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ORIGINAL PAPER

On Berezin type operators and Toeplitz operators on Bergman spaces

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Abstract

We introduce a class of integral operators called Berezin type operators. It is a generalization of the Berezin transform, and has a close relation to the Bergman–Carleson measures. The concept is partly motivated by the relationship between Hardy–Carleson measures and area operators. We mainly study the boundedness and the compactness of Berezin type operators from a Bergman space $A^{p_1}_{\alpha_1}$ to a Lebesgue space $L^{p_2}_{\alpha_2}$ with $0 < p_1, p_2 \leq \infty$ and $\alpha_1, \alpha_2 > -1$. We also show that Berezin type operators are closely related to Toeplitz operators.

Keywords Berezin type operators · Toeplitz operators · Boundedness and compactness · Bergman–Carleson measures

Mathematics Subject Classification 47G10 · 47B38 · 47B34 · 32A25

1 Introduction

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball in the nth dimensional complex Euclidean space $\mathbb{C}^n$. Let $dv(z)$ be the normalized volume measure on $\mathbb{B}_n$, such that $v(\mathbb{B}_n) = 1$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, let $L^p_\alpha := L^p(\mathbb{B}_n, dv_\alpha)$ denote the weighted

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Lebesgue space which contains measurable functions f on $\mathbb{B}_n$, such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{B}_n} |f(z)|^p \, \mathrm{d}v_\alpha(z) \right)^{1/p} < \infty,$$

where $\mathrm{d}v_\alpha(z) = c_\alpha (1-|z|^2)^\alpha \, \mathrm{d}v(z)$, and c_α is the normalized constant, such that $v_\alpha(\mathbb{B}_n) = 1$. Let $L^\infty := L^\infty(\mathbb{B}_n, \mathrm{d}v)$ be the space of measurable functions on $\mathbb{B}_n$, such that

$$\|f\|_\infty = \operatorname*{ess\,sup}_{z \in \mathbb{B}_n} |f(z)| < \infty.$$

Let $H(\mathbb{B}_n)$ be the space of all holomorphic functions on $\mathbb{B}_n$. We denote by $A^p_\alpha = L^p_\alpha \cap H(\mathbb{B}_n)$ the weighted Bergman space on $\mathbb{B}_n$ for $0 < p < \infty$, and denote by $H^\infty := L^\infty \cap H(\mathbb{B}_n)$ the holomorphic bounded function space.

Let $\beta > -1$ and let μ be a positive Borel measure on $\mathbb{B}_n$. For $f \in H(\mathbb{B}_n)$, we consider the following sublinear operator:

$$B^\beta_\mu f(z) = \int_{\mathbb{B}_n} \frac{|f(w)|}{|1 - \langle z, w \rangle|^{n+1+\beta}} \mathrm{d}\mu(w).$$

We call B^β_μ a *Berezin type operator*. Note that, if $\beta = n + 1 + 2\alpha$ and $\mathrm{d}\mu(w) = \mathrm{d}v_\alpha(w)$, then $(1-|z|^2)^{n+1+\alpha} B^\beta_\mu(f)(z)$ is the α-Berezin transform of the function $|f|$. If $\beta = s + \alpha$ and $f = 1$, we denote

$$B_{s,\alpha}(\mu)(z) := (1-|z|^2)^s B^\beta_\mu(1)(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^s}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} \, \mathrm{d}\mu(w).$$

This is called a Berezin type transform for the measure μ. We refer to [13] for more information about Berezin transforms.

Let $\mathbb{D}$ be the unit disk, let $\partial\mathbb{D}$ be the unit circle, and let μ be a nonnegative measure on $\mathbb{D}$. The area operator on the Hardy space H^p is a sublinear operator defined by

$$A_\mu(f)(\zeta) = \int_{\Gamma(\zeta)} \frac{|f(z)|}{1 - |z|} \, \mathrm{d}\mu(z), \qquad \forall \zeta \in \partial\mathbb{D},$$

where $\Gamma(\zeta)$ is a non-tangential approach region in $\mathbb{D}$ with vertex $\zeta \in \partial\mathbb{D}$ defined by

$$\Gamma(\zeta) = \{z \in \mathbb{D} : |\zeta - z| < 2(1 - |z|)\}.$$

It has been proved in [1] that the A_μ is bounded from the Hardy space H^p to $L^p(\partial\mathbb{D})$ if and only if μ is a Hardy–Carleson measure. The result has been generalized to the case $A_\mu : H^p \to L^q(\partial\mathbb{D})$ for possibly different p, q in [3]. Note that $A_\mu(1)$ is used to characterize Hardy–Carleson measures; see, for example [5, 8, 9]. It is also known that Berezin type transforms for measures are also used to characterize Bergman–Carleson measures; see [6, 11]. This observation is one of our motivations to consider the Berezin type operators on Bergman spaces.

The purpose of this paper is to study the boundedness and the compactness of the Berezin type operator from one Bergman space $A^{p_1}_{\alpha_1}$ to a Lebesgue space $L^{p_2}_{\alpha_2}$. It turns out that our characterizations are the same for Toeplitz operators.

Recall that, given $\beta > -1$ and a positive Borel measure μ on $\mathbb{B}_n$, the Toeplitz operator T^β_μ is defined by

$$T^\beta_\mu(f)(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1-\langle z, w\rangle)^{n+1+\beta}} \mathrm{d}\mu(w), \qquad z \in \mathbb{B}_n. \tag{1.1}$$

It is clear that $|T^\beta_\mu f| \le B^\beta_\mu f$ for $f \in H(\mathbb{B}_n)$. Therefore, boundedness of a Berezin type operator implies boundedness of the corresponding Toeplitz operator.

Our results will heavily depend on Carleson measures. For $\lambda > 0$ and $\alpha > -1$, we say μ is a (λ, α)-*Bergman–Carleson measure* if, for any two positive numbers p and q with $q/p = \lambda$, there is a positive constant $C > 0$, such that

$$\int_{\mathbb{B}_n} |f(z)|^q \,\mathrm{d}\mu(z) \le C\|f\|^q_{p,\alpha}$$

for any $f \in A^p_\alpha$. We also denote by

$$\|\mu\|_{\lambda,\alpha} = \sup_{f\in A^p_\alpha, \|f\|_{p,\alpha}\le 1} \int_{\mathbb{B}_n} |f(z)|^q \,\mathrm{d}\mu(z).$$

We say a positive Borel measure μ is a vanishing (λ, α)-Bergman–Carleson measure if for any two positive numbers p and q satisfying $q/p = \lambda$ and any sequence $\{f_k\}$ in A^p_α with $\|f_k\|_{p,\alpha} \le 1$ and $f_k(z) \to 0$ uniformly on any compact subset of $\mathbb{B}_n$

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |f_k|^q \mathrm{d}\mu(z) = 0.$$

For convenience, we assume that $-1 < \alpha_1, \alpha_2, \beta < \infty$ throughout the paper. We list the following conditions and notations which will be used in our main results:

$$n + 1 + \beta > n \max\left(1, \frac{1}{p_1}\right) + \frac{1+\alpha_1}{p_1}, \tag{C-1}$$

$$n+1+\beta \; > n \max\left(1, \frac{1}{p_2}\right) + \frac{1+\alpha_2}{p_2} \tag{C-2}$$

and

$$\lambda = 1 + \frac{1}{p_1} - \frac{1}{p_2}, \qquad \gamma = \frac{1}{\lambda}\left(\beta + \frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}\right). \tag{1.2}$$

Our first two main results are for the case $0 < p_1 \le p_2 < \infty$.

Theorem 1.1 *Let $0 < p_1 \le p_2 < \infty$, and let $-1 < \alpha_1, \alpha_2, \beta < \infty$ satisfy (C-1) and (C-2). Let λ, γ be given by (1.2). Then, the following statements are equivalent:*

(i) *B_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to $L_{\alpha_2}^{p_2}$.*
(ii) *T_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$.*
(iii) *The measure μ is a (λ, γ)-Bergman–Carleson measure.*

Moreover, we have

$$\|B_\mu^\beta\|_{A_{\alpha_1}^{p_1} \to L_{\alpha_2}^{p_2}} \asymp \|T_\mu^\beta\|_{A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}} \asymp \|\mu\|_{\lambda,\gamma}.$$

Theorem 1.2 *Let $0 < p_1 \le p_2 < \infty$, and let $-1 < \alpha_1, \alpha_2, \beta < \infty$ satisfy (C-1) and (C-2). Let λ, γ be given by (1.2). Then, the following statements are equivalent:*

(i) *B_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $L_{\alpha_2}^{p_2}$.*
(ii) *T_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$.*
(iii) *The measure μ is a vanishing (λ, γ)-Bergman–Carleson measure.*

Our next main result is for the case $0 < p_2 < p_1 < \infty$. For this result, we need a well-known result on decomposition of the unit ball $\mathbb{B}_n$.

For any $a \in \mathbb{B}_n$ with $a \ne 0$, we denote by $\varphi_a(z)$ the Möbius transformation on $\mathbb{B}_n$ that interchanges the points 0 and a. It is known that φ_a satisfies the following properties: $\varphi_a \circ \varphi_a(z) = z$, and

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2}, \qquad z, a \in \mathbb{B}_n. \tag{1.3}$$

For $z, w \in \mathbb{B}_n$, the *pseudo-hyperbolic distance* between z and w is defined by

$$\rho(z, w) = |\varphi_z(w)|,$$

and the *hyperbolic distance* on $\mathbb{B}_n$ between z and w induced by the Bergman metric is given by

$$\beta(z, w) = \tanh\, \rho(z, w) = \frac{1}{2} \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}.$$

Throughout the paper, for $z \in \mathbb{B}_n$ and $r > 0$, let $D(z, r)$ denote the *Bergman metric ball* at z which is given by

$$D(z, r) = \{w \in \mathbb{B}_n : \beta(z, w) < r\}.$$

It is known that, for a fixed $r > 0$, the weighted volume

$$v_\alpha(D(z, r)) \asymp (1 - |z|^2)^{n+1+\alpha}. \tag{1.4}$$

We refer to [12] for the above facts.

A sequence of points $\{a_k\}$ in $\mathbb{B}_n$ is called a *separated sequence* (in the Bergman metric) if there exists $\delta > 0$, such that $\beta(z_i, z_j) > \delta$ for any $i \neq j$.

Lemma 1.3 [12, Theorem 2.23] *There exists a positive integer N, such that for any $0 < r < 1$, we can find a sequence $\{a_j\}$ in $\mathbb{B}_n$ with the following properties:*

(i) $\mathbb{B}_n = \cup_j D(a_j, r)$.
(ii) *The sets $D(a_j, r/4)$ are mutually disjoint.*
(iii) *Each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $D(a_j, 4r)$.*

Any sequence $\{a_j\}$ satisfying the conditions of the above lemma is called a *lattice* (or an r-*lattice* if one wants to stress the dependence on r) in the Bergman metric. Obviously, any r-lattice is separated. For convenience, we will denote by $D_j = D(a_j, r)$ and $\tilde{D}_j = D(a_j, 4r)$ throughout the paper. Then, Lemma 1.3 says that $\mathbb{B}_n = \cup_{k=1}^{\infty} D_j$ and there is an positive integer N, such that every point z in $\mathbb{B}_n$ belongs to at most N of sets $\tilde{D}_j$.

Theorem 1.4 *Let $0 < p_2 < p_1 < \infty$, and let $-1 < \alpha_1, \alpha_2, \beta < \infty$ satisfy (C-1) and (C-2). Let λ, γ be given by (1.2). Given $0 < r < 1$, let $\{a_j\}$ be an r-lattice in $\mathbb{B}_n$, and let D_j and $\tilde{D}_j$ be the associated Bergman metric balls given by Lemma 1.3. Then, the following statements are equivalent:*

(i) *B_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $L_{\alpha_2}^{p_2}$.*
(ii) *B_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to $L_{\alpha_2}^{p_2}$.*
(iii) *T_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$.*
(iv) *T_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$.*
(v)

$$\{\mu_j\} := \left\{ \frac{\mu(D_j)}{(1 - |a_j|^2)^{(n+1+\gamma)\lambda}} \right\} \in l^{1/(1-\lambda)}.$$

Moreover, we have

$$\|B_\mu^\beta\|_{A_{\alpha_1}^{p_1} \to L_{\alpha_2}^{p_2}} \asymp \|T_\mu^\beta\|_{A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}} \asymp \|\{\mu_j\}\|_{l^{1/(1-\lambda)}}.$$

Remark 1.5 In the above theorems, the most parts of the results on Toeplitz operators have been proved by Pau and the second author in [6]. The only exception is condition

(v) in Theorem 1.4. In [6], it used the condition that "μ is a (λ, γ)-Bergman–Carleson measure" instead of (v) above. However, for the case $0 < p_2 < p_1 < \infty$, it may happen that $\lambda \leq 0$. In this case, the (λ, γ)-Bergman–Carleson measure condition does not make sense, while (v) is still valid.

Our work here is mostly built up from the work in [6]. Our main contributions are proofs of (iii)⇒(i) in Theorems 1.1 and 1.2 and (v)⇒(i) in Theorem 1.4. We would like to point out that our proofs are different from the proof of Theorem 1.2 in [6]. The key ingredients of our proofs are two technical results, Lemma 4.1 and Lemma 4.2, which allow us to treat all cases together. By comparison, in the proof of Theorem 1.2 in [6], for the case $1 < p_2 < \infty$, it used a new characterization of Bergman–Carleson measures discovered in that paper. For proving compactness results for B_μ^β, we have to be more careful, since we are dealing with a sublinear operator. We also gave a detailed proof of a characterization of compactness of $T_\mu^\beta : A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}$ for $0 < p_1 \leq p_2 < \infty$ (Proposition 3.2), which seems to be a folklore, but we could not find a proof. The proof of Proposition 3.2 for the case $0 < p_1 \leq 1$ is actually surprisingly involved. Besides, we have also discussed the cases when $p_1 = \infty$ or/and $p_2 = \infty$.

The paper is organized as follows. In Sect. 2, we recall some notations and preliminary results which will be used later. In Sect. 3, we develop some tools for characterizing compactness of Berezin type operators and Toeplitz operators. We give the proofs of Theorems 1.1, 1.2 and 1.4 in Sect. 4. In Sect. 5 and Sect. 6, we study the boundedness and the compactness of $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L_{\alpha_2}^{p_2}$ and $T_\mu^\beta : A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}$ for the remaining cases when $p_1 = \infty$ or/and $p_2 = \infty$.

Throughout the paper, the notation $A \lesssim B$ means that there is a positive constant C, such that $A \leq CB$, and the notation $A \asymp B$ means that both $A \lesssim B$ and $B \lesssim A$ are satisfied.

2 Preliminaries

2.1 Carleson measures

The following result was obtained by several authors and can be found, for example, in [11, Theorem 50], [11, p.71] and the references therein.

Theorem A. *Suppose $1 \leq \lambda < \infty$ and $-1 < \alpha < \infty$, the following statements are equivalent:*

(i) *μ is a (λ, α)-Bergman–Carleson measure.*
(ii) *For any real number r with $0 < r < 1$ and any $z \in \mathbb{B}_n$*

$$\mu(D(z, r)) \lesssim (1 - |z|^2)^{(n+1+\alpha)\lambda}.$$

(iii) *For some (every) $s > 0$, the Berezin type transform of μ*

$$B_{s,(n+1+\alpha)\lambda-n-1}(\mu) \in L^\infty(\mathbb{B}_n),$$

that is, there is a constant $C > 0$

$$\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(1-|a|^2)^s}{|1-\langle z,a\rangle|^{(n+1+\alpha)\lambda+s}}\,d\mu(z)\le C.$$

Especially, if $\lambda = 1$*, we get that a positive Borel measure* μ *on* $\mathbb{B}_n$ *is a* $(1,\alpha)$*-Bergman–Carleson measure if and only if* $B_{s,\alpha}(\mu)\in L^\infty(\mathbb{B}_n)$ *for some (every)* $s>0$.

Theorem B. *Suppose* $1\le\lambda<\infty$ *and* $-1<\alpha<\infty$*, the following statements are equivalent:*

(i) μ *is a vanishing* (λ,α)*-Bergman–Carleson measure.*
(ii) *For some(any)* $s>0$

$$\lim_{|a|\to 1}\int_{\mathbb{B}_n}\frac{(1-|a|^2)^s}{|1-\langle z,a\rangle|^{(n+1+\alpha)\lambda+s}}d\mu(z)=0.$$

(iii) *For any real number r with* $0<r<1$ *and any* $a\in\mathbb{B}_n$

$$\lim_{|a|\to 1}\frac{\mu(D(a,r))}{(1-|a|^2)^{(n+1+\alpha)\lambda}}=0.$$

Lemma 2.1 *Let* $1\le\lambda<\infty$ *and* $-1<\gamma<\infty$*. Let* μ *be a* (λ,γ)*-Bergman–Carleson measure on* $\mathbb{B}_n$*. Then, for any* $f\in H(\mathbb{B}_n)$ *and any* $0<p<\infty$*, we have*

$$\int_{\mathbb{B}_n}|f(z)|^p\,d\mu(z)\lesssim\int_{\mathbb{B}_n}|f(z)|^p(1-|z|^2)^{(n+1+\gamma)\lambda-(n+1)}\,dv(z).$$

Proof By [12, Lemma 2.24], we know that for $0<r<1$, we have

$$|f(z)|^p\le\frac{1}{(1-|z|^2)^{n+1}}\int_{D(z,r)}|f(w)|^p\,dv(w).$$

Hence, by Fubini's theorem, the fact that $(1-|z|^2)\approx(1-|w|^2)$ for $z\in D(w,r)$, and Theorem A, we have that

$$\begin{aligned}\int_{\mathbb{B}_n}|f(z)|^p\,\mathrm{d}\mu(z)&\le\int_{\mathbb{B}_n}\frac{1}{(1-|z|^2)^{n+1}}\int_{D(z,r)}|f(w)|^p\,\mathrm{d}v(w)\,\mathrm{d}\mu(z)\\&=\int_{\mathbb{B}_n}|f(w)|^p\int_{D(w,r)}\frac{\mathrm{d}\mu(z)}{(1-|z|^2)^{n+1}}\,\mathrm{d}v(w)\end{aligned}$$

$$\approx \int\limits_{\mathbb{B}_n} |f(w)|^p \int\limits_{D(w,r)} \frac{\mathrm{d}\mu(z)}{(1-|w|^2)^{n+1}} \,\mathrm{d}v(w)$$
$$= \int\limits_{\mathbb{B}_n} |f(w)|^p \frac{\mu(D(w,r))}{(1-|w|^2)^{n+1}} \,\mathrm{d}v(w)$$
$$\lesssim \int\limits_{\mathbb{B}_n} |f(w)|^p (1-|w|^2)^{(n+1+\gamma)\lambda-(n+1)} \,\mathrm{d}v(w).$$

The proof is complete. □

2.2 Some useful estimates

The following estimate is well known, and can be found, for example, in [7, Proposition 1.4.10], [12, Theorem 1.12] and [4, Sect. 1.2].

Lemma 2.2 *Suppose $z \in \mathbb{B}_n$, $t > -1$, and c is real. The integral*

$$I_{c,t}(z) = \int\limits_{\mathbb{B}_n} \frac{(1-|w|^2)^t}{|1-\langle z, w\rangle|^c} dv(w)$$

has the following asymptotic behavior as $|z| \to 1$.

(i) *If $c < n+1+t$, then $I_{c,t}(z) \asymp 1$.*
(ii) *If $c = n+1+t$, then $I_{c,t}(z) \asymp \log \frac{1}{1-|z|^2}$.*
(iii) *If $c > n+1+t$, then $I_{c,t}(z) \asymp (1-|z|^2)^{n+1+t-c}$.*

Lemma 2.3 [6, Lemma C] *Let $\{z_k\}$ be a separated sequence in $\mathbb{B}_n$, and $n < t < s$. Then*

$$\sum_{k=1}^{\infty} \frac{(1-|z_k|^2)^t}{|1-\langle z, z_k\rangle|^s} \le C(1-|z|^2)^{t-s}, \qquad z \in \mathbb{B}_n.$$

Lemma 2.4 *Suppose $0 < p < \infty$, $\alpha > -1$. Let $\{c_j\}$ be a positive sequence, and let $\{a_j\}$ be a separated sequence in $\mathbb{B}_n$. If $s \in \mathbb{R}$, such that*

$$s > n \max\left(1, \frac{1}{p}\right) + \frac{1+\alpha}{p},$$

and f is a measurable function on $\mathbb{B}_n$, such that

$$|f(z)| \le \sum_{j=1}^{\infty} \frac{c_j}{|1-\langle z, a_j\rangle|^s},$$

then $f \in L^p_\alpha$ *and*

$$\|f\|_{p,\alpha}^p \lesssim \sum_{j=1}^{\infty} \frac{c_j^p}{(1-|a_j|^2)^{sp-(n+1+\alpha)}}.$$

Proof If $0 < p \le 1$, then we have that

$$|f(z)|^p \le \sum_{j=1}^{\infty} \frac{c_j^p}{|1-\langle z, a_j\rangle|^{sp}}.$$

By Lemma 2.2, we have that

$$\begin{aligned}\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}v_\alpha(z) &\le \sum_{j=1}^{\infty} c_j^p \int_{\mathbb{B}_n} \frac{1}{|1-\langle z, a_j\rangle|^{sp}} \mathrm{d}v_\alpha(z) \\ &\lesssim \sum_{j=1}^{\infty} \frac{c_j^p}{(1-|a_j|^2)^{sp-(n+1+\alpha)}}.\end{aligned}$$

If $p > 1$, then $s > n + \frac{1+\alpha}{p}$. Let p' be the conjugate exponent of p, such that $1/p + 1/p' = 1$. By Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}|f(z)|^p &= \left(\sum_{j=1}^{\infty} \frac{c_j}{|1-\langle z, a_j\rangle|^s}\right)^p \\ &\le \left(\sum_{j=1}^{\infty} \frac{(1-|a_j|^2)^{s-(1+\alpha)/p}}{|1-\langle z, a_j\rangle|^s}\right)^{p-1} \left(\sum_{j=1}^{\infty} \frac{c_j^p(1-|a_j|^2)^{s(1-p)+(1+\alpha)/p'}}{|1-\langle z, a_j\rangle|^s}\right) \\ &\lesssim (1-|z|^2)^{-(1+\alpha)/p'} \left(\sum_{j=1}^{\infty} \frac{c_j^p(1-|a_j|^2)^{s(1-p)+(1+\alpha)/p'}}{|1-\langle z, a_j\rangle|^s}\right).\end{aligned}$$

Hence

$$\|f\|_{p,\alpha}^p \lesssim \sum_{j=1}^{\infty} c_j^p (1-|a_j|^2)^{s(1-p)+(1+\alpha)/p'} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{-(1+\alpha)/p'}}{|1-\langle z, a_j\rangle|^s} \mathrm{d}v_\alpha(z).$$

Since $\alpha - (1+\alpha)/p' = (1+\alpha)/p - 1 > -1$, and

$$s - (n+1+\alpha) + (1+\alpha)/p' = s - n - (1+\alpha)/p > 0,$$

the typical integral estimate in Lemma 2.2 gives the result. □

3 Compactness of B_μ^β and T_μ^β

Recall that, for a bounded linear operator T between two Banach spaces X and Y, we say that T is *compact* if T maps any bounded set in X to a relative compact set in Y. We also recall that a bounded linear operator $T : X \to Y$ is called *completely continuous* if, for every weakly convergent sequence (x_n) from X, the sequence (Tx_n) is norm-convergent in Y.

Let $-1 < \beta < \infty$. Since B_μ^β is a sublinear operator, there may be different ways to define its compactness. In this paper, following the above definition, we say that $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L_{\alpha_2}^{p_2}$ is *compact* if it maps any bounded set in $A_{\alpha_1}^{p_1}$ to a relative compact set in $L_{\alpha_2}^{p_2}$, where $0 < p_1, p_2 < \infty$, and $-1 < \alpha_1, \alpha_2 < \infty$. It is clear that $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L_{\alpha_2}^{p_2}$ is compact if and only if for any bounded sequence $\{f_n\}$ in $A_{\alpha_1}^{p_1}$, the image sequence $\{B_\mu^\beta f_n\}$ has a convergent subsequence in $L_{\alpha_2}^{p_2}$.

We first give the following sufficient condition for the compactness of $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L_{\alpha_2}^{p_2}$ for $0 < p_1, p_2 < \infty$.

Proposition 3.1 *Let $0 < p_1, p_2 < \infty$, and $-1 < \alpha_1, \alpha_2, \beta < \infty$. Assume that $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L_{\alpha_2}^{p_2}$ is a bounded sublinear operator. Suppose that, for every bounded sequence $\{f_k\}$ in $A_{\alpha_1}^{p_1}$, such that $f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$, we have*

$$\lim_{k\to\infty} \|B_\mu^\beta f_k\|_{p_2,\alpha_2} = 0.$$

Then, B_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $L_{\alpha_2}^{p_2}$.

Proof Let $\{f_k\}$ be a bounded sequence in $A_{\alpha_1}^{p_1}$. Then, there is a constant $M > 0$, such that $\|f_k\|_{p_1,\alpha_1} \le M$ for all $k \ge 1$. By [12, Theorem 2.1], $\{f_k\}$ is uniformly bounded on every compact subsets of $\mathbb{B}_n$. By Montel's Theorem, there is a subsequence of $\{f_k\}$, denoted by $\{f_{k_j}\}$, $j = 1, 2, 3...$, such that $f_{k_j} \to f$ uniformly on every compact subsets of $\mathbb{B}_n$ for some holomorphic function f on $\mathbb{B}_n$, as $j \to \infty$. By Fatou's Lemma

$$\begin{aligned}\int_{\mathbb{B}_n} |f(z)|^{p_1}\, dv_{\alpha_1}(z) &= \int_{\mathbb{B}_n} \lim_{j\to\infty} |f_{k_j}(z)|^{p_1}\, dv_{\alpha_1}(z)\\ &\le \lim_{j\to\infty} \int_{\mathbb{B}_n} |f_{k_j}(z)|^{p_1}\, dv_{\alpha_1}(z)\\ &\le \lim_{j\to\infty} \|f_{k_j}\|_{p_1,\alpha_1}^{p_1} \le M.\end{aligned}$$

Thus, $f \in A_{\alpha_1}^{p_1}$. Therefore, we get that $f_{k_j} - f \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $j \to \infty$. By our assumption, we get that

$$\lim_{j\to\infty} \|B_\mu^\beta (f_{k_j} - f)\|_{p_2,\alpha_2} = 0.$$

We can easily check that

$$\|B_\mu^\beta f_{k_j} - B_\mu^\beta f\|_{p_2,\alpha_2} \le \|B_\mu^\beta (f_{k_j} - f)\|_{p_2,\alpha_2}.$$

From this inequality, we obtain that

$$\lim_{j\to\infty} \|B_\mu^\beta f_{k_j} - B_\mu^\beta f\|_{p_2,\alpha_2} = 0,$$

which implies that $B_\mu^\beta f \in L_{\alpha_2}^{p_2}$. Thus, $\{B_\mu^\beta f_k\}$ has a convergent subsequence in $L_{\alpha_2}^{p_2}$, and so B_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $L_{\alpha_2}^{p_2}$. □

The following characterization for compactness of $T_\mu^\beta : A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}$ for $0 < p_1 \le p_2 < \infty$ may be well known, but we cannot find a reference, so we give a proof here. The result contains the case when $0 < p_1 < 1$ or $0 < p_2 < 1$, in which we still define the compactness of T_μ^β in the same way as before, that is, we say that $T_\mu^\beta : A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}$ is compact if it maps a bounded set in $A_{\alpha_1}^{p_1}$ to a relatively compact set in $A_{\alpha_2}^{p_2}$. For the case when $0 < p_1 \le 1$, the proof below is surprisingly involved.

Proposition 3.2 *Let $0 < p_1 \le p_2 < \infty$, and let $-1 < \alpha_1, \alpha_2, \beta < \infty$. Suppose that p_2, α_2 and β satisfy (C-2), and suppose that T_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$. Then, the following statements are equivalent:*

(i) *T_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$.*
(ii) *For every bounded sequence $\{f_k\}$ in $A_{\alpha_1}^{p_1}$, such that $f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$, we have*

$$\lim_{k\to\infty} \|T_\mu^\beta f_k\|_{p_2,\alpha_2} = 0.$$

We need several lemmas to prove Proposition 3.2.

Lemma 3.3 *Suppose that $1 < p < \infty$. Then, $f_k \to 0$ weakly in A_α^p if and only if $\{f_k\}$ is bounded in A_α^p and $f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$.*

This result is well known and can be easily proved, so we omit the proof here. For the case of the unit disk, see Problem 1 of Exercise 4.7 in [13].

Lemma 3.4 *Let $0 < p < \infty$, and let $-1 < \alpha < \infty$. Let $1 \le \lambda < \infty$ and $-1 < \gamma < \infty$ satisfy that*

$$(n + 1 + \gamma)\lambda > n \max\left(1, \frac{1}{p}\right) + \frac{1+\alpha}{p}. \tag{3.1}$$

Let μ be a (λ, γ)-Bergman–Carleson measure on $\mathbb{B}_n$. If $\{f_k\}$ is a bounded sequence in A_α^p, such that $f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$, then we also have that $B_\mu^\beta f_k \to 0$ and $T_\mu^\beta f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$.

Proof It suffices for us to prove $B_\mu^\beta f_k \to 0$ on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$, since $|T_\mu f_k| \le B_\mu f_k$. For convenience, denote $\eta = (n+1+\gamma)\lambda - (n+1)$. Since $\lambda \ge 1$ and $\gamma > -1$, we have $\eta > -1$. Let $\{f_k\}$ be a bounded sequence in A_α^p. Then, there is a constant $M > 0$, such that $\|f_k\|_{p,\alpha} \le M$ for all $k \ge 1$.

First, consider the case $0 < p \le 1$. In this case, (3.1) becomes

$$\frac{n+1+\alpha}{p} < n+1+\eta, \tag{3.2}$$

Then, there exists A, such that

$$\frac{n+1+\alpha}{p} < A < n+1+\eta. \tag{3.3}$$

Since $0 < p \le 1$, we get

$$\frac{1+\alpha}{p} \le \frac{n+1+\alpha}{p} - n < A - n < 1+\eta.$$

Therefore, there exists a constant B, such that $A - n < B < \min\{1+\eta, A\}$, which implies that

$$\frac{1+\alpha}{p} < B < 1+\eta. \tag{3.4}$$

Let

$$q = \frac{n}{A-B}, \qquad s = \frac{nB}{A-B} - 1.$$

Then, since $0 < A - B < n$, we see that $q > 1$ and $s > -1$. Also, it is easy to check that

$$\frac{n+1+s}{q} = A, \qquad \frac{1+s}{q} = B.$$

Hence, by (3.3) and (3.4), we obtain that

$$\frac{n+1+\alpha}{p} < \frac{n+1+s}{q} < n+1+\eta \tag{3.5}$$

and

$$\frac{1+\alpha}{p} < \frac{1+s}{q} < 1+\eta. \tag{3.6}$$

By [11, Theorem 69], (3.5) implies that $A^p_\alpha \subseteq A^q_s$, and so $\|f_k\|_{q,s} \lesssim \|f_k\|_{p,\alpha} \leq M$. Using (3.6), we get that

$$\left(\eta - \frac{s}{q}\right) q' > -1,$$

where q' is the conjugate index of q, that is, it satisfies $1/q + 1/q' = 1$. Hence,

$$\int_{\mathbb{B}_n} (1 - |w|^2)^{(\eta - s/q)q'} \, \mathrm{d}v(w) < \infty.$$

Therefore, for any $\varepsilon > 0$, there exists a constant $r \in (0, 1)$, such that

$$\int_{\mathbb{B}_n \setminus \overline{D}_r} (1 - |w|^2)^{(\eta - s/q)q'} \, \mathrm{d}v(w) < \varepsilon^{q'}, \tag{3.7}$$

where $D_r = \{w \in \mathbb{B}_n : |w| < r\}$. By Lemma 2.1, we get that

$$\begin{aligned} |B^\beta_\mu f_k(z)| &\lesssim \int_{\mathbb{B}_n} \frac{|f_k(w)|(1 - |w|^2)^{(n+1+\gamma)\lambda - (n+1)}}{|1 - \langle z, w\rangle|^{n+1+\beta}} \, \mathrm{d}v(w) \\ &= \left(\int_{\mathbb{B}_n \setminus \overline{D}_r} + \int_{\overline{D}_r} \right) \frac{|f_k(w)|(1 - |w|^2)^{\eta}}{|1 - \langle z, w\rangle|^{n+1+\beta}} \, \mathrm{d}v(w) \\ &= I_1(k, r) + I_2(k, r). \end{aligned} \tag{3.8}$$

Using Hölder's inequality and (3.7), we get that

$$\begin{aligned} I_1(k, r) &\leq \left(\int_{\mathbb{B}_n \setminus \overline{D}_r} |f_k(w)|^q (1 - |w|^2)^s \, \mathrm{d}v(w) \right)^{1/q} \\ &\quad \times \left(\int_{\mathbb{B}_n \setminus \overline{D}_r} \frac{(1 - |w|^2)^{(\eta - s/q)q'}}{|1 - \langle z, w\rangle|^{(n+1+\beta)q'}} \, \mathrm{d}v(w) \right)^{1/q'} \\ &\lesssim \frac{\|f_k\|_{q,s}}{(1 - |z|)^{(n+1+\beta)q'}} \left(\int_{\mathbb{B}_n \setminus \overline{D}_r} (1 - |w|^2)^{(\eta - s/q)q'} \, \mathrm{d}v(w) \right)^{1/q'} \\ &\leq \frac{M\varepsilon}{(1 - |z|)^{(n+1+\beta)q'}}. \end{aligned}$$

Take any compact subset K in $\mathbb{B}_n$. Then, for any $z \in K$, there is a constant $M' > 0$, such that $1/(1-|z|)^{(n+1+\beta)q'} \leq M'$. Thus

$$I_1(k,r) \lesssim MM'\varepsilon. \tag{3.9}$$

Since $f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$, there exists an integer $N > 0$, such that for any $k \geq N$ and any $w \in \overline{D_r}$

$$|f_k(w)| < \varepsilon.$$

Remembering that $\eta > -1$, we have for any $z \in K$

$$I_2(k,r) \leq \frac{\varepsilon}{(1-|z|)^{n+1+\beta}} \int_{\overline{D_r}} (1-|w|^2)^{\eta}\, dv(w) \lesssim M'\varepsilon. \tag{3.10}$$

By (3.9) and (3.10), we get that for any $k \geq N$ and any $z \in K$

$$|B_{\mu}^{\beta} f_k(z)| \leq I_1(k,r) + I_2(k,r) \lesssim \varepsilon.$$

Thus, $B_{\mu}^{\beta} f_k(z) \to 0$ uniformly on any compact subsets of $\mathbb{B}_n$ as $k \to \infty$ for $0 < p \leq 1$.

Next, consider the case $p > 1$. In this case, (3.1) becomes

$$(n+1+\gamma)\lambda > n + \frac{1+\alpha}{p},$$

Recall that $\eta = (n+1+\gamma)\lambda - (n+1)$. Thus

$$\eta + 1 = (n+1+\gamma)\lambda - n > \frac{1+\alpha}{p},$$

which implies that

$$\left(\eta - \frac{\alpha}{p}\right) p' > -1,$$

where $1/p + 1/p' = 1$. Hence

$$\int_{\mathbb{B}_n} (1-|w|^2)^{(\eta-\alpha/p)p'}\, dv(w) < \infty.$$

The rest of the proof is the same as the proof of the case $0 < p \leq 1$, except that we use p to replace q, and use α to replace s. We omit the details. □

Lemma 3.5 *Let* $0 < p_1 \le p_2 < \infty$, *let* $-1 < \alpha_1, \alpha_2, \beta < \infty$, *and let* γ, λ *be given by (1.2). Suppose that* p_2, α_2 *and* β *satisfy (C-2). Then, we have*

$$(n+1+\gamma)\lambda > n \max\left(1, \frac{1}{p_1}\right) + \frac{1+\alpha_1}{p_1}. \tag{3.11}$$

Proof Since $0 < p_1 \le p_2 < \infty$, it follows that $\lambda = 1 + 1/p_1 - 1/p_2 \ge 1$. We get the following two inequalities:

$$n+1+\beta > \frac{n+1+\alpha_2}{p_2} \tag{3.12}$$

and

$$1+\beta > \frac{1+\alpha_2}{p_2} \tag{3.13}$$

by (C-2). Bearing in mind the definitions of λ, γ in (1.2), then (3.13) gives that

$$(1+\gamma)\lambda = (1+\beta) + \frac{1+\alpha_1}{p_1} - \frac{1+\alpha_2}{p_2} > \frac{1+\alpha_1}{p_1} > 0,$$

Thus, $\gamma > -1$, and

$$\begin{aligned}(n+1+\gamma)\lambda &> n + \frac{n+1+\alpha_1}{p_1} - \frac{n}{p_2}\\ &\ge n + \frac{1+\alpha_1}{p_1},\end{aligned}$$

since $0 < p_1 \le p_2 < \infty$. Furthermore, the inequality (3.12) implies that

$$(n+1+\gamma)\lambda = (n+1+\beta) + \frac{n+1+\alpha_1}{p_1} - \frac{n+1+\alpha_2}{p_2} > \frac{n+1+\alpha_1}{p_1}.$$

The proof is complete. □

Proof of Proposition 3.2 It follows from [2, Proposition 3.3 in Chapter VI] that $T_\mu : A^{p_1}_{\alpha_1} \to A^{p_2}_{\alpha_2}$ is compact if and only if $T_\mu : A^{p_1}_{\alpha_1} \to A^{p_2}_{\alpha_2}$ is completely continuous for $1 < p_1 < \infty$. By Lemma 3.3, we know that (ii) is equivalent to that $T_\mu : A^{p_1}_{\alpha_1} \to A^{p_2}_{\alpha_2}$ is completely continuous for $1 < p_1 < \infty$. Therefore, it suffices for us to prove for the case $0 < p_1 \le 1$.

Let $0 < p_1 \le 1$, and let $0 < p_1 \le p_2 < \infty$. The proof of (ii)⇒(i) follows from the same discussion as in the proof of Proposition 3.1. Thus, we only need to prove that (i)⇒(ii). Suppose that (i) holds, i.e., that $T^\beta_\mu : A^{p_1}_{\alpha_1} \to A^{p_2}_{\alpha_2}$ is compact. Then, $T^\beta_\mu : A^{p_1}_{\alpha_1} \to A^{p_2}_{\alpha_2}$ is bounded. It follows from Theorem 1 that μ is a (λ, γ)-Bergman–Carleson measure on $\mathbb{B}_n$, where λ, γ are given by (1.2). Let $\{f_k\}$ be a bounded sequence in $A^{p_1}_{\alpha_1}$, such that $f_k \to 0$ uniformly on every compact subsets of

$\mathbb{B}_n$ as $k \to \infty$. Suppose, on the contrary, that (ii) is not true. Then, there exist an $\varepsilon > 0$ and a subsequence $\{f_{k_j}\}$ of $\{f_k\}$, such that

$$\|T_\mu^\beta f_{k_j}\|_{p_2,\alpha_2} \geq \varepsilon, \quad \text{for all } j = 1, 2, 3.... \tag{3.14}$$

Since T_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$, we can find a further subsequence $\{f_{k_{j_m}}\}$, $m = 1, 2, 3...$, and $g \in A_{\alpha_2}^{p_2}$, such that

$$\lim_{m\to\infty} \|T_\mu^\beta f_{k_{j_m}} - g\|_{p_2,\alpha_2} = 0. \tag{3.15}$$

By [12, Theorem 2.1], we have that

$$|T_\mu^\beta f_{k_{j_m}}(z) - g(z)| \leq \frac{\|T_\mu^\beta f_{k_{j_m}} - g\|_{p_2,\alpha_2}}{(1-|z|^2)^{(n+1+\alpha_2)/p_2}} \tag{3.16}$$

for all $m \geq 1$. Hence

$$|T_\mu^\beta f_{k_{j_m}}(z) - g(z)| \to 0 \tag{3.17}$$

uniformly on every compact subsets of $\mathbb{B}_n$, as $m \to \infty$.

By the definitions of λ, γ given in (1.2), and by lemma 3.5, we have that

$$(n + 1 + \gamma)\lambda > \frac{n + 1 + \alpha_1}{p_1}$$

for $0 < p_1 \leq 1$. Since $\{f_k\}$ is a bounded sequence in $A_{\alpha_1}^{p_1}$ and $f_{k_{j_m}}(z) \to 0$ uniformly on every compact subset of $\mathbb{B}_n$ as $m \to \infty$, it follows from Lemma 3.4 that $T_\mu^\beta f_{k_{j_m}}(z) \to 0$ uniformly on compact subsets of $\mathbb{B}_n$. Thus, we must have $g = 0$ by (3.17). Therefore, by (3.15), we get that

$$\lim_{m\to\infty} \|T_\mu^\beta f_{k_{j_m}}\|_{p_2,\alpha_2} = 0,$$

which contradicts to (3.14). Hence, (ii) must be true. The proof is complete. □

4 Proofs of the main theorems

Lemma 4.1 *Let $0 < p_1, p_2 < \infty$, let $-1 < \alpha_1, \alpha_2, \beta < \infty$, and let λ and γ be given by (1.2). Suppose that p_2, α_2 and β satisfy (C-2). Given $0 < r < 1$, let $\{a_j\}$ be an r-lattice in $\mathbb{B}_n$, and let D_j and $\tilde{D}_j$ be the associated Bergman metric balls given by Lemma 1.3. Then, we have*

$$\|B_\mu^\beta(f)\|_{p_2,\alpha_2}^{p_2} \lesssim \sum_{j=1}^{\infty} \left(\frac{\mu(D_j)}{(1-|a_j|^2)^{(n+1+\gamma)\lambda}} \right)^{p_2} \left(\int_{\tilde{D}_j} |f(\zeta)|^{p_1} dv_{\alpha_1}(\zeta) \right)^{p_2/p_1} . \tag{4.1}$$

Proof Using the fact that $|1-\langle z,w\rangle| \asymp |1-\langle z,a_j\rangle|$ for $w \in D_j$, we have that

$$\begin{aligned} |B_\mu^\beta(f)(z)| &\lesssim \sum_{j=1}^{\infty} \int_{D_j} \frac{|f(w)|}{|1-\langle z,w\rangle|^{n+1+\beta}} \mathrm{d}\mu(w) \\ &\lesssim \sum_{j=1}^{\infty} \left(\sup_{w\in D_j} |f(w)| \right) \int_{D_j} \frac{1}{|1-\langle z,w\rangle|^{n+1+\beta}} \,\mathrm{d}\mu(w) \\ &\lesssim \sum_{j=1}^{\infty} \left(\sup_{w\in D_j} |f(w)| \right) \frac{\mu(D_j)}{|1-\langle z,a_j\rangle|^{n+1+\beta}}. \end{aligned}$$

By [12, Lemma 2.24], we have that

$$|f(w)| \lesssim \left(\frac{1}{(1-|a_j|^2)^{n+1+\alpha_1}} \int_{\tilde{D}_j} |f(\zeta)|^{p_1} \mathrm{d}v_{\alpha_1}(\zeta) \right)^{1/p_1}$$

for any $w \in D_j$. Denote

$$|\widehat{f}(a_j)| := \left(\frac{1}{(1-|a_j|^2)^{n+1+\alpha_1}} \int_{\tilde{D}_j} |f(\zeta)|^{p_1} \mathrm{d}v_{\alpha_1}(\zeta) \right)^{1/p_1} ,$$

we get that

$$|B_\mu^\beta(f)(z)| \lesssim \sum_{j=1}^{\infty} \frac{|\widehat{f}(a_j)|\mu(D_j)}{|1-\langle z,a_j\rangle|^{n+1+\beta}}. \tag{4.2}$$

Thus, we obtain (4.1) by taking $p = p_2, \alpha = \alpha_2, s = n+1+\beta$ and $c_j = |\widehat{f}(a_j)|\mu(D_j)$ in Lemma 2.4. □

Lemma 4.2 *Given $0 < r < 1$, let $\{a_j\}$ be an r-lattice in $\mathbb{B}_n$, let $D_j = D(a_j, r)$ be the associated Bergman metric balls given by Lemma 1.3, and let $\{b_k(a_j)\}$ be a sequence depending on a_j, such that*

$$\sup_k \sum_{j=1}^{\infty} |b_k(a_j)| < \infty. \tag{4.3}$$

Suppose, in addition, for any compact subset K of $\mathbb{B}_n$, the sequence $\{b_k(a_j)\}$ satisfies that

$$\lim_{k\to\infty} \sum_{j\in\Gamma} |b_k(a_j)| = 0, \tag{4.4}$$

where $\Gamma := \{j : a_j \in K\}$. Let $\{c(a_j)\}$ be a sequence of real numbers depending on $\{a_j\}$. Then, we have the following two results.

(i) *If $\{c(a_j)\}$ is a bounded sequence, such that $\lim_{|a_j|\to 1} c(a_j) = 0$, then*

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} c(a_j) b_k(a_j) = 0.$$

(ii) *Let $0 < t < \infty$, $0 < s < 1$ and $\gamma = t/(1-s)$. If $\{c(a_j)\} \in l^\gamma$, then*

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} c(a_j)^t b_k(a_j)^s = 0.$$

Proof (i) Since $\lim_{|a_j|\to 1} c(a_j) = 0$, it follows that for any $\varepsilon > 0$, there is an $r_1 \in (0, 1)$, such that $|c(a_j)| < \varepsilon$ for all $|a_j| > r_1$. Therefore

$$\begin{aligned} &\left|\sum_{j=1}^{\infty} c(a_j) b_k(a_j)\right| \\ &\quad\le \sum_{j:\,|a_j|\le r_1} |c(a_j) b_k(a_j)| + \sum_{j:\,|a_j|> r_1} |c(a_j) b_k(a_j)| \\ &\quad\le \sup_{j:\,|a_j|\le r_1} |c(a_j)| \sum_{j:\,|a_j|\le r_1} |b_k(a_j)| + \varepsilon \sum_{j:\,|a_j|> r_1} |b_k(a_j)|. \end{aligned} \tag{4.5}$$

Since the set $\{a : |a| \le r_1\}$ is a compact subset of $\mathbb{B}_n$, by (4.4), we get that

$$\lim_{k\to\infty} \sum_{j:\,|a_j|\le r_1} |b_k(a_j)| = 0.$$

Letting $\varepsilon \to 0$ and then letting $k \to \infty$ in (4.5), we obtain the result in (i).

(ii) Since $\{c(a_j)\} \in l^\gamma$, it follows that, for any $\varepsilon > 0$, there is an $r_2 \in (0, 1)$, such that

$$\sum_{j:\,|a_j|> r_2} |c(a_j)|^\gamma < \varepsilon.$$

Since $\{a : |a| \leq r_2\}$ is a compact subset of $\mathbb{B}_n$, by (4.4), we have

$$\lim_{k\to\infty} \sum_{j:\,|a_j|\leq r_2} |b_k(a_j)| = 0.$$

Since $1/s > 1$, by Hölder inequality, we have that

$$\begin{aligned}\left|\sum_{j=1}^{\infty} c(a_j)^t b_k(a_j)^s\right| &\leq \sum_{j:\,|a_j|\leq r_2} |c(a_j)|^t |b_k(a_j)|^s + \sum_{j:\,|a_j|>r_2} |c(a_j)|^t |b_k(a_j)|^s \\ &\leq \left(\sum_{j:\,|a_j|\leq r_2} |c(a_j)|^{\gamma}\right)^{1-s} \left(\sum_{j:\,|a_j|\leq r_2} |b_k(a_j)|\right)^{s} \\ &\quad + \left(\sum_{j:\,|a_j|>r_2} |c(a_j)|^{\gamma}\right)^{1-s} \left(\sum_{j:\,|a_j|>r_2} |b_k(a_j)|\right)^{s} \\ &\leq \left(\sum_{j:\,|a_j|\leq r_2} |c(a_j)|^{\gamma}\right)^{1-s} \left(\sum_{j:\,|a_j|\leq r_2} |b_k(a_j)|\right)^{s} \\ &\quad + \varepsilon^{1-s} \sum_{j:\,|a_j|>r_2} |b_k(a_j)|.\end{aligned}$$

Letting $\varepsilon \to 0$ and then letting $k \to \infty$, we obtain the result in (ii). □

4.1 Proofs of Theorem 1.1 and Theorem 1.2

The implications (i)⇒(ii) in Theorem 1.1 and Theorem 1.2 are obvious. The implications (ii)⇒(iii) in Theorem 1.1 and Theorem 1.2 are given by [6, Theorem 1.2] and [6, Theorem 4.2], respectively. Thus, we only need to prove that (iii)⇒(i) in these two Theorems. Given $0 < r < 1$, let $\{a_j\}$ be an r-lattice in $\mathbb{B}_n$, and let D_j and $\tilde{D}_j$ be the associated Bergman metric balls given by Lemma 1.3.

(iii)⇒(i) for Theorem 1.1. Suppose that μ is a (λ, γ)-Bergman–Carleson measure. Since the condition $0 < p_1 \leq p_2 < \infty$ implies that $\lambda > 1$ and $p_2/p_1 \geq 1$, it follows from (4.1) and Theorem A. that:

$$\begin{aligned}\|B_\mu^\beta(f)\|_{p_2,\alpha_2}^{p_2} &\lesssim \|\mu\|_{\lambda,\gamma}^{p_2} \sum_{j=1}^{\infty} \left(\int_{\tilde{D}_j} |f(\zeta)|^{p_1} dv_{\alpha_1}(\zeta)\right)^{p_2/p_1} \\ &\lesssim \|\mu\|_{\lambda,\gamma}^{p_2} \left(\sum_{j=1}^{\infty} \int_{\tilde{D}_j} |f(\zeta)|^{p_1} dv_{\alpha_1}(\zeta)\right)^{p_2/p_1} \\ &\lesssim \|\mu\|_{\lambda,\gamma}^{p_2} \|f\|_{p_1,\alpha_1}^{p_2}.\end{aligned}$$

Hence, B_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to $L_{\alpha_2}^{p_2}$.

(iii)⇒(i) for Theorem 1.2. Suppose that μ is a vanishing (λ, γ)-Bergman–Carleson measure. It follows from Proposition 3.1 that we only need to show that $\|B_\mu^\beta f_k\|_{p_2,\alpha_2} \to 0$ for any bounded sequence $\{f_k\}$ in $A_{\alpha_1}^{p_1}$ converging to 0 uniformly on compact subsets of $\mathbb{B}_n$. Let

$$c(a_j) = \left(\frac{\mu(D_j)}{(1-|a_j|^2)^{(n+1+\gamma)\lambda}}\right)^{p_2},$$

and let

$$b_k(a_j) = \left(\int_{\tilde{D}_j} |f_k(\zeta)|^{p_1} \mathrm{d}v_{\alpha_1}(\zeta)\right)^{p_2/p_1}.$$

Using (4.1) again, we get

$$\|B_\mu^\beta(f_k)\|_{p_2,\alpha_2}^{p_2} \lesssim \sum_{j=1}^{\infty} c(a_j) b_k(a_j). \tag{4.6}$$

Since $p_2/p_1 \geq 1$ and $\{f_k\}$ is a bounded sequence in $A_{\alpha_1}^{p_1}$, it follows that:

$$\begin{aligned}\sup_k \sum_{j=1}^{\infty} |b_k(a_j)| &= \sup_k \sum_{j=1}^{\infty} \left(\int_{\tilde{D}_j} |f_k(\zeta)|^{p_1} \mathrm{d}v_{\alpha_1}(\zeta)\right)^{p_2/p_1} \\ &\lesssim \sup_k \|f_k\|_{p_1,\alpha_1}^{p_2} < \infty.\end{aligned}$$

Let K be any compact subset in $\mathbb{B}_n$ and Γ be given as in Lemma 4.2. Then, Γ is a finite set. Since $\{f_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{B}_n$, it follows that:

$$\lim_{k\to\infty} \sum_{j\in\Gamma} |b_k(a_j)| = \lim_{k\to\infty} \sum_{j\in\Gamma} \left(\int_{\tilde{D}_j} |f_k(\zeta)|^{p_1} \mathrm{d}v_{\alpha_1}(\zeta)\right)^{p_2/p_1} = 0.$$

By Theorem B., we have $\lim_{|a_j|\to 1} c(a_j) = 0$. Therefore, by (i) of Lemma 4.2, we get that $\lim_{k\to\infty} \|B_\mu^\beta(f_k)\|_{p_2,\alpha_2} = 0$. The proof is complete.

4.2 Proof of Theorem 1.4

We prove this theorem by showing that

$$(\mathrm{i}) \Rightarrow (\mathrm{ii}) \Rightarrow (\mathrm{iii}) \Rightarrow (\mathrm{v}) \Rightarrow (\mathrm{i}) \Rightarrow (\mathrm{iv}) \Rightarrow (\mathrm{iii}).$$

The implications (i) ⇒ (ii) ⇒ (iii) and (i) ⇒ (iv) ⇒ (iii) are trivial. Notice that the condition $0 < p_2 < p_1 < \infty$ is equivalent to $-\infty < \lambda < 1$, and the proof of Case 2 in (i) ⇒ (ii) of [6, Theorem 1.2] actually works for showing our implication (iii) ⇒ (v), with condition (v) here replacing the (λ, γ)-Bergman–Carleson measure condition in [6]. Therefore, we only need to prove that (v)⇒(i).

(v) ⇒ (i). Given $0 < r < 1$, let $\{a_j\}$ be an r-lattice in $\mathbb{B}_n$, and let D_j and $\tilde{D}_j$ be the associated Bergman metric balls given by Lemma 1.3. Assume that (v) holds. It follows from Proposition 3.1 that we need only show that $\|B_\mu^\beta f_k\|_{p_2,\alpha_2} \to 0$ for any bounded sequence $\{f_k\}$ in $A^{p_1}_{\alpha_1}$ converging to zero uniformly on compact subsets of $\mathbb{B}_n$. We follow a similar argument as in the proof of (iii) ⇒ (i) in Theorem 1.2. Denote by

$$c(a_j) = \frac{\mu(D_j)}{(1-|a_j|^2)^{(n+1+\gamma)\lambda}}, \quad b_k(a_j) = \int\limits_{\tilde{D}_j} |f_k(\zeta)|^{p_1} \mathrm{d}v_{\alpha_1}(\zeta).$$

As in the proof of Theorem 1.2, we know that the sequence $\{b_k(a_j)\}$ satisfies (4.3) and (4.4) in Lemma 4.2. Let $t = p_2$, and let $0 < s = p_2/p_1 < 1$. Then

$$\gamma = \frac{t}{1-s} = \frac{p_1 p_2}{p_1 - p_2} = \frac{1}{1-\lambda}.$$

By (v), we see that $\{c(a_j)\} \in l^\gamma$. Thus, by Lemma 4.2, we get that

$$\lim_{k\to\infty} \|B_\mu(f_k)\|_{p_2,\alpha_2} = 0.$$

The proof is complete.

5 The case when $p_2 = \infty$

In this section, we study the boundedness and compactness of $B_\mu^\beta : A^{p_1}_{\alpha_1} \to L^\infty$ and $T_\mu^\beta : A^{p_1}_{\alpha_1} \to H^\infty$ for $0 < p_1 < \infty$.

Proposition 5.1 *Let $0 < p_1 < \infty$ and let $-1 < \alpha_1, \beta < \infty$ satisfy (C-1). Let*

$$\lambda = 1 + \frac{1}{p_1}, \qquad \gamma = \frac{1}{\lambda}\left(\beta + \frac{\alpha_1}{p_1}\right). \tag{5.1}$$

If T_μ^β is bounded from $A^{p_1}_{\alpha_1}$ to H^∞, then μ is a (λ, γ)-Bergman–Carleson measure.

Proof For any fixed $a \in \mathbb{B}_n$, take function

$$f_a(z) = \frac{(1-|a|^2)^{n+1+\beta-(n+1+\alpha_1)/p_1}}{(1-\langle z, a\rangle)^{n+1+\beta}}.$$

Then, the condition (C-1) and Lemma 2.2 give that $f \in A^{p_1}_{\alpha_1}$ and $\|f\|_{p_1,\alpha_1} \asymp 1$. Since $|1-\langle w,a\rangle| \asymp 1-|a|^2$ for any $w \in D(a,r)$, it follows that:

$$\begin{aligned} T^\beta_\mu f_a(a) &= (1-|a|^2)^{n+1+\beta-(n+1+\alpha_1)/p_1} \int_{\mathbb{B}_n} \frac{\mathrm{d}\mu(w)}{|1-\langle a,w\rangle|^{2(n+1+\beta)}} \\ &\ge (1-|a|^2)^{n+1+\beta-(n+1+\alpha_1)/p_1} \int_{D(a,r)} \frac{\mathrm{d}\mu(w)}{|1-\langle a,w\rangle|^{2(n+1+\beta)}} \\ &\ge C\frac{\mu(D(a,r))}{(1-|a|^2)^{(n+1+\gamma)\lambda}}. \end{aligned} \tag{5.2}$$

The boundedness of Toeplitz operator $T^\beta_\mu : A^{p_1}_{\alpha_1} \to H^\infty$ gives that

$$|T^\beta_\mu f_a(a)| \le \|T^\beta_\mu f_a\|_\infty \le \|T^\beta_\mu\| \|f_a\|_{p_1,\alpha_1}.$$

Therefore, we get that

$$\mu(D(a,r)) \lesssim \|T^\beta_\mu\|(1-|a|^2)^{(n+1+\gamma)\lambda}.$$

It follows from Theorem A. that μ is a (λ,γ)-Bergman–Carleson measure. □

Proposition 5.2 *Let $0 < p_1 \le 1$ and let $-1 < \alpha_1, \beta < \infty$ satisfy (C-1). Let λ, γ be given by (5.1). If the measure μ is a (λ,γ)-Bergman–Carleson measure, then B^β_μ is bounded from $A^{p_1}_{\alpha_1}$ to L^∞.*

Proof Given $0 < r < 1$, let $\{a_j\}$ be an r-lattice in $\mathbb{B}_n$, and let D_j and $\tilde{D}_j$ be the associated Bergman metric balls given in Lemma 1.3. Since $0 < p_1 \le 1$, it follows from (4.2) that:

$$\begin{aligned} \sup_{z\in\mathbb{B}_n} |B^\beta_\mu(f)(z)| &\lesssim \sum_{j=1}^\infty \frac{\mu(D_j)}{(1-|a_j|^2)^{(n+1+\gamma)\lambda}} \left(\int_{\tilde{D}_j} |f(\zeta)|^{p_1} \mathrm{d}v_{\alpha_1}(\zeta) \right)^{1/p_1} \\ &\lesssim \|\mu\|_{\lambda,\gamma} \left(\sum_{j=1}^\infty \int_{\tilde{D}_j} |f(\zeta)|^{p_1} \mathrm{d}v_{\alpha_1}(\zeta) \right)^{1/p_1} \\ &\lesssim \|\mu\|_{\lambda,\gamma} \|f\|_{p_1,\alpha_1}. \end{aligned} \tag{5.3}$$

Thus, $B^\beta_\mu : A^{p_1}_{\alpha_1} \to L^\infty$ is bounded. The proof is complete. □

Combining Proposition 5.1, Proposition 5.2, and the fact that $|T^\beta_\mu f| \le B^\beta_\mu f$, we obtain the following result.

Theorem 5.3 *Let $0 < p_1 \le 1$ and let $-1 < \alpha_1, \beta < \infty$ satisfy (C-1). Let λ, γ be given by (5.1). Then, the following statements are equivalent:*

(i) *B_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to L^∞.*
(ii) *T_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to H^∞.*
(iii) *The measure μ is a (λ, γ)-Bergman–Carleson measure.*

Moreover, we have

$$\|B_\mu^\beta\|_{A_{\alpha_1}^{p_1}\to L^\infty} \asymp \|T_\mu^\beta\|_{A_{\alpha_1}^{p_1}\to H^\infty} \asymp \|\mu\|_{\lambda,\gamma}.$$

For the case $1 < p_1 < \infty$, we have the following partial result.

Proposition 5.4 *Let $1 < p_1 < \infty$ and let $-1 < \alpha_1, \beta < \infty$ satisfy (C-1). Let λ, γ be given by (5.1). Then, the following statements are equivalent.*

(i) *For any (λ, γ)-Bergman–Carleson measure μ, B_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to L^∞.*
(ii) *The integral operator*

$$(\mathcal{S}f)(z) := \int\limits_{\mathbb{B}_n} \frac{(1-|w|^2)^{\beta+(n+1+\alpha_1)/p_1}|f(w)|}{|1-\langle z, w\rangle|^{n+1+\beta}} dv(w) \tag{5.4}$$

is bounded from $A_{\alpha_1}^{p_1}$ to L^∞.

Proof (ii)⇒(i). Suppose that $\mathcal{S} : A_{\alpha_1}^{p_1} \to L^\infty$ is bounded. Let μ be an arbitrary (λ, γ)-Bergman–Carleson measure. It follows from Lemma 2.1 that:

$$\begin{aligned}
|B_\mu^\beta f(z)| &= \int\limits_{\mathbb{B}_n} \frac{|f(w)|}{|1-\langle z, w\rangle|^{n+1+\beta}} \mathrm{d}\mu(w) \\
&\lesssim \|\mu\|_{\lambda,\gamma} \int\limits_{\mathbb{B}_n} \frac{|f(w)|}{|1-\langle z, w\rangle|^{n+1+\beta}} (1-|w|^2)^{(n+1+\gamma)\lambda-n-1} \mathrm{d}v(w) \\
&= \|\mu\|_{\lambda,\gamma} \int\limits_{\mathbb{B}_n} \frac{(1-|w|^2)^{\beta+(n+1+\alpha_1)/p_1}|f(w)|}{|1-\langle z, w\rangle|^{n+1+\beta}} \mathrm{d}v(w) \\
&= \|\mu\|_{\lambda,\gamma} (\mathcal{S}f)(w).
\end{aligned}$$

Thus, the boundedness of $\mathcal{S} : A_{\alpha_1}^{p_1} \to L^\infty$ implies the boundedness of $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L^\infty$.

(i) ⇒(ii). Suppose that $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L^\infty$ is bounded for any (λ, γ)-Bergman–Carleson measure. Consider $d\mu(z) = (1-|z|^2)^{\beta+(n+1+\alpha_1)/p_1} dv(z)$. It can be easily checked that μ is a (λ, γ)-Bergman–Carleson measure. Since

$$\begin{aligned}B_\mu^\beta f(z) &= \int_{\mathbb{B}_n} \frac{|f(w)|}{|1-\langle z,w\rangle|^{n+1+\beta}} \mathrm{d}\mu(w) \\ &= \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\beta+(n+1+\alpha_1)/p_1}|f(w)|}{|1-\langle z,w\rangle|^{n+1+\beta}} \mathrm{d}v(w) \\ &= (\mathcal{S}f)(w),\end{aligned}$$

the boundedness of $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L^\infty$ implies the boundedness of $\mathcal{S} : A_{\alpha_1}^{p_1} \to L^\infty$. □

Remark 5.5 It follows from [10, Theorem 1.3] that for $1 < p_1 < \infty$, $\mathcal{S} : L_{\alpha_1}^{p_1} \to L^\infty$ is unbounded. However, we do not know whether $\mathcal{S} : A_{\alpha_1}^{p_1} \to L^\infty$ is bounded. Also, the above proposition does not fully solve the problem about when B_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to L^∞. Therefore, we propose the following open problems.

Open Problem 1 Let $1 < p < \infty$, and let $-1 < \alpha < \infty$. Is the operator $\mathcal{S}$ bounded from the Bergman space A_α^p to L^∞?

Open Problem 2 Let $1 < p < \infty$, and let $-1 < \alpha < \infty$. How to characterize boundedness of $B_\mu^\beta : A_\alpha^p \to L^\infty$ and $T_\mu^\beta : A_\alpha^p \to H^\infty$?

Next let us consider compactness of $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L^\infty$. By the same discussion as in the proof of Proposition 3.1, we can obtain the following result.

Proposition 5.6 *Let $0 < p_1 \le 1$, let $p_2 = \infty$ and let $-1 < \alpha_1 < \infty$. Suppose that, for every bounded sequence $\{f_k\}$ in $A_{\alpha_1}^{p_1}$, such that $f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$, we have*

$$\lim_{k\to\infty} \|B_\mu^\beta f_k\|_\infty = 0.$$

Then, $B_\mu^\beta : A_{\alpha_1}^{p_1} \to L^\infty$ is compact.

We can also get the the following result on T_μ^β by a similar discussion as in the proof of Proposition 3.2 combining with Proposition 5.1.

Proposition 5.7 *Let $0 < p_1 < \infty$ and let $-1 < \alpha_1, \beta < \infty$ satisfy (C-1). Suppose that T_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to H^∞. Then, the following statements are equivalent.*

(i) *T_μ^β is compact from $A_{\alpha_1}^{p_1}$ to H^∞.*
(ii) *For every bounded sequence $\{f_k\}$ in $A_{\alpha_1}^{p_1}$, such that $f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$, we have*

$$\lim_{k\to\infty} \|T_\mu^\beta f_k\|_\infty = 0.$$

Proof The implication (ii) ⇒ (i) follows from the same discussion as in Proposition 3.1. The implication (i) ⇒ (ii) follows from a similar discussion as in Proposition 3.2.

In fact, if $T_\mu^\beta : A_{\alpha_1}^{p_1} \to H^\infty$ is compact, then $T_\mu^\beta : A_{\alpha_1}^{p_1} \to H^\infty$ is bounded. It follows from Proposition 5.1 that μ is a (λ, γ)-Bergman–Carleson measure on $\mathbb{B}_n$, where λ, γ are given by (5.1). It is easy to see that

$$(n+1+\gamma)\lambda = n+1+\beta + \frac{n+1+\alpha_1}{p_1} > \frac{n+1+\alpha_1}{p_1}.$$

Thus, a similar discussion to Lemma 3.4 and Proposition 3.2 gives that (ii) holds. The proof is complete. □

Theorem 5.8 *Let $0 < p_1 \leq 1$ and let $-1 < \alpha_1, \beta < \infty$ satisfy (C-1). Let λ, γ be given by* (5.1)*. Then, the following statements are equivalent:*

(i) *B_μ^β is compact from $A_{\alpha_1}^{p_1}$ to L^∞.*
(ii) *T_μ^β is compact from $A_{\alpha_1}^{p_1}$ to H^∞.*
(iii) *The measure μ is a vanishing (λ, γ)-Bergman–Carleson measure.*

Proof (i)⇒(ii). This is is trivial.

(ii)⇒(iii). Suppose that T_μ^β is compact from $A_{\alpha_1}^{p_1}$ to H^∞. Let $\{a_k\}$ be a sequence in $\mathbb{B}_n$ with $|a_k| \to 1$. Consider the functions

$$f_k(z) = \frac{(1-|a_k|^2)^{n+1+\beta-(n+1+\alpha_1)/p_1}}{(1-\langle z, a_k\rangle)^{n+1+\beta}}$$

for $k = 1, 2, 3, ...$ It follows from Lemma 2.2 that $\sup_k \|f_k\|_{p_1,\alpha_1} < \infty$, and it is obvious that f_k converges to zero uniformly on compact subsets of $\mathbb{B}_n$. Thus, by Proposition 5.7, we have that $\|T_\mu^\beta f_k\|_\infty \to 0$. Fix any r with $0 < r < 1$. By the same discussion as in the proof of Proposition 5.1, we get that

$$\frac{\mu(D(a_k, r))}{(1-|a_k|^2)^{(n+1+\gamma)\lambda}} \leq T_\mu^\beta f_k(a_k) \leq \|T_\mu^\beta f_k\|_\infty \to 0$$

as $k \to \infty$. Thus, μ is a vanishing (λ, γ)-Bergman–Carleson measure.

(iii)⇒(i). Suppose that μ is a vanishing (λ, γ)-Bergman–Carleson measure. By proposition 5.6, it suffices to prove that $\|B_\mu^\beta f_k\|_\infty \to 0$ for any bounded sequence $\{f_k\}$ in $A_{\alpha_1}^{p_1}$ converging to zero uniformly on compact subsets of $\mathbb{B}_n$. Let $\{a_j\}$ be an r-lattice in $\mathbb{B}_n$. Let

$$c(a_j) = \frac{\mu(D_j)}{(1-|a_j|^2)^{(n+1+\gamma)\lambda}}$$

and

$$b_k(a_j) = \left(\int_{\tilde{D}_j} |f_k(\zeta)|^{p_1} \mathrm{d}v_{\alpha_1}(\zeta)\right)^{1/p_1},$$

Since $0 < p_1 \leq 1$, it follows from inequality (5.3) that:

$$\sup_{z\in\mathbb{B}_n} |B_\mu^\beta(f_k)(z)| \lesssim \sum_{j=1}^{\infty} c(a_j)b_k(a_j).$$

By the same discussion as in the proof of Theorem 1.2, we know that the sequence $\{b_k(a_j)\}$ satisfies the condition of Lemma 4.2. Since μ is a vanishing (λ, γ)-Bergman–Carleson measure, we know that $\lim_{|a_j|\to 1} c(a_j) = 0$. Hence, by (i) of Lemma 4.2, we get that $\lim_{k\to\infty} \|B_\mu^\beta f_k\|_\infty = 0$. The proof is complete. □

Open Problem 3 Let $1 < p < \infty$, and let $-1 < \alpha < \infty$. How to characterize compactness of $B_\mu^\beta : A_\alpha^p \to L^\infty$ and $T_\mu^\beta : A_\alpha^p \to H^\infty$?

6 The case when $p_1 = \infty$

In this section, we consider the case when $p_1 = \infty$.

6.1 The case when $p_1 = \infty, p_2 = \infty$

If we follow the definition of λ and γ in (1.2), we get that in this case $\lambda = 1, \gamma = \beta$. Our first result here shows that the $(1, \beta)$-Bergman–Carleson measure condition does not characterize boundedness of $B_\mu^\beta : H^\infty \to L^\infty$.

Lemma 6.1 *Let $-1 < \beta < \infty$. There exists a $(1, \beta)$-Bergman–Carleson measure μ, such that B_μ^β is unbounded from H^∞ to L^∞.*

Proof Let $d\mu = (1-|z|^2)^\beta \, dv(z)$. It can be easily checked that μ is a $(1, \beta)$-Bergman–Carleson measure, and

$$B_\mu^\beta f(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\beta |f(w)|}{|1-\langle z, w\rangle|^{n+1+\beta}} \, dv(w).$$

Let $f = 1 \in H^\infty$. Then, $B_\mu^\beta 1 \notin L^\infty$ by lemma 2.2. The proof is complete. □

Open Problem 4 How to characterize boundedness and the compactness of $B_\mu^\beta : H^\infty \to L^\infty$ and $T_\mu^\beta : H^\infty \to H^\infty$?

6.2 The case when $p_1 = \infty, 0 < p_2 < \infty$.

We give the following sufficient condition for the boundedness of the Berezin type operator $B_\mu^\beta : H^\infty \to L_{\alpha_2}^{p_2}$.

Proposition 6.2 *Let $0 < p_2 < \infty$ and let $-1 < \alpha_2, \beta < \infty$ satisfy (C-2). Given $0 < r < 1$, let $\{a_j\}$ be any r-lattice in $\mathbb{B}_n$, and let $D_j = D(a_j, r)$ be the associated Bergman metric balls given by Lemma 1.3. Suppose that*

$$\{\nu_j\} =: \left\{ \frac{\mu(D_j)}{(1-|a_j|^2)^{n+1+\beta-(n+1+\alpha_2)/p_2}} \right\} \in l^{p_2}. \tag{6.1}$$

Then, B_μ^β is bounded from H^∞ to $L_{\alpha_2}^{p_2}$.

Proof Since $|1-\langle z, w\rangle| \asymp |1-\langle z, a_j\rangle|$ for $w \in D_j$, it follows that:

$$\begin{aligned} |B_\mu^\beta(f)(z)| &\lesssim \|f\|_\infty \sum_{j=1}^{\infty} \int_{D_j} \frac{1}{|1-\langle z, w\rangle|^{n+1+\beta}} \mathrm{d}\mu(w) \\ &\lesssim \|f\|_\infty \sum_{j=1}^{\infty} \frac{\mu(D_j)}{|1-\langle z, a_j\rangle|^{n+1+\beta}}. \end{aligned}$$

By Lemma 2.4, we get

$$\begin{aligned} \|B_\mu^\beta f\|_{p_2,\alpha_2} &\lesssim \|f\|_\infty \left(\sum_{j=1}^{\infty} \frac{\mu(D_j)^{p_2}}{(1-|a_j|^2)^{(n+1+\beta)p_2-(n+1+\alpha_2)}} \right)^{1/p_2} \\ &\lesssim \|f\|_\infty \|\{\nu_j\}\|_{l^{p_2}}. \end{aligned}$$

Therefore, we get that $B_\mu^\beta : H^\infty \to A_{\alpha_2}^{p_2}$ is bounded. □

By a similar argument as in the proof of Proposition 3.1, we can get the following sufficient condition for the compactness of $B_\mu^\alpha : H^\infty \to A_{\alpha_2}^{p_2}$ for $0 < p_2 < \infty$.

Proposition 6.3 *Let $0 < p_2 < \infty$, and let $-1 < \alpha_2 < \infty$. Assume that $B_\mu^\beta : H^\infty \to L_{\alpha_2}^{p_2}$ is bounded. Suppose that, for every bounded sequence $\{f_k\}$ in H^∞, such that $f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$, we have*

$$\lim_{k\to\infty} \|B_\mu^\beta f_k\|_{p_2,\alpha_2} = 0.$$

Then, B_μ^β is compact from H^∞ to $L_{\alpha_2}^{p_2}$.

Lemma 6.4 *Given $0 < r < 1$, let $\{a_j\}$ be an r-lattice in $\mathbb{B}_n$, let $D_j = D(a_j, r)$ be the associated Bergman metric balls given by Lemma 1.3, and let $\{b_k(a_j)\}$ be a sequence depending on a_j, such that*

$$\sup_k \sup_j |b_k(a_j)| < \infty.$$

Suppose, in addition, for any compact subset K of $\mathbb{B}_n$, the sequence $\{b_k(a_j)\}$ satisfies that

$$\lim_{k\to\infty}\sup_{j\in\Gamma}|b_k(a_j)|\to 0,$$

where $\Gamma := \{j : a_j \in K\}$. Let $\{c(a_j)\}$ be a sequence of real numbers depending on $\{a_j\}$ satisfying that $\{c(a_j)\}\in l^1$. Then

$$\lim_{k\to\infty}\sum_{j=1}^{\infty}c(a_j)b_k(a_j)=0.$$

Proof Since $\{c(a_j)\}\in l^1$, then for any given $\varepsilon>0$, there is a $r_3\in(0,1)$, such that

$$\sum_{j:\,|a_j|>r_3}|c(a_j)|<\varepsilon.$$

Thus, we have

$$\begin{aligned}\left|\sum_{j=1}^{\infty}c(a_j)b_k(a_j)\right| &\le \sum_{j:\,|a_j|\le r_3}|c(a_j)b_k(a_j)|+\sum_{j:\,|a_j|>r_3}|c(a_j)b_k(a_j)|\\ &\le \sup_{j:\,|a_j|\le r_3}|b_k(a_j)|\sum_{j:\,|a_j|<r_3}|c(a_j)|\\ &\quad+\sup_{j:\,|a_j|>r_3}|b_k(a_j)|\sum_{j:\,|a_j|>r_3}|c(a_j)|\\ &\le \sup_{j:\,|a_j|\le r_3}|b_k(a_j)|\sum_{j:\,|a_j|<r_3}|c(a_j)|+\varepsilon\sup_k\sup_j|b_k(a_j)|.\end{aligned}$$

Letting $\varepsilon\to 0$, and then letting $k\to\infty$, we get the result, since

$$\lim_{k\to\infty}\sup_{j:\,|a_j|\le r_3}|b_k(a_j)|=0.$$

□

Proposition 6.5 *Let $0<p_2<\infty$, and let $-1<\alpha_2,\beta<\infty$ satisfy (C-2). Let $\{\nu_j\}$ be the sequence given by (6.1) which satisfies that $\{\nu_j\}\in l^{p_2}$. Then, B_μ^β is compact from H^∞ to $L_{\alpha_2}^{p_2}$.*

Proof Let $\{a_j\}$ be an r-lattice. By Proposition 6.3 that it suffices for us to prove that $\lim_{k\to\infty}\|B_\mu^\beta(f_k)\|_{p_2,\alpha_2}=0$ for any bounded sequence $\{f_k\}\in H^\infty$, such that $f_k\to 0$

uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$. First, it is easy to get that

$$|B_\mu^\beta(f_k)(z)| \lesssim \sum_{j=1}^{\infty} \frac{\|f_k\chi_{D_j}\|_\infty \mu(D_j)}{|1-\langle z, a_j\rangle|^{n+1+\beta}}.$$

Let

$$c(a_j) = \left(\frac{\mu(D_j)}{(1-|a_j|^2)^{n+1+\beta-(n+1+\alpha_2)/p_2}}\right)^{p_2}$$

and

$$b_k(a_j) = \|f_k\chi_{D_j}\|_\infty^{p_2}.$$

Using Lemma 2.4, we get that

$$\|B_\mu^\beta(f_k)\|_{p_2,\alpha_2}^{p_2} \lesssim \sum_{j=1}^{\infty} c(a_j)b_k(a_j). \tag{6.2}$$

Notice that $\{c(a_j)\} = \{\nu_j^{p_2}\} \in l^1$ and

$$\sup_k \sup_j |b_k(a_j)| = \sup_k \sup_j \|f_k\chi_{D_j}\|_\infty^{p_2} \le \sup_k \|f_k\|_\infty^{p_2} < \infty.$$

Let K be any compact subset in $\mathbb{B}_n$ and $\Gamma = \{j : a_j \in K\}$. Then Γ is a finite set. Since $f_k \to 0$ uniformly on every compact subsets of $\mathbb{B}_n$ as $k \to \infty$, it follows that for any given $\varepsilon > 0$, and for all $z \in \cup_{j\in\Gamma} D_j$, we have that

$$\sup_{j\in\Gamma} |b_k(a_j)| = \sup_{j\in\Gamma} \|f_k\chi_{D_j}\|_\infty^{p_2} < \varepsilon.$$

Letting $\varepsilon \to 0$ and letting $k \to \infty$, we get that

$$\lim_{k\to\infty} \sup_{j\in\Gamma} |b_k(a_j)| \to 0.$$

It follows from Lemma 6.4 that $\lim_{k\to\infty} \|B_\mu^\beta(f_k)\|_{p_2,\alpha_2} = 0$, completing the proof. □

Finally, we propose the following open problem.

Open Problem 5 How to characterize boundedness and compactness of $B_\mu^\beta : H^\infty \to L_{\alpha_2}^{p_2}$ and $T_\mu^\beta : H^\infty \to A_{\alpha_2}^{p_2}$?

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