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Centre of a compact convex set

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Abstract

We introduce the notion centre of a convex set and study the space of continuous affine functions on a compact convex set with a centre. We show that these spaces are precisely the dual of a base normed space in which the underlying base has a (unique) centre. We also characterize the corresponding base norm space. We obtain a condition on a compact, balanced, convex subset of a locally convex space, so that the corresponding space of continuous affine functions on the convex set is an absolute order unit space. Similarly, we characterize a condition on the base with a centre of a base normed space, so that the latter becomes an absolutely base normed space.

Keywords Lead points of a convex set · Centre of a convex set · Property (S) ·
Tracial absolute order unit space · Absolutely central base normed space

Mathematics Subject Classification 46B40 · 46B20

1 Introduction

Let K be a compact, convex subset of a locally convex space X and let $A_{\mathbb{R}}(K)$ denote the space of real valued, continuous and affine functions defined on K. Then, $A_{\mathbb{R}}(K)$ is a complete order unit space and the state space of $A_{\mathbb{R}}(K)$ is affinely homeomorphic to K. Conversely, if (V, e) is a complete order unit space with its state space $S(V)$, then V is unitally order isomorphic to $A_{\mathbb{R}}(S(V))$. The self-adjoint part of an operator system (that is, a self-adjoint subspace containing identity) in a unital C^*-algebra A is an order unit space. Therefore, the above said discussion describes Kadison's functional representation theorem [4].

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Furthermore, if K and L are compact convex sets in suitable locally convex spaces, then K is affinely homeomorphic to L if and only if $A_{\mathbb{R}}(K)$ is unitally order isomorphic to $A_{\mathbb{R}}(L)$. In other words, there exists a bijective correspondence between the class of compact convex sets of locally convex spaces and the class of complete order unit spaces (see, for example, [1, Theorem II.1.8]).

In this paper, we discuss an intrinsic characterization of compact convex sets which are affine homeomorphic to balanced, convex, compact sets. We also discuss the corresponding subclass of complete order unit spaces. We introduce the notion of a centre of a convex set and prove that a compact convex set K in a locally convex space X is affine homeomorphic to a balanced, convex, compact set L in another locally convex space Y if and only if K has a (unique) centre.

Let $(V^{(\cdot)}, V^{(\cdot)+}, e)$ be the order unit space obtained by adjoining an order unit to a normed linear space V using a construction due to M. M. Day (see Theorem 2.1). Then, the corresponding cone yields a natural base $B^{(\cdot)}$, such that $(V^{(\cdot)}, V^{(\cdot)+}, B^{(\cdot)})$ is a base normed space Proposition 2.3. We prove that $B^{(\cdot)}$ is precisely a base with a centre (Theorem 4.5).

We also study the properties $A(B)$, the space of continuous affine functions on a compact convex set B with a centre b_0 in a locally convex space X. We prove that $A(B)$ is unitally order isomorphic to $(V^{(\cdot)}, e)$ for some Banach space V (Theorem 3.1). We also discuss the order unit spaces whose state space have a centre (Theorem 5.6).

In [7], the author discussed an order theoretic generalization of spin factors obtained by adjoining an order unit to a normed linear space. Let V be a normed linear space and let $(V^{(\cdot)}, e)$ is the order unit space obtained by adjoining an order unit to V. He proved that there is a canonical absolute value defined in $V^{(\cdot)}$ and that $(V^{(\cdot)}, e)$ becomes an absolute order unit space if and only if V is strictly convex [7, Theorem 2.14]. In the present paper, we discuss an absolute value in a base normed space in which the base has a centre and find a condition under which the space becomes an absolutely base normed space (Theorem 6.7).

This paper is a part of the ongoing program to study the various aspects of absolutely ordered spaces initiated by the author [5–7]. In [7] we discussed a geometric aspect of an absolute order unit space. In the present paper, we have discussed a geometric aspect of an absolutely base normed space.

A summary of the paper is as follows.

In Sect. 2, we recall the adjoining of an order unit to a normed linear space and the notion of lead points of a convex set and introduce the notion of a centre of a convex set. We study some related properties.

In Sect. 3, we consider the space of continuous affine functions on a compact convex set with a centre and describe its order and norm structures. We prove that this space can be characterized as one obtained by adjoining an order unit to a Banach space.

In Sect. 4, we study the norm and order structure of a base norm space in which the base has a (unique) centre. We produce an example to show that the base of every base normed space need not have a centre.

In Sect. 5, we consider the order unit spaces having a central state space. We obtain a characterization for a state to be a centre of the state space. We study the norm and order structure an order unit space having a central state and prove that

such a space is precisely obtained by adjoining an order unit to a normed linear space. Let us recall that such a space becomes an absolute order unit space if the underlying normed space is strictly convex [7, Theorem 3.14]. We introduce the Property (S) on a compact, balanced, convex subset of a real locally convex space and prove that an order unit space having a central state is an absolute order unit space if and only if the corresponding state space satisfies Property (S).

In Sect. 6, we describe an absolute value on a base normed space in which the base has a centre. This absolute value arises naturally. We characterize the condition on the underlying base under which the space becomes an absolutely ordered base normed space.

2 Centre of a convex set

2.1 Adjoining an order unit to a normed linear space

The following construction has been adopted from [3, 1.6.1] and is apparently due to M. M. Day.

Theorem 2.1 (M. M. Day) *Let* $(V, \| \cdot \|)$ *be a real normed linear space. Consider* $V^{(\cdot)} := V \times \mathbb{R}$ *and define*

$$V^{(\cdot)+} := \{(v, \alpha) \, : \, \|v\| \leq \alpha\}.$$

Then $(V^{(\cdot)}, V^{(\cdot)+})$ *becomes a real ordered space such that* $V^{(\cdot)+}$ *is proper, generating and Archimedean. In addition,* $e = (0, 1) \in V^{(\cdot)+}$ *is an order unit for* $V^{(\cdot)}$ *so that* $(V^{(\cdot)}, e)$ *becomes an order unit space. The corresponding order unit norm is given by*

$$\|(v, \alpha)\|_e = \|v\| + |\alpha|$$

for all $(v, \alpha) \in V^{(\cdot)}$. *In particular,* $(V^{(\cdot)}, e)$ *is isometrically isomorphic to* $V \oplus_1 \mathbb{R}$. *Thus,* V, *identified with* $\{(v, 0) \, : \, v \in V\}$, *can be identified as a closed subspace of* $V^{(\cdot)}$. *Furthermore,* $V^{(\cdot)}$ *is complete if and only if so is* V.

The following result can proved in a routine way.

Proposition 2.2 *Let* V *be a real normed linear space and consider the corresponding order unit space* $(V^{(\cdot)}, e)$. *Then, the dual of* $(V^{(\cdot)}, e)$ *is a base normed space isometrically isomorphic to* $(V^{*(\cdot)}, S(V^{(\cdot)}))$, *where*

$$S(V^{(\cdot)}) := \{(f, 1) \, : \, f \in V^* \text{ and } \|f\| \leq 1\}$$

is the state space of $(V^{(\cdot)}, e)$ *as well as the base of* $V^{*(\cdot)+}$ *which determine the dual norm as the base norm on* $V^{*(\cdot)}$.

We show that a similar construction is also available on $V^{(\cdot)}$ itself albeit with a different norm (and the same order structure).

Proposition 2.3 *Let V be a real normed linear space and consider the corresponding ordered vector space* $(V^{(\cdot)}, V^{(\cdot)+})$. *Put*

$$B^{(\cdot)} = \{(v, 1) : \|v\| \le 1\}.$$

Then $B^{(\cdot)}$ *is a base for* $V^{(\cdot)+}$, *such that* $(V^{(\cdot)}, V^{(\cdot)+}, B^{(\cdot)})$ *is a base normed space* (*which is denoted by* $V^{(1)}$). *The corresponding base norm is given by*

$$\|(v, \alpha)\|_B = \max\{\|v\|, |\alpha|\}$$

for all $(v, \alpha) \in V^{(\cdot)}$. *In particular,* $(V^{(\cdot)}, B^{(\cdot)})$ *is isometrically isomorphic to* $V \oplus_\infty \mathbb{R}$. *Thus* V, *identified with* $\{(v, 0) : v \in V\}$, *can be identified as a closed subspace of* $V^{(\cdot)}$. *Furthermore,* $V^{(\cdot)}$ *is complete if and only if so is* V.

Proof It suffices to prove that $B^{(\cdot)}$ is a base for $V^{(\cdot)+}$, such that

$$\mathrm{co}\left(B^{(\cdot)} \cup -B^{(\cdot)}\right) = \{(x, \alpha) : \|x\| \le 1, |\alpha| \le 1\}.$$

Note that $B^{(\cdot)}$ is convex. If $(x, \alpha) \in V^{(\cdot)+}$, then $\|x\| \le \alpha$. Therefore, if $(x, \alpha) \ne (0, 0)$, then $\alpha > 0$. Now, letting $x_0 = \alpha^{-1}x$, we get that $(x_0, 1) \in B$ and $(x, \alpha) = \alpha(x_0, 1)$ is a unique representation. We show that

$$\mathrm{co}\left(B^{(\cdot)} \cup -B^{(\cdot)}\right) = \{(x, \alpha) : \|x\| \le 1, |\alpha| \le 1\}.$$

Let $(x, \alpha) \in \mathrm{co}\left(B^{(\cdot)} \cup -B^{(\cdot)}\right)$, say $(x, \alpha) = \lambda(x_1, 1) - (1 - \lambda)(x_2, 1)$ for some $x_1, x_2 \in V$ with $\|x_1\| \le 1$ and $\|x_2\| \le 1$ and $\lambda \in [0, 1]$. Then, $x = \lambda x_1 - (1 - \lambda)x_2$ and $\alpha = \lambda - (1 - \lambda) = 2\lambda - 1$. Thus, $\|x\| \le 1$ and $|\alpha| \le 1$. Conversely, we assume that $x \in V$ and $\alpha \in \mathbb{R}$ with $\alpha \ne 0$ be, such that $\|x\| \le 1$ and $|\alpha| \le 1$. If $\|x\| \le |\alpha|$, then $(\alpha^{-1}x, 1) \in B^{(\cdot)}$ so that $(x, \alpha) = \alpha(\alpha^{-1}x, 1) \in \mathrm{co}\left(B^{(\cdot)} \cup -B^{(\cdot)}\right)$. Therefore, we assume that $|\alpha| < \|x\|$. Put $y = \|x\|^{-1}x$. Then, $(y, 1), (-y, 1) \in B^{(\cdot)}$. Let $2\lambda = \|x\| + \alpha$ and $2\mu = \|x\| - \alpha$. Then, $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = \|x\|$ and $\lambda - \mu = \alpha$. Thus

$$\lambda(y, 1) - \mu(-y, 1) = ((\lambda + \mu)y, \lambda - \mu) = (x, \alpha)$$

so that $(x, \alpha) \in \mathrm{co}\left(B^{(\cdot)} \cup -B^{(\cdot)}\right)$. □

In this section, we shall obtain a geometric description of $B^{(\cdot)}$. First, we recall the following notion introduced in [2].

2.2 Lead points of a convex set

Definition 2.4 [2] Let C be a convex subset of a real vector space X with $0 \in C$. An element $x \in C$ is called a *lead point* of C, if for any $y \in C$ and $\lambda \in [0, 1]$ with $x = \lambda y$, we have $\lambda = 1$ and $y = x$. The set of all lead points of C is denoted by $\mathrm{Lead}(C)$.

The following result was essentially proved in [2, Proposition 3.2].

Lemma 2.5 *Let C be a non-empty, radially compact, convex subset of a real vector space X containing 0. Then, for each $x \in C$ with $x \neq 0$, there exist a unique $x_1 \in \mathrm{Lead}(C)$ and a unique $\lambda \in (0, 1]$, such that $x = \lambda x_1$. (A non-empty subset S of X is called* radially compact, *if for any $x \in X$ with $x \neq 0$, the set $\{\alpha \in \mathbb{R} : \alpha x \in S\}$ is compact in $\mathbb{R}$.)*

We extend this result to a simple but interesting characterization of a norm on a real vector space.

Theorem 2.6 *Let C_0 be a non-empty, radially compact, absolutely convex subset of a real vector space X. Let X_0 be the linear span of C_0 so that $X_0 = \cup_{n=1}^{\infty} nC_0$.*

1. *Then for each non-zero $x \in X_0$, there exists a unique $c \in \mathrm{Lead}(C_0)$ and a unique $\alpha > 0$, such that $x = \alpha c$.*
2. *Let us write $r(x) := \alpha$ and put $r(0) = 0$. Then, $r : X_0 \to \mathbb{R}^+$ determines a norm on X_0, such that C_0 is the closed unit ball.*

Proof (1): Fix $x \in X_0$, $x \neq 0$. As C_0 is absorbing in X_0, there exists a non-zero $k \in \mathbb{R}$, such that $kx \in C_0$. Since C_0 is absolutely convex, we may take $k > 0$. As C_0 is radially compact, the set $S = \{\lambda \in \mathbb{R} : \lambda x \in C_0\}$ is a compact set in $\mathbb{R}$. In addition, $\sup S = \alpha_0 > 0$ with $\alpha_0 x \in C_0$. Put $\alpha_0 x = c$. We show that $c \in \mathrm{Lead}(C_0)$. By Lemma 2.5, we have $c = \lambda c_1$ for some $c_1 \in \mathrm{Lead}(C_0)$ and $\lambda \in (0, 1]$, so that $\alpha_0 \lambda^{-1} \in S$. As $\alpha_0 = \sup S$, we get that $\lambda = 1$ and consequently, $c \in \mathrm{Lead}(C_0)$. Next, let $x = \beta d$ for some $d \in \mathrm{Lead}(C_0)$ and $\beta > 0$. Then, $\beta^{-1} \leq \alpha_0$. In addition, we have $c = \alpha_0 \beta d$. Thus by the definition of $\mathrm{Lead}(C_0)$, we get $c = d$ and $\beta = \alpha_0^{-1}$.

(2) We note that $r(x) \leq 1$ for all $x \in C_0$ and that $r(x) = 1$ if and only if $x \in \mathrm{Lead}(C_0)$. In addition, for $x \in X_0, x \neq 0$ we have $r(x) > 0$. Let $x \in X_0$ and $\alpha \in \mathbb{R}$. Without any loss of generality, we assume that $x \neq 0$ and $\alpha \neq 0$. Let $x = \lambda c$ be the unique representation with $c \in \mathrm{Lead}(C_0)$. Then, $r(x) = \lambda$. If $\alpha > 0$, then $\alpha x = \alpha \lambda c$. Thus, by the uniqueness of representation, we get that $r(\alpha x) = \alpha \lambda = \alpha r(x)$. For $\alpha < 0$, it suffices to prove that $-c \in \mathrm{Lead}(C_0)$ whenever $c \in \mathrm{Lead}(C_0)$. Let $c \in \mathrm{Lead}(C_0)$. Then, $-c \in C_0$ as C_0 is balanced. Assume that $-c = \beta d$ for some $d \in \mathrm{Lead}(C_0)$. Then $c = \beta(-d)$ with $-d \in C_0$. Now, by the definition of $\mathrm{Lead}(C_0)$ we get $-d = c$ and $\beta = 1$. Thus, $-c \in \mathrm{Lead}(C_0)$. Finally, we show that r is sub-additive. For this, let $x, y \in X_0$. Without any loss of generality, we assume that $x \neq 0$ and $y \neq 0$. Let $x = \alpha c$ and $y = \beta d$ be the unique representations with $c, d \in \mathrm{Lead}(C_0)$ and $\alpha > 0$, $\beta > 0$. Then, $r(x) = \alpha$ and $r(y) = \beta$. Now, $x + y = (\alpha + \beta)z$, where $z = \left(\frac{\alpha}{\alpha+\beta}c + \frac{\beta}{\alpha+\beta}d \right) \in C_0$. Thus

$$r(x + y) = r((\alpha + \beta)z) = (\alpha + \beta)r(z) \leq \alpha + \beta = r(x) + r(y).$$

This completes the proof.   □

Remark 2.7 The converse of Theorem 2.6 is evident. Let $(V, \|\cdot\|)$ be a normed linear space. Then, $B = \{v \in V : \|v\| \leq 1\}$ is a non-empty, radially compact, absolutely convex set in V with $\mathrm{Lead}(B) = \{v \in B : \|v\| = 1\}$.

2.3 Centre of a convex set

Definition 2.8 Let B be non-empty convex set in a real vector space X. An element $b_0 \in B$ is said to be a *centre* of B, if for each $b \in B$, there exists a (unique) $b' \in B$, such that $b_0 = \frac{1}{2}b + \frac{1}{2}b'$.

Note that if C is an absolutely convex set, then 0 is a centre of C.

Proposition 2.9 *Let B be a non-empty convex set in a real vector space X with a centre b_0. Put $B_0 = B - b_0$ and assume that B_0 is radially compact.*

1. *Then B_0 is balanced and convex.*
2. *If $x \in \mathrm{Lead}(B_0)$, then $-x \in \mathrm{Lead}(B_0)$.*
3. *Put $K = \{x + b_0 : x \in \mathrm{Lead}(B_0)\}$. Then, for each $b \in B$ with $b \neq b_0$, there exist a unique pair $b_1, b_2 \in K$ with $\frac{1}{2}b_1 + \frac{1}{2}b_2 = b_0$ and $\frac{1}{2} < \alpha \leq 1$, such that $b = \alpha b_1 + (1 - \alpha)b_2$.*

Proof (1). As B is convex, so is B_0. In addition, $0 \in B_0$ for $b_0 \in B$. Let $x \in B_0$. Then, $x = b - b_0$ for some $b \in B$. As b_0 is a centre of B, there exists a unique $b' \in B$, such that $\frac{1}{2}b + \frac{1}{2}b' = b_0$. Thus, $-x = b' - b_0 \in B_0$. In addition, for $0 \leq \lambda \leq 1$, we have $\lambda x = \lambda x + (1 - \lambda)0 \in B_0$, so that B_0 is balanced as well.

(2). First observe that $-x \in B_0$ as B_0 is balanced. Since $x \in \mathrm{Lead}(B_0)$, we have $x \neq 0$, so that there exist a unique $x_1 \in \mathrm{Lead}(B_0)$ and a unique $\lambda \in (0, 1]$, such that $-x = \lambda x_1$. Then, $x = \lambda(-x_1)$. Now, by the definition of a lead point, we get $\lambda = 1$ and consequently, $-x = x_1 \in \mathrm{Lead}(B_0)$.

(3). Let $b = x + b_0$ for some $x \in B_0$, such that $b \neq b_0$ (or equivalently, $x \neq 0$). Thus, there exist a unique $x_1 \in \mathrm{Lead}(B_0)$ and a unique $\lambda \in (0, 1]$, such that $x = \lambda x_1$. Put $\alpha = \frac{1+\lambda}{2}$. Then, $\frac{1}{2} < \alpha \leq 1$. For $b_1 = x_1 + b_0$ and $b_2 = -x_1 + b_0$, we have $\frac{1}{2}b_1 + \frac{1}{2}b_2 = b_0$. By (2), we have $b_1, b_2 \in K$. In addition

$$b = x + b_0 = (1 - \lambda)b_0 + \lambda b_1 = (1 - \lambda)\left(\frac{1}{2}b_1 + \frac{1}{2}b_2\right) + \lambda b_1 = \alpha b_1 + (1 - \alpha)b_2.$$

Next, let $b_3, b_4 \in K$ with $\frac{1}{2}b_3 + \frac{1}{2}b_4 = b_0$ and $\beta \in \mathbb{R}$ with $\frac{1}{2} < \beta \leq 1$ be such that $b = \beta b_3 + (1 - \beta)b_4$. Let $b_3 = x_2 + b_0$ for some $x_2 \in \mathrm{Lead}(B_0)$. Then, $b_4 = -x_2 + b_0$ and we have

$$x = \beta x_2 + (1 - \beta)(-x_2) = (2\beta - 1)x_2.$$

Now, by the uniqueness of a lead representation, we get $x_2 = x_1$ and

$$2\beta - 1 = \lambda = 2\alpha - 1$$

so that $\beta = \alpha$. $\square$

Corollary 2.10 *Let B be a non-empty convex set in a real vector space X and assume that $b_0 \in B$ be a centre of B. If $B_0 = B - b_0$ be radially compact, then b_0 is the only centre of B.*

Proof Let $b_1 \in B$ be another centre of B. Put $u_0 = b_1 - b_0$. Then $u_0 \in B_0$. We show that $u_0 = 0$. As b_1 is a centre of B and $b_0 \in B$, there exists $b'_0 \in B$, such that $b_1 = \frac{1}{2}b_0 + \frac{1}{2}b'_0$. Thus, $2u_0 = b'_0 - b_0 \in B_0$. Assume that $nu_0 \in B_0$ for some $n \in \mathbb{N}$. Then, $b_2 := -nu_0 + b_0 \in B$ as B_0 is balanced. Thus, there exists $c \in B$, such that $b_1 = \frac{1}{2}b_2 + \frac{1}{2}c$. It follows that $(n+2)u_0 = (c - b_0) \in B_0$. Now, by induction, $nu_0 \in B_0$ for all $n \in \mathbb{N}$. Since B_0 is radially compact, we must have $u_0 = 0$. Hence, $b_1 = b_0$. □

3 Affine functions of a central compact convex set

Let B be a compact and convex set in a real locally convex Hausdorff space X with a centre b_0 and consider the compact, balanced, convex set $B_0 = B - b_0$. Recall that a function $f : B \to \mathbb{R}$ is called *affine*, if

$$f(\alpha b_1 + (1 - \alpha)b_2) = \alpha f(b_1) + (1 - \alpha)f(b_2)$$

for all $b_1, b_2 \in B$ and $\alpha \in [0, 1]$. Let

$$A(B) = \{f : B \to \mathbb{R} | \, f \text{ is affine and continuous}\}.$$

As B is compact and convex, $A(B)$ is an order unit space with the cone

$$A^+(B) = \{f \in A(B) : f(b) \geq 0 \text{ for all } b \in B\}$$

and the order unit $\mathbf{1} : B \to \mathbb{R}$ given by $\mathbf{1}(b) = 1$ for all $b \in B$. In addition, the order unit norm is the sup-norm on $A(B)$ as a closed subspace of $C(B)$. (For a detailed discussion on this topic, please refer to [1, Chapter II.1].)

In this section, we describe the space of real valued, continuous, affine functions on a central, compact, convex set in a real locally convex Hausdorff space.

Theorem 3.1 *Let B be central compact and convex set in a real locally convex Hausdorff space X with the centre b_0 and consider the closed subspace*

$$A_0(B) := \{f \in A(B) : f(b_0) = 0\}$$

of $A(B)$. Then, $A(B)$ is unitally order isomorphic to the order unit space $(A_0(B)^{(\cdot)}, e)$.

We shall prove this result with the help of the following lemmas.

Lemma 3.2 $A^+(B) = \{f \in A(B) : \|f - f(b_0)\mathbf{1}\|_\infty \leq f(b_0)\}$.

Proof First, we assume that $f \in A^+(B)$, so that $f(b) \geq 0$ for all $b \in B$. Fix $b \in B$ with $b \neq b_0$. Then, by Proposition 2.9(3), there exist a unique pair $b_1, b_2 \in K$ with

$\frac{1}{2}b_1 + \frac{1}{2}b_2 = b_0$ and a unique $\alpha \in \mathbb{R}$ with $\frac{1}{2} < \alpha \le 1$, such that $b = \alpha b_1 + (1 - \alpha)b_2$. (Here $K = \{x + b_0 : x \in \text{Lead}(B_0)\}$ and $B_0 = B - b_0$.) Thus

$$f(b) = \alpha f(b_1) + (1 - \alpha)f(b_2) \le \max\{f(b_1), f(b_2)\}.$$

Again, as $\frac{1}{2}b_1 + \frac{1}{2}b_2 = b_0$, we have

$$\frac{1}{2}f(b_1) + \frac{1}{2}f(b_2) = f(b_0)$$

so that

$$0 \le f(b) \le \max\{f(b_1), f(b_2)\} \le 2f(b_0).$$

Thus $|f(b) - f(b_0)| \le f(b_0)$ for all $b \in B$. Therefore, $\|f - f(b_0)\mathbf{1}\|_\infty \le f(b_0)$.

Conversely, we assume that $f \in A(B)$ with $\|f - f(b_0)\mathbf{1}\|_\infty \le f(b_0)$. Then, for each $b \in B$, we have $|f(b) - f(b_0)| \le f(b_0)$ or equivalently,

$$0 \le f(b) \le \max\{f(b_1), f(b_2)\} \le 2f(b_0).$$

Thus $f \in A(B)^+$. $\qquad\square$

Lemma 3.3 *For each $f \in A(B)$, we have*

$$\|f\|_\infty = \|f - f(b_0)\mathbf{1}\|_\infty + |f(b_0)|.$$

Proof Fix $f \in A(B)$. Then, $\|f\|_\infty \le \|f - f(b_0)\mathbf{1}\|_\infty + |f(b_0)|$. Next, as $\|f\|_\infty \mathbf{1} \pm f \in A^+(B)$, by Lemma 3.2, we have

$$\|(\|f\|_\infty \mathbf{1} \pm f) - (\|f\|_\infty \pm f(b_0))\mathbf{1}\|_\infty \le (\|f\|_\infty \pm f(b_0))$$

or equivalently, $\|f - f(b_0)\mathbf{1}\|_\infty \le (\|f\|_\infty \pm f(b_0))$. Thus

$$\begin{aligned}
&\|f - f(b_0)\mathbf{1}\|_\infty + |f(b_0)| \\
&\quad = \max\{\|f - f(b_0)\mathbf{1}\|_\infty + f(b_0), \|f - f(b_0)\mathbf{1}\|_\infty - f(b_0)\} \\
&\quad \le \|f\|_\infty.
\end{aligned}$$

Hence $\|f\|_\infty = \|f - f(b_0)\mathbf{1}\|_\infty + |f(b_0)|$. $\qquad\square$

Proof of Theorem 3.1 For $f \in A(B)$, we define $\theta(f) : B \to \mathbb{R}$ given by $\theta(f)(b) = f(b) - f(b_0)$ for all $b \in B$. Then, $\theta(f) = f - f(b_0)\mathbf{1} \in A_0(B)$, so that by Lemma 3.3, we have

$$\|\theta(f)\|_\infty = \|f - f(b_0)\mathbf{1}\|_\infty \le \|f\|_\infty.$$

Thus the map $\theta : A(B) \to A_0(B)$ given by $\theta(f) = f - f(b_0)\mathbf{1}$ for all $f \in A(B)$ is a contractive, linear surjective map. We define $\chi : A(B) \to A_0(B)^{(\cdot)}$ given by $\chi(f) = (\theta(f), f(b_0))$ for all $f \in A(B)$. Then, χ is a unital linear surjection. In addition, by Lemmas 3.2 and 3.3, χ is an order isomorphism. $\qquad\square$

Example Fix $n \in \mathbb{N}$ with $n \geq 2$ and consider the compact convex set

$$S_n = \left\{ (x_i) \in \ell_1^n : x_i \geq 0 \text{ and } \sum_{i=1}^n x_i = 1 \right\}.$$

Then $B_n := \mathrm{co}\,(S_n \bigcup -S_n)$ is the closed unit ball of the base normed space ℓ_1^n and is a compact convex set with the centre 0. We show that $A(B_n)$ is isometrically isomorphic to $(A(S_n)^{(\cdot)}, e)$. Note that $A(S_n)$ is isometrically isomorphic to ℓ_∞^n.

Let $f \in A_0(B_n)$. Then, $f(-\alpha) = -f(\alpha)$ for all $\alpha \in B_n$. In fact if $\alpha \in B_n$, then $-\alpha \in B_n$ and we have $0 = \frac{1}{2}\alpha + \frac{1}{2}(-\alpha)$. Thus

$$0 = f(0) = \frac{1}{2}f(\alpha) + \frac{1}{2}f(-\alpha)$$

so that $f(-\alpha) = -f(\alpha)$. Put $a_i = f(e_i)$, where $\{e_i\}$ is the standard unit vector basis of $\mathbb{R}^n$. We show that $f((x_i)) = \sum_{i=1}^n x_i f(e_i)$ for every $(x_i) \in B_n$. Fix $(x_i) \in B_n$. Then, $\sum_{i=1}^n |x_i| \leq 1$. Put $\epsilon_i = \mathrm{sign}(x_i)$ for $1 \leq i \leq n$. Then, $\epsilon_i = \pm 1$ and $x_i = \epsilon|x_i|$ for each i. In addition, we have $\epsilon f(e_i) = f(\epsilon_i e_i)$ for each i. Put $\lambda_i = |x_i|$ for $1 \leq i \leq n$ and $\lambda_0 = 1 - \sum_{i=1}^n |x_i|$. Then, $\lambda_i \geq 0$ for $0 \leq i \leq n$ with $\sum_{i=0}^n \lambda_i = 1$. Thus as f is affine, we have

$$\sum_{i=1}^n x_i a_i = \sum_{i=1}^n |x_i| \epsilon_i f(e_i)$$

$$= \sum_{i=1}^n |x_i| f(\epsilon_i e_i)$$

$$= \lambda_0 f(0) + \sum_{i=1}^n \lambda_i f(\epsilon_i e_i)$$

$$= f(\lambda_0 0 + \sum_{i=1}^n \lambda_i \epsilon_i e_i)$$

$$= f((x_i)).$$

Next, let $f \in A(B_n)$. Define $f_0(x) = f(x) - f(0)$ for all $x \in B_n$. Then, $f_0 \in A_0(B_n)$. Thus, for any $(x_i) \in B_n$, we have

$$f((x_i)) = f_0((x_i)) + f(0)$$

$$= \sum_{i=1}^n x_i f_0(e_i) + f(0)$$

$$= \sum_{i=1}^n x_i (f(e_i) - f(0)) + f(0).$$

Therefore, $f \mapsto (f(e_1), \ldots, f(e_n))$ is linear isomorphism from $A_0(B_n)$ onto $\mathbb{R}^n$ and $f \mapsto (f(0), f(e_1) - f(0), \ldots, f(e_n) - f(0))$ is linear isomorphism from $A(B_n)$ onto $\mathbb{R}^{n+1}$.

Next, let $f \in A(B_n)$. Then

$$
\begin{aligned}
\|f\|_\infty &= \sup\{|f((x_i))| : (x_i) \in B_n\} \\
&= \sup\left\{\left|\sum_{i=1}^n x_i(f(e_i) - f(0)) + f(0)\right| : \sum_{i=1}^n |x_i| \le 1\right\} \\
&= \sup\left\{\left|\sum_{i=1}^n x_i(f(e_i) - f(0))\right| + |f(0)| : \sum_{i=1}^n |x_i| \le 1\right\} \\
&= |f(0)| + \max\{|f(e_i) - f(0)| : 1 \le i \le n\} \\
&= |f(0)| + \|(f(e_1) - f(0), \ldots, f(e_n) - f(0))\|_\infty.
\end{aligned}
$$

Hence $A_0(B_n)$ is isometrically isomorphic to ℓ^n_∞ and $A(B_n)$ is isometrically isomorphic to $\ell^n_\infty \oplus_1 \mathbb{R}$. Since $A(S_n)$ is isometrically isomorphic to ℓ^n_∞, we get that $A(B_n) \cong A(S_n)^{(\cdot)}$ as order unit spaces.

4 Base with a centre

Let (V, B) be a base normed space [1, Proposition II.1.12]. Then there exists a unique strictly positive $e \in V^*$ with $\|e\| = 1$, such that $e(v) = \|v\|$ if and only if $v \in V^+$. In particular, $B = \{v \in V^+ : e(v) = 1\}$ [10, Lemma 9.3 and Proposition 9.4]. In this section, we describe the base normed space with a central base.

Theorem 4.1 *Let (V, B) be a base normed space. For a fixed $b_0 \in B$, the following statements are equivalent*:

1. b_0 *is a centre of* B;
2. $V^+ = \{v \in V : \|v - e(v)b_0\| \le e(v)\}$;
3. $B = \{v \in V : e(v) = 1$ *and* $\|v - b_0\| \le 1\}$.

We use the following result to prove Theorem 4.1.

Lemma 4.2 *Let (V, B) be a base normed space and assume that $b_0 \in B$ is a centre of B. Then, for each $v \in V$, we have*

$$\|v\| \ge \max\{\|v - e(v)b_0\|, |e(v)|\}.$$

Proof Since b_0 is the centre of the base B, for each $b \in B$, there exists a (unique) $b' \in B$, such that $b_0 = \frac{1}{2}b + \frac{1}{2}b'$ (Definition 2.8). Let $v = \lambda b_1 - \mu b_2$ for some $b_1, b_2 \in B$ and $\lambda, \mu \ge 0$ with $\lambda + \mu = \|v\|$. Then, $e(v) = \lambda - \mu$, so that

$$v - e(v)b_0 = \lambda b_1 - \mu b_2 - (\lambda - \mu)b_0$$
$$= \lambda(b_1 - b_0) - \mu(b_2 - b_0)$$
$$= \lambda\left(\frac{b_1 - b'_1}{2}\right) - \mu\left(\frac{b_2 - b'_2}{2}\right)$$
$$= \frac{1}{2}\left\{(\lambda b_1 - \mu b_2) - (\lambda b'_1 - \mu b'_2)\right\}.$$

Thus $\|v - e(v)b_0\| \leq \lambda + \mu = \|v\|$. Also, we have $|e(v)| = |\lambda - \mu| \leq \lambda + \mu$ which completes the proof. $\qquad\square$

Proof of Theorem 4.1 **1. implies 2.:** Let us assume that b_0 is a centre of B. If $v \in V^+$, then $e(v) = \|v\|$. Thus, by Lemma 4.2, $\|v - e(v)b_0\| \leq e(v)$. Now, assume that $\|v - e(v)b_0\| \leq e(v)$ for some $v \in V$. Find $b_1, b_2 \in B$ and $\lambda, \mu \geq 0$, such that $v - e(v)b_0 = \lambda b_1 - \mu b_2$ and $\|v - e(v)b_0\| = \lambda + \mu$. Then

$$0 = e(v - e(v)b_0) = e(\lambda b_1 - \mu b_2) = \lambda - \mu$$

so that $2\lambda = 2\mu = \|v - e(v)b_0\| \leq e(v)$. Thus

$$v = \lambda(b_1 - b_2) + e(v)b_0 = \lambda b_1 + \lambda b'_2 + (e(v) - 2\lambda)b_0 \in V^+.$$

Thus $V^+ = \{v \in V : \|v - e(v)b_0\| \leq e(v)\}$.

2. implies 3.: Next we assume that $V^+ = \{v \in V : \|v - e(v)b_0\| \leq e(v)\}$. Let $b \in B$. Then, $b \in V^+$ and $\|b\| = 1 = e(b)$, so that $\|b - b_0\| \leq 1$. Put $u = b - b_0$. Then, $e(u) = 0, \|u\| \leq 1$ and we have $b = u + b_0$. Conversely, let $u \in V$ with $e(u) = 0$ and $\|u\| \leq 1$ and put $b = u + b_0$. Then, $e(b) = 1$ and $\|b - e(b)b_0\| = \|u\| \leq 1 = e(b)$. Thus, by the assumption, $b \in V^+$, so that $\|b\| = e(b) = 1$, that is, $b \in B$. Therefore, $B = \{v \in V : e(v) = 1 \text{ and } \|v - b_0\| \leq 1\}$.

3. implies 1.: Finally, we assume that

$$B = \{v \in V : e(v) = 1 \text{ and } \|v - b_0\| \leq 1\}.$$

Let $b \in B$ and consider $b' := 2b_0 - b$. Then, $e(b') = 1$ and

$$b' - e(b')b_0 = 2b_0 - b - b_0 = b_0 - b.$$

As $b \in B$, by the assumption, we have $\|b' - e(b')b_0\| \leq 1$, so that $b' \in B$. Thus, b_0 is a centre of B. $\qquad\square$

Corollary 4.3 *Let (V, B) be a base normed space with a centre b_0 of B. Then, for each $v \in V$, we have $\|v\| = \max\{\|v - e(v)b_0\|, |e(v)|\}$.*

Proof If $v \in V^+ \cup -V^+$, then $\|v\| = |e(v)|$. Thus by Lemma 4.2, we get $\|v\| = \max\{\|v - e(v)b_0\|, |e(v)|\}$. Next, let $v \notin V^+ \cup -V^+$. Then, by Theorem 4.1, we have $|e(v)| < \|v - e(v)b_0\|$. Put $u = \frac{v - e(v)b_0}{\|v - e(v)b_0\|}$. Then, $e(u) = 0$ and $\|u\| = 1$, so that $k = u + b_0 \in B$, by Theorem 4.1. Also then, $k' = -u + b_0$. Next, put

$2\lambda = \|v - e(v)b_0\| + e(v)$ and $2\mu = \|v - e(v)b_0\| - e(v)$. Then, $\lambda, \mu > 0$ with $\lambda + \mu = \|v - e(v)b_0\|$ and $\lambda - \mu = e(v)$. Thus

$$\lambda k - \mu k' = (\lambda + \mu)u + (\lambda - \mu)b_0 = v - e(v)b_0 + e(v)b_0 = v.$$

Now, it follows that

$$\|v\| \leq \lambda + \mu = \|v - e(v)b_0\|$$

so that $\|v\| = \max\{\|v - e(v)b_0\|, |e(v)|\}$. $\square$

Corollary 4.4 *A centre of B, if it exists, is unique.*

Proof Let $b_1 \in B$ be another centre of B. Then, by Theorem 4.1, $b_1 = u_1 + b_0$ for some $u_1 \in V$ with $e(u_1) = 0$ and $\|u_1\| \leq 1$. If $b_1 \neq b_0$, then $u_1 \neq 0$. In which case, $b_2 = -\|u_1\|^{-1}u_1 + b_0 \in B$. Since b_1 is a centre, by Theorem 4.1, we must have $\|b_2 - b_1\| \leq 1$. However, this is not true as $\|b_2 - b_1\| = 1 + \|u_1\| > 1$. Hence, $b_1 = b_0$. $\square$

Combining Theorem 4.1 and Corollary 4.3, we may deduce the following

Theorem 4.5 *A base normed space (V, B) with a centre $b_0 \in B$ is isometrically isomorphic to $\left(V_0^{(\cdot)}, B_0^{(\cdot)} \right)$, where $V_0 = \{v \in V : e(v) = 0\}$ and*

$$B_0^{(\cdot)} = \{(v, 1) : v \in V_0 \text{ with } \|v\| \leq 1\}.$$

Here e is the order unit of V^* corresponding to the base B.

Example Consider $V_n = \ell_1^n(\mathbb{R})$. Then, $B_n = \{(\alpha_i) : \alpha_i \geq 0 \text{ and } \sum \alpha_i = 1\}$ is the base for V_n, so that (V_n, B_n) is a base normed space. For $n \geq 3$, B_n can not have a centre. To see this fix $b_0 = (\alpha_i^0) \in B_n$, so that $\sum \alpha_i^0 = 1$. In addition, if $\alpha_m^0 = \inf\{\alpha_i^0 : 1 \leq i \leq n\}$, then $\alpha_m^0 \leq \frac{1}{n}$. Consider the m-th coordinate vector $e_m \in V_n$. Then, $e_m \in B_n$ and we have

$$\|e_m - b_0\|_1 = 1 - \alpha_m^0 + \sum_{i \neq m} \alpha_i^0 = 2(1 - \alpha_m^0) \geq 2\left(1 - \frac{1}{n}\right) > 1.$$

Thus b_0 is not a centre of B_n.

5 Order unit spaces with a central state space

We now describe a non-dual version of $A(B)$, where B is a central, compact convex subset of a real locally convex space.

Proposition 5.1 *Let (V, e) be an order unit space and let $\tau \in S(V)$. Then, the following statements are equivalent*:

1. τ *is a centre of $S(V)$;¼/Para>*
2. $v \leq 2\tau(v)e$ *for every $v \in V^+$*;
3. $V^+ = \{v \in V : \|v - \tau(v)e\| \leq \tau(v)\}$.

In this case, we say that τ is a central state.

Proof (1) implies (2): We assume that τ is a centre of $S(V)$. Let $v \in V^+$. If $f \in S(V)$, then by assumption, $2\tau - f \in S(V)$. Thus, $f(2\tau(v)e - v) = 2\tau(v) - f(v) \geq 0$ for all $f \in S(V)$, so that $v \leq 2\tau(v)e$.

(2) implies (1): Now, we assume that $v \leq 2\tau(v)e$ for each $v \in V^+$. Let $f \in S(V)$. Put $f' = 2\tau - f$. Then, f' is linear and $f'(e) = 1$. We show that it is positive. Let $v \in V^+$. Then, by assumption, $v \leq 2\tau(v)e$. As $f \geq 0$, we have $f'(v) = 2\tau(v) - f(v) = f(2\tau(v)e - v) \geq 0$. Thus, $f' \in S(V)$.

(2) implies (3): We again assume that $v \leq 2\tau(v)e$ for each $v \in V^+$. Thus, if $v \in V^+$, then $v \leq 2\tau(v)e$, or equivalently, $\tau(v)e \pm (v - \tau(v)e) \in V^+$. Therefore, $\|v - \tau(v)e\| \leq \tau(v)$. Conversely, let $v \in V$ with $\|v - \tau(v)e\| \leq \tau(v)$. Then, $\tau(v)e \pm (v - \tau(v)e) \in V^+$, so that $2\tau(v)e - v \in V^+$. Thus, $V^+ = \{v \in V : \|v - \tau(v)e\| \leq \tau(v)\}$.

(3) implies (2): Finally we assume that $V^+ = \{v \in V : \|v - \tau(v)e\| \leq \tau(v)\}$. Then, for $v \in V^+$ we have $\|v - \tau(v)e\| \leq \tau(v)$. Thus, $\tau(v)e \pm (v - \tau(v)e) \in V^+$. In particular, $v \leq 2\tau(v)e$. $\qquad\square$

The norm condition on positive elements of V is somewhat minimal as we see in the next result.

Proposition 5.2 *Let (V, e) be an order unit space with a central state $\tau \in S(V)$. Then, for each $v \in V$, we have $\|v - \tau(v)e\| \leq \|v - \alpha e\|$ for all $\alpha \in \mathbb{R}$.*

Proof Fix $\alpha \in \mathbb{R}$. Then

$$|f(v) - \alpha| = |f(v - \alpha e)| \leq \|v - \alpha e\|$$

for every $f \in S(V)$. By Proposition 5.1, given $f \in S(V)$, there exists a unique $f' \in S(V)$, such that $f + f' = 2\tau$. Thus

$$|f(v) - f'(v)| \leq |f(v) - \alpha| + |f'(v) - \alpha| \leq 2\|v - \alpha e\|.$$

Since $f - f' = 2(f - \tau)$, we get

$$|f(v - \tau(v)e)| = |f(v) - \tau(v)| \leq \|v - \alpha e\|$$

for all $f \in S(V)$. Hence, $\|v - \tau(v)e\| \leq \|v - \alpha e\|$ for all $\alpha \in \mathbb{R}$. $\qquad\square$

Proposition 5.3 *Let (V, e) be an order unit space with a central state $\tau \in S(V)$. Then, for each $v \in V$, we have*

$$\|v\| = \|v - \tau(v)e\| + |\tau(v)|.$$

Proof Fix $v \in V$. Then, $\|v\| \le \|v - \tau(v)e\| + |\tau(v)|$. In addition, as $\|v\|e \pm v \in V^+$, by (1), we have

$$\|(\|v\|e \pm v) - \tau(\|v\|e \pm v)e\| \le \tau(\|v\|e \pm v).$$

In other words, $\|v - \tau(v)e\| \le \|v\| \pm \tau(v)$, so that $\|v - \tau(v)e\| + |\tau(v)| \le \|v\|$. Hence, $\|v\| = \|v - \tau(v)e\| + |\tau(v)|$. $\qquad\square$

Remark 5.4 Let (V, e) be an order unit space with a central state $\tau \in S(V)$. Then, τ is unique and we have

1. $V^{*+} = \{f \in V^* : \|f - f(e)\tau\| \le f(e)\} = \{f + \alpha\tau : f(e) = 0 \text{ and } \|f\| \le \alpha\};$
2. $S(V) = \{f_0 + \tau : f_0 \in V^*, f_0(e) = 0 \text{ and } \|f_0\| \le 1\};$ and
3. For each $f \in V^*$, we have $\|f\| = \max\{\|f - f(e)\tau\|, |f(e)|\}.$

Proposition 5.5 *Let (V, e) be an order unit space with a central state $\tau \in S(V)$. Then, the Banach dual of $V_0 = \{v \in V : \tau(v) = 0\}$ is isometrically isomorphic to $V_0^* := \{f \in V^* : f(e) = 0\}$.*

Proof Consider the mapping $\chi : V_0^* \to (V_0)^*$ given by $f \mapsto f|_{V_0}$. Then, χ is linear. In addition, for $f \in V_0^*$ we have

$$\begin{aligned}
\|f\| &= \sup\{|f(v)| : v \in V \text{ with } \|v\| \le 1\}\\
&= \sup\{|f(v - \tau(v)e)| : v \in V \text{ with } \|v - \tau(v)e\| + |\tau(v)| \le 1\}\\
&= \sup\{|f(v)| : v \in V_0 \text{ with } \|v\| \le 1\}.
\end{aligned}$$

Thus χ is an isometry. Furthermore, for $f_0 \in (V_0)^*$, we define $f : V \to \mathbb{R}$ by $f(v) = f_0(v - \tau(v)e)$ for all $v \in V$. Then, $f \in V_0^*$ with $\chi(f) = f_0$. $\qquad\square$

We can summarize the findings of this section in the following.

Theorem 5.6 *Let (V, e) be an order unit space with a central state $\tau \in S(V)$. Then, $V/\mathbb{R}e$ is isometrically isomorphic to $V_0 := \{v \in V : \tau(v) = 0\}$ and V is unitally and isometrically order isomorphic to $V_0 \oplus_1 \mathbb{R}$ via the map $v \mapsto (v - \tau(v), \tau(v))$.*

5.1 Strict convexity

Definition 5.7 Let B_0 be a compact, balanced, convex subset of a real, locally convex, Hausdorff space X. We say that B_0 is said to satisfy *Property (S)* with respect to X^*, if for each $x_0 \in \mathrm{Lead}(B_0)$ there exists at most one $f \in X^*$, such that

$$\sup\{|f(x)| : x \in B_0\} = f(x_0) = 1.$$

Remark 5.8 The notion of Property (S) is motivated by the characterization of strictly convex spaces by M. G. Krein in 1938. However, the same result was also

proved independently by A. F. Ruston in 1949 [9, Theorem 1]. (Please refer to [8, 5.5.1.1] for a detailed discussion on it.)

Definition 5.9 Let (V, e) be an order unit space with a central trace $\tau \in S(V)$. We say that V is *tracial*, if $S(V)_0 (:= S(V) - \tau)$ satisfies Property (S).

Theorem 5.6 can now be restated as follows.

Corollary 5.10 *Let V_0 be a strictly convex real normed linear space and consider the order unit space $V_0^{(\infty)}$ obtained by adjoining an order unit to V_0 (see, [7, Theorem 2.17]). Then, $V_0^{(\infty)}$ is precisely a tracial order unit space.*

6 Central absolutely base normed space

In [7], we described the absolute value that naturally arises out of the order structure in $V^{(\cdot)}$. We studied the corresponding absolute order unit space and showed that it is a generalization of a spin factor. In this section, we shall discuss the description of the canonical absolute value in $V^{(\cdot)}$ in the context of the corresponding base $B^{(\cdot)}$.

Let us recall the notion of absolutely ordered spaces introduced in [6]. (See also, [5].)

Definition 6.1 [6, Definition 3.4] Let (U, U^+) be a real ordered vector space and let $|\cdot| : U \to U^+$ satisfy the following conditions:

1. $|v| = v$ if $v \in U^+$;
2. $|v| \pm v \in U^+$ for all $v \in U$;
3. $|kv| = |k||v|$ for all $v \in U$ and $k \in \mathbb{R}$;
4. If $u, v, w \in U$ with $|u - v| = u + v$ and $|u - w| = u + w$, then $|u - |v \pm w|| = u + |v \pm w|$;
5. If u, v and $w \in U$ with $|u - v| = u + v$ and $0 \le w \le v$, then $|u - w| = u + w$.

Then $(U, U^+, |\cdot|)$ is called an *absolutely ordered space*.

Let (V, B) be a base normed space with a centre b_0 of B. Consider the set $K = \{u + b_0 : e(u) = 0 \text{ and } \|u\| = 1\} \subset B$. We note that if $k \in K$, then $k' \in K$. In fact, $k' = -u + b_0$ whenever $k = u + b_0$. The following results show that K determines an absolute value in V.

Lemma 6.2 *Let (V, B) be a base normed space with a centre b_0 of B and let $b \in B$ with $b \neq b_0$. Then, there exist a unique $k \in K$ and a unique $\frac{1}{2} < \alpha \le 1$, such that $b = \alpha k + (1 - \alpha)k'$.*

Proof As $b \in B$ with $b \neq b_0$, we have $0 < \|b - b_0\| \leq 1$. Put $\alpha = \frac{1+\|b-b_0\|}{2}$. Then, $\frac{1}{2} < \alpha \leq 1$. Put $u_0 = \frac{b-b_0}{\|b-b_0\|}$. Then, $e(u_0) = 0$ with $\|u_0\| = 1$. Thus, $k = u_0 + b_0 \in K$ with $k' = -u_0 + b_0$ and we have

$$\alpha k + (1 - \alpha)k' = (2\alpha - 1)u_0 + b_0 = \|b - b_0\| \left(\frac{b - b_0}{\|b - b_0\|} \right) + b_0 = b.$$

Next, let $b = \beta k_1 + (1 - \beta)k_1'$ for some $k_1 \in K$ and $\frac{1}{2} < \beta \leq 1$. Then, there exists $u_1 \in V$ with $e(u_1) = 0$ and $\|u_1\| = 1$, such that $k_1 = u_1 + b_0$ and $k_1' = -u_1 + b_0$. Then, as above $b = (2\beta - 1)u_1 + b_0$, so that $(2\alpha - 1)u_0 = (2\beta - 1)u_1$. Since $\frac{1}{2} < \alpha \leq 1$ and $\frac{1}{2} < \beta \leq 1$, we have $2\alpha > 1$ and $2\beta > 1$. Now, as $\|u_0\| = 1 = \|u_1\|$, we deduce that $2\alpha - 1 = 2\beta - 1$. Thus we have $\alpha = \beta$, so that $u_0 = u_1$, $\qquad\square$

Theorem 6.3 *Let* (V, B) *be a base normed space with a centre* b_0 *of* B *and let* $v \in V \backslash \mathbb{R}b_0$. *Then, there exist* $k \in K$ *and* $\alpha, \beta \in \mathbb{R}$, *such that* $v = \alpha k + \beta k'$. *This representation is unique in the following sense*: *if* $v = \lambda k_1 + \mu k_1'$ *for some* $k_1 \in K$ *and* $\lambda, \mu \in \mathbb{R}$, *then either* $k_1 = k$ *with* $\lambda = \alpha$ *and* $\mu = \beta$ *or* $k_1 = k'$ *with* $\lambda = \beta$ *and* $\mu = \alpha$.

Proof First consider $v \in V^+ \backslash \mathbb{R}b_0$. Then, $v = \|v\|b$ for $b := \|v\|^{-1}v \in B$, such that $b \neq b_0$. Thus, by Lemma 6.2, there exist a unique $k \in K$ and $\frac{1}{2} < \alpha_0 \leq 1$, such that $b = \alpha_0 k + (1 - \alpha_0)k'$. Therefore, $v = \alpha k + \beta k'$, where $\alpha = \|v\|\alpha_0$ and $\beta = \|v\|(1 - \alpha_0)$. A similar proof works for $v \in -V^+\backslash\mathbb{R}b_0$.

Now, let $v \notin V^+ \cup -V^+$. Then, $|e(v)| < \|v - e(v)b_0\|$. Put $u = \frac{v-e(v)b_0}{\|v-e(v)b_0\|}$. Then, $e(u) = 0$ and $\|u\| = 1$, so that $k = u + b_0 \in K$ with $k' = -u + b_0$. First, we assume that $e(v) \geq 0$. Put $2\alpha = e(v) + \|v - e(v)b_0\|$ and $2\beta = e(v) - \|v - e(v)b_0\|$, we have $\alpha > 0, \beta < 0$ with $\alpha - \beta = \|v - e(v)b_0\|$ and $\alpha + \beta = e(v)$. Thus

$$\alpha k + \beta k' = (\alpha - \beta)u + (\alpha + \beta)b_0 = v - e(v)b_0 + e(v)b_0 = v.$$

Next, let $v = \lambda k_1 + \mu k_1'$ for some $k_1 \in K$ and $\lambda, \mu \in \mathbb{R}$. Then, $e(v) = \lambda + \mu$. Let $k_1 = u_1 + b_0$ for some $u_1 \in V$ with $e(u_1) = 0$ and $\|u_1\| = 1$. Then, $k' = -u_1 + b_0$ and we have $v = (\lambda - \mu)u_1 + e(v)b_0$. Replacing u_1 by $-u_1$, if required, we get that $\lambda \geq \mu$. Thus, $\|v - e(v)b_0\| = \lambda - \mu$. Now, it follows that $\lambda = \alpha$ and $\mu = \beta$ so that $u_1 = u_0$ and consequently, $k = k_1$.

When $e(v) < 0$, we interchange α and β and replace k by k'. $\qquad\square$

Remark 6.4 Let us call the unique decomposition $v = \alpha k + \beta k'$ with $k \in K$ and $|\alpha| \geq |\beta|$ as the *K-decomposition* of $v \in V\backslash\mathbb{R}b_0$. Note that for $v = \lambda b_0$, we have $v = \alpha(k + k')$ for all $k \in K$, where $2\alpha = \lambda$. Thus, given $v \in V$ and a decomposition $v = \alpha k + \beta k'$ with $|\alpha| \geq |\beta|$, we obtain $|v| = |\alpha|k + |\beta|k' \in V^+$ which is uniquely determined by v. This defines a mapping $|\cdot| : V \to V^+$.

Proposition 6.5 *Let* (V, B) *be a base normed space with a centre* b_0 *of* B. *Then, the mapping* $|\cdot| : V \to V^+$ *satisfies the following properties*:

1. $|v| = v$ if $v \in V^+$;
2. $|v| \pm v \in V^+$ for all $v \in V$;
3. $|\lambda v| = |\lambda||v|$ for all $v \in V$ and $\lambda \in \mathbb{R}$;
4. If $u, v, w \in V$ with $|u - v| = u + v$ and $|u - w| = u + w$, then $|u - |v \pm w|| = u + |v \pm w|$.

Proof Let $v \in V$ and consider its K-decomposition $v = \alpha k + \beta k'$, where $k \in K$ and $|\alpha| \geq |\beta|$. Then, $|v| := |\alpha|k + |\beta|k'$. As $k, k' \in V^+$, verification of (1), (2) and (3) is straightforward. To prove (4), let $u, v, w \in V$ with $|u - v| = u + v$ and $|u - w| = u + w$. Then, $u, v, w \in V^+$. If $u - v \in V^+$, we have $u + v = |u - v| = u - v$, so that $v = 0$. Thus

$$|u - |v \pm w| = |u - w| = u + w = u + |v \pm w|.$$

So we may assume that $u - v, u - w \notin V^+ \bigcup -V^+$. Then, $u \neq 0$, $v \neq 0$ and $w \neq 0$. Consider the (unique) K-decomposition $u - v = \lambda_1 k_1 + \mu_1 k_1'$ for some $k_1 \in K$ and $|\lambda_1 \geq |\mu_1|$. Then, $|u - v| = |\lambda_1|k_1 + |\mu_1|k_1'$. Since $|u - v| = u + v$, we get that $u = \lambda_1 k_1$ and $v = -\mu_1 k_1'$. Thus, $\lambda_1 > 0$ and $\mu_1 < 0$. Again, as $|u - w| = u + w$, we get that $u = \lambda_2 k_2$ and $w = -\mu_2 k_2'$ for some $k_2 \in K$ and $|\lambda_2 \geq |\mu_2|$ with $\lambda_2 > 0$ and $\mu_2 < 0$. Thus, $k_1 = k_2$ and $\lambda_1 = \lambda_2$. Therefore

$$|u - |v \pm w|| = |\lambda_1 k_1 - |\mu_1 \pm \mu_2|k_1'| = \lambda_1 k_1 + |\mu_1 \pm \mu_2|k_1' = u + |v \pm w|.$$

$\square$

Remark 6.6 It follows from the proof of Proposition 6.5 that $|u - v| = u + v$ if and only if there exist $k \in K$ and positive real numbers α and β, such that $u = \alpha k$ and $v = \beta k'$.

Theorem 6.7 Let (V, B) be a base normed space with a centre b_0 of B. Consider

$$V_0 = \{v \in V : e(v) = 0\} = \{v - e(v)b_0 : v \in V\}.$$

Then V_0 is closed subspace of V and the following statements are equivalent:

1. $K = \text{ext}(B)$;
2. V_0 is strictly convex; and
3. For all $u, v, w \in V^+$ with $|u - v| = u + v$ and $0 \leq w \leq v$ we have $|u - w| = u + w$.

Proof (1) implies (2): First we assume that $K = \text{ext}(B)$. Let $u_0, u_1 \in V_0$ be, such that $u_0 \neq u_1$ and $\|u_0\| = 1 = \|u_1\|$ and assume to the contrary that $\|u_\alpha\| = 1$, where $u_\alpha = \alpha u_1 + (1 - \alpha)u_0$ for $0 < \alpha < 1$. Put $k_0 = u_0 + b_0, k_1 = u_1 + b_0$ and $k_\alpha = u_\alpha + b_0$. Then, by construction, $k_0, k_1, k_\alpha \in K$ and we have $k_\alpha = \alpha k_1 + (1 - \alpha)k_0$. Since $K = \text{Ext}(B)$, we deduce that $k_0 = k_1 = k_\alpha$. This leads to a contradiction $u_0 = u_1$. Thus, V_0 must be strictly convex.

(2) implies (3): Next we assume that V_0 is strictly convex. Consider $u, v, w \in V^+$ with $|u - v| = u + v$ and $0 \leq w \leq v$. If $u - v \in V^+$, then $v = 0$, so that $w = 0$ and we have $|u - w| = u + w$. If $v - u \in V^+$, then $u = 0$, so that $|u - w| = w = u + w$. Thus, we may assume that $u - v \notin V^+ \bigcup -V^+$. Then, $u \neq 0$ and $v \neq 0$. We may also assume that $w \neq 0$. Since $|u - v| = u + v$, following the proof of Proposition 6.5, we can find $k \in K$ and $\alpha \geq \beta > 0$, such that $u = \alpha k$ and $v = \beta k'$. Since B is a base for V^+ and since $w \in V^+$ with $w \neq 0$, there exist a unique $b \in B$ and $\lambda > 0$, such that $w = \lambda b$. By Lemma 6.2, we have $b = \gamma k_1 + (1 - \gamma)k_1'$ for some $k_1 \in K$ and $\frac{1}{2} \leq \gamma \leq 1$. Since $k, k_1 \in K$, there exist $x, x_1 \in V_0$ with $\|x\| = 1 = \|x_1\|$, such that $k = x + b_0$ and $k_1 = x_1 + b_0$. Then, $k' = -x + b_0$ and $k_1 = -x_1 + b_0$. As $w \leq v$ we have

$$\lambda\big(\gamma(x_1 + b_0) + (1 - \gamma)(-x_1 + b_0)\big) \leq \beta(-x + b_0)$$

so that $-\big(\beta x + \lambda(2\gamma - 1)x_1\big) + (\beta - \lambda)b_0 \in V^+$. Now, as $\frac{1}{2} \leq \gamma \leq 1$ we have $0 \leq 2\gamma - 1 \leq 1$. Thus, by Theorem 4.1, we get

$$\beta - \lambda \leq \beta - (2\gamma - 1)\lambda = \|\beta x\| - \|(2\gamma - 1)\lambda x_1\| \leq \|\beta x + \lambda(2\gamma - 1)x_1\| \leq \beta - \lambda.$$

Therefore, we have $\gamma = 1$ and $\|\beta x + \lambda x_1\| = \beta - \lambda$. It follows that $b = k_1$, so that $w = \lambda k_1$. Since V_0 is strictly convex, we conclude that

$$\|\beta x + \lambda x_1\|^{-1}(\beta x + \lambda x_1) = \|\lambda x_1\|^{-1}(-\lambda x_1),$$

or equivalently, $\beta x + \lambda x_1 = -(\beta - \lambda)x_1$ as $\|x\| = \|x_1\|$. Thus, $x_1 = -x$, so that $k_1 = k'$ and $w = \lambda k'$. Now

$$|u - w| = |\alpha k - \lambda k'|\alpha k + \lambda k' = u + w.$$

(3) implies (1): Finally, we assume that (3) holds. We show that $K = \text{ext}(B)$. Let $b \in B$, such that $b \notin K$. Then, there exists $x \in V_0$ with $\|x\| < 1$, such that $b = x + b_0$. As $b_0 = \frac{1}{2}k + \frac{1}{2}k' \notin \text{ext}(B)$, without any loss of generality we may assume that $x \neq 0$. Put $x_1 = \|x\|^{-1}x$. Then, $b_1 := x_1 + b_0 \in B$ and we have $x = \|x\|b_1 + (1 - \|x\|)b_0$ is a proper convex combination in B. Thus, $b \notin \text{ext}(B)$. Therefore, $\text{ext}(B) \subset K$.

Conversely, let $k \in K$ and assume that $k \notin \text{ext}(B)$. Then, there are $b, c \in B$ and $\delta \in (0, 1)$, such that $k = \delta b + (1 - \delta)c$. Let $x_0, y, z \in V_0$ with $\|x_0\| = 1$, $\|y\| \leq 1$ and $\|z\| \leq 1$, such that $k = x_0 + b_0$, $b = y + b_0$ and $c = z + b_0$. Then, $x_0 = \delta y + (1 - \delta)z$. Now

$$1 = \|x_0\| = \|\delta y + (1 - \delta)z\| \leq \delta\|y\| + (1 - \delta)\|z\| \leq \delta + (1 - \delta) = 1$$

so that $\|y\| = 1 = \|z\|$. Put $u = -x_0 + b_0$, $v = x_0 + b_0$ and $w = \delta(y + b_0)$. Then, $u, v, w \in V^+ \setminus \{0\}$ with $u \in K$ and $v = u'$. Thus, $|u - v| = u + v$. In addition, $v - w = (1 - \delta)(z + b_0) \geq 0$ but $|u - w| \neq u + w$ for $w \neq \gamma v'$ for any positive real number γ which contradicts the assumption that the statement (3) holds. Thus, $K \subset \text{ext}(B)$. This proves (1). $\qquad\square$

Definition 6.8 Let (V, B) be a base normed space with a centre b_0 of B and assume that $K := \{k \in B : \|k - b_0\| = 1\} = \text{ext}(B)$. Then, (V, B, b_0) is said to be a *central base normed space*.

Now Theorem 6.7 yields in the following result.

Corollary 6.9 *Let* (V, B, b_0) *be a central base normed spaces, so that* $V_0 = \{v \in V : e(v) = 0\}$ *is strictly convex and that V is isometrically order isomorphic to* $V_0^{(1)}$ *(see, Proposition* 2.3*).*

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