$$\int_{\mathbb{C}^n} |f_1(z)|^{q_1}\, e^{-\frac{\alpha_1}{2}|z|^2 q_1} d\mu_1(z) \le C\|f_1\|^{q_1}_{p_1,\alpha_1},$$

which is the same as

$$\int_{\mathbb{C}^n} |f_1(z)|^{q_1}\, e^{-\frac{\alpha_1}{2}|z|^2 q_1} \prod_{i=2}^{k} |h_i(z)| e^{-\frac{\alpha_i}{2}|z|^2} d\mu(z) \le C\|f_1\|^{q_1}_{p_1,\alpha_1} \prod_{i=2}^{k} \|h_i\|_{p_i/q_i,\alpha_i}. \tag{4.3}$$

Let

$$d\mu_2 = \left(|f_1|^{q_1} e^{-\frac{\alpha_1}{2}|z|^2 q_1} \prod_{i=3}^{k} |h_i| e^{-\frac{\alpha_i}{2}|z|^2} d\mu\right) \Big/ \left(\|f_1\|^{q_1}_{p_1,\alpha_1} \prod_{i=3}^{k} \|h_i\|_{p_i/q_i,\alpha_i}\right).$$

Then (4.3) is the same as

$$\int_{\mathbb{C}^n} |h_2(z)|\, e^{-\frac{\alpha_2}{2}|z|^2} d\mu_2(z) \le C\|h_2\|_{p_2/q_2,\alpha_2}.$$

This means that $\mu_2$ is a $(q_2/p_2, \alpha_2)$-Fock–Carleson measure. Therefore, for any $f_2 \in F^{p_2}_{\alpha_2}$,

$$\int_{\mathbb{C}^n} |f_2(z)|^{q_2}\, e^{-\frac{\alpha_2}{2}|z|^2 q_2} d\mu_2(z) \le C\|f_2\|^{q_2}_{p_2,\alpha_2},$$

that is

$$\begin{aligned}&\int_{\mathbb{C}^n} |f_1(z)|^{q_1}\, e^{-\frac{\alpha_1}{2}|z|^2 q_1}\, |f_2(z)|^{q_2}\, e^{-\frac{\alpha_2}{2}|z|^2 q_2} \prod_{i=3}^{k} |h_i(z)| e^{-\frac{\alpha_i}{2}|z|^2} d\mu(z)\\ &\le C\|f_1\|^{q_1}_{p_1,\alpha_1} \|f_2\|^{q_2}_{p_2,\alpha_2} \prod_{i=3}^{k} \|h_i\|_{p_i/q_i,\alpha_i}.\end{aligned}$$

Continuing this process, we eventually have

$$\int_{\mathbb{C}^n} \prod_{i=1}^{k} |f_i(z)|^{q_i} e^{-\frac{\alpha_i}{2}|z|^2 q_i} \mathrm{d}\mu(z) \le C \prod_{i=1}^{k} \| f_i \|_{p_i,\alpha_i}^{q_i} .$$

Hence, we obtain (1.2). The proof of the necessity for Theorem 1.1 is complete.

Next, we prove the sufficiency. Assume that (1.2) holds for any $f_i \in F_{\alpha_i}^{p_i/q_i}$, $i = 1, 2, \dots, k$. We aim to prove that $\mu$ is a $(\lambda, \gamma)$-Fock–Carleson measure. The proof is divided into two cases: $\lambda \ge 1$ and $0 < \lambda < 1$.

Case I: $\lambda \ge 1$. For fixed $z \in \mathbb{C}^n$, set

$$f_{i,z}(w) = \frac{e^{\alpha_i \langle w,z \rangle}}{e^{\frac{\alpha_i}{2}|z|^2}}, \ \ w \in \mathbb{C}^n,$$

where $i = 1, 2, \dots, k$. By (2.2), we have $f_{i,z} \in F_{\alpha_i}^{p_i}$ with $\| f_{i,z} \|_{p_i,\alpha_i} \simeq 1$. Thus, (1.2) implies

$$\int_{\mathbb{C}^n} \prod_{i=1}^{k} \left| f_{i,z}(w) \right|^{q_i} e^{-\frac{\alpha_i}{2}|w|^2 q_i} \mathrm{d}\mu(w) \le C \prod_{i=1}^{k} \left\| f_{i,z} \right\|_{p_i,\alpha_i}^{q_i} . \tag{4.4}$$

Let $t = \max\{q_1, q_2, \dots, q_k\}$. Then

$$\begin{aligned} \widetilde{\mu}_t(z) &\simeq \int_{\mathbb{C}^n} \left| \frac{e^{\gamma \langle w,z \rangle}}{e^{\frac{\gamma}{2}|z|^2}} e^{-\frac{\gamma}{2}|w|^2} \right|^t \mathrm{d}\mu(w) \\ &\le \int_{\mathbb{C}^n} \prod_{i=1}^{k} \left| \frac{e^{\alpha_i \langle w,z \rangle}}{e^{\frac{\alpha_i}{2}|z|^2}} e^{-\frac{\alpha_i}{2}|w|^2} \right|^{q_i} \mathrm{d}\mu(w) \\ &= \int_{\mathbb{C}^n} \prod_{i=1}^{k} \left| f_{i,z}(w) \right|^{q_i} e^{-\frac{\alpha_i}{2}|w|^2 q_i} \mathrm{d}\mu(w). \end{aligned}$$

This, together with (4.4), shows

$$\widetilde{\mu}_t(z) \le C \prod_{i=1}^{k} \| f_{i,z} \|_{p_i,\alpha_i}^{q_i} \lesssim C.$$

Thus, $\widetilde{\mu}_t(z)$ is bounded. It follows from Lemma 2.2 that $\mu$ is a $(\lambda, \gamma)$-Fock–Carleson measure.

Case II: $0 < \lambda < 1$. The proof is by induction on $k$. If $k = 1$, then (1.2) is just the definition of $(\lambda, \gamma)$-Fock–Carleson measure. Assume that the result holds for $k - 1$ functions for $k \ge 2$. Set $\lambda_k = \lambda$, $\gamma_k = \gamma$ and

$$\lambda_{k-1} = \sum_{i=1}^{k-1} \frac{q_i}{p_i}, \quad \gamma_{k-1} = \sum_{i=1}^{k-1} \alpha_i .$$

Denote

$$\mathrm{d}\mu_k(z) = |f_k(z)|^{q_k} e^{-\frac{\alpha_k}{2}|z|^2 q_k} \mathrm{d}\mu(z).$$

Then we rewrite the condition (1.2) as

$$\int_{\mathbb{C}^n} \prod_{i=1}^{k-1} |f_i(z)|^{q_i}\, e^{-\frac{\alpha_i}{2}|z|^2 q_i} \mathrm{d}\mu_k(z) \le C(f_k) \prod_{i=1}^{k-1} \|f_i\|_{p_i,\alpha_i}^{q_i}$$

with $C(f_k) = C\|f_k\|_{p_k,\alpha_k}^{q_k}$. It follows from the induction assumption that $\mu_k$ is a $(\lambda_{k-1}, \gamma_{k-1})$-Fock–Carleson measure with $\|\mu_k\| \lesssim C(f_k)$. Since $0 < \lambda_{k-1} < \lambda < 1$, it follows from Lemma 2.4 that $\widetilde{\mu}_{k,t} \in L^{1/(1-\lambda_{k-1})}$ for any $t > 0$ with

$$\|\widetilde{\mu}_{k,t}\|_{1/(1-\lambda_{k-1})} \lesssim C\|f_k\|_{p_k,\alpha_k}^{q_k},$$

where

$$\widetilde{\mu}_{k,t} = \int_{\mathbb{C}^n} e^{-\frac{t\gamma_{k-1}}{2}|z-w|^2} \mathrm{d}\mu_k(w).$$

That is,

$$\left(\int_{\mathbb{C}^n} \left|\int_{\mathbb{C}^n} e^{-\frac{\gamma_{k-1}t}{2}|z-w|^2} |f_k(w)|^{q_k} e^{-\frac{\alpha_k}{2}|w|^2 q_k} \mathrm{d}\mu(w)\right|^{1/(1-\lambda_{k-1})} \mathrm{d}v(z)\right)^{1-\lambda_{k-1}} \lesssim C\|f_k\|_{p_k,\alpha_k}^{q_k}.$$

This, together with the definition of $S_\mu^{q_k,t}$ in Theorem 1.3, implies

$$\left\|S_\mu^{q_k,t} f_k\right\|_{1/(1-\lambda_{k-1})} \le C\|f_k\|_{p_k,\alpha_k}^{q_k},$$

where $f_k \in F_{\alpha_k}^{p_k}$. Thus, by Theorem 1.3, $\mu$ is a $(\lambda^*, \gamma^*)$-Fock–Carleson measure, where

$$\lambda^* = 1 + \frac{q_k}{p_k} - (1 - \lambda_{k-1}), \quad \gamma^* = \gamma_{k-1} + \alpha_k.$$

It follows from the definitions of $\lambda_{k-1}, \gamma_{k-1}$ and (1.1) that $\lambda^* = \lambda$ and $\gamma^* = \gamma$. The proof is complete. □

***Proof of Theorem 1.2*** If $0 < \lambda < 1$, then by Lemma 2.4 we know that $\mu$ is a $(\lambda, \alpha)$-Fock–Carleson measure if and only if $\mu$ is a vanishing $(\lambda, \alpha)$-Fock–Carleson measure. It is just a consequence of Theorem 1.1. And so it suffices to prove the theorem in the case of $\lambda > 1$. Since (ii)$\Longrightarrow$(iii) is obvious, the theorem will be proved by showing (i) $\Longrightarrow$(ii) and (iii)$\Longrightarrow$(i).

(i) $\Longrightarrow$(ii). Assume that $\mu$ is a vanishing $(\lambda, \gamma)$-Fock–Carleson measure. Let $\{f_{1,l}\}$ be any bounded sequence in $F_{\alpha_1}^{p_1}$ and $f_{1,l} \to 0$ uniformly on each compact subset of

$\mathbb{C}^n$ as $l \to \infty$. Suppose that $\{f_i\}$ is an arbitrary sequence in $F^{p_i}_{\alpha_i}$ with $\|f_i\|_{p_i,\alpha_i} \le 1$ for $i = 2, 3, \dots, k$. For $r > 0$, denote $B_r := B(0, r)$, and $\mu_r$ the restriction of $\mu$ to $\mathbb{C}^n \backslash B_r$. Then $\mu_r$ is also a $(\lambda, \gamma)$-Fock–Carleson measure, and

$$\lim_{r\to\infty} \|\mu_r\| = 0.$$

On one hand, by Theorem 1.1, we obtain that for any $\varepsilon > 0$ small enough, there exists an $r$ large enough such that

$$\begin{aligned}
&\int_{\mathbb{C}^n\backslash B_r} \left|f_{1,l}(z)e^{-\frac{\alpha_1}{2}|z|^2}\right|^{q_1} \prod_{i=2}^{k} \left|f_i(z)e^{-\frac{\alpha_i}{2}|z|^2}\right|^{q_i} \mathrm{d}\mu(z)\\
&= \int_{\mathbb{C}^n} \left|f_{1,l}(z)e^{-\frac{\alpha_1}{2}|z|^2}\right|^{q_1} \prod_{i=2}^{k} \left|f_i(z)e^{-\frac{\alpha_i}{2}|z|^2}\right|^{q_i} \mathrm{d}\mu_r(z)\\
&\le \|\mu_r\| \cdot \|f_{1,l}\|^{q_1}_{p_1,\alpha_1} \cdot \prod_{i=2}^{k} \|f_i\|^{q_i}_{p_i,\alpha_i} \lesssim \varepsilon.
\end{aligned}$$

Fix this $r$. Since $\left\{f_{1,l}\right\}$ converges to 0 uniformly on each compact subset of $\mathbb{C}^n$, there is a constant $K > 0$ such that for any $l > K$, $\left|f_{1,l}(z)\right| < \varepsilon$ for any $z \in B_r$. Therefore, using Theorem 1.1 again, we have

$$\begin{aligned}
&\int_{B_r} \left|f_{1,l}(z)e^{-\frac{\alpha_1}{2}|z|^2}\right|^{q_1} \prod_{i=2}^{k} \left|f_i(z)e^{-\frac{\alpha_i}{2}|z|^2}\right|^{q_i} \mathrm{d}\mu(z)\\
&\le \varepsilon \int_{\mathbb{C}^n} 1 \cdot e^{-\frac{\alpha_1}{2}|z|^2 q_1} \cdot \prod_{i=2}^{k} \left|f_i(z)e^{-\frac{\alpha_i}{2}|z|^2}\right|^{q_i} \mathrm{d}\mu(z)\\
&\lesssim \varepsilon \|1\|^{q_1}_{p_1,\alpha_1} \prod_{i=2}^{k} \|f_i\|^{q_i}_{p_i,\alpha_i} \lesssim \varepsilon
\end{aligned}$$

for $l > K$. Thus, we get

$$\begin{aligned}
&\limsup_{l\to\infty} \int_{\mathbb{C}^n} \left|f_{1,l}(z)e^{-\frac{\alpha_1}{2}|z|^2}\right|^{q_1} \prod_{i=2}^{k} \left|f_i(z)e^{-\frac{\alpha_i}{2}|z|^2}\right|^{q_i} \mathrm{d}\mu(z)\\
&= \limsup_{l\to\infty} \left(\int_{B_r} + \int_{\mathbb{C}^n\backslash B_r}\right) \left|f_{1,l}(z)e^{-\frac{\alpha_1}{2}|z|^2}\right|^{q_1} \prod_{i=2}^{k} \left|f_i(z)e^{-\frac{\alpha_i}{2}|z|^2}\right|^{q_i} \mathrm{d}\mu(z) \lesssim \varepsilon.
\end{aligned}$$

It follows from the arbitrariness of $\varepsilon$ that $\lim_{l\to\infty} F(l) = 0$.

(iii) $\Longrightarrow$ (i). For fixed $a \in \mathbb{C}^n$, take

$$f_{i,a}(z) = \frac{e^{\alpha_i\langle z,a\rangle}}{e^{\frac{\alpha_i}{2}|a|^2}}, \quad z \in \mathbb{C}^n,$$

where $i = 1, 2, \ldots, k$. By (2.2), we know that $f_{i,a} \in F^{p_i}_{\alpha_i}$ with $\|f_{i,a}\|_{p_i,\alpha_i} \simeq 1$. Furthermore, it is easy to check that $f_{i,a} \to 0$ uniformly on any compact subset of $\mathbb{C}^n$ as $|a| \to \infty$. Thus, (iii) implies

$$\lim_{|a|\to\infty} \int_{\mathbb{C}^n} \prod_{i=1}^{k} \left| \frac{e^{\alpha_i \langle z,a\rangle}}{e^{\frac{\alpha_i}{2}|a|^2}} e^{-\frac{\alpha_i}{2}|z|^2} \right|^{q_i} \mathrm{d}\mu(z) = 0.$$

Let $t = \max\{q_1, q_2, \ldots, q_k\}$. Then

$$\lim_{|a|\to\infty} \int_{\mathbb{C}^n} e^{-\frac{\gamma t}{2}|z-a|^2} \mathrm{d}\mu(z) \le \lim_{|a|\to\infty} \int_{\mathbb{C}^n} \prod_{i=1}^{k} \left| \frac{e^{\alpha_i \langle z,a\rangle}}{e^{\frac{\alpha_i}{2}|a|^2}} e^{-\frac{\alpha_i}{2}|z|^2} \right|^{q_i} \mathrm{d}\mu(z) = 0.$$

This shows $\mu$ is a vanishing $(\lambda, \gamma)$-Fock–Carleson measure by Lemma 2.3. The proof is complete. □

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