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ORIGINAL PAPER

Boundedness of maximal function for weighted Choquet integrals

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Abstract
We study the boundedness of Hardy–Littlewood maximal function on the spaces defined in terms of Choquet integrals associated with weighted Bessel and Riesz capacities. As a consequence, we obtain a class of weighted Sobolev inequalities.

Keywords Choquet integrals · Bessel capacities · Maximal function

Mathematics Subject Classification 31C15 · 42B25

1 Introduction and statements of main results

Let $\alpha > 0$ and $s > 1$. The Bessel potential spaces $H^{\alpha,s}(\mathbb{R}^n)$, $n \in \mathbb{N}$, are defined to be the completion of $\mathscr{D}(\mathbb{R}^n)$ with respect to the norm:

$$\|u\|_{H^{\alpha,s}(\mathbb{R}^n)} = \left\| \mathcal{F}^{-1}\left(\left(1+|\xi|^2\right)^{\frac{\alpha}{2}} \mathcal{F}u\right)\right\|_{L^s(\mathbb{R}^n)},$$

where $\mathscr{D}(\mathbb{R}^n)$ is the space of compactly supported smooth functions on $\mathbb{R}^n$, and $\mathcal{F}$ is the distributional Fourier transform. Note that a function u belongs to $H^{\alpha,s}(\mathbb{R}^n)$ if and only if

$$u = G_\alpha * f$$

for some $f \in L^s(\mathbb{R}^n)$ and $\|u\|_{H^{\alpha,s}(\mathbb{R}^n)} = \|f\|_{L^s(\mathbb{R}^n)}$, where $G_\alpha(\cdot)$ is the Bessel kernel defined by $G_\alpha(x) = \mathcal{F}^{-1}\left((1+|\xi|^2)^{-\frac{\alpha}{2}}\right)(x)$, $x \in \mathbb{R}^n$.

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Recall that the Bessel capacities $\mathrm{Cap}_{\alpha,s}(\cdot)$ associated with $H^{\alpha,s}(\mathbb{R}^n)$ are defined to be

$$\mathrm{Cap}_{\alpha,s}(E)=\inf\left\{\|f\|_{L^s(\mathbb{R}^n)}^s: f\geq 0,\ G_\alpha * f\geq \chi_E\right\},\quad E\subseteq\mathbb{R}^n.$$

Subsequently, a function $f:\mathbb{R}^n\to[-\infty,\infty]$ is said to be defined quasi-everywhere (q.e.) with respect to $\mathrm{Cap}_{\alpha,s}(\cdot)$ if it is defined everywhere on $\mathbb{R}^n$ except for a set of zero capacity $\mathrm{Cap}_{\alpha,s}(\cdot)$ (the notion of q.e. with respect to a set function $\mathcal{C}(\cdot)$ is defined in the same fashion). In which case, we define the Choquet integral of f associated with Bessel capacity by

$$\int_{\mathbb{R}^n} f\mathrm{dCap}_{\alpha,s}=\int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x\in\mathbb{R}^n:|f(x)|>t\})\mathrm{d}t.$$

Besides that, we define the local Hardy–Littlewood maximal function $\mathbf{M}^{\mathrm{loc}}$ by

$$\mathbf{M}^{\mathrm{loc}}f(x)=\sup_{0<r\leq 1}\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|\mathrm{d}y=\sup_{0<r\leq 1}\fint_{B_r(x)}|f(y)|\mathrm{d}y,\quad x\in\mathbb{R}^n,$$

where $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If the supremum above is taken over $r\in(0,\infty)$, then we obtain the standard Hardy–Littlewood maximal function $\mathbf{M}$.

One of the main results in [5] concerns the weak-type estimate that

$$\sup_{t>0}\left(t^{\frac{n-\alpha s}{n}}\cdot\mathrm{Cap}_{\alpha,s}\left(\left\{x\in\mathbb{R}^n:\left|\mathbf{M}^{\mathrm{loc}}f(x)\right|>t\right\}\right)\right)\leq C_{n,\alpha,s}\int_{\mathbb{R}^n}|f|^{\frac{n-\alpha s}{n}}\mathrm{dCap}_{\alpha,s},\tag{1.1}$$

where $\alpha s<n$. As noted in [5, Remark 2.6], the exponent $(n-\alpha s)/n$ is sharp. For the strong-type estimate, we have

$$\int_{\mathbb{R}^n}\left|\mathbf{M}^{\mathrm{loc}}f\right|^p\mathrm{dCap}_{\alpha,s}\leq C_{n,\alpha,s,p}\int_{\mathbb{R}^n}|f|^p\mathrm{dCap}_{\alpha,s}\tag{1.2}$$

for $p>(n-\alpha s)/n$. The present paper is to obtain the weighted version of (1.1) and (1.2), respectively. To begin with, let us introduce the weighted capacities. A non-negative function ω on $\mathbb{R}^n$ is said to be a weight if $\omega\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\omega(x)>0$ a.e.. Subsequently, the weight ω is called an A_p weight for $1\leq p<\infty$ if there exists a constant $A>0$, such that for every ball B of $\mathbb{R}^n$

$$\left(\fint_B\omega(x)\mathrm{d}x\right)\left(\fint_B\omega(x)^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1}\leq A,\quad 1<p<\infty;$$
$$\left(\fint_B\omega(x)\mathrm{d}x\right)\cdot\left\|\omega^{-1}\right\|_{L^\infty(B)}\leq A,\quad p=1.$$

The infimum of all such constants A is denoted by $[\omega]_{A_p}$. A necessary and sufficient condition for ω to be an A_1 weight is given by

$$\mathbf{M}\omega(x) \leq c_n \cdot [\omega]_{A_1} \cdot \omega(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

We refer the readers [9, Chapter 1] for a quick review of basic properties of A_p weights. For any weight ω, we define the weighted local Riesz capacities $R^{\omega}_{\alpha,s;\rho}(\cdot)$ by

$$R^{\omega}_{\alpha,s;\rho}(E) = \inf\left\{\|f\|^s_{L^s(\omega)} : f \geq 0,\ \mathcal{I}_{\alpha,\rho} f \geq \chi_E\right\}, \quad E \subseteq \mathbb{R}^n,$$

where $\alpha s < n$, $0 < \rho < \infty$, $\mathcal{I}_{\alpha,\rho} f = I_{\alpha,\rho} * f$,

$$I_{\alpha,\rho}(x) = \begin{cases} |x|^{-(n-\alpha)}, & |x| < \rho; \\ 0, & |x| \geq \rho, \end{cases}$$

and

$$\|f\|_{L^s(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^s \omega(x) \mathrm{d}x\right)^{\frac{1}{s}}.$$

Similarly, one defines the weighted Riesz capacities $R^{\omega}_{\alpha,s}(\cdot)$ by

$$R^{\omega}_{\alpha,s}(E) = \inf\left\{\|f\|^s_{L^s(\omega)} : f \geq 0,\ \mathcal{I}_{\alpha} f \geq \chi_E\right\}, \quad E \subseteq \mathbb{R}^n,$$

where $\alpha s < n$, $\mathcal{I}_\alpha f = I_\alpha * f$, and $I_\alpha(x) = |x|^{-(n-\alpha)}$ is the usual Riesz kernel. Under the assumption that $\omega \in A_s$, the weighted local Riesz capacities $R^{\omega}_{\alpha,s;\rho}(\cdot)$ are equivalent to the weighted Bessel capacities $B^{\omega}_{\alpha,s}(\cdot)$ defined by

$$B^{\omega}_{\alpha,s}(E) = \inf\left\{\|f\|^s_{L^s(\omega)} : f \geq 0,\ G_\alpha * f \geq \chi_E\right\}, \quad E \subseteq \mathbb{R}^n$$

More precisely, we have

$$C^{-1}_{n,\alpha,s,\rho,\omega} \cdot B^{\omega}_{\alpha,s}(E) \leq R^{\omega}_{\alpha,s;\rho}(E) \leq C_{n,\alpha,s,\rho,\omega} \cdot B^{\omega}_{\alpha,s}(E), \quad E \subseteq \mathbb{R}^n. \tag{1.3}$$

(see [9, Theorem 3.3.7 and Lemma 3.3.8]). For technical reason, we prefer to work with the weighted local Riesz capacities in the sequel. Our first result is the weighted weak-type estimate, to wit:

Theorem 1.1 *Let $\alpha > 0$, $s > 1$, $\alpha s < n$, and $\omega \in A_1$. For any Lebesgue measurable $R^{\omega}_{\alpha,s;1}(\cdot)$-q.e. defined function f on $\mathbb{R}^n$, it holds that*

$$\sup_{t>0}\left(t^{\frac{n-\alpha s}{n}} \cdot R^{\omega}_{\alpha,s;1}\left(\left\{x \in \mathbb{R}^n : \mathbf{M}^{\mathrm{loc}} f(x) > t\right\}\right)\right) \leq C_{n,\alpha,s,\omega} \int_{\mathbb{R}^n} |f|^{\frac{n-\alpha s}{n}} \mathrm{d}R^{\omega}_{\alpha,s;1}.$$

The weighted strong-type estimate is given by the following theorem.

Theorem 1.2 *Let $\alpha > 0$, $s > 1$, $\alpha s < n$, and $\omega \in A_1$. For any Lebesgue measurable $R^{\omega}_{\alpha,s;1}(\cdot)$-q.e. defined function f on $\mathbb{R}^n$, it holds that*

$$\int_{\mathbb{R}^n} \left(\mathbf{M}^{\mathrm{loc}} f\right)^p \mathrm{d}R^{\omega}_{\alpha,s;1} \leq C_{n,\alpha,s,\omega,p} \int_{\mathbb{R}^n} |f|^p \mathrm{d}R^{\omega}_{\alpha,s;1}$$

for $p > (n-\alpha s)/n$.

For the results regarding to weighted Riesz capacities $R^{\omega}_{\alpha,s}(\cdot)$, one replaces the local Hardy–Littlewood maximal function $\mathbf{M}^{\mathrm{loc}}$ by the standard Hardy–Littlewood maximal function $\mathbf{M}$ and the local weighted Riesz capacities $R^{\omega}_{\alpha,s;1}(\cdot)$ by $R^{\omega}_{\alpha,s}(\cdot)$ in Theorems 1.1 and 1.2 (see Sect. 4 for details).

Our results reminisce of the celebrated classical weighted norm estimates given by Muckenhoupt [4]. It is standard that a necessary and sufficient condition for ω to be an A_p weight can given by

$$\int_{\mathbb{R}^n} (\mathbf{M}f(x))^p \,\omega(x)\mathrm{d}x \leq C_{n,\omega,p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\mathrm{d}x, \quad 1 < p < \infty;$$

$$\sup_{t>0} \left(t \cdot \left|\{x \in \mathbb{R}^n : \mathbf{M}f(x) > t\}\right|\right) \leq C_{n,\omega} \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\mathrm{d}x, \quad p = 1.$$

Although Theorems 1.1 and 1.2 require stronger assumption that ω being A_1, our weighted estimates are actually valid for certain exponents $p < 1$. On the other hand, the authors in [10] and [8] concern the weighted estimates in terms of Hausdorff content. The major techniques in their proofs use covering lemmas and certain properties of dyadic cubes decomposition, which appear also in the work of [6]. Note that the Hausdorff content is defined in terms of covering. It is difficult to see how the techniques in their works can be adapted to our setting since capacities are not defined in terms of covering. Our method of proof to Theorems 1.1 and 1.2 mainly uses the nonlinear potential theory.

Sophisticated readers may realize that Theorem 1.2 can be obtained by Theorem 1.1 through the Marcinkiewicz's interpolation technique. Indeed, if ω is a weight, such that the Lebesgue measure is absolutely continuous with respect to $R^{\omega}_{\alpha,s;1}(\cdot)$:

$$R^{\omega}_{\alpha,s;1}(E) = 0 \quad \text{entails} \quad |E| = 0, \tag{1.4}$$

then $|f(x)| \leq C$ q.e. with respect to $R^{\omega}_{\alpha,s;1}(\cdot)$ implies $\mathbf{M}^{\mathrm{loc}} f(x) \leq C$ everywhere and hence

$$\left\|\mathbf{M}^{\mathrm{loc}} f\right\|_{L^{\infty}(R^{\omega}_{\alpha,s;\rho})} \leq \|f\|_{L^{\infty}(R^{\omega}_{\alpha,s;\rho})}, \tag{1.5}$$

where

$$\|u\|_{L^{\infty}(R^{\omega}_{\alpha,s;1})} = \inf\left\{C > 0 : |u(x)| \leq C \text{ q.e. with respect to } R^{\omega}_{\alpha,s;1}(\cdot)\right\}.$$

Note that if $\omega \in A_s$, then the absolute continuity given in (1.4) holds (see [9, Lemma 4.4.3]). Combining (1.5) and the weak-type estimate in Theorem 1.1, one can easily modify the proof of Marcinkiewicz's interpolation theorem given in [2, Theorem 1.3.2] to obtain the strong-type estimate in Theorem 1.2. Nevertheless, we will give an independent proof of Theorem 1.2 without appealing to Theorem 1.1 and the interpolation technique. In fact, the crucial step in proving the strong-type estimate in Theorem 1.2 is given by

$$\int_{\mathbb{R}^n} \left(\mathbf{M}^{\mathrm{loc}} f\right)^p \mathrm{d}R^{\omega}_{\alpha,s;1} \le C_{n,\alpha,s,\omega,p} \int_{\mathbb{R}^n} |f|^p \mathrm{d}R^{\omega}_{\alpha,s;1}, \quad \frac{n-\alpha s}{n} < p \le 1. \tag{1.6}$$

For the exponent $p > 1$, one observes that

$$\left(\mathbf{M}^{\mathrm{loc}} f(x)\right)^p \le \mathbf{M}^{\mathrm{loc}}(f^p)(x), \quad p > 1, \quad x \in \mathbb{R}^n,$$

and the strong-type estimate in Theorem 1.2 for $p > 1$ then reduces to the case where $p = 1$.

In the sequel, for q.e. everywhere defined (with respect to $R^{\omega}_{\alpha,s;1}(\cdot)$) function u on $\mathbb{R}^n$ and $0 < q < \infty$, we write

$$\|u\|_{L^q\left(R^{\omega}_{\alpha,s;1}\right)} = \left(\int_{\mathbb{R}^n} |u|^q \mathrm{d}R^{\omega}_{\alpha,s;1}\right)^{\frac{1}{q}},$$

$$\|u\|_{L^{q,\infty}\left(R^{\omega}_{\alpha,s;1}\right)} = \sup_{t>0}\left(t \cdot R^{\omega}_{\alpha,s;1}(\{x \in \mathbb{R}^n : |u(x)| > t\})^{\frac{1}{q}}\right).$$

Denote by $L^q(R^{\omega}_{\alpha,s;1})$ and $L^{q,\infty}(R^{\omega}_{\alpha,s;1})$ the spaces consisting of all functions u with finite quantities $\|u\|_{L^q(R^{\omega}_{\alpha,s;1})}$ and $\|u\|_{L^{q,\infty}(R^{\omega}_{\alpha,s;1})}$, respectively. An obvious notational modification will do for $L^q(R^{\omega}_{\alpha,s})$ and $L^{q,\infty}(R^{\omega}_{\alpha,s})$ with the corresponding $\|\cdot\|_{L^q(R^{\omega}_{\alpha,s})}$ and $\|\cdot\|_{L^{q,\infty}(R^{\omega}_{\alpha,s})}$. To illustrate an application of Theorems 1.1 and 1.2, let us first address a type of homogeneous weighted Sobolev inequalities.

Theorem 1.3 *Let $s > 1$, $0 < \alpha < n/s$, $q \ge (n-\alpha s)/n$, $0 < \beta < (n-\alpha s)/q$, and*

$$q^* = \frac{(n-\alpha s)q}{n-\alpha s-\beta q}.$$

For any Lebesgue measurable $R^{\omega}_{\alpha,s}(\cdot)$-q.e. defined function $f \in L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s}\right)$, $\omega \in A_1$, it holds that

$$\left\|\mathcal{I}_\beta(f\omega)\right\|_{L^{q^*,\infty}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s}\right)} \le C_{n,\alpha,s,\omega,\beta} \cdot \|f\|^{\frac{\beta}{n}}_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s}\right)} \cdot \|f\omega\|^{1-\frac{\beta}{n}}_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s}\right)} \tag{1.7}$$

provided that $q = (n-\alpha s)/n$, *and*

$$\left\|\mathcal{I}_\beta\left(f\omega^{\frac{n-\alpha s}{nq}}\right)\right\|_{L^{q^*}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s}\right)} \leq C_{n,\alpha,s,\omega,\beta,q}\cdot\|f\|^{\frac{\beta q}{n-\alpha s}}_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s}\right)}\cdot\left\|f\omega^{\frac{n-\alpha s}{nq}}\right\|^{1-\frac{\beta q}{n-\alpha s}}_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s}\right)} \tag{1.8}$$

provided that $q > (n-\alpha s)/n$.

To obtain the corresponding inhomogeneous weighted Sobolev inequalities, we impose an extra condition on the weight ω that

$$\sup_{z\in\mathbb{Z}^n}\omega(B_1(z)) < \infty. \tag{1.9}$$

Theorem 1.4 *Let* $s > 1$, $0 < \alpha < n/s$, $\alpha \in \mathbb{N}$, $q \geq (n-\alpha s)/n$, $0 < \beta < (n-\alpha s)/q$, *and*

$$q^* = \frac{(n-\alpha s)q}{n-\alpha s-\beta q}.$$

For any Lebesgue measurable $R^{\omega}_{\alpha,s;1}(\cdot)$*-q.e. defined function* $f \in L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)$, $\omega \in A_1$ *which satisfies* (1.9)*, it holds that*

$$\left\|\mathcal{G}_\beta(f\omega)\right\|_{L^{q^*,\infty}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} \leq C_{n,\alpha,s,\omega,\beta}\left(\|f\|^{\frac{\beta}{n}}_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}\cdot\|f\omega\|^{1-\frac{\beta}{n}}_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}+\|f\|_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}\right)$$

provided that $q = (n-\alpha s)/n$, *and*

$$\left\|\mathcal{G}_\beta\left(f\omega^{\frac{n-\alpha s}{nq}}\right)\right\|_{L^{q^*}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} \leq C_{n,\alpha,s,\omega,\beta,q}\left(\|f\|^{\frac{\beta q}{n-\alpha s}}_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}\cdot\left\|f\omega^{\frac{n-\alpha s}{nq}}\right\|^{1-\frac{\beta q}{n-\alpha s}}_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}+\|f\|_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}\right)$$

provided that $q > (n-\alpha s)/n$.

Remark 1.5 Consider the power weights $\omega(x) = |x|^{-\eta}$ for $0 \leq \eta < n$. Certainly, $\omega \in A_1$ and satisfies (1.9). Nevertheless, there are A_1 weights ω which fail to satisfy (1.9). For instance

$$\omega(x) = \begin{cases}\omega_i(x-4^i), & x \in [4^i - i^2, 4^i + i^2] \quad \text{for some } i \geq 3;\\ 1, & \text{otherwise,}\end{cases}$$

where $\omega_i(x) = \max\{i|x|^{-1/2}, 1\}$, $i \in \mathbb{N}$, and $x \in \mathbb{R}$.

In what follows, we assume that $\alpha > 0$, $s > 1$, $n \in \mathbb{N}$, and $\alpha s < n$. We denote by $\mathscr{D}(\mathbb{R}^n)$ the space of compactly supported infinitely differentiable functions, while $\mathscr{S}'(\mathbb{R}^n)$ refers to the class of tempered distributions. The characteristic function of a set $E \subseteq \mathbb{R}^n$ is denoted by χ_E. While $B_r(x)$ denotes the open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$.

2 Preliminaries on the weighted capacities

Let ω be a weight on $\mathbb{R}^n$. We say that ω satisfies the strong doubling property if there exist some constants $C > 0$ and $\varepsilon > 0$, such that for every ball B of $\mathbb{R}^n$ and Lebesgue measurable subset E of B, it holds that

$$\omega(B) \leq C \left(\frac{|B|}{|E|}\right)^{\varepsilon} \omega(E). \tag{2.1}$$

It is a standard fact that if $\omega \in A_p$, $1 \leq p < \infty$, then one may take $C = [\omega]_{A_p}$ and $\varepsilon = p$ in the above inequality (see [9, Proposition 1.2.7]).

As noted in the first section, the weighted local Riesz $R^{\omega}_{\alpha,s;\rho}(\cdot)$ and weighted Bessel capacities $B^{\omega}_{\alpha,s}(\cdot)$ are equivalent for $\omega \in A_s$. Let us introduce a variant weighted local Riesz capacities $\mathcal{R}^{\omega}_{\alpha,s;\rho}(\cdot)$ by

$$\mathcal{R}^{\omega}_{\alpha,s;\rho}(E) = \inf\left\{\|f\|^s_{L^s_{\omega}(\mathbb{R}^n\times(0,\rho))} : f \geq 0,\ \int_0^{\rho}\left(\int_{|\cdot-y|}\frac{f(y,t)}{t^{n-\alpha}}\omega(y)^{-\frac{1}{s-1}}\mathrm{d}y\right)\frac{\mathrm{d}t}{t} \geq \chi_E\right\},$$

where $E \subseteq \mathbb{R}^n$ and $0 < \rho < \infty$ (see [9, Section 3.6.1]). Under the same assumption that $\omega \in A_s$, we have the equivalence that

$$C^{-1}_{n,\alpha,s,\rho,\omega} \cdot R^{\omega}_{\alpha,s;\rho}(E) \leq \mathcal{R}^{\omega}_{\alpha,s;\rho}(E) \leq C_{n,\alpha,s,\rho,\omega} \cdot R^{\omega}_{\alpha,s;\rho}(E), \quad E \subseteq \mathbb{R}^n \tag{2.2}$$

(see [9, Corollary 3.6.7]). Combining with (1.3), we have

$$C^{-1}_{n,\alpha,s,\rho_1,\rho_2,\omega} \cdot \mathcal{R}^{\omega}_{\alpha,s;\rho_2}(E) \leq \mathcal{R}^{\omega}_{\alpha,s;\rho_1}(E) \leq C_{n,\alpha,s,\rho_1,\rho_2,\omega} \cdot \mathcal{R}^{\omega}_{\alpha,s;\rho_2}(E), \quad E \subseteq \mathbb{R}^n, \tag{2.3}$$

where $0 < \rho_1, \rho_2 < \infty$ are arbitrary positive numbers. Note again that the following absolute continuity holds for $\omega \in A_s$ that

$$R^{\omega}_{\alpha,s;\rho}(E) = 0 \quad \text{entails} \quad |E| = 0 \tag{2.4}$$

(see [9, Lemma 4.4.3]). In view of (2.2), one may replace $R^{\omega}_{\alpha,s;\rho}(\cdot)$ by $\mathcal{R}^{\omega}_{\alpha,s;\rho}(\cdot)$ in (2.4). On the other hand, we define the nonlinear potential $\mathcal{V}^{\mu}_{\omega;\rho}$ of a positive measure μ on $\mathbb{R}^n$ by

$$\mathcal{V}^{\mu}_{\omega;\rho}(x) = \int_0^{\rho}\int_{|x-y|<t}\left(\frac{\mu(B_t(y))}{t^{n-\alpha}}\right)^{\frac{1}{s-1}}\frac{\omega(y)^{-\frac{1}{s-1}}}{t^{n-\alpha}}\mathrm{d}y\frac{\mathrm{d}t}{t}, \quad x \in \mathbb{R}^n.$$

As a consequence, following [9, Proposition 3.6.4], if E is an arbitrary subset of $\mathbb{R}^n$ and $\omega \in A_s$, then there exists a positive measure μ^E supported in $\overline{E}$, such that

$$\begin{aligned} &\mathcal{V}_{\omega;\rho}^{\mu^E}(x) \geq 1 \quad \text{q.e. with respect to } R_{\alpha,s;\rho}^{\omega}(\cdot) \text{ for } x \in E, \\ &\mathcal{V}_{\omega;\rho}^{\mu^E}(x) \leq 1 \quad \text{for every } x \in \operatorname{supp}\left(\mu^E\right), \end{aligned} \tag{2.5}$$

and

$$\mu^E\left(\overline{E}\right) = \mathcal{R}_{\alpha,s;\rho}^{\omega}(E). \tag{2.6}$$

We may compare the nonlinear potential $\mathcal{V}_{\omega;\rho}^{\mu}$ with the Wolff potential $W_{\omega;\rho}^{\mu}$ defined by

$$W_{\omega;\rho}^{\mu}(x) = \int_0^{\rho} \left(\frac{t^{\alpha s}\mu(B_t(x))}{\omega(B_t(x))}\right)^{\frac{1}{s-1}} \frac{\mathrm{d}t}{t}, \quad x \in \mathbb{R}^n.$$

To this end, up to a constant depending on n and s, we prefer to express the Wolff potential by

$$W_{\omega;\rho}^{\mu}(x) = \int_0^{\rho} \left(\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}} \left(\fint_{B_t(x)} \omega(y)\mathrm{d}y\right)^{-\frac{1}{s-1}} \frac{\mathrm{d}t}{t}.$$

We observe that $B_t(y) \subseteq B_{2t}(x)$ for $|x-y| < t$, then $\omega \in A_s$ yields

$$\begin{aligned} \mathcal{V}_{\omega;\rho}^{\mu}(x) &\leq \int_0^{\rho} \int_{|x-y|<2t} \left(\frac{\mu(B_{2t}(x))}{t^{n-\alpha}}\right)^{\frac{1}{s-1}} \frac{\omega(y)^{-\frac{1}{s-1}}}{t^{n-\alpha}} \mathrm{d}y \frac{\mathrm{d}t}{t} \\ &\leq C_n \cdot [\omega]_{A_s} \int_0^{\rho} \left(\frac{\mu(B_{2t}(x))}{t^{n-\alpha}}\right)^{\frac{1}{s-1}} \left(\fint_{|x-y|<2t} \omega(y)\mathrm{d}y\right)^{-\frac{1}{s-1}} t^{\alpha} \frac{\mathrm{d}t}{t} \\ &= C_{n,\alpha,s} \cdot [\omega]_{A_s} \int_0^{2\rho} \left(\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}} \left(\fint_{B_t(x)} \omega(y)\mathrm{d}y\right)^{-\frac{1}{s-1}} \frac{\mathrm{d}t}{t} \\ &= C_{n,\alpha,s} \cdot [\omega]_{A_s} \cdot W_{\omega;2\rho}^{\mu}(x). \end{aligned}$$

Besides that, by writing $1 = \omega^{1/s} \cdot \omega^{-1/s}$, Hölder's inequality gives

$$1 \leq \left(\fint_{B_t(x)} \omega(y)\mathrm{d}y\right) \left(\fint_{B_t(x)} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y\right)^{s-1.}$$

Since $B_t(x) \subseteq B_{2t}(y)$ for $|x-y|<t$, we have

$$\begin{aligned}\mathcal{V}^{\mu}_{\omega;\rho}(x) &= C_{n,\alpha,s}\int_0^{\frac{\rho}{2}}\int_{|x-y|<2t}\left(\frac{\mu(B_{2t}(y))}{t^{n-\alpha}}\right)^{\frac{1}{s-1}}\frac{\omega(y)^{-\frac{1}{s-1}}}{t^{n-\alpha}}\mathrm{d}y\frac{\mathrm{d}t}{t}\\ &\geq C_{n,\alpha,s}\int_0^{\frac{\rho}{2}}\left(\frac{\mu(B_t(x))}{t^{n-\alpha}}\right)^{\frac{1}{s-1}}\left(\fint_{B_t(x)}\omega(y)\mathrm{d}y\right)^{-\frac{1}{s-1}}t^{\alpha}\frac{\mathrm{d}t}{t}\\ &= C_{n,\alpha,s}\int_0^{\frac{\rho}{2}}\left(\frac{t^{\alpha s}\mu(B_t(x))}{\omega(B_t(x))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\\ &= C_{n,\alpha,s}\cdot W^{\mu}_{\omega;\frac{\rho}{2}}(x).\end{aligned}$$

Therefore, we obtain the estimate that

$$c_{n,\alpha,s}\cdot W^{\mu}_{\omega;\frac{\rho}{2}} \leq \mathcal{V}^{\mu}_{\omega;\rho} \leq C_{n,\alpha,s}\cdot[\omega]_{A_s}\cdot W^{\mu}_{\omega;2\rho}. \tag{2.7}$$

The Wolff potential possesses the bounded maximum principle.

Lemma 2.1 *Let $\alpha>0$, $s>1$, $\alpha s<n$, $\rho>0$, and μ be a positive measure on $\mathbb{R}^n$. If ω is a weight satisfying (2.1), then*

$$W^{\mu}_{\omega;\rho}(x) \leq \frac{C^{\frac{1}{s-1}}}{2^{\frac{\alpha s}{s-1}}}\cdot\sup\{W^{\mu}_{\omega;2\rho}(z): z\in\operatorname{supp}(\mu)\},\quad x\in\mathbb{R}^n,$$

where the constant $C>0$ is as in (2.1).

Proof Let $x\notin\operatorname{supp}(\mu)$ and $x_0\in K=\operatorname{supp}(\mu)$ be the point that minimizes the distance from x to $\operatorname{supp}(\mu)$. If $B_t(x)\cap K\neq\emptyset$, then $t>|x-x_0|$, which in turn implies that $B_t(x)\subseteq B_{2t}(x_0)$. Consequently, (2.1) implies that

$$\begin{aligned}W^{\mu}_{\omega;\rho}(x) &\leq \int_0^{\rho}\left(\frac{t^{\alpha s}\mu(B_{2t}(x_0))}{\omega(B_t(x))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\\ &\leq C^{\frac{1}{s-1}}\int_0^{\rho}\left(\frac{t^{\alpha s}\mu(B_{2t}(x_0))}{\omega(B_{2t}(x_0))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\\ &= \frac{C^{\frac{1}{s-1}}}{2^{\frac{\alpha s}{s-1}}}\cdot W^{\mu}_{\omega;2\rho}(x_0),\end{aligned}$$

and the bounded maximum principle follows. ◻

Now, we show that the Wolff potential satisfies the weak-type estimate.

Proposition 2.2 *Let $\alpha>0$, $s>1$, $\alpha s<n$, $\rho>0$, $\omega\in A_s$, and μ be a positive measure on $\mathbb{R}^n$. Then*

$$\mathcal{R}^{\omega}_{\alpha,s;\rho}(\{W^{\mu}_{\omega;\rho}>t\}) \leq C_{n,\alpha,s,\rho,\omega}\cdot\frac{\mu(\mathbb{R}^n)}{t^{s-1}},\quad 0<t<\infty.$$

Proof Let γ be the measure associated with a compact subset K of $\{W^{\mu}_{\omega;\rho} > t\}$, such that $\gamma(K) = \mathcal{R}^{\omega}_{\alpha,s;2\rho}(K)$ as in (2.6). Suppose that $x \in \operatorname{supp}(\gamma)$ and we let

$$\mathbf{M}_{\gamma}\mu(x) = \sup_{t>0} \frac{\mu(B_t(x))}{\gamma(B_t(x))}.$$

Then

$$\begin{aligned} W^{\mu}_{\omega;\rho}(x) &= \int_0^{\rho} \left(\frac{t^{\alpha s}\mu(B_t(x))}{\omega(B_t(x))}\right)^{\frac{1}{s-1}} \frac{\mathrm{d}t}{t} \\ &\leq \mathbf{M}_{\gamma}\mu(x)^{\frac{1}{s-1}} \cdot W^{\gamma}_{\omega;\rho}(x) \\ &\leq C_{n,\alpha,s,\omega} \cdot \mathbf{M}_{\gamma}\mu(x)^{\frac{1}{s-1}}, \end{aligned}$$

since $W^{\gamma}_{\omega;\rho}(x) \leq C_{n,\alpha,s,\omega} \cdot \mathcal{V}^{\gamma}_{\omega;2\rho}(x) \leq C_{n,\alpha,s,\omega}$ on $\operatorname{supp}(\gamma)$ by (2.5) and (2.7). Thus,

$$\operatorname{supp}(\gamma) \subseteq \left\{\mathbf{M}_{\gamma}\mu > C_{n,\alpha,s,\omega} \cdot t^{s-1}\right\}.$$

By Besicovitch covering theorem, there are c_n collections of balls $A_i = \{B_{n_i}\}$, $i = 1, \ldots, c_n$, such that A_i is disjoint and

$$\operatorname{supp}(\gamma) \subseteq \bigcup_{i=1}^{c_n} \bigcup_{B \in A_i} B, \quad \frac{\mu(B)}{\gamma(B)} > C_{n,\alpha,s,\omega} \cdot t^{s-1}, \quad B \in A_i.$$

As a consequence,

$$\mathcal{R}^{\omega}_{\alpha,s;2\rho}(K) = \gamma(K) \leq \sum_{i=1}^{c_n} \sum_{B \in A_i} \gamma(B) \leq \frac{C_{n,\alpha,s,\omega}}{t^{s-1}} \sum_{i=1}^{c_n} \sum_{B \in A_i} \mu(B) \leq \frac{C_{n,\alpha,s,\omega}}{t^{s-1}} \cdot \mu(\mathbb{R}^n).$$

The result follows by noting that $\mathcal{R}^{\omega}_{\alpha,s;\rho}(\cdot) \leq C_{n,\alpha,s,\rho,\omega} \cdot \mathcal{R}^{\omega}_{\alpha,s;2\rho}(\cdot)$ as in (2.3). □

Remark 2.3 As noted in the remark following [1, Proposition 6.3.12], the weak-type estimate in Proposition 2.2 is false for the Wolff potential replaced by the nonlinear Bessel potential that $V^{\mu}_{\alpha,s} = G_{\alpha} * (G_{\alpha} * \mu)^{\frac{1}{s-1}}$, $1 < s \leq 2 - \alpha/n$.

The following is known to be the capacitary strong-type estimate, which will be used in justifying the normability of the space $L^1(R^{\omega}_{\alpha,s;1})$.

Proposition 2.4 *Let $\alpha > 0$, $s > 1$, $\alpha s < n$, and $\omega \in A_s$. Then*

$$\int_0^{\infty} R^{\omega}_{\alpha,s;1}(\{\mathcal{I}_{\alpha,1} f \geq t\})\mathrm{d}t^s \leq C_{n,\alpha,s,\omega} \int_{\mathbb{R}^n} f(x)^s \omega(x)\mathrm{d}x$$

for all $f \geq 0$.

Proof In view of (2.2), we will show that

$$\int_0^\infty \mathcal{R}^\omega_{\alpha,s;1}(\{\mathcal{I}_{\alpha,1}f \geq t\})\mathrm{d}t^s \leq C_{n,\alpha,s,\omega}\int_{\mathbb{R}^n} f(x)^s\omega(x)\mathrm{d}x.$$

Denote by $s' = s/(s-1)$. Assume for the moment that $f \in C_0(\mathbb{R}^n)$ is a compactly supported continuous function on $\mathbb{R}^n$. Let

$$J = \int_0^\infty \mathcal{R}^\omega_{\alpha,s;1}(\{\mathcal{I}_{\alpha,1}f \geq t\})\mathrm{d}t^s < \infty.$$

Suppose that μ_t is a measure associated with the compact set $\{\mathcal{I}_{\alpha,1}f \geq t\}$ which satisfies (2.6) with respect to $\mathcal{R}^\omega_{\alpha,s;32}(\cdot)$, that is,

$$\begin{aligned}&\mathcal{V}^{\mu_t}_{\omega;32} \leq 1 \quad \text{on } \{\mathcal{I}_{\alpha,1}f \geq t\},\\ &\mu_t(\mathbb{R}^n) = \mathcal{R}^\omega_{\alpha,s;32}(\{\mathcal{I}_{\alpha,1}f \geq t\}).\end{aligned}$$

Using (2.7) and Lemma 2.1, one obtains the boundedness of Wolff potential $W^{\mu_t}_{\omega;8}$ that

$$W^{\mu_t}_{\omega;8} \leq \sup_{\{\mathcal{I}_{\alpha,1}f\geq t\}} W^{\mu_t}_{\omega;16} \leq C_{n,\alpha,s,\omega} \sup_{\{\mathcal{I}_{\alpha,1}f\geq t\}} \mathcal{V}^{\mu_t}_{\omega;32} \leq C_{n,\alpha,s,\omega}. \tag{2.8}$$

We have

$$\begin{aligned}J &\leq s\int_0^\infty \int_{\mathbb{R}^n} \mathcal{I}_{\alpha,1}f(x)\mathrm{d}\mu_t(x)t^{s-2}\mathrm{d}t\\ &= s\int_{\mathbb{R}^n}\int_0^\infty \mathcal{I}_{\alpha,1}\mu_t(y)t^{s-2}\mathrm{d}tf(y)\mathrm{d}y\\ &\leq s\cdot \|f\|_{L^s(\omega)}\cdot L^{\frac{1}{s'}},\end{aligned}$$

where

$$L = \int_{\mathbb{R}^n}\left(\int_0^\infty \mathcal{I}_{\alpha,1}\mu_t(y)t^{s-2}\mathrm{d}t\right)^{s'}\omega(y)^{-\frac{1}{s-1}}\mathrm{d}y.$$

To conclude the proof, we will show that $L \leq C_{n,\alpha,s,\omega}\cdot J$. Assume for the moment that $s \geq 2$. Let

$$\lambda_u(E) = \int_u^\infty t^{s-2}\mu_t(E)\mathrm{d}t, \quad 0 \leq u < \infty.$$

We have

$$
\begin{aligned}
L &= \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{|y-z|<1} \frac{1}{|y-z|^{n-\alpha}} \mathrm{d}\mu_t(z) t^{s-2} \mathrm{d}t \right)^{s'} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&= C_{n,\alpha,s} \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{|y-z|<1} \int_{|y-z|}^{2|y-z|} \frac{1}{r^{n-\alpha}} \frac{\mathrm{d}r}{r} \mathrm{d}\mu_t(z) t^{s-2} \mathrm{d}t \right)^{s'} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&= C_{n,\alpha,s} \int_{\mathbb{R}^n} \left(\int_0^\infty \int_0^2 \int_{|y-z|<r} \mathrm{d}\mu_t(z) \frac{1}{r^{n-\alpha}} \frac{\mathrm{d}r}{r} t^{s-2} \mathrm{d}t \right)^{s'} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&= C_{n,\alpha,s} \int_{\mathbb{R}^n} \left(\int_0^2 \int_0^\infty \frac{\mu_t(B_r(y))}{r^{n-\alpha}} t^{s-2} \mathrm{d}t \frac{\mathrm{d}r}{r} \right)^{s'} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&= C_{n,\alpha,s} \int_{\mathbb{R}^n} \left(\int_0^2 \frac{\lambda_0(B_r(y))}{r^{n-\alpha}} \frac{\mathrm{d}r}{r} \right)^{s'} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&= C_{n,\alpha,s} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \int_0^2 \chi_{|y-z|<r} \frac{1}{r^{n-\alpha}} \frac{\mathrm{d}r}{r} \mathrm{d}\lambda_0(z) \right)^{s'} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&\le C_{n,\alpha,s} \int_{\mathbb{R}^n} \left(\int_{|y-z|<2} \int_{|y-z|}^\infty \frac{1}{r^{n-\alpha}} \frac{\mathrm{d}r}{r} \mathrm{d}\lambda_0(z) \right)^{s'} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&= C_{n,\alpha,s} \int_{\mathbb{R}^n} \left(I_{\alpha,2} * \lambda_0 \right)(y)^{s'} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&\le C_{n,\alpha,s} \int_{\mathbb{R}^n} \sup_{0<t\le 2} \left(\frac{\lambda_0(B_t(y))}{t^{n-\alpha}} \right)^{s'} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \qquad (2.9) \\
&\le C_{n,\alpha,s} \int_{\mathbb{R}^n} \sup_{0<t\le 2} \int_t^{2t} \left(\frac{\lambda_0(B_r(y))}{r^{n-\alpha}} \right)^{s'} \frac{\mathrm{d}r}{r} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&\le C_{n,\alpha,s} \int_{\mathbb{R}^n} \int_0^4 \left(\frac{\lambda_0(B_r(y))}{r^{n-\alpha}} \right)^{s'} \frac{\mathrm{d}r}{r} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y,
\end{aligned}
$$

where we have used [9, Theorem 3.1.2] in (2.9). On the other hand, integration by parts gives

$$
\lambda_0(B_r(y))^{s'} = s' \int_0^\infty \lambda_u(B_r(y))^{s'-1} \mu_u(B_r(y)) u^{s-2} \mathrm{d}u.
$$

Express $\mu_u(B_r(y)) = \mu_u(B_r(y))^{(2-s')s'} \cdot \mu_u(B_r(y))^{(s'-1)^2}$. Using Hölder's inequality with respect to the exponents that $1/(2-s')^{-1} + 1/(s'-1)^{-1} = 1$, we obtain $L \le C_{n,\alpha,s} \cdot L_1^{2-s'} \cdot L_2^{s'-1}$, where

$$
L_1 = \int_{\mathbb{R}^n} \int_0^4 \int_0^\infty \left(\frac{\mu_u(B_r(y))}{r^{n-\alpha}} \right)^{s'} u^{s-1} \mathrm{d}u \frac{\mathrm{d}r}{r} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y,
$$

and

$$L_2 = \int_{\mathbb{R}^n} \int_0^4 \int_0^\infty r^{(\alpha-n)s'} \mu_u(B_r(y))^{s'-1} \lambda_u(B_r(y)) \mathrm{d}u \frac{\mathrm{d}r}{r} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y.$$

We have

$$\begin{aligned}
L_1 &= \int_0^\infty \int_{\mathbb{R}^n} \int_0^4 \left(\frac{\mu_u(B_r(y))}{r^{n-\alpha}} \right)^{\frac{1}{s-1}} \int_{B_r(y)} \mathrm{d}\mu_u(z) \frac{\omega(y)^{-\frac{1}{s-1}}}{r^{n-\alpha}} \mathrm{d}y \frac{\mathrm{d}r}{r} u^{s-1} \mathrm{d}u \\
&= \int_0^\infty \int_{\mathbb{R}^n} \int_0^4 \int_{|y-z|<r} \left(\frac{\mu_u(B_r(y))}{r^{n-\alpha}} \right)^{\frac{1}{s-1}} \frac{\omega(y)^{-\frac{1}{s-1}}}{r^{n-\alpha}} \mathrm{d}y \frac{\mathrm{d}r}{r} \mathrm{d}\mu_u(z) u^{s-1} \mathrm{d}u \\
&= \int_0^\infty \int_{\mathbb{R}^n} \mathcal{V}^{\mu_u}_{\omega;4}(z) \mathrm{d}\mu_u(z) u^{s-1} \mathrm{d}u \\
&\le C_{n,\alpha,s,\omega} \int_0^\infty \int_{\mathbb{R}^n} W^{\mu_u}_{\omega;8}(z) \mathrm{d}\mu_u(z) u^{s-1} \mathrm{d}u \qquad (2.10) \\
&\le C_{n,\alpha,s,\omega} \int_0^\infty \mu_u(\mathbb{R}^n) u^{s-1} \mathrm{d}u \qquad (2.11) \\
&= C_{n,\alpha,s,\omega} \int_0^\infty \mathcal{R}^\omega_{\alpha,s;32}(\{\mathcal{I}_{\alpha,1} f \ge u\}) \mathrm{d}u^s \\
&\le C_{n,\alpha,s,\omega} \int_0^\infty \mathcal{R}^\omega_{\alpha,s;1}(\{\mathcal{I}_{\alpha,1} f \ge u\}) \mathrm{d}u^s \qquad (2.12) \\
&= C_{n,\alpha,s,\omega} \cdot J,
\end{aligned}$$

where we have used (2.7), (2.8), and (2.3) in (2.10), (2.11), and (2.12), respectively. Similarly

$$\begin{aligned}
L_2 &= \int_{\mathbb{R}^n} \int_0^4 \int_0^\infty r^{(\alpha-n)s'} \mu_u(B_r(y))^{s'-1} \int_u^\infty t^{s-2} \mu_t(B_r(y)) \mathrm{d}t \mathrm{d}u \frac{\mathrm{d}r}{r} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \\
&= \int_0^\infty \int_u^\infty t^{s-2} \int_{\mathbb{R}^n} \int_0^4 r^{(\alpha-n)s'} \mu_u(B_r(y))^{s'-1} \int_{B_r(y)} \mathrm{d}\mu_t(z) \frac{\mathrm{d}r}{r} \omega(y)^{-\frac{1}{s-1}} \mathrm{d}y \mathrm{d}t \mathrm{d}u \\
&= \int_0^\infty \int_u^\infty t^{s-2} \int_{\mathbb{R}^n} \int_0^4 \int_{|y-z|<r} \left(\frac{\mu_u(B_r(y))}{r^{n-\alpha}} \right)^{\frac{1}{s-1}} \frac{\omega(y)^{-\frac{1}{s-1}}}{r^{n-\alpha}} \mathrm{d}y \frac{\mathrm{d}r}{r} \mathrm{d}\mu_t(z) \mathrm{d}t \mathrm{d}u \\
&= \int_0^\infty \int_u^\infty t^{s-2} \int_{\mathbb{R}^n} \mathcal{V}^{\mu_u}_{\omega;4}(z) \mathrm{d}\mu_t(z) \mathrm{d}t \mathrm{d}u \\
&\le C_{n,\alpha,s,\omega} \int_0^\infty \int_u^\infty t^{s-2} \int_{\mathbb{R}^n} W^{\mu_u}_{\omega;8}(z) \mathrm{d}\mu_t(z) \mathrm{d}t \mathrm{d}u \\
&\le C_{n,\alpha,s,\omega} \int_0^\infty \int_u^\infty t^{s-2} \mu_t(\mathbb{R}^n) \mathrm{d}t \mathrm{d}u \\
&= C_{n,\alpha,s,\omega} \int_0^\infty t^{s-1} \mu_t(\mathbb{R}^n) \mathrm{d}t \\
&\le C_{n,\alpha,s,\omega} \cdot J.
\end{aligned}$$

Now, we consider the case that $1 < s < 2$. We write

$$\lambda^u(E) = \int_0^u t^{s-2}\mu_t(E)\mathrm{d}t, \quad 0 < u \leq \infty,$$

and integration by parts gives

$$\lambda^\infty(B_r(y))^{s'} = s' \int_0^\infty \lambda^u(B_r(y))^{s'-1}\mu_u(B_r(y))u^{s-2}\mathrm{d}u.$$

Thus, a similar estimate as before yields

$$\begin{aligned} L &= C_{n,\alpha,s} \int_{\mathbb{R}^n} \left(\int_0^2 \frac{\lambda^\infty(B_r(y))}{r^{n-\alpha}} \frac{\mathrm{d}r}{r}\right)^{s'} \omega(y)^{-\frac{1}{s-1}}\mathrm{d}y \\ &\leq C_{n,\alpha,s} \int_{\mathbb{R}^n} (I_{\alpha,2} * \lambda^\infty)(y)^{s'} \omega(y)^{-\frac{1}{s-1}}\mathrm{d}y \\ &\leq C_{n,\alpha,s,\omega} \int_{\mathbb{R}^n} \int_0^4 \left(\frac{\lambda^\infty(B_r(y))}{r^{n-\alpha}}\right)^{s'} \frac{\mathrm{d}r}{r}\omega(y)^{-\frac{1}{s-1}}\mathrm{d}y \\ &= C_{n,\alpha,s,\omega} \int_0^\infty \left\|\lambda^u\left(B_{(\cdot)}(\cdot)\right)\right\|_{L^{s'-1}(\sigma_u)}^{s'-1} u^{s-2}\mathrm{d}u, \end{aligned}$$

where

$$\mathrm{d}\sigma_s(y,r) = \chi_{0<r<4} \cdot r^{(\alpha-n)s'}\mu_u(B_r(y))\omega(y)^{-\frac{1}{s-1}}\mathrm{d}y\frac{\mathrm{d}r}{r}.$$

Note that

$$\begin{aligned} &\left\|\lambda^u\left(B_{(\cdot)}(\cdot)\right)\right\|_{L^{s'-1}(\sigma_u)} \\ &\leq \int_0^u t^{s-2}\left\|\mu_t\left(B_{(\cdot)}(\cdot)\right)\right\|_{L^{s'-1}(\sigma_s)}\mathrm{d}t \\ &= \int_0^u t^{s-2}\left(\int_0^4\int_{\mathbb{R}^n}\left(\frac{\mu_t(B_r(y))}{r^{n-\alpha}}\right)^{\frac{1}{s-1}}\int_{B_r(y)}\mathrm{d}\mu_u(z)\frac{\omega(y)^{-\frac{1}{s-1}}}{r^{n-\alpha}}\mathrm{d}y\frac{\mathrm{d}r}{r}\right)^{s-1}\mathrm{d}t \\ &= \int_0^u t^{s-2}\left(\int_{\mathbb{R}^n}\mathcal{V}_{\omega;4}^{\mu_t}(z)\mathrm{d}\mu_u(z)\right)^{s-1}\mathrm{d}t \\ &\leq C_{n,\alpha,s,\omega}\int_0^u t^{s-2}\left(\int_{\mathbb{R}^n}W_{\omega;8}^{\mu_t}(z)\mathrm{d}\mu_u(z)\right)^{s-1}\mathrm{d}t \\ &\leq C_{n,\alpha,s,\omega}\cdot\mu_u(\mathbb{R}^n)^{s-1}u^{s-1}, \end{aligned}$$

which yields

$$L \leq C_{n,\alpha,s,\omega}\int_0^\infty \mathcal{R}_{\alpha,s;32}^{\omega}(\{\mathcal{I}_{\alpha,1}f \geq u\})u^{s-1}\mathrm{d}u \leq C_{n,\alpha,s,\omega}\cdot J,$$

and the result holds for the case where $0 \le f \in C_0(\mathbb{R}^n)$. For general $f \in L^s(\omega)$, we approximate f by a sequence $\{f_j\}$ of functions in $C_0(\mathbb{R}^n)$. Indeed, if $f_j \to f$ in $L^s(\omega)$ and $f_j(x) \to f(x)$ a.e., then

$$\mathcal{R}^{\omega}_{\alpha,s;1}(\{\mathcal{I}_{\alpha,1}f \ge t\}) \le \liminf_{j\to\infty} \mathcal{R}^{\omega}_{\alpha,s;1}(\{\mathcal{I}_{\alpha,1}f_j \ge t\}),$$

which finishes the proof. □

Let us introduce an auxiliary functional that

$$\gamma(f) = \inf\left\{\|\varphi\|^s_{L^s(\omega)} : \varphi \ge 0,\ \mathcal{I}_{\alpha,\rho}\varphi(x) \ge |f(x)|^{\frac{1}{s}} \text{ q.e. with respect to } R^{\omega}_{\alpha,s;\rho}(\cdot)\right\}.$$

Proposition 2.5 *Let $\alpha > 0$, $s > 1$, $\alpha s < n$, and $\omega \in A_s$. The functional $\gamma(\cdot)$ is sublinear. Furthermore,*

$$C^{-1}_{n,\alpha,s,\omega} \cdot \gamma(f) \le \int_0^\infty R^{\omega}_{\alpha,s;1}(\{x \in \mathbb{R}^n : |f(x)| > t\})\mathrm{d}t \le C_{n,\alpha,s,\omega} \cdot \gamma(f). \quad (2.13)$$

In other words, the space $L^1(R^{\omega}_{\alpha,s;1})$ is normable.

Proof We first show the sublinearity of γ. To this end, let $\mathcal{H}$ be the set of all non-negative functions f on $\mathbb{R}^n$, such that $\mathcal{I}_{\alpha,1}\varphi(x) \ge f(x)^{1/s}$ q.e. with respect to $R^{\omega}_{\alpha,s;1}(\cdot)$ for some non-negative function $\varphi \in L^s(\omega)$ with $\|\varphi\|_{L^s(\omega)} \le 1$. We claim that $\mathcal{H}$ is convex. Suppose that $f_1, f_2 \in \mathcal{H}$ and $0 < c < 1$. Then there are non-negative $\varphi_1, \varphi_2 \in L^s(\omega)$, such that $\mathcal{I}_{\alpha,1}\varphi_i(x) \ge f_i(x)^{1/s}$ q.e. with respect to $R^{\omega}_{\alpha,s;1}(\cdot)$ and $\|\varphi_i\|_{L^s(\omega)} \le 1$ for $i = 1, 2$. It follows by the reverse Minkowski's inequality that

$$\begin{aligned}
&\left(\mathcal{I}_{\alpha,1}\left(\left(c\varphi_1^s + (1-c)\varphi_2^s\right)^{\frac{1}{s}}\right)(x)\right)^s \\
&= \left(\int_{\mathbb{R}^n} I_{\alpha,1}(x-y)\left(c\varphi_1(y)^s + (1-c)\varphi_2(y)^s\right)^{\frac{1}{s}}\mathrm{d}y\right)^s \\
&\ge \left(\int_{\mathbb{R}^n} I_{\alpha,1}(x-y)\left(c\varphi_1(y)^s\right)^{\frac{1}{s}}\mathrm{d}y\right)^s + \left(\int_{\mathbb{R}^n} I_{\alpha,1}(x-y)\left((1-c)\varphi_2(y)^s\right)^{\frac{1}{s}}\mathrm{d}y\right)^s \\
&= c\left(\mathcal{I}_{\alpha,1}\varphi_1(x)\right)^s + (1-c)\left(\mathcal{I}_{\alpha,1}\varphi_2(x)\right)^s \\
&\ge (cf_1 + (1-c)f_2)(x).
\end{aligned}$$

Subsequently, it also holds that

$$\begin{aligned}
\left\|(c\varphi_1^s + (1-c)\varphi_2^s)^{\frac{1}{s}}\right\|_{L^s(\omega)} &= \left(\int_{\mathbb{R}^n}\left(c\varphi_1(x)^s + (1-c)\varphi_2(x)^s\right)\omega(x)\mathrm{d}x\right)^{\frac{1}{s}} \\
&= \left(c\|\varphi_1\|^s_{L^s(\omega)} + (1-c)\|\varphi_2\|^s_{L^s(\omega)}\right)^{\frac{1}{s}} \\
&\le (c + (1-c))^{\frac{1}{s}} \\
&= 1.
\end{aligned}$$

The convexity of $\mathcal{H}$ follows. Now, it is routine to check that

$$\gamma(f)=\inf\{c>0: f\in c\mathcal{H}\},$$

from which the sublinearity of $\gamma(\cdot)$ follows.

Now, we show for (2.13). Proposition 2.4 gives the $\leq$ direction of (2.13). On the other hand, since

$$R^{\omega}_{\alpha,s;1}(\{|f|=\infty\})\leq\frac{1}{N}\int_0^{\infty}R^{\omega}_{\alpha,s;1}(\{|f|>t\})\mathrm{d}t,\quad N=1,2,\ldots,$$

one has $R^{\omega}_{\alpha,s;1}(\{|f|=\infty\})=0$. Then we have by the sublinearity of $\gamma(\cdot)$ that

$$\begin{aligned}\gamma(f)&=\gamma(f\chi_{\{|f|<\infty\}})\\&=\gamma\left(\sum_{i\in\mathbb{Z}}|f|\chi_{\{2^i\leq|f|<2^{i+1}\}}\right)\\&\leq\sum_{i\in\mathbb{Z}}\gamma\left(f\chi_{\{2^i\leq|f|<2^{i+1}\}}\right)\\&\leq\sum_{i\in\mathbb{Z}}2^{i+1}\cdot R^{\omega}_{\alpha,s;1}(\{2^i\leq|f|<2^{i+1}\}).\end{aligned}$$

However

$$\begin{aligned}\int_0^{\infty}R^{\omega}_{\alpha,s;1}(\{x\in\mathbb{R}^n:|f(x)|>t\})\mathrm{d}t&=\sum_{i\in\mathbb{Z}}\int_{2^{i-1}}^{2^i}R^{\omega}_{\alpha,s;1}(\{x\in\mathbb{R}^n:|f(x)|>t\})\mathrm{d}t\\&\geq\sum_{i\in\mathbb{Z}}2^{i-1}\cdot R^{\omega}_{\alpha,s;1}(\{2^i\leq|f|<2^{i+1}\}),\end{aligned}$$

which yields

$$\gamma(f)\leq 4\int_0^{\infty}R^{\omega}_{\alpha,s;1}(\{x\in\mathbb{R}^n:|f(x)|>t\})\mathrm{d}t,$$

and hence the $\geq$ direction of (2.13). □

We claim that the spaces $L^p(R^{\omega}_{\alpha,s;1})$ and $L^{p,\infty}(R^{\omega}_{\alpha,s;1})$ satisfy the p-convexity for $0<p<1$.

Corollary 2.6 *Let $\alpha>0$, $s>1$, $\alpha s<n$, $0<p\leq 1$, and $\omega\in A_s$. Then*

$$\left\|\sum_k|f_k|\right\|_{L^p\left(R^{\omega}_{\alpha,s;1}\right)}\leq C_{n,\alpha,s,\omega}\left(\sum_k\|f_k\|^p_{L^p\left(R^{\omega}_{\alpha,s;1}\right)}\right)^{\frac{1}{p}}.$$

Proof

$$\left\|\sum_k |f_k|\right\|_{L^p\left(R^{\omega}_{\alpha,s;1}\right)} = \sup_{N\in\mathbb{N}} \left\|\sum_{k=1}^N |f_k|\right\|_{L^p\left(R^{\omega}_{\alpha,s;1}\right)} \tag{2.14}$$

$$= \sup_{N\in\mathbb{N}} \left\|\left(\sum_{k=1}^N |f_k|\right)^p\right\|^{\frac{1}{p}}_{L^1\left(R^{\omega}_{\alpha,s;1}\right)}$$

$$\le \sup_{N\in\mathbb{N}} \left\|\sum_{k=1}^N |f_k|^p\right\|^{\frac{1}{p}}_{L^1\left(R^{\omega}_{\alpha,s;1}\right)}$$

$$\le C_{n,\alpha,s,\omega}\left(\sum_k \left\||f_k|^p\right\|_{L^1\left(R^{\omega}_{\alpha,s;1}\right)}\right)^{\frac{1}{p}} \tag{2.15}$$

$$= C_{n,\alpha,s,\omega}\left(\sum_k \|f_k\|^p_{L^p\left(R^{\omega}_{\alpha,s;1}\right)}\right)^{\frac{1}{p}},$$

where we have used Proposition 2.5 for the normability of $L^1(R^{\omega}_{\alpha,s;1})$ in (2.15). □

Corollary 2.7 *Let $\alpha > 0$, $s > 1$, $\alpha s < n$, $0 < p < 1$, and $\omega \in A_s$. Then*

$$\left\|\sum_k |f_k|\right\|_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)} \le C_{n,\alpha,s,\omega,p}\left(\sum_k \|f_k\|^p_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)}\right)^{\frac{1}{p}}.$$

Proof As in [2, Exercise 1.1.14], the countable subadditivity of $R^{\omega}_{\alpha,s;1}(\cdot)$ implies

$$\left\|\max_{1\le k\le N} |f_k|\right\|^p_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)} \le \sum_{k=1}^N \|f_k\|^p_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)}, \quad N\in\mathbb{N}. \tag{2.16}$$

It is a standard fact that

$$R^{\omega}_{\alpha,s;1}\left(\bigcup_{N\in\mathbb{N}} E_N\right) = \sup_{N\in\mathbb{N}} R^{\omega}_{\alpha,s;1}(E_N)$$

for any increasing sequence $\{E_N\}_{N=1}^{\infty}$ of subsets of $\mathbb{R}^n$. As a consequence, for any $t > 0$, one obtains

$$\begin{aligned} & R^{\omega}_{\alpha,s;1}\left(\left\{\sum_k |f_k| > t\right\}\right) \\ & \leq R^{\omega}_{\alpha,s;1}\left(\left\{\sum_k |f_k| > t, \sup_k |f_k| \leq t\right\}\right) + R^{\omega}_{\alpha,s;1}\left(\sup_k |f_k| > t\right) \\ & = \sup_{N\in\mathbb{N}}\left(R^{\omega}_{\alpha,s;1}\left(\left\{\sum_{k=1}^{N} |f_k| > t, \sup_k |f_k| \leq t\right\}\right) + R^{\omega}_{\alpha,s;1}\left(\max_{1\leq k\leq N} |f_k| > t\right)\right) \end{aligned}$$

Let

$$I = \sup_{t>0} t^p \cdot \sup_{N\in\mathbb{N}} R^{\omega}_{\alpha,s;1}\left(\left\{\sum_{k=1}^{N} |f_k| > t, \sup_k |f_k| \leq t\right\}\right).$$

It follows by (2.16) that

$$\left\|\sum_k |f_k|\right\|^p_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)} \leq I + \sum_k \|f_k\|^p_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)}. \tag{2.17}$$

Now, we estimate I. Note that for any $t > 0$, we have

$$\begin{aligned} & t^p \cdot \sup_{N\in\mathbb{N}} R^{\omega}_{\alpha,s;1}\left(\left\{\sum_{k=1}^{N} |f_k| > t, \sup_k |f_k| \leq t\right\}\right) \\ & \leq t^{p-1}\int_0^t \sup_{N\in\mathbb{N}} R^{\omega}_{\alpha,s;1}\left(\left\{\sum_{k=1}^{N} |f_k| > \lambda\right\} \cap \left\{\sup_k |f_k| \leq t\right\}\right) \mathrm{d}\lambda \\ & \leq t^{p-1} \sup_{N\in\mathbb{N}} \int_0^{\infty} R^{\omega}_{\alpha,s;1}\left(\left\{x \in \left\{\sup_k |f_k| \leq t\right\} : \sum_{k=1}^{N} |f_k(x)| > \lambda\right\}\right) \mathrm{d}\lambda \\ & \leq C_{n,\alpha,s,\omega} \cdot t^{p-1} \sum_k \int_0^{\infty} R^{\omega}_{\alpha,s;1}\left(\left\{x \in \left\{\sup_k |f_k| \leq t\right\} : |f_k(x)| > \lambda\right\}\right) \mathrm{d}\lambda, \end{aligned} \tag{2.18}$$

where we have used see Proposition 2.5 for the normability of $L^1(R^{\omega}_{\alpha,s;1})$ in (2.18). As a consequence

$$
\begin{aligned}
& t^p \cdot R^{\omega}_{\alpha,s;1}\left(\left\{\sum_k |f_k| > t, \sup_k |f_k| \le t\right\}\right) \\
&\le C_{n,\alpha,s} \cdot t^{p-1} \sum_k \int_0^{\infty} R^{\omega}_{\alpha,s;1}\left(\{x \in \{|f_k| \le t\} : |f_k(x)| > \lambda\}\right) \mathrm{d}\lambda \\
&= C_{n,\alpha,s} \cdot t^{p-1} \sum_k \int_0^{t} R^{\omega}_{\alpha,s;1}(\{x \in \mathbb{R}^n : |f_k(x)| > \lambda\}) \mathrm{d}\lambda \\
&\le C_{n,\alpha,s,p} \sum_k \|f_k\|^p_{L^{p,\infty}(R^{\omega}_{\alpha,s;1})}. \qquad (2.19)
\end{aligned}
$$

The proof is complete by combining (2.17) and (2.19). □

3 Proofs of main results

We begin by introducing a technical lemma.

Lemma 3.1 *For any $\alpha > 0$, $s > 1$, $\alpha s < n$, $\omega \in A_1$, and positive measure μ on $\mathbb{R}^n$, it holds that*

$$
\|W^{\mu}_{\omega;4}\|_{L^{\eta,\infty}(B_r(x_0))} \le C_{n,\alpha,s,\omega} \cdot |B_r(x_0)|^{\frac{1}{\eta}} W^{\mu}_{\omega;8}(x_0),
$$

where $\eta = (s-1)n/(n-\alpha s)$, $0 < r \le 1$, and the $L^{\eta,\infty}(B_r(x_0))$ indicates the weak L^{η} space over the ball $B_r(x_0)$.

Proof Let

$$
P_1 = \left\| \int_0^r \left(\frac{\mu(B_t(\cdot))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}} \left(⨍_{B_t(\cdot)} \omega(y)\mathrm{d}y\right)^{-\frac{1}{s-1}} \frac{\mathrm{d}t}{t} \right\|_{L^{\eta,\infty}(B_r(x_0))}
$$

and

$$
P_2 = \left\| \int_r^4 \left(\frac{\mu(B_t(\cdot))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}} \left(⨍_{B_t(\cdot)} \omega(y)\mathrm{d}y\right)^{-\frac{1}{s-1}} \frac{\mathrm{d}t}{t} \right\|_{L^{\eta,\infty}(B_r(x_0))}.
$$

To bound P_1, we may assume that μ is supported in $B_{2r}(x_0)$. First note that if $x \in B_r(x_0)$ and $0 < t < r$, then $B_t(x) \subseteq B_{4r}(x_0)$. The strong doubling property (2.1) of $\omega \in A_1$ gives

$$
⨍_{B_{4r}(x_0)} \omega(y)\mathrm{d}y \le [\omega]_{A_1} ⨍_{B_t(x)} \omega(y)\mathrm{d}y.
$$

As a consequence,

$$\begin{aligned}&\int_0^r\left(\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}}\left(\fint_{B_t(x)}\omega(y)\mathrm{d}y\right)^{-\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\\&\quad\le[\omega]_{A_1}\left(\fint_{B_{4r}(x_0)}\omega(y)\mathrm{d}y\right)^{-\frac{1}{s-1}}\int_0^r\left(\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}.\end{aligned}$$

Now, we claim that

$$\int_0^r\left(\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\le C_{n,\alpha,s}\cdot\mu(B_{2r}(x_0))^{\frac{\alpha s}{(s-1)n}}\mathbf{M}(\mu)(x)^{\frac{1}{\eta}},\quad x\in\mathbb{R}^n,\tag{3.1}$$

where

$$\mathbf{M}(\mu)(x)=\sup_{\varepsilon>0}\frac{\mu(B_\varepsilon(x))}{\varepsilon^n},\quad x\in\mathbb{R}^n.$$

Let $0<\delta\le r$ to be determined later. We write

$$\begin{aligned}\int_0^r\left(\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}&=\int_0^\delta\left(\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}+\int_\delta^r\left(\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\\&=I_1+I_2.\end{aligned}$$

We have

$$I_1\le\mathbf{M}(\mu)(x)^{\frac{1}{s-1}}\int_0^\delta t^{\frac{\alpha s}{s-1}}\frac{\mathrm{d}t}{t}=C_{\alpha,s}\cdot\mathbf{M}(\mu)(x)^{\frac{1}{s-1}}\delta^{\frac{\alpha s}{s-1}}.$$

On the other hand, we also have

$$I_2\le\mu(\mathbb{R}^n)^{\frac{1}{s-1}}\int_\delta^\infty\frac{1}{t^{\frac{n-\alpha s}{s-1}}}\frac{\mathrm{d}t}{t}=C_{n,\alpha,s}\cdot\mu(\mathbb{R}^n)^{\frac{1}{s-1}}\frac{1}{\delta^{\frac{n-\alpha s}{s-1}}}.$$

In proving (3.1), the left-sided of (3.1) allows us to assume also that μ is supported in $B_r(x)$. As a consequence,

$$\mathbf{M}(\mu)(x)\ge\frac{\mu(B_r(x))}{r^n}=\frac{\mu(\mathbb{R}^n)}{r^n}=\frac{\mu(B_{2r}(x_0))}{r^n}.$$

By letting

$$\delta=\left(\frac{\mu(B_{2r}(x_0))}{\mathbf{M}(\mu)(x)}\right)^{\frac{1}{n}},$$

then $0<\delta\le r$, and (3.1) follows by routine simplification of I_1 and I_2.

Subsequently, the weak-type (1, 1) boundedness of $\mathbf{M}$ implies that

$$\begin{aligned} P_1 &\le [\omega]_{A_1} \left(\fint_{B_{4r}(x_0)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \left\| \int_0^r \left(\frac{\mu(B_t(\cdot))}{t^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{\mathrm{d}t}{t} \right\|_{L^{\eta,\infty}(B_r(x_0))} \\ &\le C_{n,\alpha,s,\omega} \left(\fint_{B_{4r}(x_0)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \mu(B_{2r}(x_0))^{\frac{\alpha s}{(s-1)n}} \left\| \mathbf{M}(\mu)^{\frac{1}{\eta}} \right\|_{L^{\eta,\infty}(B_r(x_0))} \\ &= C_{n,\alpha,s,\omega} \left(\fint_{B_{4r}(x_0)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \mu(B_{2r}(x_0))^{\frac{\alpha s}{(s-1)n}} \|\mathbf{M}(\mu)\|_{L^{1,\infty}(B_r(x_0))}^{\frac{1}{\eta}} \\ &\le C_{n,\alpha,s,\omega} \cdot \mu(B_{2r}(x_0))^{\frac{\alpha s}{(s-1)n}} \left(\fint_{B_{4r}(x_0)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \mu(B_{2r}(x_0))^{\frac{1}{\eta}} \\ &= C_{n,\alpha,s,\omega} \cdot |B_r(x_0)|^{\frac{1}{\eta}} \left(\fint_{B_{4r}(x_0)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \left(\frac{\mu(B_{2r}(x_0))}{r^{n-\alpha s}} \right)^{\frac{1}{s-1}} \\ &\le C_{n,\alpha,s,\omega} \cdot |B_r(x_0)|^{\frac{1}{\eta}} \int_{2r}^{4r} \left(\fint_{B_t(x_0)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \left(\frac{\mu(B_t(x_0))}{t^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{\mathrm{d}t}{t} \\ &\le C_{n,\alpha,s,\omega} \cdot |B_r(x_0)|^{\frac{1}{\eta}} W^{\mu}_{\omega;4}(x_0). \end{aligned}$$

Now, we bound for P_2. Observe that if $x \in B_r(x_0)$ and $t \ge r$, then $B_t(x) \subseteq B_{2t}(x_0)$ and again the strong doubling property (2.1) of $\omega \in A_1$ yields

$$\fint_{B_{2t}(x_0)} \omega(y)\mathrm{d}y \le [\omega]_{A_1} \fint_{B_t(x)} \omega(y)\mathrm{d}y,$$

which implies that

$$\begin{aligned} P_2 &\le \left\| \int_r^4 \left(\frac{\mu(B_{2t}(x_0))}{t^{n-\alpha s}} \right)^{\frac{1}{s-1}} \left(\fint_{B_{2t}(x_0)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \frac{\mathrm{d}t}{t} \right\|_{L^{\eta,\infty}(B_r(x_0))} \\ &\le C_{n,\alpha,s,\omega} \cdot |B_r(x_0)|^{\frac{1}{\eta}} W^{\mu}_{\omega;8}(x_0), \end{aligned}$$

and the lemma now follows by combining the estimates P_1 and P_2. □

Now, we prove Theorem 1.1 for the case where f is the characteristic function of a Lebesgue measurable subset E of $\mathbb{R}^n$.

Proposition 3.2 *Let $\alpha > 0$, $s > 1$, $\alpha s < n$, $\omega \in A_1$, and E be a measurable subset of $\mathbb{R}^n$. Then*

$$\left\| \mathbf{M}^{\mathrm{loc}}(\chi_E) \right\|_{L^{\frac{n-\alpha s}{n},\infty}\left(R^{\omega}_{\alpha,s;1}\right)} \le C_{n,\alpha,s,\omega} \cdot R^{\omega}_{\alpha,s;1}(E)^{\frac{n}{n-\alpha s}}.$$

Proof By choosing the nonlinear potential $\mathcal{V}^{\mu}_{\omega;2}$ as in (2.5), we have

$$\mu(\mathbb{R}^n) = \mu\left(\overline{E}\right) = \mathcal{R}^{\omega}_{\alpha,s;2}(E),$$
$$\mathcal{V}^{\mu}_{\omega;2}(x) \geq 1 \quad \text{q.e. with respect to } \mathcal{R}^{\omega}_{\alpha,s;2}(\cdot) \text{ on } E.$$

Since $\omega \in A_1 \subseteq A_s$, the absolute continuity (2.4) entails

$$\chi_E(x) \leq (\mathcal{V}^{\mu}_{\omega;2}(x))^{\frac{(s-1)n}{n-\alpha s}} \quad \text{a.e.}$$

Thus, for any $x_0 \in \mathbb{R}^n$ and $0 < r \leq 1$, we have by (2.7) and Lemma 3.1 that

$$\begin{aligned}\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \chi_E(y)\mathrm{d}y &= \frac{1}{|B_r(x_0)|}\|\chi_E\|_{L^{1,\infty}(B_r(x_0))} \\ &\leq \frac{1}{|B_r(x_0)|}\left\|\mathcal{V}^{\mu}_{\omega;2}\right\|^{\frac{(s-1)n}{n-\alpha s}}_{L^{\frac{(s-1)n}{n-\alpha s},\infty}(B_r(x_0))} \\ &\leq C_{n,\alpha,s,\omega}\left(W^{\mu}_{\omega;8}(x_0)\right)^{\frac{(s-1)n}{n-\alpha s}},\end{aligned}$$

which yields

$$\mathbf{M}^{\mathrm{loc}}(\chi_E)(x_0) \leq C_{n,\alpha,s,\omega}\left(W^{\mu}_{\omega;8}(x_0)\right)^{\frac{(s-1)n}{n-\alpha s}}.$$

As a consequence, for any $t > 0$, we have by Proposition 2.2 that

$$\begin{aligned}\mathcal{R}^{\omega}_{\alpha,s;8}\left(\{\mathbf{M}^{\mathrm{loc}}(\chi_E) > t\}\right) &\leq \mathcal{R}^{\omega}_{\alpha,s;8}\left(\left\{W^{\mu}_{\omega;8} > (t/C_{n,\alpha,s,\omega})^{\frac{n-\alpha s}{(s-1)n}}\right\}\right) \\ &\leq C_{n,\alpha,s,\omega}\cdot\frac{\mu(\mathbb{R}^n)}{t^{\frac{n-\alpha s}{n}}} \\ &= C_{n,\alpha,s,\omega}\cdot\frac{\mathcal{R}^{\omega}_{\alpha,s;2}(E)}{t^{\frac{n-\alpha s}{n}}}\end{aligned}$$

and the proof is complete by appealing to the equivalence of the capacities that (2.2) and (2.3). ☐

Proof of Theorem 1.1 First of all, Corollary 2.7 entails

$$\left\|\sum_{k\in\mathbb{Z}}|f_k|\right\|_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)} \leq C_{n,\alpha,s}\left(\sum_{k\in\mathbb{Z}}\|f_k\|^p_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)}\right)^{\frac{1}{p}} \tag{3.2}$$

with $p = (n-\alpha s)/n \in (0,1)$. Suppose that $f \in L^p(R^{\omega}_{\alpha,s;1})$. Then

$$R^{\omega}_{\alpha,s;1}(\{|f| = \infty\}) \leq \frac{1}{N^p}\int_{\mathbb{R}^n}|f|^p\mathrm{d}R^{\omega}_{\alpha,s;1}, \quad N = 1, 2, \ldots$$

gives $R^{\omega}_{\alpha,s;1}(\{|f|=\infty\})=0$. As a result, we can write

$$f=\sum_{k\in\mathbb{Z}} f\chi_{\{2^{k-1}<|f|\leq 2^k\}} \quad \text{q.e. with respect to } R^{\omega}_{\alpha,s;1}(\cdot),$$

and hence

$$|f|\leq\sum_{k\in\mathbb{Z}} 2^k\chi_{\{2^{k-1}<|f|\leq 2^k\}} \quad \text{q.e. with respect to } R^{\omega}_{\alpha,s;1}(\cdot).$$

The absolutely continuity (2.4) implies

$$\mathbf{M}^{\mathrm{loc}}(f)(x)\leq\sum_{k\in\mathbb{Z}} 2^k\cdot\mathbf{M}^{\mathrm{loc}}\left(\chi_{\{2^{k-1}<|f|\leq 2^k\}}\right)(x),\quad x\in\mathbb{R}^n.$$

We obtain by (3.2) and Proposition 3.2 that

$$\begin{aligned}\left\|\mathbf{M}^{\mathrm{loc}}(f)\right\|_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)}&\leq C_{n,\alpha,s}\left(\sum_{k\in\mathbb{Z}} 2^{kp}\left\|\mathbf{M}^{\mathrm{loc}}\left(\chi_{\{2^{k-1}<|f|\leq 2^k\}}\right)\right\|^p_{L^p\left(R^{\omega}_{\alpha,s;1}\right)}\right)^{\frac{1}{p}}\\&\leq C_{n,\alpha,s,\omega}\left(\sum_{k\in\mathbb{Z}} 2^{kp}\cdot R^{\omega}_{\alpha,s;1}\left(\left\{2^{k-1}<|f|\leq 2^k\right\}\right)\right)^{\frac{1}{p}},\end{aligned}$$

which yields

$$\begin{aligned}\left\|\mathbf{M}^{\mathrm{loc}}(f)\right\|_{L^{p,\infty}\left(R^{\omega}_{\alpha,s;1}\right)}&\leq C_{n,\alpha,s,\omega}\left(\sum_{k\in\mathbb{Z}}\int_{2^{k-2}}^{2^{k-1}} t^p\cdot R^{\omega}_{\alpha,s;1}(\{2^{k-1}<|f|\leq 2^k\})\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}\\&=C_{n,\alpha,s,\omega}\left(\int_0^{\infty} t^p\cdot R^{\omega}_{\alpha,s;1}(\{2^{k-1}<|f|\leq 2^k\})\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}\\&=C_{n,\alpha,s,\omega}\cdot\|f\|_{L^p\left(R^{\omega}_{\alpha,s;1}\right)},\end{aligned}$$

as expected. □

For any positive measure μ on $\mathbb{R}^n$ and $\rho>0$, we denote by

$$\mathbf{M}_{\rho}(\mu)(x)=\sup_{0<r\leq\rho}\frac{\mu(B_r(x))}{r^n},\quad x\in\mathbb{R}^n.$$

Lemma 3.3 *Let $\alpha>0$, $s>1$, $\alpha s<n$, $\rho>0$, $\omega\in A_1$, $x\in\mathbb{R}^n$, and μ be a compactly supported positive measure on $\mathbb{R}^n$. Then*

$$W^{\mu}_{\omega;\rho}(x)\leq C_{n,\alpha,s,\omega}\cdot\mu(\mathbb{R}^n)^{\frac{\alpha s}{n(s-1)}}\left(⨍_{B_{\rho}(x)}\omega(y)\mathrm{d}y\right)^{-\frac{1}{s-1}}\mathbf{M}_{\rho}(\mu)(x)^{\frac{n-\alpha s}{n(s-1)}}.$$

Proof Let $0 < \delta \le \rho$ to be determined later. We write

$$W^{\mu}_{\omega;\rho}(x) = \int_0^{\delta} \left(\frac{t^{\alpha s}\mu(B_t(x))}{\omega(B_t(x))} \right)^{\frac{1}{s-1}} \frac{\mathrm{d}t}{t} + \int_{\delta}^{\rho} \left(\frac{t^{\alpha s}\mu(B_t(x))}{\omega(B_t(x))} \right)^{\frac{1}{s-1}} \frac{\mathrm{d}t}{t} = I_1 + I_2.$$

Note that the strong doubling property (2.1) of $\omega \in A_1$ gives

$$\fint_{B_\rho(x)} \omega(y)\mathrm{d}y \le [\omega]_{A_1} \fint_{B_t(x)} \omega(y)\mathrm{d}y, \quad 0 < t < \delta.$$

We have

$$\begin{aligned} I_1 &= \int_0^{\delta} t^{\frac{\alpha s}{s-1}} \left(\frac{\mu(B_t(x))}{t^n} \right)^{\frac{1}{s-1}} \left(\fint_{B_t(x)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \frac{\mathrm{d}t}{t} \\ &\le [\omega]_{A_1}^{\frac{1}{s-1}} \cdot \mathbf{M}_\rho(\mu)(x)^{\frac{1}{s-1}} \left(\fint_{B_\rho(x)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \int_0^{\delta} t^{\frac{\alpha s}{s-1}} \frac{\mathrm{d}t}{t} \\ &= C_{\alpha,s,\omega} \cdot \mathbf{M}_\rho(\mu)(x)^{\frac{1}{s-1}} \left(\fint_{B_\rho(x)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \delta^{\frac{\alpha s}{s-1}}. \end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned} I_2 &\le \mu(\mathbb{R}^n)^{\frac{1}{s-1}} \left(\fint_{B_\rho(x)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \int_{\delta}^{\infty} \frac{1}{t^{\frac{n-\alpha s}{s-1}}} \frac{\mathrm{d}t}{t} \\ &= C_{n,\alpha,s} \cdot \mu(\mathbb{R}^n)^{\frac{1}{s-1}} \left(\fint_{B_\rho(x)} \omega(y)\mathrm{d}y \right)^{-\frac{1}{s-1}} \frac{1}{\delta^{\frac{n-\alpha s}{s-1}}}. \end{aligned} \tag{3.4}$$

Note that we may assume that μ is supported in $B_\rho(x)$, this gives

$$\mathbf{M}_\rho(\mu)(x) \ge \frac{\mu(B_\rho(x))}{\rho^n} = \frac{\mu(\mathbb{R}^n)}{\rho^n}.$$

If we choose

$$\delta = \left(\frac{\mu(\mathbb{R}^n)}{\mathbf{M}_\rho(\mu)(x)} \right)^{\frac{1}{n}},$$

then $0 < \delta \le \rho$ and

$$\mathbf{M}_\rho(\mu)(x)^{\frac{1}{s-1}} \delta^{\frac{\alpha s}{s-1}} = \mu(\mathbb{R}^n)^{\frac{1}{s-1}} \frac{1}{t^{\frac{n-\alpha s}{s-1}}},$$

the proof is complete by routine simplification of (3.3) and (3.4). □

Proposition 3.4 *Let* $\alpha > 0$, $s > 1$, $\alpha s < n$, $\rho > 0$, $0 < \delta < n(s-1)/(n-\alpha s)$, $\omega \in A_1$, *and* $x \in \mathbb{R}^n$. *Then*

$$\mathbf{M}^{\mathrm{loc}}\left((W^{\mu}_{\omega;\rho})^{\delta}\right)(x) \leq C_{n,\alpha,s,\omega,\delta}(W^{\mu}_{\omega;3\rho}(x))^{\delta}.$$

Proof Let $0 < r \leq 1$ be given. We need to estimate that

$$I = \frac{1}{r^n}\int_{B_r(x)}\left(\int_0^{\rho}\left(\frac{t^{\alpha s}\mu(B_t(y))}{\omega(B_t(y))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\right)^{\delta}\mathrm{d}y \leq C_{\delta}\cdot(I_1+I_2),$$

where

$$I_1 = \frac{1}{r^n}\int_{B_r(x)}\left(\int_0^{\min\{\rho,r\}}\left(\frac{t^{\alpha s}\mu(B_t(y))}{\omega(B_t(y))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\right)^{\delta}\mathrm{d}y,$$
$$I_2 = \frac{1}{r^n}\int_{B_r(x)}\left(\int_{\min\{\rho,r\}}^{\rho}\left(\frac{t^{\alpha s}\mu(B_t(y))}{\omega(B_t(y))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\right)^{\delta}\mathrm{d}y$$

with the convention that $I_2 = 0$ if $\rho \leq r$. Assume without loss of generality that $\rho > r$. For the integral I_1, we may assume that μ is supported in $B_{2r}(x)$. In view of Lemma 3.3, it suffices to estimate

$$\int_{B_r(x)}\mathbf{M}_r(\mu)(y)^{\frac{\delta(n-\alpha s)}{n(s-1)}}\mathrm{d}y.$$

Let $p = \delta(n-\alpha s)/(n(s-1)) < 1$. We appeal to the estimate that

$$\int_E |F(y)|^p\mathrm{d}y \leq C_{n,p}\cdot|E|^{1-p}\|F\|^p_{L^{1,\infty}(E)}$$

for any measurable set $E \subseteq \mathbb{R}^n$ with $|E| < \infty$ and $F \in L^{1,\infty}(E)$, here $L^{1,\infty}(E)$ is the weak Lebesgue space on E (see [2, Exercise 1.1.11]). As a consequence,

$$\begin{aligned}
I_1 &= \frac{1}{r^n}\int_{B_r(x)}W^{\mu}_{\omega;r}(y)^{\delta}\mathrm{d}y && (3.5)\\
&\leq C_{n,\alpha,s,\omega}\cdot\mu(\mathbb{R}^n)^{\frac{\delta\alpha s}{n(s-1)}}\left(\fint_{B_r(x)}\omega(y)\mathrm{d}y\right)^{-\frac{\delta}{s-1}}\frac{1}{r^n}\int_{B_r(x)}\mathbf{M}_r(\mu)(y)^{\frac{\delta(n-\alpha s)}{n(s-1)}}\mathrm{d}y\\
&\leq C_{n,\alpha,s,\omega}\cdot\mu(\mathbb{R}^n)^{\frac{\delta\alpha s}{n(s-1)}}\left(\fint_{B_r(x)}\omega(y)\mathrm{d}y\right)^{-\frac{\delta}{s-1}}r^{-n}|B_r(x)|^{1-p}\|\mathbf{M}_r(\mu)\|^p_{L^{1,\infty}(\mathbb{R}^n)}\\
&\leq C_{n,\alpha,s,\omega}\cdot r^{-\frac{\delta(n-\alpha s)}{s-1}}\left(\fint_{B_r(x)}\omega(y)\mathrm{d}y\right)^{-\frac{\delta}{s-1}}\mu(\mathbb{R}^n)^{\frac{\delta\alpha s}{n(s-1)}+p} && (3.6)
\end{aligned}$$

$$
\begin{aligned}
&= C_{n,\alpha,s,\omega}\left(\frac{r^{\alpha s}\mu(B_{2r}(x))}{\omega(B_r(x))}\right)^{\frac{\delta}{s-1}} \\
&\leq C_{n,\alpha,s,\omega}\left(\int_{2r}^{3r}\left(\frac{t^{\alpha s}\mu(B_t(x))}{\omega(B_t(x))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\right)^{\delta} \\
&\leq C_{n,\alpha,s,\omega}(W^{\mu}_{\omega;3\rho}(x))^{\delta}, \qquad (3.7)
\end{aligned}
$$

where we have used and the weak-type (1, 1) boundedness of $\mathbf{M}(\cdot)$ in (3.6) that

$$
\mathbf{M}(\mu)(\cdot) = \sup_{r>0}\frac{\mu(B_r(\cdot))}{r^n}
$$

with $\mathbf{M}_r(\cdot) \leq \mathbf{M}(\cdot)$ for all $r > 0$, and the strong doubling property (2.1) of $\omega \in A_1$ in (3.7). For the integral I_2, observe that $B_t(y) \subseteq B_{2t}(x)$ for $y \in B_r(x)$ and $t > r$. Using the strong doubling property (2.1) of $\omega \in A_1$ again, one obtains

$$
I_2 \leq C_{s,\omega,\delta}\left(\int_{r}^{\rho}\left(\frac{t^{\alpha s}\mu(B_{2t}(x))}{\omega(B_t(x))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}\right)^{\delta} \leq C_{\alpha,s,\omega,\delta}(W^{\mu}_{\omega;2\rho}(x))^{\delta}. \qquad (3.8)
$$

The proof is complete by combining (3.5) and (3.8). □

Remark 3.5 Assume that the weighted Riesz capacities $R^{\omega}_{\alpha,s}(\cdot)$ are taken into account. Following the similar argument given in the proof of Proposition 3.4, one may show that the Wolff potential W^{μ}_{ω} defined by

$$
W^{\mu}_{\omega}(x) = \int_{0}^{\infty}\left(\frac{t^{\alpha s}\mu(B_t(x))}{\omega(B_t(x))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t}
$$

satisfies that

$$
\mathbf{M}\left((W^{\mu}_{\omega})^{\delta}\right)(x) \leq C_{n,\alpha,s,\omega,\delta}(W^{\mu}_{\omega}(x))^{\delta}
$$

for $x \in \mathbb{R}^n$ and $0 < \delta < n(s-1)/(n-\alpha s)$. In other words, $(W^{\mu}_{\omega})^{\delta}$ is an A_1 weight for those admissible exponents δ.

Next we prove Theorem 1.2 for the case where f is the characteristic function of a Lebesgue measurable subset E of $\mathbb{R}^n$.

Proposition 3.6 *Let $\alpha > 0$, $s > 1$, $\alpha s < n$, $\omega \in A_1$, and E be a measurable subset of $\mathbb{R}^n$. For any $p > (n-\alpha s)/n$, it holds that*

$$
\left\|\mathbf{M}^{\mathrm{loc}}(\chi_E)\right\|_{L^p\left(R^{\omega}_{\alpha,s;1}\right)} \leq C_{n,\alpha,s,\omega,p}\cdot R^{\omega}_{\alpha,s;1}(E)^{\frac{1}{p}}.
$$

Proof For any $p > (n-\alpha s)/n$, choose an $\varepsilon = \varepsilon_p > 0$, such that

$$p > \frac{n-\alpha s+\varepsilon}{n}.$$

Let $\delta = (s-1)n/(n-\alpha s+\varepsilon)$. By choosing the nonlinear potential $\mathcal{V}^{\mu}_{\omega;1}$ as in (2.5) and using (2.7), one has

$$\begin{aligned} &\mu(\mathbb{R}^n) = \mu\left(\overline{E}\right) = \mathcal{R}^{\omega}_{\alpha,s;1}(E),\\ &\mathcal{V}^{\mu}_{\omega;1}(x) \geq 1 \quad \text{q.e. with respect to } \mathcal{R}^{\omega}_{\alpha,s;1}(\cdot) \text{ on } E. \end{aligned}$$

Since $\omega \in A_1 \subseteq A_s$, the absolute continuity (2.4) and (2.7) entail

$$\chi_E(x) \leq (\mathcal{V}^{\mu}_{\omega;1}(x))^{\delta} \leq C_{n,\alpha,s,\omega,p}(W^{\mu}_{\omega;2}(x))^{\delta} \quad \text{a.e.,}$$

then Proposition 3.4 yields

$$\mathbf{M}^{\mathrm{loc}}(\chi_E)(x) \leq C_{n,\alpha,s,\omega,p} \cdot \mathbf{M}\left((W^{\mu}_{\omega;2})^{\delta}\right)(x) \leq C_{n,\alpha,s,\omega,p}(W^{\mu}_{\omega;6}(x))^{\delta}, \quad x \in \mathbb{R}^n.$$

Note that $\mathbf{M}^{\mathrm{loc}}(\chi_E) \leq 1$ and $(s-1)/(p\delta) < 1$. Using the equivalence of capacities that (2.2) and (2.3), we obtain by Proposition 2.2 that

$$\begin{aligned} \int_0^{\infty} R^{\omega}_{\alpha,s;1}\left(\left\{\left|\mathbf{M}^{\mathrm{loc}}(\chi_E)\right|^p > t\right\}\right) \mathrm{d}t &= \int_0^1 R^{\omega}_{\alpha,s;1}\left(\left\{\left|\mathbf{M}^{\mathrm{loc}}(\chi_E)\right|^p > t\right\}\right) \mathrm{d}t\\ &\leq \int_0^1 R^{\omega}_{\alpha,s;1}\left(\left\{W^{\mu}_{\omega;6} > \left(t/C_{n,\alpha,s,\omega,p}\right)^{\frac{1}{p\delta}}\right\}\right) \mathrm{d}t\\ &\leq C_{n,\alpha,s,\omega} \int_0^{C^{-1}_{n,\alpha,s,\omega,p}} \mathcal{R}^{\omega}_{\alpha,s;6}\left(\left\{W^{\mu}_{\omega;6} > t^{\frac{1}{p\delta}}\right\}\right) \mathrm{d}t\\ &\leq C_{n,\alpha,s,\omega} \int_0^{C^{-1}_{n,\alpha,s,\omega,p}} \frac{\mu(\mathbb{R}^n)}{t^{\frac{s-1}{p\delta}}} \mathrm{d}t\\ &= C_{n,\alpha,s,\omega,p} \cdot \mathcal{R}^{\omega}_{\alpha,s;1}(E)\\ &\leq C_{n,\alpha,s,\omega,p} \cdot R^{\omega}_{\alpha,s;1}(E), \end{aligned}$$

which completes the proof. □

Proof of Theorem 1.2 It has been noted in (1.6) that we only need to prove the estimate for the exponent that $(n-\alpha s)/n < p \leq 1$. By using Corollary 2.6 and Proposition 3.6, one obtains the result by repeating the argument given in the proof of Theorem 1.1. □

Proof of Theorem 1.3 We appeal to the weighted Sobolev embedding that

$$\left(\int_{\mathbb{R}^n} |I_\alpha * f(x)|^{s^*} \omega(x) \mathrm{d}x\right)^{\frac{1}{s^*}} \leq C_{n,\alpha,s,\omega}\left(\int_{\mathbb{R}^n} |f(x)|^s \omega(x)^{\frac{n-\alpha s}{n}} \mathrm{d}x\right)^{\frac{1}{s}}, \tag{3.9}$$

where $\omega \in A_q$, $q = s^*/s' + 1$, $1/s^* = 1/s - \alpha/n$ (see [9, Theorem 2.2.1]). The assumption that $\omega \in A_1$ entails $\omega \in A_q$ and $\omega^{\frac{n-\alpha s}{n}} \in A_1 \subseteq A_s$. By recalling the definition of $R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}(\cdot)$, one obtains immediately that

$$\omega(E)^{\frac{1}{s^*}} \leq C_{n,\alpha,s,\omega} \cdot R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}(E)^{\frac{1}{s}}, \quad E \subseteq \mathbb{R}^n,$$

and hence

$$\omega(E)^{\frac{n-\alpha s}{n}} \leq C_{n,\alpha,s,\omega} \cdot R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}(E), \quad E \subseteq \mathbb{R}^n.$$

As a consequence, we have

$$\begin{aligned} \|f\|_{L^q\left(R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}\right)}^q &= q\int_0^\infty t^{q-1} \cdot R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}(\{|f| > t\})\mathrm{d}t \\ &\geq C_{n,\alpha,s,\omega,q}\int_0^\infty t^{q-1}\omega(\{|f|>t\})^{(n-\alpha s)/n}\mathrm{d}t \\ &= C_{n,\alpha,s,\omega,q} \cdot \|f\|_{L^{\frac{nq}{n-\alpha s},q}(\omega)}^q \\ &\geq C_{n,\alpha,s,\omega,q} \cdot \|f\|_{L^{\frac{nq}{n-\alpha s}}(\omega)}^q, \end{aligned} \tag{3.10}$$

where $L^{\frac{nq}{n-\alpha s},q}(\omega)$ is the Lorentz space.

Let $\delta > 0$ to be determined later. For any $1 \leq p < n/\beta$, we compute that

$$\begin{aligned} I_\beta * \left(f\omega^{\frac{1}{p}}\right)(x) &= \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\beta}} f(y)\omega(y)^{\frac{1}{p}}\mathrm{d}y \\ &= \int_0^\infty \frac{1}{t^{n-\beta}}\int_{B_t(x)} f(y)\omega(y)^{\frac{1}{p}}\mathrm{d}y\frac{\mathrm{d}t}{t} \\ &= \int_0^\delta \frac{\int_{B_t(x)} f(y)\omega(y)^{\frac{1}{p}}\mathrm{d}y}{t^{n-\beta}}\frac{\mathrm{d}t}{t} + \int_\delta^\infty \frac{\int_{B_t(x)} f(y)\omega(y)^{\frac{1}{p}}\mathrm{d}y}{t^{n-\beta}}\frac{\mathrm{d}t}{t}. \end{aligned}$$

Consequently, Hölder's inequality gives

$$\left|I_\beta * \left(f\omega^{\frac{1}{p}}\right)(x)\right| \leq C_{n,\beta,p}\left(\mathbf{M}\left(f\omega^{\frac{1}{p}}\right)(x)\cdot\delta^\beta + \|f\|_{L^p(\omega)}\cdot\delta^{\beta-\frac{n}{p}}\right).$$

By choosing

$$\delta = \delta(x) = \left(\frac{\|f\|_{L^p(\omega)}}{\mathbf{M}\left(f\omega^{\frac{1}{p}}\right)(x)}\right)^{\frac{p}{n}},$$

we obtain

$$\left|I_\beta * \left(f\omega^{\frac{1}{p}}\right)(x)\right| \leq C_{n,\beta,p} \cdot \|f\|_{L^p(\omega)}^{\frac{\beta p}{n}} \cdot \mathbf{M}\left(f\omega^{\frac{1}{p}}\right)(x)^{1-\frac{\beta p}{n}}. \tag{3.11}$$

Let $p = nq/(n-\alpha s)$. Note that $q^*(1-\beta q/(n-\alpha s)) = q$. If $q = (n-\alpha s)/n$, then Theorem 1.1 (in terms of weighted Riesz capacities $R_{\alpha,s}^{\omega}(\cdot)$) implies that

$$\begin{aligned}
&\left\|\mathcal{I}_\beta\left(f\omega^{\frac{n-\alpha s}{nq}}\right)\right\|_{L^{q^*,\infty}\left(R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}\right)} \\
&\leq C_{n,\alpha,s,\beta} \cdot \|f\|_{L_\omega^{\frac{nq}{n-\alpha s}}(\mathbb{R}^n)}^{\frac{\beta q}{n-\alpha s}} \cdot \left\|\mathbf{M}\left(f\omega^{\frac{n-\alpha s}{nq}}\right)\right\|_{L^{q^*\left(1-\frac{\beta q}{n-\alpha s}\right),\infty}\left(R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}\right)}^{1-\frac{\beta q}{n-\alpha s}} \\
&\leq C_{n,\alpha,s,\omega,\beta} \cdot \|f\|_{L^q\left(R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}\right)}^{\frac{\beta q}{n-\alpha s}} \cdot \left\|\mathbf{M}\left(f\omega^{\frac{n-\alpha s}{nq}}\right)\right\|_{L^{q,\infty}\left(R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}\right)}^{1-\frac{\beta q}{n-\alpha s}} \\
&\leq C_{n,\alpha,s,\omega,\beta} \cdot \|f\|_{L^q\left(R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}\right)}^{\frac{\beta q}{n-\alpha s}} \cdot \left\|f\omega^{\frac{n-\alpha s}{nq}}\right\|_{L^q\left(R_{\alpha,s}^{\omega^{(n-\alpha s)/n}}\right)}^{1-\frac{\beta q}{n-\alpha s}},
\end{aligned}$$

and the estimate (1.7) follows. For $q > (n-\alpha s)/n$, the same argument will prove for the estimate (1.8) by using Theorem 1.2. The proof is now complete. □

Let us address a localization principle. Assume that $\gamma \in \mathscr{D}(\mathbb{R}^n)$ satisfies

$$\sum_{z\in\mathbb{Z}^n} \gamma(x-z) = 1, \quad x \in \mathbb{R}^n.$$

Then [7, Theorem 2.21] shows that

$$\|f\|_{F_{s,q}^{\alpha,\omega}} \approx \left(\sum_{z\in\mathbb{Z}^n} \|\gamma(\cdot - z)f\|_{F_{s,q}^{\alpha,\omega}}^p\right)^{\frac{1}{p}}, \quad f \in \mathscr{S}'(\mathbb{R}^n), \quad \omega \in \bigcup_{1\leq s<\infty} A_s, \tag{3.12}$$

where $F_{s,q}^{\alpha,\omega}$ are the weighted Triebel-Lizorkin spaces (see [7, Definition 2.4] for precise definition). In particular, whenever $\alpha \in \mathbb{N}$, [7, Theorem 2.20] gives

$$\|f\|_{F_{s,q}^{\alpha,\omega}} \approx \sum_{|\gamma|\leq\alpha} \|D^\gamma f\|_{F_{s,q}^{0,\omega}} \tag{3.13}$$

If we let $q = 2$ in (3.12) and (3.13), then one can use [7, Theorem 1.10] and [9, Theorem 3.3.4] to deduce the quasi-additivity of $R_{\alpha,s;1}^{\omega}(\cdot)$ that

$$\sum_{z\in\mathbb{Z}^n} R_{\alpha,s;1}^{\omega}(E \cap B_1(z)) \leq C_{n,\alpha,s,\omega} \cdot R_{\alpha,s;1}^{\omega}(E) \tag{3.14}$$

for any subset E of $\mathbb{R}^n$.

Proof of Theorem 1.4 By replacing $I_\alpha(\cdot)$ with $I_{\alpha,1}(\cdot)$ in (3.9), one has

$$\omega(E)^{\frac{n-\alpha s}{n}} \leq C_{n,\alpha,s,\omega} \cdot R_{\alpha,s;1}^{\omega^{(n-\alpha s)/n}}(E), \quad E \subseteq \mathbb{R}^n.$$

We deduce similarly as in (3.10) that

$$\|f\|^q_{L^q\left(R_{\alpha,s;1}^{\omega^{(n-\alpha s)/n}}\right)} \geq C_{n,\alpha,s,\omega,q} \cdot \|f\|^q_{L^{\frac{nq}{n-\alpha s}}(\omega)}. \tag{3.15}$$

Now, we recall the pointwise behavior of the Bessel kernel $G_\beta(\cdot)$ that

$$G_\beta(x) = c_{n,\beta} \cdot |x|^{-(n-\beta)} + o\left(|x|^{-(n-\beta)}\right), \quad |x| \to 0, \quad 0 < \beta < n;$$
$$G_\beta(x) = O\left(e^{-\frac{|x|}{2}}\right), \quad |x| \to \infty, \quad \beta > 0,$$

(see [1, Section 1.2.4]). For $1 \leq p < n/\beta$, we have

$$\begin{aligned}
&\left|G_\beta * \left(f\omega^{\frac{1}{p}}\right)(x)\right| \\
&= \left|\int_{|x-y|\leq\frac{1}{2}} G_\beta(x-y)f(y)\omega(y)^{\frac{1}{p}}\mathrm{d}y + \int_{|x-y|>\frac{1}{2}} G_\beta(x-y)f(y)\omega(y)^{\frac{1}{p}}\mathrm{d}y\right| \\
&\leq C_{n,\beta}\left(\int_0^1 \frac{1}{t^{n-\beta}}\int_{B_t(x)} |f(y)|\omega(y)^{\frac{1}{p}}\mathrm{d}y\frac{\mathrm{d}t}{t} + \int_{\mathbb{R}^n} e^{-\frac{|x-y|}{2}}|f(y)|\omega(y)^{\frac{1}{p}}\mathrm{d}y\right) \\
&= C_{n,\beta}\left(\mathcal{I}(x) + \mathcal{J}(x)\right).
\end{aligned}$$

Arguing as in (3.11), one has

$$\mathcal{I}(x) \leq C_{n,\beta,p} \cdot \|f\|^{\frac{\beta p}{n}}_{L^p(\omega)} \cdot \mathbf{M}^{\mathrm{loc}}\left(f\omega^{\frac{1}{p}}\right)(x)^{1-\frac{\beta p}{n}}.$$

In the sequel, denote by $p = nq/(n-\alpha s)$. By Theorems 1.1 and 1.2, we deduce that for $q = (n-\alpha s)/n$,

$$\begin{aligned}
&\|\mathcal{I}(\cdot)\|_{L^{q^*,\infty}\left(R_{\alpha,s;1}^{\omega^{(n-\alpha s)/n}}\right)} \\
&\quad \leq C_{n,\alpha,s,\omega,\beta} \cdot \|f\|^{\frac{\beta q}{n-\alpha s}}_{L^q\left(R_{\alpha,s;1}^{\omega^{(n-\alpha s)/n}}\right)} \cdot \left\|f\omega^{\frac{n-\alpha s}{nq}}\right\|^{1-\frac{\beta q}{n-\alpha s}}_{L^q\left(R_{\alpha,s;1}^{\omega^{(n-\alpha s)/n}}\right)},
\end{aligned}$$

and $L^{q^*,\infty}\left(R_{\alpha,s;1}^{\omega^{(n-\alpha s)/n}}\right)$ is replaced by $L^{q^*}\left(R_{\alpha,s;1}^{\omega^{(n-\alpha s)/n}}\right)$ whenever $q > (n-\alpha s)/n$. To bound $\mathcal{J}(\cdot)$, we observe by Hölder's inequality that

$$\mathcal{J}(x) \leq C_p\left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|}{2}}|f(y)|^p\omega(y)\mathrm{d}y\right)^{\frac{1}{p}}.$$

Consider the partition $\{Q_z\}_{z\in\mathbb{Z}^n}$ of $\mathbb{R}^n$ defined by $Q_z = z + [0, 1/2)^n$. Then

$$\mathcal{J}(x) \leq C_p \left(\sum_{z\in\mathbb{Z}^n} \int_{Q_z} e^{-\frac{\operatorname{dist}(x,Q_z)}{2}} |f(y)|^p \omega(y) \mathrm{d}y \right)^{\frac{1}{p}}.$$

Note that $q/p = (n - \alpha s)/n < 1$. Using Corollary 2.6, one obtains

$$\begin{aligned} \|\mathcal{J}(\cdot)\|^q_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} &\leq C_{n,\alpha,s} \left\| \sum_{z\in\mathbb{Z}^n} \left(\int_{Q_z} |f(y)|^p \omega(y)\mathrm{d}y \right) e^{-\frac{\operatorname{dist}(\cdot,Q_z)}{2}} \right\|^{\frac{q}{p}}_{L^{\frac{q}{p}}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} \\ &\leq C_{n,\alpha,s,\omega} \sum_{z\in\mathbb{Z}^n} \left(\int_{Q_z} |f(y)|^p \omega(y)\mathrm{d}y \right)^{\frac{q}{p}} \left\| e^{-\frac{\operatorname{dist}(\cdot,Q_z)}{2}} \right\|^{\frac{q}{p}}_{L^{\frac{q}{p}}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} \end{aligned}$$

Using the exponential decay of $e^{-\operatorname{dist}(\cdot,Q_z)/2}$, it is not hard to compute that

$$\left\| e^{-\frac{\operatorname{dist}(\cdot,Q_z)}{2}} \right\|^{\frac{q}{p}}_{L^{\frac{q}{p}}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} \leq C_{n,\alpha,s} \sup_{u\in\mathbb{Z}^n} R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}(B_1(u)).$$

Recall that $\omega \in A_1$ and hence $\omega^{(n-\alpha s)/n} \in A_1 \subseteq A_s$. We use the size estimate that

$$R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}(B_1(u)) \leq C_{n,\alpha,s,\omega} \cdot \omega^{(n-\alpha s)/n}(B_1(u))$$

(see [9, Lemma 3.3.12]), then assumption (1.9) yields

$$\begin{aligned} \|\mathcal{J}(\cdot)\|^q_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} &\leq C_{n,\alpha,s,\omega} \sum_{z\in\mathbb{Z}^n} \left(\int_{Q_z} |f(y)|^p \omega(y)\mathrm{d}y \right)^{\frac{q}{p}} \sup_{u\in\mathbb{Z}^n} R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}(B_1(z)) \\ &\leq C_{n,\alpha,s,\omega} \sum_{z\in\mathbb{Z}^n} \left(\int_{Q_z} |f(y)|^p \omega(y)\mathrm{d}y \right)^{\frac{q}{p}} \sup_{u\in\mathbb{Z}^n} \omega^{(n-\alpha s)/n}(B_1(u)) \\ &\leq C_{n,\alpha,s,\omega} \sum_{z\in\mathbb{Z}^n} \left(\int_{Q_z} |f(y)|^p \omega(y)\mathrm{d}y \right)^{\frac{q}{p}} \sup_{u\in\mathbb{Z}^n} \omega(B_1(u))^{\frac{n-\alpha s}{n}} \\ &\leq C_{n,\alpha,s,\omega} \sum_{z\in\mathbb{Z}^n} \left(\int_{Q_z} |f(y)|^p \omega(y)\mathrm{d}y \right)^{\frac{q}{p}}. \end{aligned}$$

Now, we use (3.15) and the quasi-additivity (3.14) to deduce that

$$\begin{aligned}\|\mathcal{J}(\cdot)\|^q_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} &\le C_{n,\alpha,s,\omega}\sum_{z\in\mathbb{Z}^n}\|f\chi_{Q_z}\|^q_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} \\ &= C_{n,\alpha,s,\omega}\int_0^\infty\sum_{z\in\mathbb{Z}^n}R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\left(\left\{|f|^q>t\right\}\cap Q_z\right)\mathrm{d}t \\ &\le C_{n,\alpha,s,\omega}\int_0^\infty R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\left(\left\{|f|^q>t\right\}\right)\mathrm{d}t \\ &= C_{n,\alpha,s,\omega}\cdot\|f\|^q_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}.\end{aligned} \tag{3.16}$$

Similarly, for any $r>p$, we obtain

$$\begin{aligned}&\|\mathcal{J}(\cdot)\|^r_{L^r\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)} \\ &\le C_{n,\alpha,s,\omega}\left\|\sum_{z\in\mathbb{Z}^n}\left(\int_{Q_z}|f(y)|^p\omega(y)\mathrm{d}y\right)e^{-\frac{\mathrm{dist}(\cdot,Q_z)}{2}}\right\|^{\frac{r}{p}}_{L^{\frac{r}{p}}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}\end{aligned} \tag{3.17}$$

$$\le C_{n,\alpha,s,\omega}\left(\sum_{z\in\mathbb{Z}^n}\int_{Q_z}|f(y)|^p\omega(y)\mathrm{d}y\left\|e^{-\frac{\mathrm{dist}(\cdot,Q_z)}{2}}\right\|_{L^{\frac{r}{p}}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}\right)^{\frac{r}{p}} \tag{3.18}$$

$$\le C_{n,\alpha,s,\omega}\left(\int_{\mathbb{R}^n}|f(y)|^p\omega(y)\mathrm{d}y\right)^{\frac{r}{p}}$$

$$\le C_{n,\alpha,s,\omega}\cdot\|f\|^r_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}, \tag{3.19}$$

where we have used the normability of $L^r\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)$ for $r>1$ in (3.18) (see [3, Theorem 1.2]), and (3.15) in (3.19). Note that $q^*>q$ and $p>q$. Using (3.16), (3.17), and interpolation theorem, one has

$$\|\mathcal{J}(\cdot)\|_{L^{q^*}\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}\le C_{n,\alpha,s,\omega,\beta,q}\cdot\|f\|_{L^q\left(R^{\omega^{(n-\alpha s)/n}}_{\alpha,s;1}\right)}.$$

The proof is now complete. □

4 A note on Theorems 1.1 and 1.2 for weighted Riesz capacities

In this section, we show briefly that Theorems 1.1 and 1.2 hold for weighted Riesz capacities $R^{\omega}_{\alpha,s}(\cdot)$ and the usual Hardy–Littlewood maximal function $\mathbf{M}$ in place of

$R^{\omega}_{\alpha,s;1}(\cdot)$ and $\mathbf{M}^{\text{loc}}$, respectively. More precisely, we have

$$\sup_{t>0}\left(t^{\frac{n-\alpha s}{n}}\cdot R^{\omega}_{\alpha,s}\left(\left\{x\in\mathbb{R}^n:\mathbf{M}f(x)>t\right\}\right)\right)\leq C_{n,\alpha,s,\omega}\int_{\mathbb{R}^n}|f|^{\frac{n-\alpha s}{n}}\mathrm{d}R^{\omega}_{\alpha,s},\quad \omega\in A_1,\tag{4.1}$$

$$\int_{\mathbb{R}^n}(\mathbf{M}f)^p\,\mathrm{d}R^{\omega}_{\alpha,s}\leq C_{n,\alpha,s,\omega,q}\int_{\mathbb{R}^n}|f|^p\mathrm{d}R^{\omega}_{\alpha,s},\quad p>\frac{n-\alpha s}{n},\quad \omega\in A_1.\tag{4.2}$$

We introduce a variant weighted Riesz capacities $\mathcal{R}^{\omega}_{\alpha,s}(\cdot)$ defined by

$$\mathcal{R}^{\omega}_{\alpha,s}(E)=\inf\left\{\|f\|^s_{L^s_{\omega}(\mathbb{R}^n\times(0,\infty))}:f\geq 0,\ \int_0^{\infty}\left(\int_{|\cdot-y|}\frac{f(y,t)}{t^{n-\alpha}}\omega(y)^{-\frac{1}{s-1}}\mathrm{d}y\right)\frac{\mathrm{d}t}{t}\geq\chi_E\right\}.$$

The corresponding nonlinear potential $\mathcal{V}^{\mu}_{\omega}$ and Wolff potential W^{μ}_{ω} of a positive measure μ on $\mathbb{R}^n$ are defined by

$$\mathcal{V}^{\mu}_{\omega}(x)=\int_0^{\infty}\int_{|x-y|<t}\left(\frac{\mu(B_t(y))}{t^{n-\alpha}}\right)^{\frac{1}{s-1}}\frac{\omega(y)^{-\frac{1}{s-1}}}{t^{n-\alpha}}\mathrm{d}y\frac{\mathrm{d}t}{t},\quad x\in\mathbb{R}^n,$$
$$W^{\mu}_{\omega}(x)=\int_0^{\infty}\left(\frac{t^{\alpha s}\mu(B_t(x))}{\omega(B_t(x))}\right)^{\frac{1}{s-1}}\frac{\mathrm{d}t}{t},\quad x\in\mathbb{R}^n.$$

If $\omega\in A_s$, then we have

$$c_{n,\alpha,s}\cdot W^{\mu}_{\omega}\leq\mathcal{V}^{\mu}_{\omega}\leq C_{n,\alpha,s}\cdot[\omega]_{A_s}\cdot W^{\mu}_{\omega},$$
$$C^{-1}_{n,\alpha,s,\omega}\cdot R^{\omega}_{\alpha,s}(E)\leq\mathcal{R}^{\omega}_{\alpha,s}(E)\leq C_{n,\alpha,s,\omega}\cdot R^{\omega}_{\alpha,s}(E),\quad E\subseteq\mathbb{R}^n.$$

The Wolff potential W^{μ}_{ω} satisfies the bounded maximum principle

$$W^{\mu}_{\omega}\leq C_{n,\alpha,s,\omega}\sup_{\text{supp}(\mu)}W^{\mu}_{\omega}$$

for all weights ω satisfying (2.1). Besides that, we have the following weak-type estimate of Wolff potential that

$$\sup_{t>0}t^{s-1}\cdot\mathcal{R}^{\omega}_{\alpha,s}(\{W^{\mu}_{\alpha,s}>t\})\leq C_{n,\alpha,s,\omega}\cdot\mu(\mathbb{R}^n),\quad \omega\in A_s.$$

The capacitary strong-type estimate for $R^{\omega}_{\alpha,s}(\cdot)$ reads as

$$\int_0^{\infty}R^{\omega}_{\alpha,s}(\{\mathcal{I}_{\alpha}f\geq t\})\mathrm{d}t^s\leq C_{n,\alpha,s,\omega}\int_{\mathbb{R}^n}f(x)^s\omega(x)\mathrm{d}x,\quad f\geq 0,\quad \omega\in A_s.$$

By introducing the auxiliary functional

$$\gamma(f)=\inf\left\{\|\varphi\|^s_{L^s(\omega)}:\varphi\geq 0,\ \mathcal{I}_{\alpha}\varphi(x)\geq|f(x)|^{\frac{1}{s}}\text{ q.e. with respect to }R^{\omega}_{\alpha,s}(\cdot)\right\},$$

one proves the normability of $L^1(R^\omega_{\alpha,s})$ by noting that

$$C^{-1}_{n,\alpha,s,\omega}\cdot\gamma(f)\le\int_0^\infty R^\omega_{\alpha,s}(\{x\in\mathbb{R}^n:|f(x)|>t\})\mathrm{d}t\le C_{n,\alpha,s,\omega}\cdot\gamma(f),\quad \omega\in A_s.$$

As a result, the following p-convexity is obtained.

$$\left\|\sum_k|f_k|\right\|_{L^p(R^\omega_{\alpha,s})}\le C_{n,\alpha,s,\omega}\left(\sum_k\|f_k\|^p_{L^p(R^\omega_{\alpha,s})}\right)^{\frac{1}{p}},\quad 0<p\le 1,\quad \omega\in A_s,$$
$$\left\|\sum_k|f_k|\right\|_{L^{p,\infty}(R^\omega_{\alpha,s})}\le C_{n,\alpha,s,\omega,p}\left(\sum_k\|f_k\|^p_{L^{p,\infty}(R^\omega_{\alpha,s})}\right)^{\frac{1}{p}},\quad 0<p<1,\quad \omega\in A_s.$$

We have similarly as in Lemma 3.1 that

$$\|W^\mu_\omega\|_{L^{\eta,\infty}(B_r(x_0))}\le C_{n,\alpha,s,\omega}\cdot|B_r(x_0)|^{\frac{1}{\eta}}W^\mu_\omega(x_0),\quad r>0,\quad x_0\in\mathbb{R}^n,$$

where $\eta=(s-1)n/(n-\alpha s)$. Consequently,

$$\|\mathbf{M}(\chi_E)\|_{L^{\frac{n-\alpha s}{n},\infty}(R^\omega_{\alpha,s})}\le C_{n,\alpha,s,\omega}\cdot R^\omega_{\alpha,s}(E),\quad E\subseteq\mathbb{R}^n,\quad \omega\in A_1.$$

As a result, the estimate (4.1) is proved by using the p-convexity. To obtain the estimate (4.2), we have by Remark 3.5 that

$$\mathbf{M}\left((W^\mu_\omega)^\delta\right)(x)\le C_{n,\alpha,s,\omega,\delta}(W^\mu_\omega(x))^\delta,\quad x\in\mathbb{R}^n,\quad 0<\delta<\frac{n(s-1)}{n-\alpha s},\quad \omega\in A_1,$$

Consequently, for $p>(n-\alpha s)/n$, we have

$$\|\mathbf{M}(\chi_E)\|_{L^p(R^\omega_{\alpha,s})}\le C_{n,\alpha,s,\omega,p}\cdot R^\omega_{\alpha,s}(E)^{\frac{1}{p}},\quad E\subseteq\mathbb{R}^n,\quad \omega\in A_1.$$

Finally, the estimate (4.2) is obtained by using the p-convexity again.

Data Availability Not applicable.

References

1. Adams, D.R., Hedberg, L.I.: Function spaces and potential theory. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 314. Springer, Berlin (1996)
2. Grafakos, L.: Classical Fourier Analysis. Graduate Texts in Mathematics, vol. 249. Springer, New York (2014)
3. Kalton, N.J., Peck, N.T., Robert, J.W.: An F-Space Sampler. London Mathematical Society Lecture Note Series, vol. 89. Cambridge University Press, Cambridge (1984)
4. Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. **165**, 207–226 (1972)

5. Ooi, K.H., Phuc, N.C.: The Hardy-Littlewood maximal function, Choquet integrals, and embeddings of Sobolev type. Math. Ann. **382**(3–4), 1865–1879 (2022)
6. Orobitg, J., Verdera, J.: Choquet integrals, Hausdorff content and the Hardy–Littlewood maximal operator. Bull. Lond. Math. Soc. **30**, 145–150 (1998)
7. Rychkov, V.S.: Littlewood-Paley theory and function spaces with A_p^{loc} weights. Math. Nachr. **224**, 145–180 (2001)
8. Saito, H., Tanaka, H., Watanabe, T.: Fractional maximal operators with weighted Hausdorff content. Positivity **23**(1), 125–138 (2019)
9. Turesson, B.O.: Nonlinear Potential Theory and Weighted Sobolev Spaces. Lecture Notes in Mathematics, vol. 1736. Springer, Berlin (2000)
10. Tang, L.: Choquet integrals, weighted Hausdorff content and maximal operators. Georgian Math. J. **18**(3), 587–596 (2011)