Banach J. Math. Anal. (2023) 17:56
https://doi.org/10.1007/s43037-023-00283-0

Tusi Mathematical Research Group

ORIGINAL PAPER

Difference of composition operators on the weighted Bergman spaces over the half-plane

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Received: 5 March 2023 / Accepted: 15 June 2023 / Published online: 1 July 2023

Abstract

In this paper, based on the characterization of Carleson measure, we study bounded difference of composition operators from Bergman spaces with Békollé weight to Lebesgue spaces over the half-plane. We also obtain a characterization for the Carleson measure by products of functions.

Keywords Békollé weight · Carleson measure · Composition operator · Weighted Bergman space

Mathematics Subject Classification 47B32 · 46E30

1 Introduction

Let Π^+ be the upper half of the complex plane, i.e., $\Pi^+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, and dA be the Lebesgue area measure on Π^+. Given a positive Lebesgue function ω on the Π^+, for $0 < p < \infty$, the weighted Bergman space $A^p(\omega)$ is the space of

Communicated by Manuel Maestre.

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holomorphic functions f over Π^+ with

$$\|f\|_{A^p(\omega)} = \left(\int_{\Pi^+} |f(z)|^p \omega(z) \mathrm{d}A(z) \right)^{\frac{1}{p}} < \infty.$$

If $\omega(z) = c_\alpha (\mathrm{Im} z)^\alpha$ for $\alpha > -1$, then $A^p(\omega)$ is the standard weighted Bergman space $A^p_\alpha(\Pi^+)$, where $c_\alpha = \frac{2^\alpha(\alpha+1)}{\pi}$. Let $S(\Pi^+)$ be the set of all holomorphic self-maps on Π^+. The composition operator C_φ on $A^p(\omega)$ induced by $\varphi \in S(\Pi^+)$ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in A^p(\omega).$$

Composition operators on various analytic function spaces have been extensively studied (see the monographs [4, 17, 21]). One of the most important topics in the study of composition operators is to characterize properties of the difference of composition operators, especially the compactness (see [6, 8, 11, 12, 16, 19] and the references therein). Different from the unit disk case, there exist unbounded composition operators and there are no compact composition operators on $A^p_\alpha(\Pi^+)$ [10, 18]. In [7], Choe et al. characterized bounded and compact difference of composition operators on $A^p_\alpha(\Pi^+)$. In [14], Pang and Wang extended the results in [7] to the composition operators from $A^p_\alpha(\Pi^+)$ to Lebesgue spaces $L^q(\mu)$ for all $0 < p, q < \infty$. Here, μ is a positive Borel measure on Π^+ and $L^q(\mu)$ is the space of all measurable functions f defined on Π^+ with "norm"

$$\|f\|_{L^q(\mu)} = \left(\int_{\Pi^+} |f|^q \mathrm{d}\mu \right)^{\frac{1}{q}} < \infty.$$

In this paper, we consider the bounded difference of composition operators from $A^p(\omega)$ into $L^q(\mu)$ for $\frac{\omega(z)}{(\mathrm{Im} z)^\alpha} \in B_{p_0}(\alpha)$ with $p_0 > 1$ and $\alpha > -1$.

Let $p_0 > 1$ and $\alpha > -1$. Recall that the class $B_{p_0}(\alpha)$ consists of all positive locally integrable functions ω on Π^+ satisfying

$$\sup_I \frac{\int_{Q_I} \omega \mathrm{d}A_\alpha}{\int_{Q_I} \mathrm{d}A_\alpha} \left(\frac{\int_{Q_I} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A_\alpha}{\int_{Q_I} \mathrm{d}A_\alpha} \right)^{\frac{p_0}{p_0'}} < \infty,$$

where I is an interval in $\mathbb{R}$, $Q_I = I \times [0, |I|]$ ($|I|$ denotes the length of I) is the Carleson square associated to I and p_0' is the conjugate index of p_0. Since $\frac{|\mathrm{Im} z|}{|I|} \leq 1$ for $z \in Q_I$, we see that $B_{p_0}(\alpha) \subset B_{p_0}(\beta)$ if $-1 < \alpha < \beta$.

In order to state our main results, we introduce more terminology and notation.

Let ρ be the pseudo-hyperbolic distance on Π^+, that is

$$\rho(z, \xi) = \left| \frac{z - \xi}{z - \bar{\xi}} \right|, \quad z, \xi \in \Pi^+.$$

For $z \in \Pi^+, 0 < \delta < 1$, $E_\delta(z)$ denotes the pseudo-hyperbolic disk centered at z with radius δ. That is, $E_\delta(z) = \{\xi \in \Pi^+, \rho(z, \xi) < \delta\}$. A sequence $\{z_n\} \subset \Pi^+$ is called δ-separated if $\{E_\delta(z_n)\}$ are pairwise disjoint, and is called a δ-lattice if it is $\frac{\delta}{2}$-separated and $\Pi^+ = \bigcup_{n=1}^{\infty} E_\delta(z_n)$. A δ-lattice on the upper half plane exists and can be explicitly constructed by using almost the same argument as that on the unit disk [21, Lemma 4.8].

The Borel measure μ is called an (ω, p, q)-Carleson measure if there exists a constant $C > 0$ such that for any $f \in A^p(\omega)$,

$$\|f\|_{L^q(\mu)} \leq C\|f\|_{A^p(\omega)}.$$

Denote

$$H_{\omega,\mu,\delta}(z) = \frac{\mu(E_\delta(z))}{\omega(E_\delta(z))^{\frac{q}{p}}}, \quad G_{\omega,\mu,\delta}(z) = \frac{\mu(E_\delta(z))}{\omega(E_\delta(z))}, \qquad z \in \Pi^+.$$

Our first result gives the characterization of (ω, p, q)-Carleson measure.

Theorem 1.1 *Let* $0 < p, q < \infty$, $\alpha > -1$, $p_0 > \max\{1, p\}$, $\frac{\omega(z)}{(\mathrm{Im} z)^\alpha} \in B_{p_0}(\alpha)$ *and* μ *be a positive Borel measure on* Π^+.

(1) *If* $0 < p \leq q < \infty$, *then* μ *is an* (ω, p, q)*-Carleson measure if and only if*

$$\sup_{z\in\Pi^+} H_{\omega,\mu,\delta}(z) < \infty.$$

(2) *If* $0 < q < p < \infty$, *then the following statements are equivalent.*

(a) μ *is an* (ω, p, q)*-Carleson measure*;
(b) $\{H_{\omega,\mu,2\delta}(z_n)\} \in l^{\frac{p}{p-q}}$ *for any* δ*-lattice* $\{z_n\} \subset \Pi^+$ *with* $0 < \delta < \frac{1}{3}$;
(c) $\{H_{\omega,\mu,2\delta}(z_n)\} \in l^{\frac{p}{p-q}}$ *for some* δ*-lattice* $\{z_n\} \subset \Pi^+$ *with* $0 < \delta < \frac{1}{3}$;
(d) $G_{\omega,\mu,\delta} \in L^{\frac{p}{p-q}}(\omega \mathrm{d}A)$ *for some* $0 < \delta < 1$.

For $\varphi, \psi \in S(\Pi^+)$ and $0 < \delta < 1$, let

$$\sigma(z) := \sigma_{\varphi,\psi}(z) = \rho(\varphi(z), \psi(z)), \quad z \in \Pi^+.$$

The joint pullback measure $\mu_{\varphi,\psi,q}$ is defined for any Borel set $E \subset \Pi^+$ as

$$\mu_{\varphi,\psi,q}(E) = \int_{\varphi^{-1}(E)} \sigma^q \mathrm{d}\mu + \int_{\psi^{-1}(E)} \sigma^q \mathrm{d}\mu.$$

Based on Theorem 1.1, we characterize the bounded difference of composition operators from $A^p(\omega)$ to $L^q(\mu)$.

Theorem 1.2 *Let* $0 < p, q < \infty$, $\alpha > -1$, $p_0 > \max\{1, p\}$, $\frac{\omega(z)}{(\mathrm{Im} z)^\alpha} \in B_{p_0}(\alpha)$ *and* μ *be a positive Borel measure on* Π^+. *Suppose that* $\varphi, \psi \in S(\Pi^+)$. *Then* $C_\varphi - C_\psi$ *is bounded from* $A^p(\omega)$ *to* $L^q(\mu)$ *if and only if* $\mu_{\varphi,\psi,q}$ *is an* (ω, p, q)*-Carleson measure.*

Let $\lambda = \frac{q}{p}$, then $\frac{p}{p-q} = \frac{1}{1-\lambda}$. By Theorem 1.1, we see that the (ω, p, q)-Carleson measure depends only on the ratio $\lambda = \frac{q}{p}$. So we introduce the following definition. A Borel measure μ is called an (ω, λ)-Carleson measure if there exists a constant $C > 0$ such that for all $0 < p, q < \infty$ with $\lambda = \frac{q}{p}$ and any $f \in A^p(\omega)$,

$$\|f\|_{L^q(\mu)} \le C\|f\|_{A^p(\omega)}.$$

Finally, we give a characterization for (ω, λ)-Carleson measure by using products of functions in $A^p(\omega)$.

Theorem 1.3 *Let $\alpha > -1$, $p_0 > 1$, $\frac{\omega(z)}{(\mathrm{Im}z)^\alpha} \in B_{p_0}(\alpha)$ and μ be a positive Borel measure on Π^+. For any integer $k \ge 1$ and $i = 1, 2, \dots, k$, let*

$$0 < p_i, q_i < \infty, \quad \lambda = \sum_{i=1}^{k} \frac{q_i}{p_i}.$$

Then μ is an (ω, λ)-Carleson measure if and only if there exists a positive constant C such that for any $f_i \in A^{p_i}(\omega)$, $i = 1, 2, \dots, k$,

$$\int_{\Pi^+} \prod_{i=1}^{k} |f_i(z)|^{q_i} \mathrm{d}\mu(z) \le C \prod_{i=1}^{k} \|f_i\|^{q_i}_{A^{p_i}(\omega)}. \tag{1.1}$$

The paper is organized as follows. In Sect. 2, we discuss the class $B_{p_0}(\alpha)$ and prove a collection of preliminary results which will be used. In Sects. 3–5, we give the proofs of Theorem 1.1, 1.2 and 1.3 respectively.

Throughout this paper, the notation $A \lesssim B$ means that there is a positive constant C which is independent of $z \in \Pi^+$ and $f \in A^p(\omega)$ such that $A \le CB$, and the notation $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2 Preliminaries

In this section, we present some results about the class $B_{p_0}(\alpha)$ and the weighted Bergman spaces $A^p(\omega)$. Some technical lemmas used throughout the paper are proved.

Lemma 2.1 [7] *Let $z \in \Pi^+$, $0 < \delta < 1$. Then for all $\xi \in E_\delta(z)$ and $a \in \Pi^+$,*

$$\mathrm{Im}\xi \approx \mathrm{Im}z, \ |z - \bar{\xi}| \approx 2\mathrm{Im}z, \quad |z - \bar{a}| \approx |\xi - \bar{a}|.$$

For $p_0 > 1, 0 < \delta < 1$, we say that a weight ω belongs to the $C_{p_0}(\delta)$ class if

$$\sup_{z\in\Pi^+} \frac{\int_{E_\delta(z)} \omega \mathrm{d}A}{\int_{E_\delta(z)} \mathrm{d}A} \left(\frac{\int_{E_\delta(z)} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A}{\int_{E_\delta(z)} \mathrm{d}A} \right)^{\frac{p_0}{p_0'}} < \infty.$$

Let E be a measurable subset in Π^+, denote $|E| = \int_E \mathrm{d}A$. Then for any Carleson square Q_I,

$$\int_{Q_I} \mathrm{d}A_\alpha \approx |Q_I|^{1+\frac{\alpha}{2}} = |I|^{2+\alpha}.$$

Given a pseudo-hyperbolic disk $E_\delta(z)$. $E_\delta(z)$ is actually a Euclidean disk centered at $x + \mathrm{i}\frac{1+\delta^2}{1-\delta^2}y$ with radius $\frac{2\delta}{1-\delta^2}y$, where $z = x + \mathrm{i}y, x = \mathrm{Re}z, y = \mathrm{Im}z$ [7]. Let $z' = x + \mathrm{i}\frac{1+\delta}{1-\delta}y$ and $Q(z') = \{\xi \in \Pi^+ : |\mathrm{Re}\xi - x| < \frac{1}{2}\mathrm{Im}z', 0 < \mathrm{Im}\xi < \mathrm{Im}z'\}$. Then, $Q(z')$ is a Carleson square with side length $\frac{1+\delta}{1-\delta}y$. Obviously, $E_\delta(z) \subset Q(z')$ and

$$\frac{|E_\delta(z)|}{|Q(z')|} = \frac{\pi(\frac{2\delta}{1-\delta^2}y)^2}{(\frac{1+\delta}{1-\delta}y)^2} = \pi\left(\frac{2\delta}{(1+\delta)^2}\right)^2.$$

That is, $|E_\delta(z)| \approx |Q(z')|$. The following lemma shows that $B_{p_0}(\alpha) \subset C_{p_0}(\delta)$ for any $0 < \delta < 1$.

Lemma 2.2 *Let $\alpha > -1, 0 < \delta < 1$. Then $B_{p_0}(\alpha) \subset C_{p_0}(\delta)$. Furthermore, if $\frac{\omega(z)}{(\mathrm{Im}z)^\alpha} \in B_{p_0}(\alpha)$, then $\omega \in C_{p_0}(\delta)$.*

Proof Let $\omega \in B_{p_0}(\alpha)$. Then, by Lemma 2.1,

$$\begin{aligned}
&\frac{\int_{E_\delta(z)} \omega \mathrm{d}A}{\int_{E_\delta(z)} \mathrm{d}A}\left(\frac{\int_{E_\delta(z)} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A}{\int_{E_\delta(z)} \mathrm{d}A}\right)^{\frac{p_0}{p_0'}} \\
&\approx \frac{\int_{E_\delta(z)} \frac{\omega(\xi)}{(\mathrm{Im}z)^\alpha} \mathrm{d}A_\alpha(\xi)}{|E_\delta(z)|}\left(\frac{\int_{E_\delta(z)} \frac{\omega(\xi)^{-\frac{p_0'}{p_0}}}{(\mathrm{Im}z)^\alpha} \mathrm{d}A_\alpha(\xi)}{|E_\delta(z)|}\right)^{\frac{p_0}{p_0'}} \\
&\lesssim \left(\frac{1}{(\mathrm{Im}z)^\alpha}\right)^{1+\frac{p_0}{p_0'}} \frac{\int_{Q(z')} \omega \mathrm{d}A_\alpha}{|Q(z')|}\left(\frac{\int_{Q(z')} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A_\alpha}{|Q(z')|}\right)^{\frac{p_0}{p_0'}} \\
&\approx \frac{\int_{Q(z')} \omega \mathrm{d}A_\alpha}{\int_{Q(z')} \mathrm{d}A_\alpha}\left(\frac{\int_{Q(z')} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A_\alpha}{\int_{Q(z')} \mathrm{d}A_\alpha}\right)^{\frac{p_0}{p_0'}} \le C.
\end{aligned}$$

The last "≈" follows from the fact that $\int_{Q(z')} \mathrm{d}A_\alpha \approx |Q(z')|^{1+\frac{\alpha}{2}} \approx (\mathrm{Im}z)^{\alpha+2}$. Thus, $\omega \in C_{p_0}(\delta)$ and $B_{p_0}(\alpha) \subset C_{p_0}(\delta)$.

Suppose $\frac{\omega(z)}{(\mathrm{Im} z)^\alpha} \in B_{p_0}(\alpha)$. Then, we have $\frac{\omega(z)}{(\mathrm{Im} z)^\alpha} \in C_{p_0}(\delta)$. By Lemma 2.1,

$$\begin{aligned}
&\sup_{z\in\Pi^+} \frac{\int_{E_\delta(z)} \omega \mathrm{d}A}{\int_{E_\delta(z)} \mathrm{d}A} \left(\frac{\int_{E_\delta(z)} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A}{\int_{E_\delta(z)} \mathrm{d}A} \right)^{\frac{p_0}{p_0'}} \\
&\approx \sup_{z\in\Pi^+} \frac{\int_{E_\delta(z)} \frac{\omega(\xi)}{(\mathrm{Im}\xi)^\alpha} (\mathrm{Im} z)^\alpha \mathrm{d}A(\xi)}{|E_\delta(z)|} \left(\frac{\int_{E_\delta(z)} (\frac{\omega(\xi)}{(\mathrm{Im}\xi)^\alpha})^{-\frac{p_0'}{p_0}} (\mathrm{Im} z)^{-\alpha\frac{p_0'}{p_0}} \mathrm{d}A(\xi)}{|E_\delta(z)|} \right)^{\frac{p_0}{p_0'}} \\
&= \sup_{z\in\Pi^+} \frac{\int_{E_\delta(z)} \frac{\omega(\xi)}{(\mathrm{Im}\xi)^\alpha} \mathrm{d}A(\xi)}{|E_\delta(z)|} \left(\frac{\int_{E_\delta(z)} \left(\frac{\omega(\xi)}{(\mathrm{Im}\xi)^\alpha}\right)^{-\frac{p_0'}{p_0}} \mathrm{d}A(\xi)}{|E_\delta(z)|} \right)^{\frac{p_0}{p_0'}} < \infty.
\end{aligned}$$

Thus, $\omega \in C_{p_0}(\delta)$. □

Lemma 2.3 *Let* $\alpha > -1,\ p_0 > 1$ *and* $\frac{\omega(z)}{(\mathrm{Im} z)^\alpha} \in B_{p_0}(\alpha)$. *If* $\xi \in E_\delta(z)$, *then*

$$\omega(E_\delta(\xi)) \approx \omega(E_\delta(z)).$$

Proof Take $0 < \delta_1, \delta_2 < 1$. We first show that $\omega(E_{\delta_1}(z)) \approx \omega(E_{\delta_2}(z))$.

Without loss of generality, we assume $\delta_1 \le \delta_2$. Then, $E_{\delta_1}(z) \subset E_{\delta_2}(z)$. Hence,

$$\omega(E_{\delta_1}(z)) \le \omega(E_{\delta_2}(z)), \quad |E_{\delta_1}(z)| \approx |E_{\delta_2}(z)|.$$

On the other hand, by Lemma 2.2, $\omega \in C_{p_0}(\delta_2)$. So

$$\begin{aligned}
\omega(E_{\delta_2}(z)) &= \int_{E_{\delta_2}(z)} \omega \mathrm{d}A \\
&\lesssim \left(\int_{E_{\delta_2}(z)} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A \right)^{-\frac{p_0}{p_0'}} \left(\int_{E_{\delta_2}(z)} \mathrm{d}A \right)^{p_0} \\
&\lesssim \left(\int_{E_{\delta_1}(z)} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A \right)^{-\frac{p_0}{p_0'}} \left(\int_{E_{\delta_1}(z)} \mathrm{d}A \right)^{p_0} \\
&\le \left(\int_{E_{\delta_1}(z)} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A \right)^{-\frac{p_0}{p_0'}} \left(\int_{E_{\delta_1}(z)} \omega \mathrm{d}A \right) \left(\int_{E_{\delta_1}(z)} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A \right)^{\frac{p_0}{p_0'}} \\
&\lesssim \int_{E_{\delta_1}(z)} \omega \mathrm{d}A = \omega(E_{\delta_1}(z)).
\end{aligned}$$

The first "$\lesssim$" and "$\le$" in the formula above follow from the definition of the class $C_{p_0}(\delta)$ and Hölder's inequality respectively. We obtain that $\omega(E_{\delta_1}(z)) \approx \omega(E_{\delta_2}(z))$.

Since $\xi \in E_\delta(z)$, $E_{\frac{1-\delta}{2}}(\xi) \subset E_{\frac{1-\delta}{2}+\delta}(z)$, $E_{\frac{1-\delta}{2}}(z) \subset E_{\frac{1-\delta}{2}+\delta}(\xi)$. Hence,

$$\begin{aligned}\omega(E_{\frac{1-\delta}{2}+\delta}(\xi)) &\approx \omega(E_\delta(\xi)) \approx \omega(E_{\frac{1-\delta}{2}}(\xi))\\ &\le \omega(E_{\frac{1-\delta}{2}+\delta}(z)) \approx \omega(E_\delta(z)) \approx \omega(E_{\frac{1-\delta}{2}}(z))\\ &\le \omega(E_{\frac{1-\delta}{2}+\delta}(\xi)).\end{aligned}$$

Therefore, we have $\omega(E_\delta(\xi)) \approx \omega(E_\delta(z))$. □

Applying Hölder's inequality, it is easy to verify that $B_{p_0}(\alpha) \subset B_{p_1}(\alpha)$, $C_{p_0}(\delta) \subset C_{p_1}(\delta)$ if $p_0 < p_1$.

In the following, we discuss the properties of functions in $A^p(\omega)$ with $\frac{\omega(z)}{(\mathrm{Im}z)^\alpha} \in B_{p_0}(\alpha)$. These properties are the extension of the corresponding properties of functions in standard weighted Bergman spaces $A^p_\alpha(\Pi^+)$.

Lemma 2.4 *Suppose that* $0 < p < \infty, \alpha > -1$, $p_0 > 1$ *and* $\frac{\omega(z)}{(\mathrm{Im}z)^\alpha} \in B_{p_0}(\alpha)$. *Let* f *be any analytic function on* Π^+ *and* $z \in \Pi^+$.

(1) $|f(z)|^p \lesssim \frac{1}{\omega(E_\delta(z))}\int_{E_\delta(z)} |f|^p \omega \mathrm{d}A,\ z \in \Pi^+$. *In particular*,

$$|f(z)|^p \lesssim \frac{1}{\omega(E_\delta(z))}\int_{\Pi^+} |f|^p \omega \mathrm{d}A;$$

(2) *Let* $0 < \delta' < \delta$. *For* $\xi \in E_{\delta'}(z)$,

$$|f(z) - f(\xi)|^p \lesssim \frac{\rho(z,\xi)^p}{\omega(E_\delta(z))}\int_{E_\delta(z)} |f|^p \omega \mathrm{d}A.$$

Proof. (1) By Hölder's inequality and submean value type inequality with respect to the Lebesgue measure dA [9, Lemma 3.6], we have

$$\begin{aligned}|f(z)|^{\frac{p}{p_0}} &\lesssim \frac{1}{|E_\delta(z)|}\int_{E_\delta(z)} |f|^{\frac{p}{p_0}}\,\mathrm{d}A\\ &\le \frac{1}{|E_\delta(z)|}\left(\int_{E_\delta(z)} |f|^p\,\omega \mathrm{d}A\right)^{\frac{1}{p_0}}\left(\int_{E_\delta(z)} \omega^{-\frac{p_0'}{p_0}}\mathrm{d}A\right)^{\frac{1}{p_0'}}.\end{aligned}$$

Since $\frac{\omega(z)}{(\mathrm{Im}z)^\alpha} \in B_{p_0}(\alpha)$, it follows from Lemma 2.2 that $\omega \in C_{p_0}(\delta)$ and hence

$$\left(\int_{E_\delta(z)} \omega^{-\frac{p_0'}{p_0}}\mathrm{d}A\right)^{\frac{1}{p_0'}} \lesssim \frac{|E_\delta(z)|}{\omega(E_\delta(z))^{\frac{1}{p_0}}}. \tag{2.1}$$

Therefore,

$$|f(z)|^{\frac{p}{p_0}} \lesssim \frac{1}{\omega(E_\delta(z))^{\frac{1}{p_0}}}\left(\int_{E_\delta(z)} |f|^p\,\omega \mathrm{d}A\right)^{\frac{1}{p_0}},$$

and

$$|f(z)|^p \lesssim \frac{1}{\omega(E_\delta(z))} \int_{E_\delta(z)} |f|^p \, \omega \mathrm{d}A.$$

(2) By [7, Lemma 3.2], Hölder's inequality and (2.1), we obtain

$$\begin{aligned}
&|f(z) - f(\xi)|^{\frac{p}{p_0}} \\
&\lesssim \frac{\rho(z,\xi)^{\frac{p}{p_0}}}{|E_\delta(z)|} \int_{E_\delta(z)} |f|^{\frac{p}{p_0}} \mathrm{d}A \\
&\lesssim \frac{\rho(z,\xi)^{\frac{p}{p_0}}}{|E_\delta(z)|} \left(\int_{E_\delta(z)} |f|^p \omega \mathrm{d}A \right)^{\frac{1}{p_0}} \left(\int_{E_\delta(z)} \omega^{-\frac{p_0'}{p_0}} \mathrm{d}A \right)^{\frac{1}{p_0'}} \\
&\lesssim \frac{\rho(z,\xi)^{\frac{p}{p_0}}}{\omega(E_\delta(z))^{\frac{1}{p_0}}} \left(\int_{E_\delta(z)} |f|^p \omega \mathrm{d}A \right)^{\frac{1}{p_0}}.
\end{aligned}$$

Thus,

$$|f(z) - f(\xi)|^p \lesssim \frac{\rho(z,\xi)^p}{\omega(E_\delta(z))} \int_{E_\delta(z)} |f|^p \omega \mathrm{d}A.$$

□

For $\alpha > -1$, let K_α be the reproducing kernel functions of $A^2_\alpha(\Pi^+)$, i.e.,

$$K_\alpha(z,\xi) = \frac{1}{(\xi - \bar{z})^{\alpha+2}}, \quad z, \xi \in \Pi^+.$$

The integral operators P_α and P_α^+ are defined as

$$P_\alpha f(z) = \int_{\Pi^+} \frac{f(\xi)}{(z - \bar{\xi})^{\alpha+2}} \mathrm{d}A_\alpha(\xi), \quad P_\alpha^+ f(z) = \int_{\Pi^+} \frac{|f(\xi)|}{|z - \bar{\xi}|^{\alpha+2}} \mathrm{d}A_\alpha(\xi).$$

The following result shows that the class $B_{p_0}(\alpha)$ plays a special role in the theory of function spaces.

Theorem 2.5 [13, Theorem 1.3] *Let $\alpha > -1$, $p_0 > 1$ and ω be a positive locally integrable function. The following statements are equivalent*:

(1) *P_α is bounded from $L^{p_0}(\omega \mathrm{d}A)$ to $L^{p_0}(\omega \mathrm{d}A)$;*
(2) *P_α^+ is bounded from $L^{p_0}(\omega \mathrm{d}A)$ to $L^{p_0}(\omega \mathrm{d}A)$;*
(3) $\frac{\omega(z)}{(\mathrm{Im} z)^\alpha} \in B_{p_0}(\alpha)$.

Note that the class $B_{p_0}(\alpha)$ was firstly studied by Békollé and Bonami in the setting of the unit disk (or the unit ball) [1, 2]. We will see that the class $B_{p_0}(\alpha)$ in the upper half plane shares similar properties as that in the unit disk [3, 5, 20].

Lemma 2.6 *Let* $\alpha > -1$, $p_0 > 1$, $0 < p < \infty$ *and* $\frac{\omega(z)}{(\mathrm{Im} z)^\alpha} \in B_{p_0}(\alpha)$. *Then,*

$$\|K_\alpha(z,\cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^p \approx \frac{\omega(E_\delta(z))}{(\mathrm{Im} z)^{p_0(\alpha+2)}}.$$

Proof. By the submean value type inequality [9], we have

$$\begin{aligned}
|K_\alpha(z,\xi)| &= \left|\frac{1}{(\bar{\xi}-z)^{\alpha+2}}\right| \\
&\lesssim \frac{1}{(\mathrm{Im} z)^{\alpha+2}} \int_{E_\delta(z)} \frac{1}{|(\bar{\xi}-\eta)^{\alpha+2}|} \mathrm{d}A_\alpha(\eta) \\
&= \frac{1}{(\mathrm{Im} z)^{\alpha+2}} \int_{\Pi^+} \frac{\chi_{E_\delta(z)}(\eta)}{|(\xi-\bar{\eta})^{\alpha+2}|} \mathrm{d}A_\alpha(\eta) \\
&= \frac{1}{(\mathrm{Im} z)^{\alpha+2}} (P_\alpha^+ \chi_{E_\delta(z)})(\xi).
\end{aligned} \tag{2.2}$$

It follows from Theorem 2.5 that

$$\begin{aligned}
\|K_\alpha(z,\cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^p &= \int_{\Pi^+} |K_\alpha(z,\xi)|^{p_0} \omega(\xi) \mathrm{d}A(\xi) \\
&\lesssim \frac{1}{(\mathrm{Im} z)^{p_0(\alpha+2)}} \int_{\Pi^+} [(P_\alpha^+ \chi_{E_\delta(z)})(\xi)]^{p_0} \omega(\xi) \mathrm{d}A(\xi) \\
&\lesssim \frac{1}{(\mathrm{Im} z)^{p_0(\alpha+2)}} \|\chi_{E_\delta(z)}\|_{L^{p_0}(\omega \mathrm{d}A)}^{p_0} \\
&= \frac{\omega(E_\delta(z))}{(\mathrm{Im} z)^{p_0(\alpha+2)}}.
\end{aligned}$$

On the other hand, by Lemma 2.1,

$$\begin{aligned}
\frac{\omega(E_\delta(z))}{(\mathrm{Im} z)^{p_0(\alpha+2)}} &\approx \int_{E_\delta(z)} \frac{\omega(\xi)}{|\xi-\bar{z}|^{p_0(\alpha+2)}} \mathrm{d}A(\xi) \\
&\le \int_{\Pi^+} |K_\alpha(z,\xi)|^{p_0} \omega(\xi) \mathrm{d}A(\xi) = \|K_\alpha(z,\cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^p.
\end{aligned}$$

Therefore, we obtain

$$\|K_\alpha(z,\cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^p \approx \frac{\omega(E_\delta(z))}{(\mathrm{Im} z)^{p_0(\alpha+2)}}.$$

□

The following lemma is a modification of [9, Lemma 4.2].

Lemma 2.7 *Let* $0 < \delta < \frac{1}{3}$ *and* $s = 1, 2$. *If* $\{z_n\} \subset \Pi^+$ *is a* δ*-lattice, then there exists a positive integer* $N = N(s,\delta)$ *such that no more than* N *of the balls* $E_{s\delta}(z_n)$ *contain a common point.*

Let $r_n : [0,1] \to [-1,1]$ be the Rademacher functions defined as

$$r_n(t) = \operatorname{sgn}(\sin(2^n \pi t)).$$

Khinchine's inequality says that for $0 < p < \infty$, there are constants $0 < A_p \le B_p < \infty$ such that

$$A_p \left(\sum_{n=1}^{m} |c_n|^2 \right)^{\frac{p}{2}} \le \int_0^1 \Big| \sum_{n=1}^{m} c_n r_n(t) \Big|^p \mathrm{d}t \le B_p \left(\sum_{n=1}^{m} |c_n|^2 \right)^{\frac{p}{2}}$$

for all natural numbers m and all complex numbers $c_1, c_2, \ldots, c_m$ [14].

Lemma 2.8 *Let* $0 < p < \infty, \alpha > -1, p_0 > \max\{1, p\}$ *and* $\frac{\omega(z)}{(\mathrm{Im} z)^\alpha} \in B_{p_0}(\alpha)$. *Suppose that* $0 < \delta < \frac{1}{3}$. *Then, for any* δ*-lattice* $\{z_n\} \subset \Pi^+$ *and* $\{c_n\} \in l^p$,

$$f_t(z) = \sum_{n=1}^{\infty} c_n r_n(t) \frac{K_\alpha(z_n, z)^{\frac{p_0}{p}}}{\|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}} \in A^p(\omega) \quad \textit{and} \quad \|f_t\|_{A^p(\omega)} \lesssim \|\{c_n\}\|_{l^p},$$

where $\{r_n(t)\}$ *are the Rademacher functions.*

Proof Since $p_0 > p$,

$$\begin{aligned} |f_t(z)|^{\frac{p}{p_0}} &\le \left(\sum_{n=1}^{\infty} |c_n| \frac{|K_\alpha(z_n, z)|^{\frac{p_0}{p}}}{\|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}} \right)^{\frac{p}{p_0}} \\ &\le \sum_{n=1}^{\infty} |c_n|^{\frac{p}{p_0}} \frac{|K_\alpha(z_n, z)|}{\|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{\frac{p}{p_0}}}. \end{aligned}$$

By (2.2), we have

$$\begin{aligned} |f_t(z)|^{\frac{p}{p_0}} &\lesssim \sum_{n=1}^{\infty} |c_n|^{\frac{p}{p_0}} \frac{(P_\alpha^+ \chi_{E_\delta(z_n)})(z)}{(\mathrm{Im} z_n)^{\alpha+2} \|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{\frac{p}{p_0}}} \\ &= P_\alpha^+ \left(\sum_{n=1}^{\infty} |c_n|^{\frac{p}{p_0}} \frac{\chi_{E_\delta(z_n)}(z)}{(\mathrm{Im} z_n)^{\alpha+2} \|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{\frac{p}{p_0}}} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\Pi^+} |f_t(z)|^p \omega(z) \mathrm{d}A(z) \\ &\lesssim \int_{\Pi^+} \left(P_\alpha^+ \Big(\sum_{n=1}^{\infty} |c_n|^{\frac{p}{p_0}} \frac{\chi_{E_\delta(z_n)}(z)}{(\mathrm{Im} z_n)^{\alpha+2} \|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{\frac{p}{p_0}}} \Big) \right)^{p_0} \omega(z) \mathrm{d}A(z) \end{aligned}$$

$$
\begin{aligned}
&= \left\| P_\alpha^+ \left(\sum_{n=1}^{\infty} |c_n|^{\frac{p}{p_0}} \frac{\chi_{E_\delta(z_n)}(z)}{(\mathrm{Im} z_n)^{\alpha+2} \|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{\frac{p}{p_0}}} \right) \right\|_{L^{p_0}(\omega \mathrm{d}A)}^{p_0} \\
&\lesssim \left\| \sum_{n=1}^{\infty} |c_n|^{\frac{p}{p_0}} \frac{\chi_{E_\delta(z_n)}(z)}{(\mathrm{Im} z_n)^{\alpha+2} \|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{\frac{p}{p_0}}} \right\|_{L^{p_0}(\omega \mathrm{d}A)}^{p_0} \\
&= \int_{\Pi^+} \left(\sum_{n=1}^{\infty} |c_n|^{\frac{p}{p_0}} \frac{\chi_{E_\delta(z_n)}(z)}{(\mathrm{Im} z_n)^{\alpha+2} \|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{\frac{p}{p_0}}} \right)^{p_0} \omega(z) \mathrm{d}A(z).
\end{aligned}
$$

By Lemma 2.7,

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} |c_n|^{\frac{p}{p_0}} \frac{\chi_{E_\delta(z_n)}(z)}{(\mathrm{Im} z_n)^{\alpha+2} \|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{\frac{p}{p_0}}} \right)^{p_0} \\
&\quad \lesssim \sum_{n=1}^{\infty} |c_n|^p \frac{\chi_{E_\delta(z_n)}(z)}{(\mathrm{Im} z_n)^{p_0(\alpha+2)} \|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{p}}.
\end{aligned}
$$

It follows from Lemma 2.6 that

$$
\begin{aligned}
&\int_{\Pi^+} |f_t(z)|^p \omega(z) \mathrm{d}A(z) \\
&\quad \lesssim \int_{\Pi^+} \sum_{n=1}^{\infty} |c_n|^p \frac{\chi_{E_\delta(z_n)}(z)}{(\mathrm{Im} z_n)^{p_0(\alpha+2)} \|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}^{p}} \omega(z) \mathrm{d}A(z) \\
&\quad \lesssim \sum_{n=1}^{\infty} |c_n|^p \frac{(\mathrm{Im} z_n)^{p_0(\alpha+2)}}{(\mathrm{Im} z_n)^{p_0(\alpha+2)} \omega(E_\delta(z_n))} \int_{E_\delta(z_n)} \omega(z) \mathrm{d}A(z) \\
&\quad = \sum_{n=1}^{\infty} |c_n|^p < \infty.
\end{aligned}
$$

Therefore, $f_t \in A^p(\omega)$ and $\|f_t\|_{A^p(\omega)} \lesssim \|\{c_n\}\|_{l^p}$. □

The following lemma provides an estimate of the difference of our test functions in terms of the pseudo-hyperbolic distance.

Lemma 2.9 [14, Lemma 2.12] *Let* $0 < p < \infty$, $p_0 > 1$ *and* $0 < \delta < 1$, *then*

$$
|K_\alpha(z, \xi)^{\frac{p_0}{p}} - K_\alpha(z, \eta)^{\frac{p_0}{p}}| \gtrsim |K_\alpha(z, \xi)^{\frac{p_0}{p}}| \quad \rho(\xi, \eta)
$$

for all $z, \eta \in \Pi^+$ *and* $\xi \in E_\delta(z)$.

3 Characterization of (ω, p, q)-Carleson measure

In this section, we give the proof of Theorem 1.1. Recall that

$$H_{\omega,\mu,\delta}(z)=\frac{\mu(E_\delta(z))}{\omega(E_\delta(z))^{\frac{q}{p}}},\quad G_{\omega,\mu,\delta}(z)=\frac{\mu(E_\delta(z))}{\omega(E_\delta(z))},\quad z\in\Pi^+.$$

Proof of Theorem 1.1 (1) For sufficiency, assume that

$$\sup_{z\in\Pi^+}H_{\omega,\mu,\delta}(z)<\infty.$$

For any $f\in A^p(\omega)$, by Lemma 2.4(1) and Fubini's theorem,

$$\begin{aligned}
&\int_{\Pi^+}|f(\xi)|^q\mathrm{d}\mu(\xi)\\
&\lesssim\int_{\Pi^+}\left(\frac{1}{\omega(E_\delta(\xi))}\int_{E_\delta(\xi)}|f(z)|^q\omega(z)\mathrm{d}A(z)\right)\mathrm{d}\mu(\xi)\\
&=\int_{\Pi^+}\left(\frac{1}{\omega(E_\delta(\xi))}\int_{\Pi^+}|f(z)|^q\chi_{E_\delta(\xi)}(z)\omega(z)\mathrm{d}A(z)\right)\mathrm{d}\mu(\xi)\\
&=\int_{\Pi^+}\int_{\Pi^+}\frac{\chi_{E_\delta(z)}(\xi)}{\omega(E_\delta(\xi))}\mathrm{d}\mu(\xi)|f(z)|^q\omega(z)\mathrm{d}A(z)\\
&=\int_{\Pi^+}\int_{E_\delta(z)}\frac{1}{\omega(E_\delta(\xi))}\mathrm{d}\mu(\xi)|f(z)|^q\omega(z)\mathrm{d}A(z).
\end{aligned}$$

For $\xi\in E_\delta(z)$, by Lemma 2.3, we have $\omega(E_\delta(\xi))\approx\omega(E_\delta(z))$. Thus

$$\begin{aligned}
\int_{\Pi^+}|f(\xi)|^q\mathrm{d}\mu(\xi)&\lesssim\int_{\Pi^+}\int_{E_\delta(z)}\frac{1}{\omega(E_\delta(z))}\mathrm{d}\mu(\xi)|f(z)|^q\omega(z)\mathrm{d}A(z)\\
&=\int_{\Pi^+}\frac{\mu(E_\delta(z))}{\omega(E_\delta(z))}|f(z)|^q\omega(z)\mathrm{d}A(z)\\
&=\int_{\Pi^+}G_{\omega,\mu,\delta}(z)|f(z)|^{q-p}|f(z)|^p\omega(z)\mathrm{d}A(z).
\end{aligned}$$

It follows from Lemma 2.4 that $|f(z)|^{q-p}\lesssim\frac{1}{\omega(E_\delta(z))^{\frac{q-p}{p}}}\|f\|_{A^p(\omega)}^{q-p}$. Therefore,

$$\begin{aligned}
\int_{\Pi^+}|f(\xi)|^q\mathrm{d}\mu(\xi)&\lesssim\int_{\Pi^+}\frac{G_{\omega,\mu,\delta}(z)}{\omega(E_\delta(z))^{\frac{q-p}{p}}}\|f\|_{A^p(\omega)}^{q-p}|f(z)|^p\omega(z)\mathrm{d}A(z)\\
&=\|f\|_{A^p(\omega)}^{q-p}\int_{\Pi^+}H_{\omega,\mu,\delta}(z)|f(z)|^p\omega(z)\mathrm{d}A(z)\\
&\le\sup_{z\in\Pi^+}H_{\omega,\mu,\delta}(z)\|f\|_{A^p(\omega)}^q.
\end{aligned}$$

Therefore, μ is an (ω, p, q)-Carleson measure.

For necessity, assume that μ is an (ω, p, q)-Carleson measure.

For any $z \in \Pi^+$, let $f_z(\xi) = \frac{K_\alpha(z,\xi)^{\frac{p_0}{p}}}{\|K_\alpha(z,\cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}}$. Then, $\|f_z\|_{A^p(\omega)} = 1$. By Lemma 2.6,

$$|f_z(\xi)|^q \approx |K_\alpha(z,\xi)|^{p_0\frac{q}{p}} \frac{(\mathrm{Im}z)^{p_0\frac{q}{p}(\alpha+2)}}{\omega(E_\delta(z))^{\frac{q}{p}}}.$$

By Lemma 2.1,

$$\begin{aligned}
\int_{E_\delta(z)} |f_z(\xi)|^q \mathrm{d}\mu(\xi) &\approx \frac{(\mathrm{Im}z)^{p_0\frac{q}{p}(\alpha+2)}}{\omega(E_\delta(z))^{\frac{q}{p}}} \int_{E_\delta(z)} \frac{1}{|\xi-\bar{z}|^{p_0\frac{q}{p}(\alpha+2)}} \mathrm{d}\mu(\xi)\\
&\approx \frac{(\mathrm{Im}z)^{p_0\frac{q}{p}(\alpha+2)}}{\omega(E_\delta(z))^{\frac{q}{p}}} \int_{E_\delta(z)} \frac{1}{(\mathrm{Im}z)^{p_0\frac{q}{p}(\alpha+2)}} \mathrm{d}\mu(\xi)\\
&= \frac{\mu(E_\delta(z))}{\omega(E_\delta(z))^{\frac{q}{p}}}\\
&= H_{\omega,\mu,\delta}(z).
\end{aligned}$$

Therefore,

$$H_{\omega,\mu,\delta}(z) \approx \int_{E_\delta(z)} |f_z(\xi)|^q \mathrm{d}\mu(\xi) \le \int_{\Pi^+} |f_z(\xi)|^q \mathrm{d}\mu(\xi) \lesssim \|f_z\|^q_{A^p(\omega)}.$$

Thus, $\sup_{z\in\Pi^+} H_{\omega,\mu,\delta}(z) < \infty$.

(2) (a) $\Rightarrow$ (b). Suppose that μ is an (ω, p, q)-Carleson measure. Let

$$f_t(z) = \sum_{n=1}^{\infty} c_n r_n(t) \frac{K_\alpha(z_n, z)^{\frac{p_0}{p}}}{\|K_\alpha(z_n,\cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}},$$

be the functions as in Lemma 2.8. Then, $f_t \in A^p(\omega)$ and $\|f_t\|_{A^p(\omega)} \lesssim \|\{c_n\}\|_{l^p}$. Thus,

$$\int_{\Pi^+} |f_t(z)|^q \mathrm{d}\mu(z) \lesssim \|f_t\|^q_{A^p(\omega)} \lesssim \|\{c_n\}\|^q_{l^p}.$$

By Khinchine's inequality and Fubini's theorem,

$$\begin{aligned}
&\int_{\Pi^+} \left(\sum_{n=1}^{\infty} \left|c_n \frac{K_\alpha(z_n, z)^{\frac{p_0}{p}}}{\|K_\alpha(z_n,\cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}}\right|^2\right)^{\frac{q}{2}} \mathrm{d}\mu(z)\\
&\lesssim \int_{\Pi^+} \left(\int_0^1 \left|\sum_{n=1}^{\infty} c_n r_n(t) \frac{K_\alpha(z_n, z)^{\frac{p_0}{p}}}{\|K_\alpha(z_n,\cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}}\right|^q \mathrm{d}t\right) \mathrm{d}\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Pi^+} \int_0^1 |f_t(z)|^q \mathrm{d}t \; \mathrm{d}\mu(z) \\
&= \int_0^1 \int_{\Pi^+} |f_t(z)|^q \mathrm{d}\mu(z) \; \mathrm{d}t \\
&\lesssim \|\{c_n\}\|_{l^p}^q. \qquad (3.1)
\end{aligned}
$$

By Lemmas 2.1 and 2.3,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} |c_n|^q H_{\omega,\mu,2\delta}(z_n) \\
&= \sum_{n=1}^{\infty} |c_n|^q \frac{\mu(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\frac{q}{p}}} \approx \sum_{n=1}^{\infty} |c_n|^q \frac{\int_{E_{2\delta}(z_n)} \mathrm{d}\mu(z)}{\omega(E_\delta(z_n))^{\frac{q}{p}}} \\
&\approx \sum_{n=1}^{\infty} |c_n|^q \int_{E_{2\delta}(z_n)} \frac{1}{\omega(E_\delta(z_n))^{\frac{q}{p}}} \left(\frac{\mathrm{Im} z_n}{|\bar{z} - z_n|} \right)^{\frac{p_0}{p}(\alpha+2)q} \mathrm{d}\mu(z) \\
&= \int_{\Pi^+} \sum_{n=1}^{\infty} \left(|c_n| \frac{\chi_{E_{2\delta}(z_n)}(z)}{\omega(E_\delta(z_n))^{\frac{1}{p}}} \left(\frac{\mathrm{Im} z_n}{|\bar{z} - z_n|} \right)^{\frac{p_0}{p}(\alpha+2)} \right)^q \mathrm{d}\mu(z). \qquad (3.2)
\end{aligned}
$$

By Lemmas 2.7 and 2.6, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \left(|c_n| \frac{\chi_{E_{2\delta}(z_n)}(z)}{\omega(E_\delta(z_n))^{\frac{1}{p}}} \left(\frac{\mathrm{Im} z_n}{|\bar{z} - z_n|} \right)^{\frac{p_0}{p}(\alpha+2)} \right)^q \\
&\quad \lesssim \left(\sum_{n=1}^{\infty} \left(\frac{|c_n|}{\omega(E_\delta(z_n))^{\frac{1}{p}}} \left(\frac{\mathrm{Im} z_n}{|\bar{z} - z_n|} \right)^{\frac{p_0}{p}(\alpha+2)} \right)^2 \right)^{\frac{q}{2}} \\
&\quad \approx \left(\sum_{n=1}^{\infty} \left(|c_n| \frac{|K_\alpha(z_n, z)|^{\frac{p_0}{p}}}{\|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}} \right)^2 \right)^{\frac{q}{2}}.
\end{aligned}
$$

Integrate both sides of this formula, by (3.1), we get

$$
\begin{aligned}
&\int_{\Pi^+} \sum_{n=1}^{\infty} \left(|c_n| \frac{\chi_{E_{2\delta}(z_n)}(z)}{\omega(E_\delta(z_n))^{\frac{1}{p}}} \left(\frac{\mathrm{Im} z_n}{|\bar{z} - z_n|} \right)^{\frac{p_0}{p}(\alpha+2)} \right)^q \mathrm{d}\mu(z) \\
&\quad \lesssim \int_{\Pi^+} \left(\sum_{n=1}^{\infty} \left(|c_n| \frac{|K_\alpha(z_n, z)|^{\frac{p_0}{p}}}{\|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}} \right)^2 \right)^{\frac{q}{2}} \mathrm{d}\mu(z) \\
&\quad \lesssim \|\{c_n\}\|_{l^p}^q.
\end{aligned}
$$

Combining with (3.2), we obtain

$$\sum_{n=1}^{\infty} |c_n|^q H_{\omega,\mu,2\delta}(z_n) \lesssim \|\{c_n\}\|_{l^p}^q.$$

Since $\{c_n\} \in l^p$ if and only if $\{|c_n|^q\} \in l^{\frac{p}{q}}$, we deduce that

$$\{H_{\omega,\mu,2\delta}(z_n)\} \in l^{(\frac{p}{q})'} = l^{\frac{p}{p-q}}.$$

(b) $\Rightarrow$ (c). It is trivial.

(c) $\Rightarrow$ (d). Suppose that there exists a constant $0 < \delta < \frac{1}{3}$ such that $\{H_{\omega,\mu,2\delta}(z_n)\} \in l^{\frac{p}{p-q}}$, where $\{z_n\} \subset \Pi^+$ is a δ-lattice.

For $z \in E_\delta(z_n)$, we have $E_\delta(z) \subset E_{2\delta}(z_n)$ and $\omega(E_\delta(z)) \approx \omega(E_{2\delta}(z_n))$. Then,

$$G_{\omega,\mu,\delta}(z) = \frac{\mu(E_\delta(z))}{\omega(E_\delta(z))} \lesssim \frac{\mu(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))} = G_{\omega,\mu,2\delta}(z_n).$$

Thus,

$$\begin{aligned}
&\int_{\Pi^+} |G_{\omega,\mu,\delta}(z)|^{\frac{p}{p-q}} \omega(z)\mathrm{d}A(z) \\
&\lesssim \sum_{n=1}^{\infty} \int_{E_\delta(z_n)} |G_{\omega,\mu,2\delta}(z_n)|^{\frac{p}{p-q}} \omega(z)\mathrm{d}A(z) \\
&\leq \sum_{n=1}^{\infty} \left(\frac{G_{\omega,\mu,2\delta}(z_n)}{\omega(E_{2\delta}(z_n))^{\frac{q-p}{p}}} \right)^{\frac{p}{p-q}} \\
&= \sum_{n=1}^{\infty} H_{\omega,\mu,2\delta}(z_n)^{\frac{p}{p-q}} < \infty.
\end{aligned}$$

Therefore, $G_{\omega,\mu,\delta} \in L^{\frac{p}{p-q}}(\omega \mathrm{d}A)$.

(d) $\Rightarrow$ (a). Suppose that there exists $0 < \delta < 1$ such that $G_{\omega,\mu,\delta} \in L^{\frac{p}{p-q}}(\omega \mathrm{d}A)$. For any $f \in A^p(\omega)$, by Lemma 2.4(1), Fubini's theorem, Lemma 2.3 and Hölder's inequality,

$$\begin{aligned}
&\int_{\Pi^+} |f(z)|^q \mathrm{d}\mu(z) \\
&\lesssim \int_{\Pi^+} \left(\frac{1}{\omega(E_\delta(z))} \int_{E_\delta(z)} |f(\xi)|^q \omega(\xi)\mathrm{d}A(\xi) \right) \mathrm{d}\mu(z) \\
&= \int_{\Pi^+} \left(\frac{1}{\omega(E_\delta(z))} \int_{\Pi^+} |f(\xi)|^q \chi_{E_\delta(z)}(\xi)\omega(\xi)\mathrm{d}A(\xi) \right) \mathrm{d}\mu(z) \\
&= \int_{\Pi^+} \int_{E_\delta(\xi)} \frac{1}{\omega(E_\delta(z))} \mathrm{d}\mu(z) |f(\xi)|^q \omega(\xi)\mathrm{d}A(\xi)
\end{aligned}$$

$$
\begin{aligned}
&\approx \int_{\Pi^+}\int_{E_\delta(\xi)} \frac{1}{\omega(E_\delta(\xi))}\mathrm{d}\mu(z)|f(\xi)|^q\omega(\xi)\mathrm{d}A(\xi)\\
&= \int_{\Pi^+} \frac{\mu(E_\delta(\xi))}{\omega(E_\delta(\xi))}|f(\xi)|^q\omega(\xi)\mathrm{d}A(\xi)\\
&= \int_{\Pi^+} G_{\omega,\mu,\delta}(\xi)|f(\xi)|^q\omega(\xi)\mathrm{d}A(\xi)\\
&\le \left(\int_{\Pi^+} G_{\omega,\mu,\delta}(\xi)^{\frac{p}{p-q}}\omega(\xi)\mathrm{d}A(\xi)\right)^{\frac{p-q}{p}}\left(\int_{\Pi^+}|f(\xi)|^p\omega(\xi)\mathrm{d}A(\xi)\right)^{\frac{q}{p}}\\
&= \|G_{\omega,\mu,\delta}\|_{L^{\frac{p}{p-q}}(\omega\mathrm{d}A)}\|f\|^q_{A^p(\omega)},
\end{aligned}
$$

which implies that

$$
\int_{\Pi^+}|f(z)|^q\mathrm{d}\mu(z) \lesssim \|f\|^q_{A^p(\omega)}.
$$

Thus, μ is an (ω, p, q)-Carleson measure. □

4 Bounded difference of composition operators

In this section, we give the proof of Theorem 1.2. Recall that for $\varphi, \psi \in S(\Pi^+)$ and $0 < \delta < 1$,

$$
\sigma(z) := \sigma_{\varphi,\psi}(z) = \rho(\varphi(z), \psi(z)), \quad \Omega_\delta := \{z \in \Pi^+ : \sigma(z) < \delta\}.
$$

For any Borel set $E \subset \Pi^+$,

$$
\mu_{\varphi,\psi,q}(E) = \int_{\varphi^{-1}(E)}\sigma^q\mathrm{d}\mu + \int_{\psi^{-1}(E)}\sigma^q\mathrm{d}\mu.
$$

In the following, we write $\mu_{\varphi,\psi,q}$ as μ_q for simplicity.

Proof of Theorem 1.2 For sufficiency, assume that μ_q is an (ω, p, q)-Carleson measure.

Let $0 < \delta < 1$. For any $f \in A^p(\omega)$,

$$
\begin{aligned}
&\|(C_\varphi - C_\psi)f\|^q_{L^q(\mu)}\\
&\quad= \int_{\Pi^+}|(C_\varphi - C_\psi)f|^q\mathrm{d}\mu\\
&\quad= \int_{\Omega_{\frac{\delta}{2}}}|f(\varphi) - f(\psi)|^q\mathrm{d}\mu + \int_{\Pi^+\setminus\Omega_{\frac{\delta}{2}}}|f(\varphi) - f(\psi)|^q\mathrm{d}\mu\\
&\quad\lesssim \int_{\Omega_{\frac{\delta}{2}}}|f(\varphi) - f(\psi)|^q\mathrm{d}\mu + \int_{\Pi^+\setminus\Omega_{\frac{\delta}{2}}}\big(|f(\varphi)|^q + |f(\psi)|^q\big)\mathrm{d}\mu
\end{aligned}
$$

$$=: \mathrm{I}(f) + \mathrm{II}(f). \tag{4.1}$$

For $z \in \Pi^+ \setminus \Omega_{\frac{\delta}{2}}$, $\sigma(z) \geq \frac{\delta}{2}$. Thus,

$$\begin{aligned}
\mathrm{II}(f) &\leq \frac{2^q}{\delta^q} \int_{\Pi^+ \setminus \Omega_{\frac{\delta}{2}}} (|f(\varphi)|^q + |f(\psi)|^q)\sigma^q \mathrm{d}\mu \\
&\lesssim \int_{\Pi^+} (|f(\varphi)|^q + |f(\psi)|^q)\sigma^q \mathrm{d}\mu \\
&= \int_{\Pi^+} |f|^q [(\sigma^q \mathrm{d}\mu) \circ \varphi^{-1} + (\sigma^q \mathrm{d}\mu) \circ \psi^{-1}] \\
&= \int_{\Pi^+} |f|^q \mathrm{d}\mu_q.
\end{aligned}$$

Since μ_q is an (ω, p, q)-Carleson measure,

$$\mathrm{II}(f) \lesssim \|f\|^q_{A^p(\omega)}. \tag{4.2}$$

Furthermore, we estimate the first term $\mathrm{I}(f)$. By Lemma 2.4(2) and Fubini's Theorem,

$$\begin{aligned}
\mathrm{I}(f) &= \int_{\Omega_{\frac{\delta}{2}}} |f(\varphi) - f(\psi)|^q \mathrm{d}\mu \\
&\lesssim \int_{\Omega_{\frac{\delta}{2}}} \left(\frac{\sigma(z)^q}{\omega(E_\delta(\varphi(z)))} \int_{E_\delta(\varphi(z))} |f(\xi)|^q \omega(\xi) \mathrm{d}A(\xi) \right) \mathrm{d}\mu(z) \\
&= \int_{\Pi^+} \chi_{\Omega_{\frac{\delta}{2}}}(z) \left(\frac{\sigma(z)^q}{\omega(E_\delta(\varphi(z)))} \int_{\Pi^+} \chi_{E_\delta(\varphi(z))}(\xi) |f(\xi)|^q \omega(\xi) \mathrm{d}A(\xi) \right) \mathrm{d}\mu(z) \\
&= \int_{\Pi^+} \int_{\Pi^+} \chi_{\Omega_{\frac{\delta}{2}} \bigcap \varphi^{-1}(E_\delta(\xi))}(z) \frac{\sigma(z)^q}{\omega(E_\delta(\varphi(z)))} \mathrm{d}\mu(z) |f(\xi)|^q \omega(\xi) \mathrm{d}A(\xi) \\
&= \int_{\Pi^+} \int_{\Omega_{\frac{\delta}{2}} \bigcap \varphi^{-1}(E_\delta(\xi))} \frac{\sigma(z)^q}{\omega(E_\delta(\varphi(z)))} \mathrm{d}\mu(z) |f(\xi)|^q \omega(\xi) \mathrm{d}A(\xi) \\
&\leq \int_{\Pi^+} \int_{\varphi^{-1}(E_\delta(\xi))} \frac{\sigma(z)^q}{\omega(E_\delta(\varphi(z)))} \mathrm{d}\mu(z) |f(\xi)|^q \omega(\xi) \mathrm{d}A(\xi) \\
&\approx \int_{\Pi^+} \frac{1}{\omega(E_\delta(\xi))} \int_{\varphi^{-1}(E_\delta(\xi))} \sigma(z)^q \mathrm{d}\mu(z) |f(\xi)|^q \omega(\xi) \mathrm{d}A(\xi).
\end{aligned}$$

The same estimate holds when the roles of φ and ψ are interchanged. Thus

$$\begin{aligned}
\mathrm{I}(f) &\lesssim \int_{\Pi^+} \frac{|f(\xi)|^q \omega(\xi)}{\omega(E_\delta(\xi))} \left(\int_{\varphi^{-1}(E_\delta(\xi))} \sigma(z)^q \mathrm{d}\mu(z) + \int_{\psi^{-1}(E_\delta(\xi))} \sigma^q(z) \mathrm{d}\mu(z) \right) \mathrm{d}A(\xi) \\
&= \int_{\Pi^+} \frac{1}{\omega(E_\delta(\xi))} \mu_q(E_\delta(\xi)) |f(\xi)|^q \omega(\xi) \mathrm{d}A(\xi)
\end{aligned}$$

$$= \int_{\Pi^+} G_{\omega,\mu_q,\delta}(\xi)|f(\xi)|^q \omega(\xi)\mathrm{d}A(\xi).$$

If $0 < q < p < \infty$, then by Hölder's inequality,

$$\begin{aligned}\mathrm{I}(f) &\lesssim \left(\int_{\Pi^+} G_{\omega,\mu_q,\delta}(\xi)^{\frac{p}{p-q}} \omega(\xi)\mathrm{d}A(\xi)\right)^{\frac{p-q}{p}} \left(\int_{\Pi^+} |f(\xi)|^p \omega(\xi)\mathrm{d}A(\xi)\right)^{\frac{q}{p}} \\ &= \left(\int_{\Pi^+} G_{\omega,\mu_q,\delta}(\xi)^{\frac{p}{p-q}} \omega(\xi)\mathrm{d}A(\xi)\right)^{\frac{p-q}{p}} \|f\|^q_{A^p(\omega)}.\end{aligned}$$

Therefore, by Theorem 1.1 and (4.2), we have

$$\mathrm{I}(f) + \mathrm{II}(f) \lesssim \|f\|^q_{A^p(\omega)}.$$

Therefore, $C_\varphi - C_\psi$ is bounded from $A^p(\omega)$ to $L^q(\mu)$ by (4.1).

If $0 < p \le q < \infty$, then

$$\begin{aligned}\mathrm{I}(f) &\lesssim \int_{\Pi^+} G_{\omega,\mu_q,\delta}(\xi)|f(\xi)|^q \omega(\xi)\mathrm{d}A(\xi) \\ &= \int_{\Pi^+} G_{\omega,\mu_q,\delta}(\xi)|f(\xi)|^{q-p}|f(\xi)|^p \omega(\xi)\mathrm{d}A(\xi).\end{aligned}$$

By Lemma 2.4(1), we have $|f(\xi)|^{q-p} \lesssim \frac{1}{\omega(E_\delta(\xi))^{\frac{q-p}{p}}} \|f\|^{q-p}_{A^p(\omega)}$. Then,

$$\begin{aligned}\mathrm{I}(f) &\lesssim \int_{\Pi^+} \frac{G_{\omega,\mu_q,\delta}(\xi)}{\omega(E_\delta(\xi))^{\frac{q-p}{p}}} |f(\xi)|^p \omega(\xi)\mathrm{d}A(\xi)\|f\|^{q-p}_{A^p(\omega)} \\ &= \int_{\Pi^+} H_{\omega,\mu_q,\delta}(\xi)|f(\xi)|^p \omega(\xi)\mathrm{d}A(\xi)\|f\|^{q-p}_{A^p(\omega)} \\ &\le \sup_{\xi\in\Pi^+} H_{\omega,\mu_q,\delta}(\xi)\|f\|^q_{A^p(\omega)}.\end{aligned}$$

Thus by Theorem 1.1 and (4.2), we have

$$\mathrm{I}(f) + \mathrm{II}(f) \lesssim \|f\|^q_{A^p(\omega)}.$$

Therefore, $C_\varphi - C_\psi$ is bounded from $A^p(\omega)$ to $L^q(\mu)$ by (4.1).

Therefore, if μ_q is an (ω, p, q)-Carleson measure, then $C_\varphi - C_\psi$ is bounded from $A^p(\omega)$ to $L^q(\mu)$ for $0 < p, q < \infty$.

For necessity, assume that $C_\varphi - C_\psi$ is bounded from $A^p(\omega)$ to $L^q(\mu)$.

Assume $0 < q < p < \infty$. Let

$$f_t(z) = \sum_{n=1}^{\infty} c_n r_n(t) \frac{K_\alpha(z_n, z)^{\frac{p_0}{p}}}{\|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}}.$$

be the functions in Lemma 2.8. Here $\{z_n\} \subset \Pi^+$ is a δ-lattice with $0 < \delta < \frac{1}{3}$. Then $f_t \in A^p(\omega)$ and $\|f_t\|_{A^p(\omega)} \lesssim \|\{c_n\}\|_{l^p}$.

For $\varphi(z) \in E_{2\delta}(z_n)$, by Lemma 2.9, we have

$$|K_\alpha(z_n, \varphi(z))^{\frac{p_0}{p}} - K_\alpha(z_n, \psi(z))^{\frac{p_0}{p}}| \gtrsim |K_\alpha(z_n, \varphi(z))^{\frac{p_0}{p}}|\sigma(z).$$

Therefore, by Lemma 2.1,

$$\begin{aligned}\sigma(z)^q &\lesssim \frac{|K_\alpha(z_n, \varphi(z))^{\frac{p_0}{p}} - K_\alpha(z_n, \psi(z))^{\frac{p_0}{p}}|^q}{|K_\alpha(z_n, \varphi(z))^{\frac{p_0}{p}}|^q} \\ &\approx (\mathrm{Im} z_n)^{\frac{p_0 q}{p}(\alpha+2)}|(C_\varphi - C_\psi)(K_\alpha(z_n, z)^{\frac{p_0}{p}})|^q.\end{aligned}$$

Thus, by Lemma 2.6 and Khinchine's inequality,

$$\begin{aligned}&\sum_{n=1}^\infty |c_n|^q \frac{1}{\omega(E_{2\delta}(z_n))^{\frac{q}{p}}} \int_{\varphi^{-1}(E_{2\delta}(z_n))} \sigma^q \mathrm{d}\mu \\ &= \int_{\Pi^+} \sum_{n=1}^\infty |c_n|^q \frac{\chi_{\varphi^{-1}(E_{2\delta}(z_n))}(z)}{\omega(E_{2\delta}(z_n))^{\frac{q}{p}}} \sigma^q(z)\mathrm{d}\mu(z) \\ &\lesssim \int_{\Pi^+} \sum_{n=1}^\infty |c_n|^q \chi_{\varphi^{-1}(E_{2\delta}(z_n))}(z) \frac{|(C_\varphi - C_\psi)(K_\alpha(z_n, z)^{\frac{p_0}{p}})|^q}{\|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|^q_{A^p(\omega)}} \mathrm{d}\mu(z) \\ &\lesssim \int_{\Pi^+} \left(\sum_{n=1}^\infty |c_n|^2 \left|\frac{(C_\varphi - C_\psi)(K_\alpha(z_n, z)^{\frac{p_0}{p}})}{\|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}}\right|^2\right)^{\frac{q}{2}} \mathrm{d}\mu(z) \\ &\approx \int_{\Pi^+} \left(\int_0^1 \left|\sum_{n=1}^\infty c_n r_n(t) \frac{(C_\varphi - C_\psi)(K_\alpha(z_n, z)^{\frac{p_0}{p}})}{\|K_\alpha(z_n, \cdot)^{\frac{p_0}{p}}\|_{A^p(\omega)}}\right|^q \mathrm{d}t\right) \mathrm{d}\mu(z) \\ &= \int_{\Pi^+} \left(\int_0^1 |(C_\varphi - C_\psi) f_t(z)|^q \mathrm{d}t\right) \mathrm{d}\mu(z) \\ &= \int_0^1 \left(\int_{\Pi^+} |(C_\varphi - C_\psi) f_t(z)|^q \mathrm{d}\mu(z)\right) \mathrm{d}t \\ &\lesssim \int_0^1 \|f_t\|^q_{A^p(\omega)} \mathrm{d}t \\ &\lesssim \|\{c_n\}\|^q_{l^p}.\end{aligned}$$

The same estimate holds when we replace φ by ψ,

$$\sum_{n=1}^\infty |c_n|^q \frac{1}{\omega(E_{2\delta}(z_n))^{\frac{q}{p}}} \int_{\psi^{-1}(E_{2\delta}(z_n))} \sigma^q \mathrm{d}\mu \lesssim \|\{c_n\}\|^q_{l^p}.$$

Since

$$\sum_{n=1}^{\infty}|c_n|^q H_{\omega,\mu_q,2\delta}(z_n)=\sum_{n=1}^{\infty}|c_n|^q\frac{\mu_q(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\frac{q}{p}}}$$
$$=\sum_{n=1}^{\infty}|c_n|^q\frac{1}{\omega(E_{2\delta}(z_n))^{\frac{q}{p}}}\bigg(\int_{\varphi^{-1}(E_{2\delta}(z_n))}\sigma^q\mathrm{d}\mu+\int_{\psi^{-1}(E_{2\delta}(z_n))}\sigma^q\mathrm{d}\mu\bigg),$$

we have

$$\sum_{n=1}^{\infty}|c_n|^q H_{\omega,\mu_q,2\delta}(z_n)\lesssim\|\{c_n\}\|_{l^p}^q.$$

It follows from the arbitrary of $\{c_n\}\in l^p$ that

$$\{H_{\omega,\mu_q,2\delta}(z_n)\}\in l^{(\frac{p}{q})'}=l^{\frac{p}{p-q}}.$$

By Theorem 1.1, μ_q is an (ω, p, q)-Carleson measure.

If $0<p\le q<\infty$, by Lemmas 2.1 and 2.9, for $\xi\in\Pi^+$, $0<\delta<1$, $z\in\varphi^{-1}(E_\delta(\xi))$, we have

$$\begin{aligned}&|(C_\varphi-C_\psi)(K_\alpha(\xi,z)^{\frac{p_0}{p}})|^q\\&=|K_\alpha(\xi,\varphi(z))^{\frac{p_0}{p}}-K_\alpha(\xi,\psi(z))^{\frac{p_0}{p}}|^q\\&\gtrsim|K_\alpha(\xi,\varphi(z))^{\frac{p_0}{p}}|^q\sigma(z)^q\\&=\left|\frac{1}{(\varphi(z)-\bar{\xi})^{\frac{p_0}{p}(\alpha+2)}}\right|^q\sigma(z)^q\\&\approx\frac{\sigma(z)^q}{(\mathrm{Im}\xi)^{\frac{p_0}{p}(\alpha+2)q}}.\end{aligned}$$

Thus

$$\begin{aligned}&\int_{\Pi^+}|(C_\varphi-C_\psi)(K_\alpha(\xi,z)^{\frac{p_0}{p}})|^q\mathrm{d}\mu(z)\\&\ge\int_{\varphi^{-1}(E_\delta(\xi))}|(C_\varphi-C_\psi)(K_\alpha(\xi,z)^{\frac{p_0}{p}})|^q\mathrm{d}\mu(z)\\&\gtrsim\int_{\varphi^{-1}(E_\delta(\xi))}\frac{\sigma(z)^q}{(\mathrm{Im}\xi)^{\frac{p_0}{p}(\alpha+2)q}}\mathrm{d}\mu(z)\\&=\frac{1}{(\mathrm{Im}\xi)^{\frac{p_0}{p}(\alpha+2)q}}\int_{\varphi^{-1}(E_\delta(\xi))}\sigma(z)^q\mathrm{d}\mu(z).\end{aligned}$$

The same estimate holds when the roles of φ and ψ are interchanged. So we have

$$\begin{aligned}&\int_{\Pi^+}|(C_\varphi-C_\psi)(K_\alpha(\xi,z)^{\frac{p_0}{p}})|^q\mathrm{d}\mu(z)\\&\gtrsim\frac{1}{(\mathrm{Im}\xi)^{\frac{p_0}{p}(\alpha+2)q}}\left(\int_{\varphi^{-1}(E_\delta(\xi))}\sigma(z)^q\mathrm{d}\mu(z)+\int_{\psi^{-1}(E_\delta(\xi))}\sigma(z)^q\mathrm{d}\mu(z)\right)\\&=\frac{\mu_q(E_\delta(\xi))}{(\mathrm{Im}\xi)^{\frac{p_0}{p}(\alpha+2)q}}.\end{aligned}$$

By Lemma 2.6,

$$\begin{aligned}\frac{\|(C_\varphi-C_\psi)(K_\alpha(\xi,\cdot)^{\frac{p_0}{p}})\|^q_{L^q(\mu)}}{\|K_\alpha(\xi,\cdot)^{\frac{p_0}{p}}\|^q_{A^p(\omega)}}&\gtrsim\frac{\mu_q(E_\delta(\xi))}{(\mathrm{Im}\xi)^{\frac{p_0}{p}(\alpha+2)q}}\frac{(\mathrm{Im}\xi)^{\frac{p_0}{p}(\alpha+2)q}}{\omega(E_\delta(\xi))^{q/p}}\\&=\frac{\mu_q(E_\delta(\xi))}{\omega(E_\delta(\xi))^{\frac{q}{p}}}\\&=H_{\omega,\mu_q,\delta}(\xi).\end{aligned}$$

Since $C_\varphi-C_\psi$ is bounded from $A^p(\omega)$ to $L^q(\mu)$, we obtain

$$H_{\omega,\mu_q,\delta}(\xi)\lesssim\frac{\|(C_\varphi-C_\psi)K_\alpha(\xi,\cdot)^{\frac{p_0}{p}}\|^q_{L^q(\mu)}}{\|K_\alpha(\xi,\cdot)^{\frac{p_0}{p}}\|^q_{A^p(\omega)}}\lesssim 1.$$

By Theorem 1.1, μ_q is an (ω, p, q)-Carleson measure. □

5 Characterization of (ω, λ)-Carleson measure

In this section, we give the proof of Theorem 1.3. Some ideas are derived from [15].

Lemma 5.1 *Let ω be a positive Lebesgue measurable function on Π^+. For any integer $k\geq 1$ and $i=1,2,\dots,k$, let $0<p_i,q_i<\infty$, $f_i\in A^{p_i/q_i}(\omega)$ and $\lambda=\Sigma_{i=1}^k\frac{q_i}{p_i}$. Then $\prod_{i=1}^k f_i\in A^{\frac{1}{\lambda}}(\omega)$ and*

$$\left\|\prod_{i=1}^k f_i\right\|_{A^{\frac{1}{\lambda}}(\omega)}\leq\prod_{i=1}^k\|f_i\|_{A^{p_i/q_i}(\omega)}.$$

The proof of Lemma 5.1 is similar to the proof of [15, Lemma 3.1], we omit it.

Proof of Theorem 1.3 For necessity, assume that μ is an (ω,λ)-Carleson measure. If $k=1$, then it is just the definition of Carleson measure.

Assume that $k \geq 2$. Let $h_i \in A^{p_i/q_i}(\omega)$, $i = 1, 2, \ldots, k$. By Lemma 5.1, $\prod_{i=1}^{k} h_i \in A^{\frac{1}{\lambda}}(\omega)$ and

$$\left\| \prod_{i=1}^{k} h_i \right\|_{A^{\frac{1}{\lambda}}(\omega)} \leq \prod_{i=1}^{k} \|h_i\|_{A^{p_i/q_i}(\omega)}.$$

Since μ is an (ω, λ)-Carleson measure,

$$\int_{\Pi^+} \left| \prod_{i=1}^{k} h_i(z) \right| \mathrm{d}\mu(z) \lesssim \left\| \prod_{i=1}^{k} h_i \right\|_{A^{1/\lambda}(\omega)} \leq \prod_{i=1}^{k} \|h_i\|_{A^{p_i/q_i}(\omega)}. \tag{5.1}$$

Let

$$\mathrm{d}\mu_1 = \left(\prod_{i=2}^{k} |h_i| \mathrm{d}\mu \right) \Big/ \left(\prod_{i=2}^{k} \|h_i\|_{A^{p_i/q_i}(\omega)} \right).$$

Then (5.1) is equivalent to

$$\int_{\Pi^+} |h_1(z)| \mathrm{d}\mu_1(z) \lesssim \|h_1\|_{A^{p_1/q_1}(\omega)}.$$

This means that μ_1 is an $(\omega, \frac{q_1}{p_1})$-Carleson measure. Thus for any $f_1 \in A^{p_1}(\omega)$, we have

$$\int_{\Pi^+} |f_1(z)|^{q_1} \mathrm{d}\mu_1(z) \lesssim \|f_1\|_{A^{p_1}(\omega)}^{q_1},$$

that is

$$\int_{\Pi^+} |f_1(z)|^{q_1} \prod_{i=2}^{k} |h_i(z)| \mathrm{d}\mu(z) \lesssim \|f_1\|_{A^{p_1}(\omega)}^{q_1} \prod_{i=2}^{k} \|h_i\|_{A^{p_i/q_i}(\omega)}. \tag{5.2}$$

Let

$$\mathrm{d}\mu_2 = \left(|f_1|^{q_1} \prod_{i=3}^{k} |h_i| \mathrm{d}\mu \right) \Big/ \left(\|f_1\|_{A^{p_1}(\omega)}^{q_1} \prod_{i=3}^{k} \|h_i\|_{A^{p_i/q_i}(\omega)} \right).$$

Then (5.2) is equivalent to

$$\int_{\Pi^+} |h_2(z)| \mathrm{d}\mu_2(z) \lesssim \|h_2\|_{A^{p_2/q_2}(\omega)}.$$

This means that μ_2 is an $(\omega, \frac{q_2}{p_2})$-Carleson measure. Thus for any $f_2 \in A^{p_2}(\omega)$, we have

$$\int_{\Pi^+} |f_2(z)|^{q_2} \mathrm{d}\mu_2(z) \lesssim \|f_2\|_{A^{p_2}(\omega)}^{q_2},$$

which is the same as

$$\int_{\Pi^+} |f_1(z)|^{q_1} |f_2(z)|^{q_2} \prod_{i=3}^{k} |h_i(z)| \mathrm{d}\mu(z) \lesssim \|f_1\|_{A^{p_1}(\omega)}^{q_1} \|f_2\|_{A^{p_2}(\omega)}^{q_2} \prod_{i=3}^{k} \|h_i\|_{A^{p_i/q_i}(\omega)}.$$

Continuing this process, we will eventually get

$$\int_{\Pi^+} \prod_{i=1}^{k} |f_i(z)|^{q_i} \mathrm{d}\mu(z) \lesssim \prod_{i=1}^{k} \|f_i\|_{A^{p_i}(\omega)}^{q_i}.$$

For sufficiency. Assume first that $\lambda \geq 1$. Let

$$f_{i,\xi}(z) = \frac{(\mathrm{Im}\xi)^{(2+\alpha)p_0/p_i} \omega(E_\delta(\xi))^{-\frac{1}{p_i}}}{(z - \bar{\xi})^{(2+\alpha)p_0/p_i}}, \quad i = 1, 2, \ldots, k.$$

By Lemma 2.6, $f_{i,\xi} \in A^{p_i}(\omega)$ and $\|f_{i,\xi}\|_{A^{p_i}(\omega)} \lesssim 1$, $i = 1, 2, \ldots, k$. Thus,

$$\int_{\Pi^+} \prod_{i=1}^{k} |f_{i,\xi}(z)|^{q_i} \mathrm{d}\mu(z) \lesssim \prod_{i=1}^{k} \|f_{i,\xi}\|_{A^{p_i}(\omega)}^{q_i} \lesssim 1. \tag{5.3}$$

Since

$$\begin{aligned} \prod_{i=1}^{k} |f_{i,\xi}(z)|^{q_i} &= \prod_{i=1}^{k} \frac{(\mathrm{Im}\xi)^{p_0(2+\alpha)q_i/p_i} \omega(E_\delta(\xi))^{-\frac{q_i}{p_i}}}{|z - \bar{\xi}|^{p_0(2+\alpha)q_i/p_i}} \\ &= \frac{(\mathrm{Im}\xi)^{p_0(2+\alpha)\lambda} \omega(E_\delta(\xi))^{-\lambda}}{|z - \bar{\xi}|^{p_0(2+\alpha)\lambda}}, \end{aligned}$$

we have

$$\begin{aligned} \int_{\Pi^+} \prod_{i=1}^{k} |f_{i,\xi}(z)|^{q_i} \mathrm{d}\mu(z) &= \frac{(\mathrm{Im}\xi)^{p_0(2+\alpha)\lambda}}{\omega(E_\delta(\xi))^{\lambda}} \int_{\Pi^+} \frac{1}{|z - \bar{\xi}|^{p_0(2+\alpha)\lambda}} \mathrm{d}\mu(z) \\ &\geq \frac{(\mathrm{Im}\xi)^{p_0(2+\alpha)\lambda}}{\omega(E_\delta(\xi))^{\lambda}} \int_{E_\delta(\xi)} \frac{1}{|z - \bar{\xi}|^{p_0(2+\alpha)\lambda}} \mathrm{d}\mu(z) \\ &\approx \frac{(\mathrm{Im}\xi)^{p_0(2+\alpha)\lambda}}{\omega(E_\delta(\xi))^{\lambda}} \int_{E_\delta(\xi)} \frac{1}{(\mathrm{Im}\xi)^{p_0(2+\alpha)\lambda}} \mathrm{d}\mu(z) \end{aligned}$$

$$= \frac{\mu(E_\delta(\xi))}{\omega(E_\delta(\xi))^\lambda}.$$

It follows from Theorem 1.1 and (5.3) that μ is an (ω, λ)-Carleson measure.

Next we consider the case $0 < \lambda < 1$.

If $k = 1$, then (1.1) is just the definition of the Carleson measure. Let $k \geq 2$ and assume that the result holds for $k - 1$. Let

$$\lambda_{k-1} = \Sigma_{i=1}^{k-1} \frac{q_i}{p_i}, \quad \lambda = \lambda_{k-1} + \frac{q_k}{p_k}.$$

Considering the measure

$$\mathrm{d}\mu_k(z) = \frac{|f_k(z)|^{q_k}}{\|f_k\|_{A^{p_k}(\omega)}^{q_k}} \mathrm{d}\mu(z).$$

Then, the condition (1.1) is equivalent to the condition

$$\int_{\Pi^+} \prod_{i=1}^{k-1} |f_i(z)|^{q_i} \mathrm{d}\mu_k(z) \lesssim \prod_{i=1}^{k-1} \|f_i\|_{A^{p_i}(\omega)}^{q_i}.$$

By induction, μ_k is an (ω, λ_{k-1})-Carleson measure. Since $0 < \lambda_{k-1} < \lambda < 1$, by Theorem 1.1, for any δ-lattice $\{z_n\}$ with $0 < \delta < \frac{1}{3}$,

$$\left\{ \frac{\mu_k(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\lambda_{k-1}}} \right\} \in l^{\frac{1}{1-\lambda_{k-1}}}.$$

Let

$$f_k(z) = \frac{1}{(z - \bar{z_n})^{p_0(\alpha+2)/p_k}}.$$

By Lemma 2.6,

$$\|f_k\|_{A^{p_k}(\omega)} \approx \frac{\omega(E_{2\delta}(z_n))^{1/p_k}}{(\mathrm{Im} z_n)^{p_0(\alpha+2)/p_k}}.$$

Therefore,

$$\begin{aligned}
\mu_k(E_{2\delta}(z_n)) &= \int_{E_{2\delta}(z_n)} \frac{|f_k(z)|^{q_k}}{\|f_k\|_{A^{p_k}(\omega)}^{q_k}} \mathrm{d}\mu(z) \\
&\approx \frac{(\mathrm{Im} z_n)^{p_0(2+\alpha)q_k/p_k}}{\omega(E_{2\delta}(z_n))^{q_k/p_k}} \int_{E_{2\delta}(z_n)} \frac{1}{|z - \bar{z_n}|^{p_0(2+\alpha)q_k/p_k}} \mathrm{d}\mu(z) \\
&\approx \frac{(\mathrm{Im} z_n)^{p_0(2+\alpha)q_k/p_k}}{\omega(E_{2\delta}(z_n))^{q_k/p_k}} \int_{E_{2\delta}(z_n)} \frac{1}{(\mathrm{Im} z_n)^{p_0(2+\alpha)q_k/p_k}} \mathrm{d}\mu(z)
\end{aligned}$$

$$= \frac{\mu(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{q_k/p_k}}.$$

So we have

$$\frac{\mu_k(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\lambda_{k-1}}} \approx \frac{\mu(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\lambda_{k-1}+q_k/p_k}} = \frac{\mu(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\lambda}}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{\mu(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\lambda}}\right)^{\frac{1}{1-\lambda}} &\approx \sum_{n=1}^{\infty}\left(\frac{\mu_k(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\lambda_{k-1}}}\right)^{\frac{1}{1-\lambda_{k-1}}\cdot\frac{1-\lambda_{k-1}}{1-\lambda}} \\ &\lesssim \left(\sum_{n=1}^{\infty}\left(\frac{\mu_k(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\lambda_{k-1}}}\right)^{\frac{1}{1-\lambda_{k-1}}}\right)^{\frac{1-\lambda_{k-1}}{1-\lambda}} \\ &< \infty.\end{aligned}$$

We obtain

$$\left\{\frac{\mu(E_{2\delta}(z_n))}{\omega(E_{2\delta}(z_n))^{\lambda}}\right\} \in l^{\frac{1}{1-\lambda}}$$

It follows from Theorem 1.1 that μ is an (ω, λ)-Carleson measure. □

Acknowledgements The author would like to thank the referees for their helpful suggestions and careful reading, which has improved the presentation of this paper. This work was completed with the support of the National Natural Science Foundation of China (Grant no. 12271134), Shanxi Scholarship Council of China (Grant no. 2020-089), and Fund Program for the Scientific Activities of Selected Returned Overseas Professionals in Shanxi Province (Grant no. 20200019).

Data availability Not applicable.

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