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ORIGINAL PAPER

Φ-moment martingale inequalities on Lorentz spaces with variable exponents

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Abstract
In this paper, with the help of some new atomic decomposition theorems, some Φ-moment martingale inequalities in the framework of Lorentz spaces with variable exponents are proved. The results obtained here generalize the previous results in variable martingale Lorentz–Hardy spaces and various classical martingale Hardy spaces.

Keywords Martingale · Lorentz space with variable exponent · Atomic decomposition · Φ-moment inequality

Mathematics Subject Classification 60G46 · 60G42 · 46E30

1 Introduction

Martingale inequalities occupy an important position in martingale space theory. The topic we shall touch here is the Φ-moment martingale inequalities in the framework of Lorentz spaces with variable exponents. In 1970, Burkholder and Gundy [6] first discussed the Φ-moment inequalities for martingales. Then the Φ-moment version of the Burkholder–Davis–Gundy inequality was proved in [5]. Kikuchi [29] proved the Φ-

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moment martingale inequalities in the framework of rearrangement invariant Banach function spaces. It should be noticed that the functions Φ in the articles mentioned above are convex functions. In 2012, Jiao and Yu [24] proved some Φ-moment martingale inequalities associated with concave functions. Later, Peng and Li [40] extended the results in [24] to the framework of Lorentz spaces. Wu et al. [47] deduced some modular martingale inequalities in the framework of Orlicz–Karamata spaces. Jiao et al. [18] proved the Φ-moment version of Burkholder's inequalities in rearrangement invariant spaces. Some Φ-moment inequalities for noncommutative martingales (see [4]) and independent and freely independent random variables (see [19]) were also deduced. We point out that the results on Φ-moment martingale inequalities mentioned above are all in the framework of rearrangement invariant spaces. Hence one natural question arises, that is,

Do Φ-moment martingale inequalities also hold in the non-rearrangement invariant setting?

In this article, we give an affirmative answer. More precisely, we shall extend the Φ-moment martingale inequalities to the framework of function spaces of variable exponents. As is well known, Lebesgue space with variable exponent is an important kind of non-rearrangement invariant spaces and has been widely used in elasticity, fluid dynamics, calculus of variations, differential equations and so on, see, for example, [2, 12, 42, 51, 52]. Such spaces were first introduced by Orlicz [39] in 1931. Kováčik and Rákosník [30], Fan and Zhao [13] investigated various properties of variable Lebesgue spaces and variable Sobolev spaces. A fundamental breakthrough of the study of Lebesgue spaces with variable exponents is due to Diening [9, 10], who proposed the so-called log-Hölder continuity condition to obtain the boundedness of the Hardy–Littlewood maximal operator. Since then, much progress has been made in variable exponent function space theory. Many important results such as atomic decomposition, boundedness of singular integral operator, Littlewood–Paley characterization, dual theory, and so on, have been extended to the variable exponent setting, see, for example, [7, 8, 11, 38, 43, 49]. As generalizations of variable Lebesgue spaces and variable Hardy spaces, respectively, variable Lorentz spaces and variable Lorentz–Hardy spaces have also been studied by many authors. Kempka and Vybíral [28] showed many important properties of Lorentz spaces with variable exponents. Jiao et al. [27] proved the maximal function characterizations, atomic decompositions, interpolation results, duality results, Littlewood–Paley function characterizations, boundedness of singular integral operators for variable Lorentz–Hardy spaces. We refer the reader to [1, 31–33, 48, 53] for more information about variable Lorentz spaces and variable Lorentz–Hardy spaces.

Inspired by the considerable progress of function space theory in the variable exponent setting, the martingale space theory in the variable exponent setting has gained a lot of interests in recent years. Aoyama [3] proved the Doob maximal inequality under some strict restrictions on the variable exponent $p(\cdot)$. This is the first attempt to study martingale space theory in the variable exponent setting. However, the condition in [3] is too strong. Indeed, Nakai and Sadasue [37] gave a counterexample to show that the condition in [3] is not necessary for the boundedness of the Doob maximal operator on variable Lebesgue spaces. The major difficulties to study the martingale space theory in the variable exponent setting are, on one hand, abstract probability spaces do not enjoy natural metric structure, and thus the log-Hölder continuity condition cannot be well defined any more; on the other hand, the arguments in the classical setting are no longer efficient and the essential reason is that the variable Lebesgue spaces and variable Lorentz spaces are not rearrangement invariant spaces. To overcome these difficulties, Jiao et al. [25] introduced a condition without metric characterization to replace the log-Hölder continuity condition in some sense. Under this new condition, they proved the weak-type and strong-type estimates of the Doob maximal operator, established the atomic decompositions and obtained duality theorems as well as John–Nirenberg inequalities for the martingale Hardy spaces with variable exponents. Subsequently, the variable martingale Hardy spaces and variable martingale Lorentz–Hardy spaces were systematically studied in [21]. We refer the reader to [15, 20, 22, 26, 45, 46] for more information about martingale Hardy spaces and martingale Lorentz–Hardy spaces in the variable exponent setting.

We consider the Φ-moment martingale inequalities associated with concave functions in the framework of variable Lorentz spaces. Our main method is to establish some new atomic decomposition theorems by simple atoms. As far as we know, this is the first paper which deals with the Φ-moment martingale inequalities in the variable exponent setting. It should be pointed out that the atomic decomposition theorems of this paper improve the atomic decomposition theorems in [21, 22]. Furthermore, let $p(\cdot) \equiv p$, our results greatly broaden the scope of the atomic decomposition theorems in [40]. In [40], the authors just considered the atomic decomposition theorems under the conditions of $0 < q \le 1$ and $q \le p < \infty$. They guessed the restricted conditions may be removed via the method in [23]. But the method in [23] is only applicable for "∞-atom decompositions". In this paper, we present "r-atom decompositions" ($1 < r \le \infty$) and remove the restricted conditions of $0 < q \le 1$ and $q \le p < \infty$ in [40] via some refinement techniques.

The paper is organized as follows. In the next section, some preliminaries are introduced. Atomic decompositions for martingale Lorentz–Hardy spaces with variable exponents are presented in Sect. 3. In the final section, with the help of atomic decompositions, the Φ-moment martingale inequalities in the framework of Lorentz spaces with variable exponents are deduced.

At the end of this section, we make some conventions. Throughout this paper, the set of nonnegative integers, the set of integers, the set of real numbers and the set of complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. We use C to denote the positive constant that is independent of the essential variables involved but whose value may vary from line to line. The symbol $f \lesssim g$ stands for the inequality $f \le Cg$. If $f \lesssim g \lesssim f$, then we write $f \approx g$.

2 Preliminaries

In this section, we give preliminaries necessary to the whole paper.

2.1 Variable Lebesgue spaces and variable Lorentz spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{\mathcal{F}_n\}_{n\geq 0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\big(\bigcup_{n\geq 0} \mathcal{F}_n\big)$. A measurable function $p(\cdot) : \Omega \to (0, \infty)$ is called a variable exponent. For $A \in \mathcal{F}$, we denote

$$p_-(A) = \inf_{x\in A} p(x), \quad p_+(A) = \sup_{x\in A} p(x)$$

and for convenience

$$p_- = p_-(\Omega), \quad p_+ = p_+(\Omega).$$

Let $\mathcal{P} = \mathcal{P}(\Omega)$ denote the collection of all variable exponents $p(\cdot)$ such that $0 < p_- \leq p_+ < \infty$. The variable Lebesgue space $L_{p(\cdot)} = L_{p(\cdot)}(\Omega, \mathcal{F}, \mathbb{P})$ is defined as the space of all measurable functions f such that for some $\lambda > 0$,

$$\rho\left(\frac{f}{\lambda}\right) = \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}\mathbb{P} < \infty.$$

Then, the space $L_{p(\cdot)}$ becomes a (quasi-)Banach function space when it is equipped with the (quasi-)norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

Let $\underline{p} = \min\{p_-, 1\}$. For $p_- \geq 1$, the conjugate variable exponent $p'(\cdot)$ is defined pointwise by

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

The following facts are well known (see [38]).

(i) (Positivity) $\|f\|_{p(\cdot)} \geq 0$ and $\|f\|_{p(\cdot)} = 0 \Leftrightarrow f = 0$;
(ii) (Homogeneity) $\|cf\|_{p(\cdot)} = |c| \cdot \|f\|_{p(\cdot)}$, $(c \in \mathbb{C})$;
(iii) (The θ-triangle inequality) $\|f + g\|^\theta_{p(\cdot)} \leq \|f\|^\theta_{p(\cdot)} + \|g\|^\theta_{p(\cdot)}$, $(0 < \theta \leq \underline{p})$.

We collect some useful lemmas, which will be used in the sequel.

Lemma 2.1 (See [8]) *Let $p(\cdot) \in \mathcal{P}$ and $s > 0$. Then for $f \in L_{sp(\cdot)}$, we have*

$$\| |f|^s \|_{p(\cdot)} = \|f\|^s_{sp(\cdot)}.$$

Lemma 2.2 (See [7]) *Let $p(\cdot) \in \mathcal{P}$. Then for all $f \in L_{p(\cdot)}$ and $\|f\|_{p(\cdot)} \neq 0$, we have*

$$\int_\Omega \left|\frac{f(w)}{\|f\|_{p(\cdot)}}\right|^{p(w)} \mathrm{d}\mathbb{P} = 1.$$

Lemma 2.3 (See [7, 13]) *Let $p(\cdot) \in \mathcal{P}$ and $f \in L_{p(\cdot)}$. Then we have*

(1) $\|f\|_{p(\cdot)} \leq 1(=1)$ *if and only if* $\int_\Omega |f|^{p(\cdot)} \mathrm{d}\mathbb{P} \leq 1(=1)$;
(2) If $\|f\|_{p(\cdot)} > 1$, *then* $\rho(f)^{1/p_+} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}$;
(3) If $0 < \|f\|_{p(\cdot)} \leq 1$, *then* $\rho(f)^{1/p_-} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_+}$.

Lemma 2.4 (Hölder's inequality, see [7]) *Let $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}$ such that*

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}.$$

Then for all $f \in L_{p(\cdot)}$ and $g \in L_{q(\cdot)}$, we have $fg \in L_{r(\cdot)}$ and

$$\|fg\|_{r(\cdot)} \leq C\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

We now present the definition of Lorentz spaces with variable exponents. For more information about such spaces, see [28].

Definition 2.5 Let $p(\cdot) \in \mathcal{P}$ and $0 < q \leq \infty$. The variable Lorentz space $L_{p(\cdot),q} = L_{p(\cdot),q}(\Omega, \mathcal{F}, \mathbb{P})$ is defined as the space of all measurable functions f such that

$$\|f\|_{p(\cdot),q} = \begin{cases} \left(\int_0^\infty t^q \|\chi_{\{|f|>t\}}\|_{p(\cdot)}^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup_{t>0} t\|\chi_{\{|f|>t\}}\|_{p(\cdot)} & \text{if } q = \infty, \end{cases}$$

is finite.

Note that $L_{p(\cdot),q}$ is a quasi-Banach space for $p(\cdot) \in \mathcal{P}$ and $0 < q \leq \infty$ (see [28]). Obviously, $L_{p(\cdot),q}$ is the generalization of classical Lorentz space $L_{p,q}$ and it coincides with $L_{p,q}$ when $p(\cdot) \equiv p$. Moreover, the functional $\|\cdot\|_{p(\cdot),q}$ can be discretized as follows.

Lemma 2.6 (See [28]) *Let $p(\cdot) \in \mathcal{P}$ and $0 < q \leq \infty$. If $f \in L_{p(\cdot),q}$, then*

$$\|f\|_{p(\cdot),q} \approx \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{kq} \|\chi_{\{|f|>2^k\}}\|_{p(\cdot)}^q\right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup_{k\in\mathbb{Z}} 2^k\|\chi_{\{|f|>2^k\}}\|_{p(\cdot)} & \text{if } q = \infty. \end{cases}$$

2.2 Orlicz functions

Let $\mathcal{G}$ be the set of all functions $\Phi : [0, \infty) \to [0, \infty)$ such that Φ is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t > 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. We refer the reader to [16, 36, 41, 50] for more information about properties of functions in $\mathcal{G}$ and related functions. We have the following simple but useful lemma.

Lemma 2.7 *(See [35, 40]) Let $\Phi \in \mathcal{G}$ be concave.*

(1) If $0 < s \leq 1$ *and* $t \geq 0$, *then* $\Phi(st) \geq s\Phi(t)$;

(2) If $s \geq 1$ and $t \geq 0$, then $\Phi(st) \leq s\Phi(t)$.

Moreover, Φ is subadditive, continuous and bijective from $[0, \infty)$ to $[0, \infty)$.

We now give the following lemma which will be used in the proof of the atomic decomposition theorems in the next section.

Lemma 2.8 *Let $\Phi \in \mathcal{G}$ be concave and $0 < p \leq r \leq \infty$. For $f \in L_r$, there exists $A \in \mathcal{F}$ with $\mathbb{P}(A) \neq 0$ such that $\{f \neq 0\} \subset A$, then*

$$\|\Phi(|f|)\|_p \lesssim \mathbb{P}(A)^{\frac{1}{p}} \Phi\left(\frac{\|f\|_r}{\mathbb{P}(A)^{\frac{1}{r}}}\right).$$

Proof It follows from Lemma 2.7 and Hölder's inequality that

$$\begin{aligned}
\|\Phi(|f|)\|_p &\leq \left\| \Phi\left(|f| + \frac{\|f\|_r}{\mathbb{P}(A)^{\frac{1}{r}}}\right) \chi_A \right\|_p \qquad (2.1)\\
&\leq \left\| \left(\frac{|f|\mathbb{P}(A)^{\frac{1}{r}}}{\|f\|_r} + 1\right) \Phi\left(\frac{\|f\|_r}{\mathbb{P}(A)^{\frac{1}{r}}}\right) \chi_A \right\|_p \\
&= \Phi\left(\frac{\|f\|_r}{\mathbb{P}(A)^{\frac{1}{r}}}\right) \left\| \left(\frac{|f|\mathbb{P}(A)^{\frac{1}{r}}}{\|f\|_r} + 1\right) \chi_A \right\|_p \\
&\lesssim \Phi\left(\frac{\|f\|_r}{\mathbb{P}(A)^{\frac{1}{r}}}\right) \left(\left\| \frac{|f|\mathbb{P}(A)^{\frac{1}{r}}}{\|f\|_r} \chi_A \right\|_p + \mathbb{P}(A)^{\frac{1}{p}} \right) \\
&\leq \Phi\left(\frac{\|f\|_r}{\mathbb{P}(A)^{\frac{1}{r}}}\right) \left(\frac{\mathbb{P}(A)^{\frac{1}{r}}}{\|f\|_r} \|f\|_r \mathbb{P}(A)^{\frac{1}{p}-\frac{1}{r}} + \mathbb{P}(A)^{\frac{1}{p}} \right) \\
&\lesssim \mathbb{P}(A)^{\frac{1}{p}} \Phi\left(\frac{\|f\|_r}{\mathbb{P}(A)^{\frac{1}{r}}}\right).
\end{aligned}$$

□

2.3 Martingales

Let us recall some standard notations from martingale theory. We refer the reader to [14, 34, 44] for the theory of classical martingale space theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_n\}_{n\geq 0}$ be stated as in Sect. 2.1. The expectation operator and the conditional expectation operator related to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence of measurable functions $f = (f_n)_{n\geq 0} \subset L_1(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$ if $\mathbb{E}_n(f_{n+1}) = f_n$ for every $n \geq 0$. Let $\mathcal{M}$ be the set of all martingale $f = (f_n)_{n\geq 0}$ relative to $\{\mathcal{F}_n\}_{n\geq 0}$ such that $f_0 = 0$. For $f = (f_n)_{n\geq 0} \in \mathcal{M}$, denote its martingale difference by $\mathrm{d}f_n = f_n - f_{n-1}$ ($n \geq 0$, with convention $f_{-1} = 0$). Define the maximal function, the square function and the conditional square function of f, respectively, as follows

$$M_m(f) = \sup_{n \le m} |f_n|, \quad M(f) = \sup_{n \ge 0} |f_n|;$$
$$S_m(f) = \left(\sum_{n=0}^{m} |\mathrm{d}f_n|^2\right)^{1/2}, \quad S(f) = \left(\sum_{n=0}^{\infty} |\mathrm{d}f_n|^2\right)^{1/2};$$
$$s_m(f) = \left(\sum_{n=0}^{m} \mathbb{E}_{n-1}|\mathrm{d}f_n|^2\right)^{1/2}, \quad s(f) = \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1}|\mathrm{d}f_n|^2\right)^{1/2}.$$

For $f = (f_n)_{n\ge 0} \in \mathcal{M}$, if $f_n \in L_{p(\cdot)}$ for every $n \ge 0$, then f is called an $L_{p(\cdot)}$-martingale. Furthermore, if $\|f\|_{p(\cdot)} = \sup_{n\ge 0} \|f_n\|_{p(\cdot)} < \infty$, then f is called a bounded $L_{p(\cdot)}$-martingale and it is denoted by $f \in L_{p(\cdot)}$.

Let Λ be the collection of all sequences $(\lambda_n)_{n\ge 0}$ of nondecreasing, nonnegative and adapted functions, set $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. For $f \in \mathcal{M}$, $\Phi \in \mathcal{G}$, $p(\cdot) \in \mathcal{P}$ and $0 < q \le \infty$, let

$$\Lambda[\mathcal{Q}_{p(\cdot),q,\Phi}](f) = \left\{(\lambda_n)_{n\ge 0} \in \Lambda : S_n(f) \le \lambda_{n-1} \quad (n \ge 1),\ \Phi(\lambda_\infty) \in L_{p(\cdot),q}\right\}$$

and

$$\Lambda[\mathcal{D}_{p(\cdot),q,\Phi}](f) = \left\{(\lambda_n)_{n\ge 0} \in \Lambda : |f_n| \le \lambda_{n-1} \quad (n \ge 1),\ \Phi(\lambda_\infty) \in L_{p(\cdot),q}\right\}.$$

Set

$$\|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} = \inf_{(\lambda_n)_{n\ge 0} \in \Lambda[\mathcal{Q}_{p(\cdot),q,\Phi}](f)} \|\Phi(\lambda_\infty)\|_{p(\cdot),q},$$
$$\|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} = \inf_{(\lambda_n)_{n\ge 0} \in \Lambda[\mathcal{D}_{p(\cdot),q,\Phi}](f)} \|\Phi(\lambda_\infty)\|_{p(\cdot),q}.$$

Now let us recall the notion of the log-Hölder continuity condition. Given a function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, let $p_\infty = \lim_{x\to\infty} p(x)$, then $p(\cdot)$ is said to satisfy the log-Hölder continuity condition, if for all $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C}{\log(e + \frac{1}{|x-y|})}$$

and

$$|p(x) - p_\infty| \le \frac{C}{\log(e + |x|)}.$$

We mention that if $p_- > 1$ and $p(\cdot)$ satisfies the log-Hölder continuity condition in the Euclidean spaces $\mathbb{R}^n$, then the Hardy–Littlewood maximal operator is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$ and the inverse Hölder's inequality holds for the characteristic functions defined on cubes in $L_{p(\cdot)}(\mathbb{R}^n)$ (see [8, 38]). Compared with the Euclidean space $\mathbb{R}^n$, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has no natural metric structure. Fortunately, Jiao et al. [15, 25, 26] found the following condition without metric characterization to replace the log-Hölder continuity condition in some sense. That is, there exists an absolute constant $K_{p(\cdot)} \geq 1$ depending only on $p(\cdot)$ such that

$$\mathbb{P}(B)^{p_-(B)-p_+(B)} \leq K_{p(\cdot)}, \quad \forall\, B \in \bigcup_{n\geq 0} A(\mathcal{F}_n), \tag{2.2}$$

where $\mathcal{F}_n$ is generated by countable atoms ($B \in \mathcal{F}_n$ is called an atom if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $\mathbb{P}(A) = \mathbb{P}(B)$ or $\mathbb{P}(A) = 0$) and $A(\mathcal{F}_n)$ is the set of all atoms in $\mathcal{F}_n$. We always assume that every $\mathcal{F}_n$ is generated by at most countable many atoms in the sequel.

Lemma 2.9 (See [15, 21, 26]) *Let $p(\cdot) \in \mathcal{P}$ satisfy (2.2). Then*

(1) For $p_- > 1$ and $B \in \bigcup_{n\geq 0} A(\mathcal{F}_n)$, we have

$$\mathbb{P}(B) \approx \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)}.$$

(2) For $q(\cdot) \in \mathcal{P}$ satisfying (2.2) and $r(\cdot) \in \mathcal{P}$. If $r(\cdot)$ satisfies

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)},$$

then $r(\cdot)$ also satisfies (2.2). Moreover, for $B \in \bigcup_{n\geq 0} A(\mathcal{F}_n)$, we have

$$\|\chi_B\|_{r(\cdot)} \approx \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{q(\cdot)}.$$

(3) For $p_- > 1$ and $f \in L_{p(\cdot)}$, we have

$$\|M(f)\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}.$$

3 Atomic decompositions

In this section, we construct some new atomic decomposition theorems. The atoms we used here are simple atoms. Note that there are two notions of atoms here, one is a measurable set described in Sect. 2, the other is a measurable function defined as follows.

Definition 3.1 Let $p(\cdot) \in \mathcal{P}$, $1 < r \leq \infty$ and $\Phi \in \mathcal{G}$. A measurable function a is called a simple $(\Phi, p(\cdot), r)^s$-atom (resp. $(\Phi, p(\cdot), r)^S$-atom, $(\Phi, p(\cdot), r)^M$-atom) if there exists $I \in A(\mathcal{F}_m)$ for some $m \in \mathbb{N}$ such that

1. the support of a is contained in I;
2. $\|s(a)\|_r$ (resp. $\|S(a)\|_r$, $\|M(a)\|_r$) $\leq \mathbb{P}(I)^{1/r} \Phi^{-1}\left(\frac{1}{\|\chi_I\|_{p(\cdot)}}\right)$;
3. $\mathbb{E}_m(a) = 0$.

Given $p(\cdot) \in \mathcal{P}$ and $0 < q \leq \infty$. Denote by $\mathcal{A}^s(\Phi, p(\cdot), q, r)$ (resp. $\mathcal{A}^S(\Phi, p(\cdot), q, r)$, $\mathcal{A}^M(\Phi, p(\cdot), q, r)$) the collection of all sequences of triples $(a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}}$, where $(a^{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}}$ are simple $(\Phi, p(\cdot), r)^s$-atoms (resp. $(\Phi, p(\cdot), r)^S$-atoms, $(\Phi, p(\cdot), r)^M$-atoms), $(I_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}} \subset A(\mathcal{F}_m)$ are disjoint for fixed k and satisfying (1), (2) and (3) in Definition 3.1. Moreover,

$$\mu_{k,m,i} = \frac{2\Phi^{-1}(2^{k+1})}{\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)}$$

and

$$\sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_{i} \Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{q} < \infty, \quad (0 < q < \infty),$$

(with the usual modification for $q = \infty$).

Remark 3.2 (i) Note that if we consider the special case $\Phi(t) = t$, then the simple $(\Phi, p(\cdot), r)^\star$-atom ($\star = s, S, M$) is the same as Definition 3.1 in Jiao et al. [22]. Moreover, if $\Phi(t) = t$, $p(\cdot) \equiv p$ and $r = \infty$, then the simple $(\Phi, p(\cdot), r)^\star$-atom ($\star = s, S, M$) is the same as Definition 2.4 in Weisz [44].

(ii) It follows from [22] that if a is a simple $(\Phi, p(\cdot), r)^\star$-atom ($\star = s, S, M$, resp.) associated with $I \in A(\mathcal{F}_m)$ for some $m \in \mathbb{N}$, then

$$s(a) = s(a)\chi_I, \quad S(a) = S(a)\chi_I \quad \text{and} \quad M(a) = M(a)\chi_I.$$

(iii) Let $\Phi \in \mathcal{G}$ be concave. If $(a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i}) \in \mathcal{A}^s(\Phi, p(\cdot), q, r)$, then

$$\Phi\left(\frac{\mu_{k,m,i}\|s(a^{k,m,i})\|_r}{\mathbb{P}(I_{k,m,i})^{1/r}}\right) \leq 2^{k+2}.$$

Theorem 3.3 *Let $p(\cdot) \in \mathcal{P}$ satisfy (2.2), $\Phi \in \mathcal{G}$ be a concave function, $0 < q \leq \infty$ and $1 < r \leq \infty$. If the martingale $f = (f_n)_{n\geq 0}$ satisfies $\left\|\Phi\big(s(f)\big)\right\|_{p(\cdot),q} < \infty$, then there exists a sequence of triples $(a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}} \subset \mathcal{A}^s(\Phi, p(\cdot), q, r)$ such that for each $n \geq 0$,*

$$f_n = \sum_{k\in\mathbb{Z}} \sum_{m=0}^{n-1} \sum_{i} \mu_{k,m,i}\mathbb{E}_n(a^{k,m,i}) \quad a.e., \tag{3.1}$$

and

$$\left\{\sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_{i} \Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{q}\right\}^{1/q} \tag{3.2}$$
$$\lesssim \|\Phi(s(f))\|_{p(\cdot),q}, \quad (0 < q < \infty),$$

(with the usual modification for $q = \infty$).

Conversely, if $\max\{p_+, 1\} < r \leq \infty$ and the martingale $f = (f_n)_{n\geq 0}$ has a decomposition of type (3.1), then for $0 < q < \infty$,

$$\|\Phi(s(f))\|_{p(\cdot),q} \lesssim \inf \left\{ \sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_i \Phi\left[\mu_{k,m,i} \Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}} \right) \right] \chi_{I_{k,m,i}} \right\|_{p(\cdot)}^q \right\}^{1/q},$$

(with the usual modification for $q = \infty$), where the infimum is taken over all the preceding decompositions of f of the form (3.1).

Proof Let $f = (f_n)_{n\geq 0}$ be a martingale with $\left\|\Phi\big(s(f)\big)\right\|_{p(\cdot),q} < \infty$. For every $k \in \mathbb{Z}$, define

$$\tau_k = \inf\left\{n \in \mathbb{N} : s_{n+1}(f) > \Phi^{-1}(2^k)\right\} \quad (\inf \emptyset = \infty).$$

Apparently, τ_k is a stopping time and $\tau_k \leq \tau_{k+1}$ for each $k \in \mathbb{Z}$. It is easy to see that for each $n \in \mathbb{N}$,

$$f_n = \sum_{k\in\mathbb{Z}} \left(f_n^{\tau_{k+1}} - f_n^{\tau_k}\right) \quad \text{a.e.},$$

where $f^{\tau_k} := (f_{n\wedge\tau_k})_{n\geq 0}$. Note that for fixed $k \in \mathbb{Z}$ and $m \in \mathbb{N}$, $\{\tau_k = m\} \in \mathcal{F}_m$. Then there exist disjoint atoms $(I_{k,m,i})_i \subset A(\mathcal{F}_m)$ such that

$$\{\tau_k = m\} = \bigcup_i I_{k,m,i}. \tag{3.3}$$

For each $n \geq 0$, set

$$\mu_{k,m,i} = \frac{2\Phi^{-1}(2^{k+1})}{\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)} \quad \text{and} \quad a_n^{k,m,i} = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_{k,m,i}} \chi_{I_{k,m,i}}.$$

Hence, for each $n \geq 0$, f_n can be represented as follows

$$\begin{aligned} f_n &= \sum_{k\in\mathbb{Z}} \left(f_n^{\tau_{k+1}} - f_n^{\tau_k}\right)\chi_{\{\tau_k<n\}} \\ &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{n-1} \left(f_n^{\tau_{k+1}} - f_n^{\tau_k}\right)\chi_{\{\tau_k=m\}} \\ &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{n-1} \sum_i \left(f_n^{\tau_{k+1}} - f_n^{\tau_k}\right)\chi_{I_{k,m,i}} = \sum_{k\in\mathbb{Z}} \sum_{m=0}^{n-1} \sum_i \mu_{k,m,i} a_n^{k,m,i}. \end{aligned}$$

For fixed k, m and i, $(a_n^{k,m,i})_{n\geq 0}$ is a martingale. Moreover, in view of the definition of τ_k, we have $s(f^{\tau_k}) = s_{\tau_k}(f) \leq \Phi^{-1}(2^k)$. Hence, using the definition of $\mu_{k,m,i}$, we obtain

$$s\big((a_n^{k,m,i})_{n\geq 0}\big) \leq \frac{s(f_n^{\tau_{k+1}}) + s(f_n^{\tau_k})}{\mu_{k,m,i}} \leq \Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right).$$

That is, $(a_n^{k,m,i})_{n\geq 0}$ is an L_2-bounded martingale. Thus there exists an $a^{k,m,i} \in L_2$ such that $\mathbb{E}_n(a^{k,m,i}) = a_n^{k,m,i}$ and

$$\|s(a^{k,m,i})\|_r \leq \mathbb{P}(I_{k,m,i})^{1/r}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right).$$

Furthermore,

$$\begin{aligned}(f_n^{\tau_{k+1}} - f_n^{\tau_k})\chi_{I_{k,m,i}} &= \left(\sum_{l=0}^{n}\chi_{\{l\leq\tau_{k+1}\}}\mathrm{d}f_l - \sum_{l=0}^{n}\chi_{\{l\leq\tau_k\}}\mathrm{d}f_l\right)\chi_{I_{k,m,i}}\\ &= \chi_{I_{k,m,i}}\sum_{l=0}^{n}\mathrm{d}f_l\,\chi_{\{\tau_k<l\leq\tau_{k+1}\}}\\ &= \chi_{I_{k,m,i}}\sum_{l=m+1}^{n}\mathrm{d}f_l\,\chi_{\{\tau_k<l\leq\tau_{k+1}\}}.\end{aligned}$$

Hence, $\mathbb{E}_m(a^{k,m,i}) = 0$. Thus, we conclude that $a^{k,m,i}$ is really a simple $(\Phi, p(\cdot), r)^s$-atom.

Now we show (3.2). For $0 < q < \infty$, since $\chi_{\{\tau_k<\infty\}} = \sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}$, it follows from $\{\tau_k < \infty\} = \big\{s(f) > \Phi^{-1}(2^k)\big\}$ and Lemma 2.7 that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left\|\sum_{m=0}^{\infty}\sum_i \Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}}\right\|_{p(\cdot)}^q\\ &= \sum_{k\in\mathbb{Z}}\left\|\sum_{m=0}^{\infty}\sum_i \Phi\big(2\Phi^{-1}(2^{k+1})\big)\chi_{I_{k,m,i}}\right\|_{p(\cdot)}^q \leq \sum_{k\in\mathbb{Z}}2^{(k+2)q}\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^q\\ &= \sum_{k\in\mathbb{Z}}2^{(k+2)q}\|\chi_{\{\Phi(s(f))>2^k\}}\|_{p(\cdot)}^q \lesssim \sum_{k\in\mathbb{Z}}\|\chi_{\{\Phi(s(f))>2^k\}}\|_{p(\cdot)}^q\int_{2^{k-1}}^{2^k}y^q\frac{\mathrm{d}y}{y}\\ &\leq \sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k}\left(y\|\chi_{\{\Phi(s(f))>y\}}\|_{p(\cdot)}\right)^q\frac{\mathrm{d}y}{y} = \int_0^{\infty}\left(y\|\chi_{\{\Phi(s(f))>y\}}\|_{p(\cdot)}\right)^q\frac{\mathrm{d}y}{y}\\ &= \|\Phi(s(f))\|_{p(\cdot),q}^q.\end{aligned}$$

For $q=\infty$, similarly to the proof of the case $0<q<\infty$, we have

$$\begin{aligned}&\left\|\sum_{m=0}^{\infty}\sum_i \Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}}\right\|_{p(\cdot)}\\&\quad\le 2^{k+2}\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}=2^{k+2}\|\chi_{\{\Phi(s(f))>2^k\}}\|_{p(\cdot)}\\&\quad\lesssim \|\Phi(s(f))\|_{p(\cdot),\infty}.\end{aligned}$$

Thus, we conclude that $(a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}}\subset\mathcal{A}^s(\Phi, p(\cdot), q, r)$ and (3.2) holds.

To prove the converse part, let $f=(f_n)_{n\ge0}$ have a decomposition of type (3.1). It follows from the sublinearity of the conditional square operator s and subadditivity of Φ that

$$\begin{aligned}\Phi\big(s(f)\big)\quad&\le\Phi\left(\textstyle\sum_{k\in\mathbb{Z}}\sum_{m=0}^{\infty}\sum_i \mu_{k,m,i}s(a^{k,m,i})\right)\\&\le\textstyle\sum_{k\in\mathbb{Z}}\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)\quad\text{a.e.}\end{aligned}\tag{3.4}$$

For an arbitrary integer k_0, let

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)=T_1+T_2,$$

where

$$T_1=\sum_{k=-\infty}^{k_0-1}\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)$$

and

$$T_2=\sum_{k=k_0}^{\infty}\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big).$$

For $0<q<\infty$. Let $0<\theta<\min\{p_-, q, 1\}$. We can choose λ such that $1<\lambda<\min\{1/\theta, r/p_+\}$. Using Hölder's inequality, we obtain

$$\begin{aligned}T_1&\le\left(\sum_{k=-\infty}^{k_0-1}2^{k\sigma\lambda'}\right)^{\frac{1}{\lambda'}}\left\{\sum_{k=-\infty}^{k_0-1}2^{-k\sigma\lambda}\left[\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)\right]^{\lambda}\right\}^{\frac{1}{\lambda}}\\&=\left(\frac{2^{k_0\sigma\lambda'}}{2^{\sigma\lambda'}-1}\right)^{\frac{1}{\lambda'}}\left\{\sum_{k=-\infty}^{k_0-1}2^{-k\sigma\lambda}\left[\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)\right]^{\lambda}\right\}^{\frac{1}{\lambda}},\end{aligned}$$

where $0 < \sigma < 1 - 1/\lambda$ and $1/\lambda + 1/\lambda' = 1$. By the θ-triangle inequality and Lemma 2.1, we obtain

$$\begin{aligned}
\|\chi_{\{T_1>2^{k_0}\}}\|_{p(\cdot)} &\leq \frac{1}{2^{k_0\lambda}}\|T_1^\lambda\|_{p(\cdot)} \qquad (3.5)\\
&\lesssim 2^{k_0\lambda(\sigma-1)}\left\|\sum_{k=-\infty}^{k_0-1}2^{-k\sigma\lambda}\left[\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)\right]^\lambda\right\|_{p(\cdot)}\\
&\lesssim 2^{k_0\lambda(\sigma-1)}\left\{\sum_{k=-\infty}^{k_0-1}2^{-k\sigma\lambda\theta}\left\|\left[\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)\right]^{\lambda\theta}\right\|_{\frac{p(\cdot)}{\theta}}\right\}^{1/\theta}\\
&\leq 2^{k_0\lambda(\sigma-1)}\left\{\sum_{k=-\infty}^{k_0-1}2^{-k\sigma\lambda\theta}\left\|\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)^{\lambda\theta}\right\|_{\frac{p(\cdot)}{\theta}}\right\}^{1/\theta}.
\end{aligned}$$

Now, we first estimate

$$\Xi_{k,\alpha} := \left\|\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)^{\alpha\theta}\right\|_{\frac{p(\cdot)}{\theta}},$$

where $k \in \mathbb{Z}$ and $0 < \alpha < \min\{1/\theta, r/p_+\}$. Since $0 < \theta < \min\{p_-, q, 1\}$, we know that $\|\cdot\|_{\frac{p(\cdot)}{\theta}}$ is a norm. By the duality, there exists a nonnegative measurable function $h \in L_{\zeta(\cdot)}$ with $\|h\|_{\zeta(\cdot)} \leq 1$ such that

$$\Xi_{k,\alpha} \leq 2\int_\Omega \sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)^{\alpha\theta} h\,\mathrm{d}\mathbb{P},$$

where

$$\frac{1}{\zeta(\omega)} + \frac{\theta}{p(\omega)} = 1 \quad \text{a.e. } \omega \in \Omega.$$

By Lemma 2.9(2), we know that $\zeta(\cdot)$ satisfies (2.2). Due to $0 < \alpha < 1/\theta$, $r > 1$, Hölder's inequality and Lemma 2.8, we get

$$\begin{aligned}
\Xi_{k,\alpha} &\lesssim \sum_{m=0}^{\infty}\sum_i \left\|\Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)^{\alpha\theta}\right\|_{\frac{r}{\alpha\theta}} \left\|h\chi_{I_{k,m,i}}\right\|_{\frac{r}{r-\alpha\theta}}\\
&= \sum_{m=0}^{\infty}\sum_i \left\|\Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)\right\|_r^{\alpha\theta} \left\|h\chi_{I_{k,m,i}}\right\|_{\frac{r}{r-\alpha\theta}}\\
&\lesssim \sum_{m=0}^{\infty}\sum_i \Phi\left(\frac{\mu_{k,m,i}\|s(a^{k,m,i})\|_r}{\mathbb{P}(I_{k,m,i})^{1/r}}\right)^{\alpha\theta}\mathbb{P}(I_{k,m,i})^{\frac{\alpha\theta}{r}}\left\|h\chi_{I_{k,m,i}}\right\|_{\frac{r}{r-\alpha\theta}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \sum_{i} \Phi\left(\frac{\mu_{k,m,i}\|s(a^{k,m,i})\|_r}{\mathbb{P}(I_{k,m,i})^{1/r}}\right)^{\alpha\theta} \mathbb{P}(I_{k,m,i}) \\
&\quad \times \left(\frac{1}{\mathbb{P}(I_{k,m,i})} \int_{I_{k,m,i}} h^{\frac{r}{r-\alpha\theta}} \, d\mathbb{P}\right)^{\frac{r-\alpha\theta}{r}} \\
&\le \sum_{m=0}^{\infty} \sum_{i} \Phi\left(\frac{\mu_{k,m,i}\|s(a^{k,m,i})\|_r}{\mathbb{P}(I_{k,m,i})^{1/r}}\right)^{\alpha\theta} \int_{\Omega} \chi_{I_{k,m,i}} \left[M(h^{\frac{r}{r-\alpha\theta}})\right]^{\frac{r-\alpha\theta}{r}} \, d\mathbb{P} \\
&= \int_{\Omega} \sum_{m=0}^{\infty} \sum_{i} \Phi\left(\frac{\mu_{k,m,i}\|s(a^{k,m,i})\|_r}{\mathbb{P}(I_{k,m,i})^{1/r}}\right)^{\alpha\theta} \chi_{I_{k,m,i}} \left[M(h^{\frac{r}{r-\alpha\theta}})\right]^{\frac{r-\alpha\theta}{r}} \, d\mathbb{P} \\
&\lesssim \left\| \sum_{m=0}^{\infty} \sum_{i} \Phi\left(\frac{\mu_{k,m,i}\|s(a^{k,m,i})\|_r}{\mathbb{P}(I_{k,m,i})^{1/r}}\right)^{\alpha\theta} \chi_{I_{k,m,i}} \right\|_{\frac{p(\cdot)}{\theta}} \\
&\quad \times \left\| \left[M(h^{\frac{r}{r-\alpha\theta}})\right]^{\frac{r-\alpha\theta}{r}} \right\|_{\zeta(\cdot)}.
\end{aligned}
$$

Since $\alpha p_+ < r$, we have

$$
\operatorname*{ess\,inf}_{\omega\in\Omega} \zeta(\omega)\frac{r-\alpha\theta}{r} = \frac{p_+}{p_+ - \theta}\frac{r-\alpha\theta}{r} > 1.
$$

Using Lemma 2.1 and Lemma 2.9(3), we have

$$
\begin{aligned}
\left\| \left[M(h^{\frac{r}{r-\alpha\theta}})\right]^{\frac{r-\alpha\theta}{r}} \right\|_{\zeta(\cdot)} &= \left\| M(h^{\frac{r}{r-\alpha\theta}}) \right\|_{\zeta(\cdot)\frac{r-\alpha\theta}{r}}^{\frac{r-\alpha\theta}{r}} \lesssim \left\| h^{\frac{r}{r-\alpha\theta}} \right\|_{\zeta(\cdot)\frac{r-\alpha\theta}{r}}^{\frac{r-\alpha\theta}{r}} \\
&= \|h\|_{\zeta(\cdot)} \le 1.
\end{aligned}
$$

According to Remark 3.2(iii) and the disjointness of $(I_{k,m,i})_{m,i\in\mathbb{N}}$ for fixed $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
\Xi_{k,\alpha} &\lesssim \left\| \sum_{m=0}^{\infty} \sum_{i} \Phi\left(\frac{\mu_{k,m,i}\|s(a^{k,m,i})\|_r}{\mathbb{P}(I_{k,m,i})^{1/r}}\right)^{\alpha\theta} \chi_{I_{k,m,i}} \right\|_{\frac{p(\cdot)}{\theta}} \\
&\le 2^{(k+2)\alpha\theta} \left\| \sum_{m=0}^{\infty} \sum_{i} \chi_{I_{k,m,i}} \right\|_{\frac{p(\cdot)}{\theta}} \\
&= 2^{(k+2)\alpha\theta} \left\| \sum_{m=0}^{\infty} \sum_{i} \chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{\theta}.
\end{aligned} \tag{3.6}
$$

For $1/\lambda < \eta < 1-\sigma$, taking $\alpha = \lambda$ in (3.6), it follows from (3.5), (3.6) and Hölder's inequality that

$$\begin{aligned}
&\|\chi_{\{T_1>2^{k_0}\}}\|_{p(\cdot)} \\
&\lesssim 2^{k_0\lambda(\sigma-1)}\left\{\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma\lambda\theta}\,\Xi_{k,\lambda}\right\}^{1/\theta} \\
&\lesssim 2^{k_0\lambda(\sigma-1)}\left\{\sum_{k=-\infty}^{k_0-1} 2^{k\lambda\theta(1-\sigma)}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{\theta}\right\}^{1/\theta} \\
&= 2^{k_0\lambda(\sigma-1)}\left\{\sum_{k=-\infty}^{k_0-1} 2^{k\lambda\theta(1-\sigma-\eta)}2^{k\lambda\theta\eta}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{\theta}\right\}^{1/\theta} \\
&\le 2^{k_0\lambda(\sigma-1)}\left\{\sum_{k=-\infty}^{k_0-1} 2^{k\lambda\theta(1-\sigma-\eta)\frac{q}{q-\theta}}\right\}^{\frac{q-\theta}{q\theta}} \\
&\quad\times\left\{\sum_{k=-\infty}^{k_0-1} 2^{k\lambda q\eta}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q}\right\}^{1/q} \\
&\lesssim 2^{-k_0\lambda\eta}\left\{\sum_{k=-\infty}^{k_0-1} 2^{k\lambda q\eta}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q}\right\}^{1/q}.
\end{aligned}\tag{3.7}$$

Then, it follows from the Abel transformation and the monotonicity of Φ^{-1} that

$$\begin{aligned}
&\sum_{k_0=-\infty}^{\infty} 2^{k_0 q}\|\chi_{\{T_1>2^{k_0}\}}\|_{p(\cdot)}^{q} \\
&\lesssim \sum_{k_0=-\infty}^{\infty} 2^{k_0 q(1-\lambda\eta)}\sum_{k=-\infty}^{k_0-1} 2^{k\lambda q\eta}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q} \\
&= \sum_{k=-\infty}^{\infty} 2^{k\lambda q\eta}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q}\sum_{k_0=k+1}^{\infty} 2^{k_0 q(1-\lambda\eta)} \\
&\lesssim \sum_{k=-\infty}^{\infty} 2^{kq}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q} \\
&\lesssim \sum_{k\in\mathbb{Z}}\left\|\sum_{m=0}^{\infty}\sum_i \Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q}.
\end{aligned}\tag{3.8}$$

Now, we estimate T_2. For the above symbol θ, let $0 < \beta < \xi < 1$. Taking $\alpha = \beta$ in (3.6), it follows from Minkowski's inequality and (3.6) that

$$\begin{aligned}
\|\chi_{\{T_2>2^{k_0}\}}\|_{p(\cdot)} &\le \frac{1}{2^{k_0\beta}}\|T_2^\beta\|_{p(\cdot)} \qquad (3.9)\\
&= \frac{1}{2^{k_0\beta}}\left\{\left\|\left[\sum_{k=k_0}^{\infty}\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)\right]^{\beta\theta}\right\|_{\frac{p(\cdot)}{\theta}}\right\}^{1/\theta}\\
&\le \frac{1}{2^{k_0\beta}}\left\{\sum_{k=k_0}^{\infty}\left\|\sum_{m=0}^{\infty}\sum_i \Phi\big(\mu_{k,m,i}s(a^{k,m,i})\big)^{\beta\theta}\right\|_{\frac{p(\cdot)}{\theta}}\right\}^{1/\theta}\\
&= \frac{1}{2^{k_0\beta}}\left\{\sum_{k=k_0}^{\infty}\Xi_{k,\beta}\right\}^{1/\theta} \lesssim \frac{1}{2^{k_0\beta}}\left\{\sum_{k=k_0}^{\infty}2^{(k+2)\beta\theta}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{\theta}\right\}^{1/\theta}\\
&\approx \frac{1}{2^{k_0\beta}}\left\{\sum_{k=k_0}^{\infty}2^{k\theta(\beta-\xi)}2^{k\xi\theta}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{\theta}\right\}^{1/\theta}\\
&\le \frac{1}{2^{k_0\beta}}\left\{\sum_{k=k_0}^{\infty}2^{k\theta(\beta-\xi)\frac{q}{q-\theta}}\right\}^{\frac{q-\theta}{q\theta}}\left\{\sum_{k=k_0}^{\infty}2^{kq\xi}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q}\right\}^{1/q}\\
&\lesssim \frac{1}{2^{k_0\xi}}\left\{\sum_{k=k_0}^{\infty}2^{kq\xi}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q}\right\}^{1/q}.
\end{aligned}$$

By the Abel transformation and the monotonicity of Φ^{-1}, we have

$$\begin{aligned}
&\sum_{k_0=-\infty}^{\infty}2^{k_0q}\|\chi_{\{T_2>2^{k_0}\}}\|_{p(\cdot)}^q \qquad (3.10)\\
&\lesssim \sum_{k_0=-\infty}^{\infty}2^{k_0q(1-\xi)}\sum_{k=k_0}^{\infty}2^{kq\xi}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^q\\
&= \sum_{k=-\infty}^{\infty}2^{kq\xi}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^q\sum_{k_0=-\infty}^{k}2^{k_0q(1-\xi)}\\
&\lesssim \sum_{k=-\infty}^{\infty}2^{kq}\left\|\sum_{m=0}^{\infty}\sum_i \chi_{I_{k,m,i}}\right\|_{p(\cdot)}^q\\
&\lesssim \sum_{k\in\mathbb{Z}}\left\|\sum_{m=0}^{\infty}\sum_i \Phi\left[\mu_{k,m,i}\Phi^{-1}\Big(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\Big)\right]\chi_{I_{k,m,i}}\right\|_{p(\cdot)}^q.
\end{aligned}$$

Combining (3.4), (3.8) and (3.10), we conclude that

$$\|\Phi(s(f))\|_{p(\cdot),q}^q \approx \sum_{k_0\in\mathbb{Z}}2^{(k_0+1)q}\|\chi_{\{\Phi(s(f))>2^{k_0+1}\}}\|_{p(\cdot)}^q$$

$$\leq \sum_{k_0\in\mathbb{Z}} 2^{(k_0+1)q} \left\| \chi_{\{T_1+T_2>2^{k_0+1}\}} \right\|_{p(\cdot)}^q$$
$$\lesssim \sum_{k_0\in\mathbb{Z}} 2^{(k_0+1)q} \left(\left\| \chi_{\{T_1>2^{k_0}\}} \right\|_{p(\cdot)}^q + \left\| \chi_{\{T_2>2^{k_0}\}} \right\|_{p(\cdot)}^q \right)$$
$$\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \left\| \chi_{\{T_1>2^{k_0}\}} \right\|_{p(\cdot)}^q + \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \left\| \chi_{\{T_2>2^{k_0}\}} \right\|_{p(\cdot)}^q$$
$$\lesssim \sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_i \Phi\left[\mu_{k,m,i} \Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}} \right) \right] \chi_{I_{k,m,i}} \right\|_{p(\cdot)}^q .$$

Taking over all the admissible representations of (3.1) for f, we obtain the desired result.

For $q = \infty$. On one hand, since $0 < \sigma < 1 - \frac{1}{\lambda}$, it follows from (3.7) that

$$\begin{aligned} &2^{k_0} \|\chi_{\{T_1>2^{k_0}\}}\|_{p(\cdot)} \\ &\lesssim 2^{k_0(\lambda\sigma-\lambda+1)} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{k\lambda\theta(1-\sigma)} \left\| \sum_{m=0}^{\infty} \sum_i \chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{\theta} \right\}^{1/\theta} \\ &= 2^{k_0(\lambda\sigma-\lambda+1)} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{k\theta(\lambda-\lambda\sigma-1)} 2^{k\theta} \left\| \sum_{m=0}^{\infty} \sum_i \chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{\theta} \right\}^{1/\theta} \\ &\leq 2^{k_0(\lambda\sigma-\lambda+1)} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{k\theta(\lambda-\lambda\sigma-1)} \right\}^{1/\theta} \sup_{k\in\mathbb{Z}} 2^k \left\| \sum_{m=0}^{\infty} \sum_i \chi_{I_{k,m,i}} \right\|_{p(\cdot)} \\ &\lesssim \sup_{k\in\mathbb{Z}} 2^k \left\| \sum_{m=0}^{\infty} \sum_i \chi_{I_{k,m,i}} \right\|_{p(\cdot)} . \end{aligned} \tag{3.11}$$

On the other hand, since $0 < \beta < 1$, it follows from (3.9) that

$$\begin{aligned} &2^{k_0} \|\chi_{\{T_2>2^{k_0}\}}\|_{p(\cdot)} \\ &\lesssim 2^{k_0(1-\beta)} \left\{ \sum_{k=k_0}^{\infty} 2^{(k+2)\beta\theta} \left\| \sum_{m=0}^{\infty} \sum_i \chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{\theta} \right\}^{1/\theta} \\ &\lesssim 2^{k_0(1-\beta)} \left\{ \sum_{k=k_0}^{\infty} 2^{k\theta(\beta-1)} 2^{k\theta} \left\| \sum_{m=0}^{\infty} \sum_i \chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{\theta} \right\}^{1/\theta} \\ &\leq 2^{k_0(1-\beta)} \left\{ \sum_{k=k_0}^{\infty} 2^{k\theta(\beta-1)} \right\}^{1/\theta} \sup_{k\in\mathbb{Z}} 2^k \left\| \sum_{m=0}^{\infty} \sum_i \chi_{I_{k,m,i}} \right\|_{p(\cdot)} \\ &\lesssim \sup_{k\in\mathbb{Z}} 2^k \left\| \sum_{m=0}^{\infty} \sum_i \chi_{I_{k,m,i}} \right\|_{p(\cdot)} . \end{aligned} \tag{3.12}$$

By (3.11), (3.12) and the monotonicity of Φ^{-1}, we obtain

$$\begin{aligned}\left\|\Phi(s(f))\right\|_{p(\cdot),\infty} &\approx \sup_{k_0\in\mathbb{Z}} 2^{k_0+1}\left\|\chi_{\{\Phi(s(f))>2^{k_0+1}\}}\right\|_{p(\cdot)}\\ &\lesssim \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left\|\chi_{\{T_1>2^{k_0}\}}\right\|_{p(\cdot)} + \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left\|\chi_{\{T_2>2^{k_0}\}}\right\|_{p(\cdot)}\\ &\lesssim \sup_{k\in\mathbb{Z}} 2^{k}\left\|\sum_{m=0}^{\infty}\sum_{i}\chi_{I_{k,m,i}}\right\|_{p(\cdot)}\\ &\lesssim \sup_{k\in\mathbb{Z}}\left\|\sum_{m=0}^{\infty}\sum_{i}\Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}}\right\|_{p(\cdot)}.\end{aligned}$$

Taking over all the admissible representations of (3.1) for f, we obtain the desired result. This completes the proof of the theorem. □

Obviously, we can obtain that the converse part of Theorem 3.3 also holds for $\left\|\Phi\big(M(f)\big)\right\|_{p(\cdot),q}$ and $\left\|\Phi\big(S(f)\big)\right\|_{p(\cdot),q}$ as follows.

Theorem 3.4 *Let $p(\cdot)\in\mathcal{P}$ satisfy (2.2), $\Phi\in\mathcal{G}$ be a concave function, $0<q\le\infty$ and $\max\{p_+,1\}<r\le\infty$. If the martingale $f=(f_n)_{n\ge0}$ has a decomposition of type (3.1) with $(a^{k,m,i},I_{k,m,i},\mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}}\subset\mathcal{A}^M(\Phi,p(\cdot),q,r)$ $\big($resp. $\mathcal{A}^S(\Phi,p(\cdot),q,r)\big)$, then for $0<q<\infty$,*

$$\begin{aligned}&\left\|\Phi\big(M(f)\big)\right\|_{p(\cdot),q}\\ &\quad\lesssim \inf\left\{\sum_{k\in\mathbb{Z}}\left\|\sum_{m=0}^{\infty}\sum_{i}\Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q}\right\}^{1/q}\\ &\Bigg(\text{resp. }\left\|\Phi\big(S(f)\big)\right\|_{p(\cdot),q}\\ &\quad\lesssim \inf\left\{\sum_{k\in\mathbb{Z}}\left\|\sum_{m=0}^{\infty}\sum_{i}\Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q}\right\}^{1/q}\Bigg),\end{aligned}$$

(with the usual modification for $q=\infty$), where the infimum is taken over all the preceding decompositions of f of the form (3.1).

We now establish the ∞-atom decompositions for $\Phi(f)$ in $\mathcal{D}_{p(\cdot),q}$ and $\mathcal{Q}_{p(\cdot),q}$.

Theorem 3.5 *Let $p(\cdot)\in\mathcal{P}$ satisfy (2.2), $\Phi\in\mathcal{G}$ be a concave function and $0<q\le\infty$. If the martingale $f=(f_n)_{n\ge0}$ satisfies $\|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}}<\infty$, then there exists a sequence of triples $(a^{k,m,i},I_{k,m,i},\mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}}\subset\mathcal{A}^M(\Phi,p(\cdot),q,\infty)$ such that for each $n\ge0$, (3.1) holds and*

$$\left\{\sum_{k\in\mathbb{Z}}\left\|\sum_{m=0}^{\infty}\sum_{i}\Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}}\right\|_{p(\cdot)}^{q}\right\}^{1/q} \tag{3.13}$$

$$\lesssim \|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}}, \quad (0 < q < \infty),$$

(with the usual modification for $q = \infty$*).*

Conversely, if the martingale $f = (f_n)_{n\geq 0}$ *has a decomposition of type (3.1), then for* $0 < q < \infty$,

$$\|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} \lesssim \inf \left\{ \sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_{i} \Phi\left[\mu_{k,m,i} \Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right] \chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{q} \right\}^{1/q},$$

(with the usual modification for $q = \infty$*), where the infimum is taken over all the preceding decompositions of* f *of the form (3.1).*

Proof The proof follows the ideas in Theorem 3.3, so we omit some details. For $0 < q < \infty$. Suppose that $f = (f_n)_{n\geq 0}$ satisfies $\|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} < \infty$. We define stopping times as follows

$$\tau_k = \inf\left\{n \in \mathbb{N} : \lambda_n > \Phi^{-1}(2^k)\right\}, \qquad \inf \emptyset = \infty,$$

where $(\lambda_n)_{n\geq 0}$ is a sequence of nondecreasing, nonnegative and adapted functions such that $|f_n| \leq \lambda_{n-1}$ and $\Phi(\lambda_\infty) \in L_{p(\cdot),q}$.

Let $(a^{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}}$ and $(\mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}}$ be defined as in the proof of the Theorem 3.3. Obviously, $a^{k,m,i}$ is a simple $(\Phi, p(\cdot), \infty)^M$-atom for fixed $k \in \mathbb{Z}$, $m, i \in \mathbb{N}$. Now we present $(a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}} \subset \mathcal{A}^M(\Phi, p(\cdot), q, \infty)$. Since $\{\tau_k < \infty\} = \{\Phi(\lambda_\infty) > 2^k\}$, then we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_{i} \Phi\Big[\mu_{k,m,i} \Phi^{-1}\Big(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\Big)\Big] \chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{q} \\
&\quad \leq \sum_{k\in\mathbb{Z}} 2^{(k+2)q} \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{q} = \sum_{k\in\mathbb{Z}} 2^{(k+2)q} \|\chi_{\{\Phi(\lambda_\infty)>2^k\}}\|_{p(\cdot)}^{q} \\
&\quad \lesssim \|\Phi(\lambda_\infty)\|_{p(\cdot),q}^{q}.
\end{aligned}$$

Taking the infimum over $(\lambda_n)_{n\geq 0} \in \Lambda[\mathcal{D}_{p(\cdot),q,\Phi}](f)$, we obtain the desired result.

For the converse part, let

$$\lambda_n = \sum_{k\in\mathbb{Z}} \sum_{m=0}^{n} \sum_{i} \mu_{k,m,i} \|M(a^{k,m,i})\|_\infty \chi_{I_{k,m,i}}.$$

Clearly, $(\lambda_n)_{n\geq 0}$ is a nondecreasing, nonnegative and adapted sequence with $|f_n| \leq \lambda_{n-1}$. For an arbitrary k_0, set

$$J_1 = \sum_{k=-\infty}^{k_0-1} \sum_{m=0}^{\infty} \sum_{i} \Phi\big(\mu_{k,m,i} \|M(a^{k,m,i})\|_\infty\big) \chi_{I_{k,m,i}}$$

and

$$J_2 = \sum_{k=k_0}^{\infty} \sum_{m=0}^{\infty} \sum_i \Phi\big(\mu_{k,m,i} \|M(a^{k,m,i})\|_\infty\big) \chi_{I_{k,m,i}}.$$

By replacing T_1 (resp. T_2) in the proof of Theorem 3.3 with J_1 (resp. J_2), we get

$$\begin{aligned} \|\Phi(f)\|^q_{\mathcal{D}_{p(\cdot),q}} &\le \|\Phi(\lambda_\infty)\|^q_{p(\cdot),q} \lesssim \|J_1\|^q_{p(\cdot),q} + \|J_2\|^q_{p(\cdot),q} \\ &\lesssim \sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_i \Phi\Big[\mu_{k,m,i} \Phi^{-1}\Big(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\Big)\Big] \chi_{I_{k,m,i}} \right\|^q_{p(\cdot)}. \end{aligned}$$

Taking over all the admissible representations of (3.1) for f, we obtain the desired result. The proof of the case $q = \infty$ is analogous. This completes the proof of the theorem. □

Similar to the method of the proof above, we can present the atomic decomposition for $\Phi(f)$ under the functional of $\|\cdot\|_{\mathcal{Q}_{p(\cdot),q}}$.

Theorem 3.6 *Let $p(\cdot) \in \mathcal{P}$ satisfy (2.2), $\Phi \in \mathcal{G}$ be a concave function and $0 < q \le \infty$. If the martingale $f = (f_n)_{n\ge0}$ satisfies $\|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} < \infty$, then there exists a sequence of triples $(a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}} \subset \mathcal{A}^S(\Phi, p(\cdot), q, \infty)$ such that for each $n \ge 0$, (3.1) holds and*

$$\begin{aligned} &\left\{ \sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_i \Phi\left[\mu_{k,m,i} \Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right] \chi_{I_{k,m,i}} \right\|^q_{p(\cdot)} \right\}^{1/q} \\ &\lesssim \|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}}, \quad (0 < q < \infty), \end{aligned}$$

(with the usual modification for $q = \infty$).

Conversely, if the martingale $f = (f_n)_{n\ge0}$ has a decomposition of type (3.1), then for $0 < q < \infty$,

$$\begin{aligned} &\|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} \\ &\lesssim \inf \left\{ \sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_i \Phi\left[\mu_{k,m,i} \Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right] \chi_{I_{k,m,i}} \right\|^q_{p(\cdot)} \right\}^{1/q}, \end{aligned}$$

(with the usual modification for $q = \infty$), where the infimum is taken over all the preceding decompositions of f of the form (3.1).

4 Φ-moment martingale inequalities

In this section, using the atomic decompositions we presented above, we deduce some fundamental Φ-moment martingale inequalities on variable Lorentz spaces. We need the following lemma.

Lemma 4.1 *(See [14, 34]) Let h be a martingale. Then*

$$\|M(h)\|_2 \le 2\|S(h)\|_2 = 2\|s(h)\|_2 \le 2\|M(h)\|_2;$$
$$\|s(h)\|_r \le \sqrt{\frac{r}{2}}\,\big\|M(h)\big\|_r, \quad (r \ge 2);$$
$$\|s(h)\|_r \le \sqrt{\frac{r}{2}}\,\big\|S(h)\big\|_r, \quad (r \ge 2).$$

Moreover, if the stochastic basis $\{\mathcal{F}_n\}_{n\ge 0}$ *is regular, then*

$$\|M(h)\|_r \approx \|S(h)\|_r \approx \|s(h)\|_r, \quad (0 < r < \infty). \tag{4.1}$$

Theorem 4.2 *Let* $p(\cdot) \in \mathcal{P}$ *satisfy (2.2) and* $\Phi \in \mathcal{G}$ *be a concave function. Then*

(1) For $0 < p_+ < 2$ *and* $0 < q \le \infty$,

$$\big\|\Phi(M(f))\big\|_{p(\cdot),q} \lesssim \big\|\Phi(s(f))\big\|_{p(\cdot),q}, \quad \big\|\Phi(S(f))\big\|_{p(\cdot),q} \lesssim \big\|\Phi(s(f))\big\|_{p(\cdot),q}.$$

(2) For $0 < q \le \infty$,

$$\big\|\Phi(s(f))\big\|_{p(\cdot),q} \lesssim \big\|\Phi(f)\big\|_{\mathcal{D}_{p(\cdot),q}}, \quad \big\|\Phi(s(f))\big\|_{p(\cdot),q} \lesssim \big\|\Phi(f)\big\|_{\mathcal{Q}_{p(\cdot),q}}.$$

(3) For $0 < p_+ < 2$ *and* $0 < q \le \infty$,

$$\big\|\Phi(f)\big\|_{\mathcal{Q}_{p(\cdot),q}} \lesssim \big\|\Phi(f)\big\|_{\mathcal{D}_{p(\cdot),q}} \lesssim \big\|\Phi(f)\big\|_{\mathcal{Q}_{p(\cdot),q}}.$$

Proof (1) Let $0 < p_+ < 2$ and $0 < q \le \infty$. Suppose that $f = (f_n)_{n\ge 0}$ is a martingale with $\big\|\Phi\big(s(f)\big)\big\|_{p(\cdot),q} < \infty$. By Theorem 3.3, there exists a sequence of triples $(a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}} \subset \mathcal{A}^s(\Phi, p(\cdot), q, 2)$ such that (3.1) and (3.2) hold. Using Lemma 4.1, we have

$$\Big\|M\Big(\frac{1}{2}a^{k,m,i}\Big)\Big\|_2 \le \big\|s(a^{k,m,i})\big\|_2 \le \mathbb{P}(I_{k,m,i})^{1/2}\Phi^{-1}\Big(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\Big).$$

Hence, $\frac{1}{2}a^{k,m,i}$ is a simple $(\Phi, p(\cdot), 2)^M$-atom and

$$\Big(\frac{1}{2}a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i}\Big)_{k\in\mathbb{Z},m,i\in\mathbb{N}} \subset \mathcal{A}^M(\Phi, p(\cdot), q, 2).$$

Moreover, applying (3.1), (3.2), Lemma 2.7 and Theorem 3.4, we obtain that for each $n \geq 0$,

$$\frac{1}{2} f_n = \sum_{k\in\mathbb{Z}} \sum_{m=0}^{n-1} \sum_i \mu_{k,m,i} \mathbb{E}_n\Big(\frac{1}{2} a^{k,m,i}\Big) \quad \text{a.e.,}$$

and

$$\begin{aligned}
\big\|\Phi(M(f))\big\|_{p(\cdot),q} &\leq 2 \Big\|\Phi\Big(M\Big(\frac{1}{2}f\Big)\Big)\Big\|_{p(\cdot),q} \\
&\lesssim \left\{\sum_{k\in\mathbb{Z}} \Big\| \sum_{m=0}^{\infty} \sum_i \Phi\Big[\mu_{k,m,i} \Phi^{-1}\Big(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\Big)\Big]\chi_{I_{k,m,i}} \Big\|_{p(\cdot)}^q \right\}^{1/q} \\
&\lesssim \|\Phi(s(f))\|_{p(\cdot),q}, \quad (0 < q < \infty),
\end{aligned}$$

(with the usual modification for $q = \infty$). The second inequality of (1) can be proved analogously.

(2) Let $\max\{p_+, 2\} < r < \infty$, $0 < q \leq \infty$ and martingale $f = (f_n)_{n\geq 0}$ satisfy $\|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} < \infty$. According to Theorem 3.5, there exists a sequence of triples $(a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i})_{k\in\mathbb{Z},m,i\in\mathbb{N}} \subset \mathcal{A}^M(\Phi, p(\cdot), q, \infty)$ such that for each $n \geq 0$, (3.1) and (3.13) hold. For fixed $k \in \mathbb{Z}$, $m, i \in \mathbb{N}$, we get

$$\begin{aligned}
\sqrt{\frac{2}{r}}\, \|s(a^{k,m,i})\|_r &\leq \big\|M(a^{k,m,i})\big\|_r \leq \|\chi_{I_{k,m,i}}\|_r \big\|M(a^{k,m,i})\big\|_\infty \\
&\leq \mathbb{P}(I_{k,m,i})^{1/r} \Phi^{-1}\Big(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\Big).
\end{aligned}$$

By Lemma 4.1 and $a^{k,m,i}$ being a simple $(\Phi, p(\cdot), \infty)^M$-atom. So $\sqrt{\frac{2}{r}}a^{k,m,i}$ is a simple $(\Phi, p(\cdot), r)^s$-atom and

$$\left(\sqrt{\frac{2}{r}}a^{k,m,i}, I_{k,m,i}, \mu_{k,m,i}\right)_{k\in\mathbb{Z},m,i\in\mathbb{N}} \subset \mathcal{A}^s(\Phi, p(\cdot), q, r).$$

It follows from (3.1), (3.2), (3.13) and Lemma 2.7 that for each $n \geq 0$,

$$\sqrt{\frac{2}{r}}\, f_n = \sum_{k\in\mathbb{Z}} \sum_{m=0}^{n-1} \sum_i \mu_{k,m,i} \mathbb{E}_n\left(\sqrt{\frac{2}{r}}\, a^{k,m,i}\right) \quad \text{a.e.,}$$

and

$$\big\|\Phi(s(f))\big\|_{p(\cdot),q} \leq \sqrt{\frac{r}{2}} \left\|\Phi\left(s\left(\sqrt{\frac{2}{r}}f\right)\right)\right\|_{p(\cdot),q}$$

$$
\begin{aligned}
&\lesssim \left\{ \sum_{k\in\mathbb{Z}} \left\| \sum_{m=0}^{\infty} \sum_{i} \Phi\left[\mu_{k,m,i}\Phi^{-1}\left(\frac{1}{\|\chi_{I_{k,m,i}}\|_{p(\cdot)}}\right)\right]\chi_{I_{k,m,i}} \right\|_{p(\cdot)}^{q} \right\}^{1/q} \\
&\lesssim \left\|\Phi(f)\right\|_{\mathcal{D}_{p(\cdot),q}}, \quad (0 < q < \infty),
\end{aligned}
$$

(with the usual modification for $q = \infty$). The second inequality of (2) can be proved analogously.

(3) Let $0 < p_+ < 2$, $0 < q \leq \infty$ and $f = (f_n)_{n\geq 0}$ be a martingale with $\|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} < \infty$. For any $\epsilon > 0$, there exists $(\lambda_n)_{n\geq 0} \in \Lambda[\mathcal{Q}_{p(\cdot),q,\Phi}](f)$ such that $\|\Phi(\lambda_\infty)\|_{p(\cdot),q} \leq \|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} + \epsilon$. Then, there has

$$
\begin{aligned}
|f_n| = |f_n - f_{n-1} + f_{n-1}| &\leq |f_n - f_{n-1}| + M_{n-1}(f) \\
&\leq S_n(f) + M_{n-1}(f) \leq \lambda_{n-1} + M_{n-1}(f).
\end{aligned}
$$

Applying (1) and (2), we have

$$
\begin{aligned}
\left\|\Phi\big(\lambda_\infty + M(f)\big)\right\|_{p(\cdot),q} &\leq \left\|\Phi(\lambda_\infty) + \Phi\big(M(f)\big)\right\|_{p(\cdot),q} \\
&\lesssim \|\Phi(\lambda_\infty)\|_{p(\cdot),q} + \left\|\Phi\big(M(f)\big)\right\|_{p(\cdot),q} \\
&\lesssim \|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} + \epsilon + \left\|\Phi\big(s(f)\big)\right\|_{p(\cdot),q} \\
&\lesssim \|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} + \epsilon.
\end{aligned} \tag{4.2}
$$

Hence, $\big(\lambda_n + M_n(f)\big)_{n\geq 0} \in \Lambda[\mathcal{D}_{p(\cdot),q,\Phi}](f)$. From the definition of $\|\cdot\|_{\mathcal{D}_{p(\cdot),q}}$ and (4.2), we have

$$
\|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} \lesssim \|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} + \epsilon.
$$

Taking $\epsilon \longrightarrow 0$, we obtain the right inequality in (3).

We consider the rest part of (3) in the following. Let $f = (f_n)_{n\geq 0}$ be a martingale with $\|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} < \infty$. For any $\eta > 0$, there exists $(\gamma_n)_{n\geq 0} \in \Lambda[\mathcal{D}_{p(\cdot),q,\Phi}](f)$ such that $\|\Phi(\gamma_\infty)\|_{p(\cdot),q} \leq \|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} + \eta$. Then, this yields

$$
\begin{aligned}
S_n(f) \leq S_{n-1}(f) + |f_n - f_{n-1}| &\leq S_{n-1}(f) + 2M_n(f) \\
&\leq S_{n-1}(f) + 2\gamma_{n-1}.
\end{aligned}
$$

Combining (1) and (2), we get

$$
\begin{aligned}
\left\|\Phi\big(S(f) + 2\gamma_\infty\big)\right\|_{p(\cdot),q} &\leq \left\|\Phi\big(S(f)\big) + \Phi\big(2\gamma_\infty\big)\right\|_{p(\cdot),q} \\
&\lesssim \left\|\Phi\big(S(f)\big)\right\|_{p(\cdot),q} + \left\|\Phi(2\gamma_\infty)\right\|_{p(\cdot),q} \\
&\leq \left\|\Phi\big(S(f)\big)\right\|_{p(\cdot),q} + 2\left\|\Phi(\gamma_\infty)\right\|_{p(\cdot),q} \\
&\lesssim \left\|\Phi\big(s(f)\big)\right\|_{p(\cdot),q} + \|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} + \eta \\
&\lesssim \|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} + \eta.
\end{aligned} \tag{4.3}
$$

This means $\big(S_n(f)+2\gamma_n\big)_{n\geq 0} \in \Lambda[\mathcal{Q}_{p(\cdot),q,\Phi}](f)$. By the definition of $\|\cdot\|_{\mathcal{Q}_{p(\cdot),q}}$ and (4.3), we have

$$\|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} \lesssim \|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} + \eta.$$

Taking $\eta \longrightarrow 0$, we obtain the left inequality in (3). We conclude that

$$\|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}} \lesssim \|\Phi(f)\|_{\mathcal{Q}_{p(\cdot),q}} \lesssim \|\Phi(f)\|_{\mathcal{D}_{p(\cdot),q}}.$$

□

Remark 4.3 Let $p(\cdot) \equiv p$. Then, we obtain all the results in Peng and Li [40]. Moreover, if $p(\cdot) \equiv 1 = q$, then we obtain all the results in Jiao and Yu [24].

Remark 4.4 Let $\Phi(t) = t$. Then, we obtain some results in [21]. Moreover, if $\Phi(t) = t$ and $p(\cdot) \equiv p$, then we obtain many results in the classical martingale space theory as in [17, 44].

Furthermore, if the stochastic basis is regular, similarly to proof of Theorem 3.5 in [21], the other part of Theorem 3.4 also holds, then we can easily get the following further conclusion via (4.1) and the proof process of Theorems 4.2.

Theorem 4.5 *Let $p(\cdot) \in \mathcal{P}$ satisfy (2.2), $\Phi \in \mathcal{G}$ be a concave function and $0 < q \leq \infty$. If the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then*

$$\begin{aligned}\big\|\Phi(M(f))\big\|_{p(\cdot),q} &\approx \big\|\Phi(S(f))\big\|_{p(\cdot),q} \approx \big\|\Phi(s(f))\big\|_{p(\cdot),q} \\ &\approx \big\|\Phi(f)\big\|_{\mathcal{Q}_{p(\cdot),q}} \approx \big\|\Phi(f)\big\|_{\mathcal{D}_{p(\cdot),q}}.\end{aligned}$$

If the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, let $\Phi(t) = t$, then we obtain from Theorem 4.5 that the five variable martingale Lorentz–Hardy spaces are equivalent, see also Corollary 3.8 in [22].

Corollary 4.6 *Let $p(\cdot) \in \mathcal{P}$ satisfy (2.2) and $0 < q \leq \infty$. If the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then*

$$H^M_{p(\cdot),q} = H^S_{p(\cdot),q} = H^s_{p(\cdot),q} = \mathcal{Q}_{p(\cdot),q} = \mathcal{D}_{p(\cdot),q}$$

with equivalent quasi-norms.

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References

1. Almeida, V., Betancor, J., Rodríguez-Mesa, L.: Anisotropic Hardy–Lorentz spaces with variable exponents. Can. J. Math. **69**, 1219–1273 (2017)
2. Acerbi, E., Mingione, G.: Regularity results for a class of functionals with non-standard growth. Arch. Ration. Mech. Anal. **156**, 121–140 (2001)
3. Aoyama, H.: Lebesgue spaces with variable exponent on a probability space. Hiroshima Math. J. **39**, 207–216 (2009)
4. Bekjan, T., Chen, Z.: Interpolation and Φ-moment inequalities of noncommutative martingales. Probab. Theory Relat. Fields **152**, 179–206 (2012)
5. Burkholder, D., Davis, B., Gundy, R.: Integral inequalities for convex functions of operators on martingales. In: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (University of California, Berkeley, California, 1970/1971), vol. II. Probability theory, pp. 223–240. University of California Press, Berkeley (1972)
6. Burkholder, D., Gundy, R.: Extrapolation and interpolation of quasi-linear operators on martingales. Acta Math. **124**, 249–304 (1970)
7. Cruz-Uribe, D., Fiorenza, A.: Variable Lebesgue Spaces, Foundations and Harmonic Analysis. Birkhäuser, Basel (2013)
8. Cruz-Uribe, D., Wang, L.-A. D.: Variable Hardy spaces. Indiana Univ. Math. J. **63**, 447–493 (2014)
9. Diening, L.: Maximal functions on generalized $L^{p(\cdot)}$ spaces. Math. Inequal. Appl. **7**, 245–253 (2004)
10. Diening, L.: Maximal function on Musielak–Orlicz spaces and generalized Lebesgue spaces. Bull. Sci. Math. **129**, 657–700 (2005)
11. Diening, L., Harjulehto, P., Hästö, P., Ruzicka, M.: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics, vol. 2017. Springer, Berlin (2011)
12. Diening, L., Ruzicka, M.: Calderón–Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. J. Reine Angew. Math. **563**, 197–220 (2003)
13. Fan, X., Zhao, D.: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}$. J. Math. Anal. Appl. **263**, 424–446 (2001)
14. Garsia, A.: Martingale Inequalities: Seminar Notes on Recent Progress. Mathematics Lecture Note Series, Benjamin, New York (1973)
15. Hao, Z., Jiao, Y.: Fractional integral on martingale Hardy spaces with variable exponents. Fract. Calc. Appl. Anal. **18**, 1128–1145 (2015)
16. Harjulehto, P., Hästö, P.: Orlicz Spaces and Generalized Orlicz Spaces. Lecture Notes in Mathematics, vol. 2236. Springer, Berlin (2019)
17. Jiao, Y., Peng, L., Liu, P.: Atomic decompositions of Lorentz martingale spaces and applications. J. Funct. Spaces Appl. **7**, 153–166 (2009)
18. Jiao, Y., Quan, X., Wu, L.: Burkholder's inequalities associated with Orlicz functions in rearrangement invariant spaces. Math. Z. **289**, 943–960 (2018)
19. Jiao, Y., Sukochev, F., Xie, G., Zanin, D.: Φ-moment inequalities for independent and freely independent random variables. J. Funct. Anal. **270**, 4558–4596 (2016)
20. Jiao, Y., Weisz, F., Wu, L., Zhou, D.: Real interpolation for variable martingale Hardy spaces. J. Math. Anal. Appl. **491**, 12 (2020)
21. Jiao, Y., Weisz, F., Wu, L., Zhou, D.: Variable martingale Hardy spaces and their applications in Fourier analysis. Diss. Math. **550**, 67 (2020)
22. Jiao, Y., Weisz, F., Wu, L., Zhou, D.: Dual spaces for variable martingale Lorentz–Hardy spaces. Banach J. Math. Anal. **15**, 31 (2021)
23. Jiao, Y., Xie, G., Zhou, D.: Dual spaces and John–Nirenberg inequalities of martingale Hardy–Lorentz–Karamata spaces. Q. J. Math. **66**, 605–623 (2015)
24. Jiao, Y., Yu, W.: Some new Φ-moment martingale inequalities associated with concave functions. Nonlinear Anal. **75**, 4271–4277 (2012)
25. Jiao, Y., Zhou, D., Hao, Z., Chen, W.: Martingale Hardy spaces with variable exponents. Banach J. Math. Anal. **10**, 750–770 (2016)
26. Jiao, Y., Zhou, D., Weisz, F., Hao, Z.: Corrigendum: Fractional integral on martingale Hardy spaces with variable exponents. Fract. Calc. Appl. Anal. **20**, 1051–1052 (2017)

27. Jiao, Y., Zuo, Y., Zhou, D., Wu, L.: Variable Hardy–Lorentz spaces $H^{p(\cdot),q}(\mathbb{R}^n)$. Math. Nachr. **292**, 309–349 (2019)
28. Kempka, H., Vybíral, J.: Lorentz spaces with variable exponents. Math. Nachr. **287**, 938–954 (2014)
29. Kikuchi, M.: On some inequalities for Doob decompositions in Banach function spaces. Math. Z. **265**, 865–887 (2010)
30. Kovàčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czechoslov. Math. J. **41**, 592–618 (1991)
31. Kulak, Ö.: The inclusion theorems for variable exponent Lorentz spaces. Turk. J. Math. **40**, 605–619 (2016)
32. Liu, J., Weisz, F., Yang, D., Yuan, W.: Littlewood–Paley and finite atomic characterizations of anisotropic variable Hardy–Lorentz spaces and their applications. J. Fourier Anal. Appl. **25**, 874–922 (2019)
33. Liu, J., Yang, D., Yuan, W.: Anisotropic variable Hardy–Lorentz spaces and their real interpolation. J. Math. Anal. Appl. **456**, 356–393 (2017)
34. Long, R.: Martingale Spaces and Inequalities. Peking University Press, Beijing (1993)
35. Miyamoto, T., Nakai, E., Sadasue, G.: Martingale Orlicz–Hardy spaces. Math. Nachr. **285**, 670–686 (2012)
36. Musielak, J.: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, vol. 1034. Springer, Berlin (1983)
37. Nakai, E., Sadasue, G.: Maximal function on generalized martingale Lebesgue spaces with variable exponent. Stat. Probab. Lett. **83**, 2168–2171 (2013)
38. Nakai, E., Sawano, Y.: Hardy spaces with variable exponents and generalized Campanato spaces. J. Funct. Anal. **262**, 3665–3784 (2012)
39. Orlicz, W.: Über konjugierte Expoentenfolgen. Stud. Math. **3**, 200–211 (1931)
40. Peng, L., Li, J.: A generalization of Φ-moment martingale inequalities. Stat. Probab. Lett. **102**, 61–68 (2015)
41. Rao, M., Ren, Z.: Theory of Orlicz Spaces. Marcel Dekker, Inc., New York (1991)
42. Ruzicka, M.: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000)
43. Sawano, Y.: Atomic decomposition of Hardy spaces with variable exponents and its application to bounded linear operators. Integral Equ. Oper. Theory **77**, 123–148 (2013)
44. Weisz, F.: Martingale Hardy Spaces and Their Applications in Fourier Analysis. Lecture Notes in Mathematics, vol. 1568. Springer, Berlin (1994)
45. Weisz, F.: Characterizations of variable martingale Hardy spaces via maximal functions. Fract. Calc. Appl. Anal. **24**, 393–420 (2021)
46. Wu, L., Hao, Z., Jiao, Y.: John–Nirenberg inequalities with variable exponents on probability spaces. Tokyo J. Math. **38**, 353–367 (2015)
47. Wu, L., Zhou, D., Jiao, Y.: Modular inequalities in martingale Orlicz–Karamata spaces. Math. Nachr. **291**, 1450–1462 (2018)
48. Wu, L., Zhou, D., Zhuo, C., Jiao, Y.: Riesz transform characterizations of variable Hardy–Lorentz spaces. Rev. Mat. Complut. **31**, 747–780 (2018)
49. Yan, X., Yang, D., Yuan, W., Zhuo, C.: Variable weak Hardy spaces and their applications. J. Funct. Anal. **271**, 2822–2887 (2016)
50. Yang, D., Liang, Y., Ky, L.: Real-Variable Theory of Musielak–Orlicz Hardy Spaces. Lecture Notes in Mathematics, vol. 2182. Springer, Berlin (2017)
51. Zhikov, V.: Averaging of functionals of the calculus of variations and elasticity theory. Izv. Akad. Nauk SSSR Ser. Mat. **50**, 675–710 (1986)
52. Zhikov, V.: On Lavrentiev's phenomenon. Russ. J. Math. Phys. **3**, 249–269 (1995)
53. Zuo, Y., Saibi, K., Jiao, Y.: Variable Hardy–Lorentz spaces associated to operators satisfying Davies–Gaffney estimates. Banach J. Math. Anal. **13**, 769–797 (2019)