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ORIGINAL PAPER

Boundedness of operators generated by fractional semigroups associated with Schrödinger operators on Campanato type spaces via $T1$ theorem

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Abstract

Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator, where the nonnegative potential V belongs to the reverse Hölder class B_q. By the aid of the subordinative formula, we estimate the regularities of the fractional heat semigroup, $\{e^{-t\mathcal{L}^\alpha}\}_{t>0}$, associated with $\mathcal{L}$. As an application, we obtain the $\mathrm{BMO}^\gamma_{\mathcal{L}}$-boundedness of the maximal function, and the Littlewood–Paley g-functions associated with $\mathcal{L}$ via $T1$ theorem, respectively.

Keywords Schrödinger operators · Fractional heat semigroups · $T1$ theorem · Campanato type space

Mathematics Subject Classification 35J10 · 42B20 · 42B30

1 Introduction

In the research of harmonic analysis and partial differential equations, the maximal operators and Littlewood–Paley g-functions paly an important role and were investigated by many mathematicians extensively. For any integrable function f on $\mathbb{R}^n$, the Hardy–Littlewood maximal operator is defined as

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$$M(f)(x) := \sup_{Q} \frac{1}{|Q|} \int_{Q} |f(y)| \mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. For $f \in \mathrm{BMO}(\mathbb{R}^n)$, Bennett–DeVore–Sharpley proved in [1] that $M(f)$ is either infinite or belongs to $\mathrm{BMO}(\mathbb{R}^n)$. The boundedness result in [1] can be extended to other maximal operators. For example, let $-\Delta$ be the Laplace operator: $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Denote by M_Δ and g the maximal operator and Littlewood–Paley g-function generated by the heat semigroup $\{e^{-t(-\Delta)}\}_{t>0}$, respectively, i.e.,

$$\begin{cases} M_\Delta(f)(x) := \sup_{t>0} |e^{-t(-\Delta)}(f)(x)|; \\ g(f)(x) := \left(\int_0^\infty |e^{-t(-\Delta)}(f)(x)|^2 \frac{\mathrm{d}t}{t^{n+1}} \right)^{1/2}. \end{cases} \tag{1}$$

Due to the mean value on “large” cubes may be infinite, the $\mathrm{BMO}(\mathbb{R}^n)$-boundedness of M_Δ or g holds if $M_\Delta(f) < \infty$ or $g(f) < \infty$ for $f \in \mathrm{BMO}(\mathbb{R}^n)$.

However, if the Laplacian $-\Delta$ is replaced by other second-order differential operators, the situation becomes different. Consider the Schrödinger $\mathcal{L} = -\Delta + V$ in $\mathbb{R}^n$, $n \geq 3$, where V is a nonnegative potential belonging to the reverse Hölder class B_q for some $q > n/2$. Here a nonnegative potential V is said to belong to B_q if there exists $C > 0$ such that for every ball B,

$$\left(\frac{1}{|B|} \int_B V^q(x) \mathrm{d}x \right)^{1/q} \leq \frac{C}{|B|} \int_B V(x) \mathrm{d}x.$$

In [1], the authors pointed out that for $f \in \mathrm{BMO}(\mathbb{R}^n)$ a supremum of averages of f over “large” cubes may be infinite, see [1, page 610]. In 2005, Dziubański et al. [9] proved the square functions associated with Schrödinger operators are bounded on the BMO type space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ related with $\mathcal{L}$ which is distinguished from the case of $BMO(\mathbb{R}^n)$. See also [13, 18] for similar results in the setting of Heisenberg groups and stratified Lie groups.

Let $H = -\Delta + |x|^2$ be the harmonic oscillator. In [2], Betancor et al. introduced a $T1$ criterion for Calderón–Zygmund operators related to H on the BMO type space $\mathrm{BMO}_H(\mathbb{R}^n)$. Later, Ma et al. [15] generalized the $T1$ criterion to the case of Campanato type spaces $\mathrm{BMO}_{\mathcal{L}}^{\gamma}(\mathbb{R}^n)$ related with $\mathcal{L}$. As applications, the authors in [15] proved that the maximal operators associated with the heat semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ and with the generalized Poisson operators $\{P_t^\sigma\}_{t>0} (0 < \sigma < 1)$, the Littlewood–Paley g-functions given in terms of the heat and the Poisson semigroups are bounded on $\mathrm{BMO}_{\mathcal{L}}^{\gamma}(\mathbb{R}^n)$.

Notice that for $\sigma \in (0, 1)$, the generalized Poisson operator $\{P_t^\sigma\}_{t>0}$ is expressed as

$$P_t^\sigma(f)(x) := \frac{t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-\frac{t^2}{4r}} e^{-r\mathcal{L}}(f) \frac{\mathrm{d}r}{r^{1+\sigma}}. \tag{2}$$

Specially, for $\sigma = 1/2$, $\{P_t^{1/2}\}_{t>0}$ is corresponding to the Poisson semigroup $\{e^{-t\mathcal{L}^{1/2}}\}_{t>0}$ associated with $\mathcal{L}$. The main purpose of this paper is to derive the pointwise estimate and regularity properties of the fractional heat semigroup $\{e^{-t\mathcal{L}^\alpha}\}_{t>0}$, $\alpha > 0$, to prove the boundedness of the maximal function and the Littlewood–Paley g-functions generated by $\{e^{-t\mathcal{L}^\alpha}\}_{t>0}$ on $\mathrm{BMO}^\gamma_{\mathcal{L}}(\mathbb{R}^n)$, $0 < \gamma < \min\{2\alpha, \delta_0, 1\}$, via $T1$ theorem, respectively.

When $\mathcal{L} = -\Delta$, the kernels of the fractional heat semigroup $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$ can be defined via the Fourier transform, i.e.,

$$K_{\alpha,t}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi - t|\xi|^{2\alpha}} \mathrm{d}\xi. \tag{3}$$

For $\mathcal{L} = -\Delta + V$ with $V > 0$, the kernels of fractional heat semigroups $\{e^{-t\mathcal{L}^\alpha}\}_{t>0}$, $\alpha \in (0, 1)$, can not be defined via (3). However, for $\alpha > 0$, the subordinative formula (cf. [10]) indicates that

$$K^{\mathcal{L}}_{\alpha,t}(x, y) = \int_0^\infty \eta^\alpha_t(s) K^{\mathcal{L}}_s(x, y)\mathrm{d}s, \tag{4}$$

where $\eta^\alpha_t(\cdot)$ is a continuous function on $(0, \infty)$ satisfying (19) below. In [12], the identity (4) was applied to estimate $K^{\mathcal{L}}_{\alpha,t}(\cdot, \cdot)$ via the heat kernel $K^{\mathcal{L}}_t(\cdot, \cdot)$, see Proposition 2. Specially, for $\alpha = 1/2$, the estimates of $K^{\mathcal{L}}_{\alpha,t}(\cdot, \cdot)$ goes back to those of the Poisson kernel $P^{\mathcal{L}}_t(\cdot, \cdot)$, see [5, Lemma 3.9].

We point out that, compared with the case of $\{P^\sigma_t\}_{t>0}$, some new regularity estimates should be introduced to prove the $\mathrm{BMO}^\gamma_{\mathcal{L}}$-boundedness of the maximal function and Littlewood–Paley g-functions generated by $\{e^{-t\mathcal{L}^\alpha}\}_{t>0}$. Let $E = L^\infty((0, \infty), \mathrm{d}t)$. It follows from (2) and the Minkowski integral inequality that

$$\|P^\sigma_t(f)\|_E \le C_\sigma \int_0^\infty t^{2\sigma} e^{-\frac{t^2}{4r}} \|e^{-r\mathcal{L}}(f)\|_E \frac{\mathrm{d}r}{r^{1+\sigma}}.$$

The fact that

$$\int_0^\infty t^{2\sigma} e^{-\frac{t^2}{4r}} \frac{\mathrm{d}r}{r^{1+\sigma}} < \infty$$

ensures that the $\mathrm{BMO}^\gamma_{\mathcal{L}}$-boundedness of the maximal function $\sup_{t>0} |P^\sigma_t f(x)|$ can be deduced from that of the heat maximal function $\sup_{t>0} |e^{-t\mathcal{L}} f(x)|$, see [15, Proposition 4.7]. However, we can see from the identity (4) that this method is not applicable to the case $\{e^{-t\mathcal{L}^\alpha}\}_{t>0}$.

In this paper, we get the following results:

- In Sect. 3.1, let $\nabla_x = \left(\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n\right)$. By the perturbation theory for semigroups of operators, we deduce the pointwise estimates and the Hölder type estimates of the kernels:

$$\begin{cases} \left|\nabla_x(K_t(x-y)-K_t^{\mathcal{L}}(x,y))\right|, \\ \left|t^m\partial_t^m(K_t(x-y)-K_t^{\mathcal{L}}(x,y))\right|, \end{cases}$$

see Lemmas 8, 10 and Proposition 1, respectively.

- In Sect. 3.2, we use (3) to obtain the corresponding estimates for

$$\begin{cases} \left|K_{\alpha,t}^{\mathcal{L}}(x,y)-K_{\alpha,t}(x-y)\right|, \\ \left|t^{1/2\alpha}\nabla_x(K_{\alpha,t}(x-y)-K_{\alpha,t}^{\mathcal{L}}(x,y))\right|, \\ \left|t^m\partial_t^m(K_{\alpha,t}(x-y)-K_{\alpha,t}^{\mathcal{L}}(x,y))\right|, \end{cases}$$

see Propositions 5–10, respectively.

- In Sect. 4, as applications of the regularity estimates obtained in Sect. 3, we use the $T1$ criterion established in [15] to prove the boundedness on Campanato type spaces $\mathrm{BMO}_{\mathcal{L}}^{\gamma}(\mathbb{R}^n)$, $0<\gamma<\min\{2\alpha,\delta_0,1\}$, of the following maximal operator and g-functions:

$$\begin{cases} M_{\mathcal{L}}^{\alpha}f(x) := \sup_{t>0}|e^{-t\mathcal{L}^\alpha}f(x)|; \\ g_\alpha^{\mathcal{L}}(f)(x) := \left(\int_0^\infty |D_{\alpha,t}^{\mathcal{L},m}(f)(x)|^2\frac{\mathrm{d}t}{t}\right)^{1/2}; \\ \widetilde{g}_\alpha^{\mathcal{L}}f(x) := \left(\int_0^\infty |\widetilde{D}_{\alpha,t}^{\mathcal{L}}(f)(x)|^2\frac{\mathrm{d}t}{t}\right)^{1/2}, \end{cases}$$

see Theorems 3–5, respectively, where $D_{\alpha,t}^{\mathcal{L},m}$ and $\widetilde{D}_{\alpha,t}^{\mathcal{L}}$ are the operators with the integral kernels

$$\begin{cases} D_{\alpha,t}^{\mathcal{L},m}(x,y) := t^m\partial_t^m K_{\alpha,t}^{\mathcal{L}}(x,y), \\ \widetilde{D}_{\alpha,t}^{\mathcal{L}}(x,y) := t^{1/2\alpha}\nabla_x K_{\alpha,t}^{\mathcal{L}}(x,y), \end{cases} \tag{5}$$

respectively.

Notations We will use c and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C>1$ such that $C^{-1}\le B_1/B_2\le C$.

2 Preliminaries

2.1 Schrödinger operators and function spaces

In this paper, let $\delta_0 = 2-n/q$. At first, we list some properties of the potential V which will be used in the sequel.

Lemma 1 [16, *Lemma*1.2]

(i) *For* $0 < r < R < \infty$,

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\mathrm{d}y \le C\Big(\frac{r}{R}\Big)^{\delta}\frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\mathrm{d}y.$$

(ii) $r^{2-n}\int_{B(x,r)} V(y)\mathrm{d}y = 1$ *if* $r = \rho(x)$. $r \sim \rho(x)$ *if and only if* $r^{2-n}\int_{B(x,r)} V(y)\mathrm{d}y \sim 1$.

Lemma 2 [16, *Lemma* 1.4]

(i) *There exist* $C > 0$ *and* $k_0 \ge 1$ *such that for all* $x, y \in \mathbb{R}^n$,

$$C^{-1}\rho(x)(1+|x-y|/\rho(x))^{-k_0} \le \rho(y) \le C\rho(x)(1+|x-y|/\rho(x))^{k_0/(1+k_0)}.$$

In particular, $\rho(y) \sim \rho(x)$ if $|x-y| < C\rho(x)$.

(ii) *There exists* $l_0 > 1$ *such that*

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-2}}\mathrm{d}y \le \frac{C}{R^{n-2}}\int_{B(x,R)} V(y)\mathrm{d}y \le C\Big(1+\frac{R}{\rho(x)}\Big)^{l_0}.$$

Lemma 3 [7, *Corollary* 2.8] *For every nonnegative Schwarz function* ω, *there exist* $\delta > 0$ *and* $C > 0$ *such that*

$$\int_{\mathbb{R}^n} t^{-n/2}\omega(|x-y|/\sqrt{t})V(y)\mathrm{d}y \le \begin{cases} Ct^{-1}(\sqrt{t}/\rho(x))^{\delta}, & t < \rho(x)^2; \\ Ct^{-1}(\sqrt{t}/\rho(x))^{l_0}, & t \ge \rho(x)^2, \end{cases}$$

where l_0 *is the constant given in Lemma* 2.

It is well known that the classical Hardy space $H^1(\mathbb{R}^n)$ can be defined via the maximal function $\sup_{t>0}|e^{-t(-\Delta)}f(x)|$ (cf. [17]). In this sense, we can say that the Hardy space $H^1(\mathbb{R}^n)$ is the Hardy space associated with $-\Delta$. Since 1990s, the theory of Hardy spaces associated with operators on $\mathbb{R}^n$ has been investigated extensively. In [6], Dziubański and Zienkiewicz introduced the Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ related to Schrödinger operators $\mathcal{L}$ and obtained the atomic characterization and the Riesz transform characterization of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ via local Hardy spaces. By the aid of Campanato type spaces, the spaces $H^p_{\mathcal{L}}(\mathbb{R}^n)$ $(0 < p \le 1)$ were introduced by Dziubański and Zienkiewicz [7]. In recent years, the results of [6, 7] have been extended to other second-ordered differential operators, and various function spaces associated to operators have been established. For further information, we refer the reader to [3, 18–20] and the references therein.

For a Schrödinger operator $\mathcal{L}$, let $\{e^{-t\mathcal{L}}\}_{t>0}$ be the heat semigroup generated by $\mathcal{L}$ and denote by $K^{\mathcal{L}}_t(\cdot,\cdot)$ the integral kernel of $e^{-t\mathcal{L}}$. Because the potential $V \ge 0$, the Feynman-Kac formula implies that

$$0 \le K_t^{\mathcal{L}}(x,y) \le K_t(x-y) := (4\pi t)^{-n/2} e^{-|x-y|^2/(4t)}.$$

The Hardy type spaces $H^p_{\mathcal{L}}(\mathbb{R}^n)$, $0 < p \le 1$, are defined as follows (cf. [7]):

Definition 1 For $0 < p \le 1$, the Hardy type space $H^p_{\mathcal{L}}(\mathbb{R}^n)$ is defined as the completion of the space of compactly supported $L^1(\mathbb{R}^n)$-functions such that the maximal function

$$M_{\mathcal{L}}(f)(x) := \sup_{t>0} |e^{-t\mathcal{L}}(f)(x)|$$

belongs to $L^p(\mathbb{R}^n)$. The quasi-norm in $H^p_{\mathcal{L}}(\mathbb{R}^n)$ is defined as $\|f\|_{H^p_{\mathcal{L}}} := \|M_{\mathcal{L}}(f)\|_{L^p}$.

Let f be a locally integrable function on $\mathbb{R}^n$ and $B = B(x, r)$ be a ball. Denote by f_B the mean of f on B, i.e., $f_B = |B|^{-1} \int_B f(y)\mathrm{d}y$. Let

$$f(B, V) := \begin{cases} f_B, & r < \rho(x); \\ 0, & r \ge \rho(x), \end{cases}$$

where the auxiliary function $\rho(\cdot)$ is defined as

$$\rho(x) := \sup\left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,y)} V(y)\mathrm{d}y \le 1 \right\}.$$

Definition 2 Let $0 < \gamma \le 1$. The Campanato type space $\mathrm{BMO}^{\gamma}_{\mathcal{L}}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f satisfying

$$\|f\|_{\mathrm{BMO}^{\gamma}_{\mathcal{L}}} := \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|^{1+\gamma/n}} \int_B |f(x) - f(B, V)|\mathrm{d}x \right\} < \infty.$$

The dual space of $H^{n/(n+\gamma)}_{\mathcal{L}}(\mathbb{R}^n)$, $0 \le \gamma < 1$, is the Campanato type space $\mathrm{BMO}^{\gamma}_{\mathcal{L}}(\mathbb{R}^n)$ (cf. [14, Theorem 4.5]).

2.2 The *T*1 criterion on Campanato type spaces

We denote by $L^p_c(\mathbb{R}^n)$ the set of functions $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, whose support supp (f) is a compact subset of $\mathbb{R}^n$.

Definition 3 Let $0 \le \beta < n$, $1 < p \le q < \infty$ with $1/q = 1/p - \beta/n$. Let T be a bounded linear operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\mathrm{d}y, \quad f \in L^p_c(\mathbb{R}^n) \text{ and } a.e.\ x \notin \mathrm{supp}(f).$$

We shall say that T is a β-Schrödinger–Calderón–Zygmund operator with regularity exponent $\delta > 0$ if there exists a constant $C > 0$ such that

(i) $|K(x,y)| \leq \frac{C}{|x-y|^{n-\beta}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}$ for all $N>0$ and $x\neq y$;

(ii) $|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)| \leq C\frac{|y-z|^{\delta}}{|x-y|^{n-\beta+\delta}}$ when $|x-y|>2|y-z|$

.

The following $T1$ type criterions on Campanato type spaces were established by Ma et al. [15].

Theorem 1 [15, *Theorem*1.1] *Let T be a β-Schrödinger–Calderón–Zygmund operator,* $\beta \geq 0$, $0<\beta+\gamma<\min\{1,\delta\}$, *with smoothness exponent* δ. *Then T is a bounded operator from* $BMO_{\mathcal{L}}^{\gamma}(\mathbb{R}^n)$ *into* $BMO_{\mathcal{L}}^{\gamma+\beta}(\mathbb{R}^n)$ *if and only if there exists a constant C such that*

$$\left(\frac{\rho(x)}{r}\right)^{\gamma}\frac{1}{|B|^{1+\beta/n}}\int_B |T1(y)-(T1)_B|\mathrm{d}y \leq C$$

for every ball $B(x,r)$, $x\in\mathbb{R}^n$ *and* $0<r\leq\rho(x)/2$.

When $\gamma=0$, the authors in [15] also proved

Theorem 2 [15, *Theorem*1.2] *Let T be a β-Schrödinger–Calderón–Zygmund operator,* $0\leq\beta<\min\{1,\delta\}$, *with smoothness exponent* δ. *Then T is a bounded operator from* $BMO_{\mathcal{L}}(\mathbb{R}^n)$ *into* $BMO_{\mathcal{L}}^{\beta}(\mathbb{R}^n)$ *if and only if there exists a constant C such that*

$$\log\left(\frac{\rho(x)}{r}\right)\frac{1}{|B|^{1+\beta/n}}\int_B |T1(y)-(T1)_B|\mathrm{d}y \leq C$$

for every ball $B(x,r)$, $x\in\mathbb{R}^n$ *and* $0<r\leq\rho(x)/2$.

Lemma 4 [15, *Remark* 4.1] *Theorems 1 and 2 can be also stated in a vector-valued setting. If Tf takes values in a Banach space* $\mathbb{B}$ *and the absolute values in the conditions are replaced by the norm in* $\mathbb{B}$, *then both results hold.*

3 Regularity estimates

3.1 Regularities of heat kernels

By the fundamental solutions of Schrödinger operators, Dziubański and Zienkiewicz proved that the heat kernel $K_t^{\mathcal{L}}(\cdot,\cdot)$ satisfies the following estimates, see also [11].

Lemma 5

(i) ([6, *Theorem* 2.11]) *For any* $N > 0$, *there exist constants* $C_N, c > 0$ *such that*

$$|K_t^{\mathcal{L}}(x,y)| \le C_N t^{-n/2} e^{-c|x-y|^2/t}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

(ii) ([8, *Theorem* 4.11]) *Assume that* $0 < \delta \le \min\{1, \delta_0\}$. *For any* $N > 0$, *there exist constants* $C_N, c > 0$ *such that for all* $|h| < \sqrt{t}$,

$$\left|K_t^{\mathcal{L}}(x+h,y) - K_t^{\mathcal{L}}(x,y)\right| \le C_N\left(\frac{|h|}{\sqrt{t}}\right)^{\delta} t^{-n/2} e^{-c|x-y|^2/t}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

Lemma 6 [7, *Proposition* 2.16] *There exist constants* $C, c > 0$ *such that for* $x, y \in \mathbb{R}^n$ *and* $t > 0$,

$$\left|K_t^{\mathcal{L}}(x,y) - K_t(x-y)\right| \le C\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0} t^{-n/2} e^{-c|x-y|^2/t}.$$

In [5], under the assumption that $V \in B_q, q > n$, Duong et al. obtained the following regularity estimate for the kernel $K_t^{\mathcal{L}}(\cdot,\cdot)$.

Lemma 7 [5, *Lemma* 3.8] *Suppose that* $V \in B_q$ *for some* $q > n$. *For any* $N > 0$, *there exist constants* $C > 0$ *and* $c > 0$ *such that for all* $x, y \in \mathbb{R}^n$ *and* $t > 0$,

$$|\nabla_x K_t^{\mathcal{L}}(x,y)| \le C t^{-(n+1)/2} e^{-c|x-y|^2/t}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}. \tag{6}$$

By the perturbation theory for semigroups of operators,

$$\begin{aligned} K_t(x-y) - K_t^{\mathcal{L}}(x,y) &= \int_0^t \int_{\mathbb{R}^n} K_{t-s}(w-x)V(w)K_s^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s \\ &= \int_0^{t/2} \int_{\mathbb{R}^n} K_{t-s}(w-x)V(w)K_s^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s \\ &\quad + \int_0^{t/2} \int_{\mathbb{R}^n} K_s(w-x)V(w)K_{t-s}^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s. \end{aligned} \tag{7}$$

Similar to [7, Proposition 2.16], we can prove the following lemma.

Lemma 8 *Suppose that* $V \in B_q$ *for some* $q > n$. *There exist constants* $C, c > 0$ *such that*

$$\left|\nabla_x K_t(x-y) - \nabla_x K_t^{\mathcal{L}}(x,y)\right| \leq Ct^{-(n+1)/2}e^{-c|x-y|^2/t}\min\left\{\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}, \left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0}\right\}.$$

Proof If $t \geq \rho(y)^2$, it is easy to see that

$$\left|\nabla_x K_t(x-y) - \nabla_x K_t^{\mathcal{L}}(x,y)\right| \leq Ct^{-(n+1)/2}e^{-c|x-y|^2/t}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0}.$$

If $t < \rho(y)^2$, by (7), we get

$$\begin{aligned}\left|\nabla_x K_t(x-y) - \nabla_x K_t^{\mathcal{L}}(x,y)\right| =& \left|\int_0^t \int_{\mathbb{R}^n} \nabla_x K_{t-s}(x-w)V(w)K_s^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s\right| \\ \leq& I_1 + I_2,\end{aligned} \tag{8}$$

where

$$\begin{cases} I_1 := \int_0^{t/2} \int_{\mathbb{R}^n} |\nabla_x K_{t-s}(x-w)|V(w)K_s^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s; \\ I_2 := \int_0^{t/2} \int_{\mathbb{R}^n} |\nabla_x K_s(x-w)|V(w)K_{t-s}^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s. \end{cases}$$

For I_1, it follows from Lemmas 7 and 3 that

$$\begin{aligned} I_1 =& \int_0^{t/2} \int_{|w-y|<|x-y|/2} |\nabla_x K_{t-s}(x-w)|V(w)K_s^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s \\ &+ \int_0^{t/2} \int_{|w-y|\geq|x-y|/2} |\nabla_x K_{t-s}(x-w)|V(w)K_s^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s \\ \leq& Ct^{-(n+1)/2}e^{-c|x-y|^2/t}\int_0^{t/2} \int_{|w-y|<|x-y|/2} V(w)s^{-n/2}e^{-c|w-y|^2/s}\mathrm{d}w\mathrm{d}s \\ &+ Ct^{-(n+1)/2}\int_0^{t/2} \int_{|w-y|\geq|x-y|/2} V(w)s^{-n/2}e^{-c(|w-y|^2+|x-y|^2)/s}\mathrm{d}w\mathrm{d}s \\ \leq& Ct^{-(n+1)/2}e^{-c|x-y|^2/t}\int_0^{t/2} \frac{1}{s}\left(\frac{\sqrt{s}}{\rho(y)}\right)^{\delta_0}\mathrm{d}s \\ =& Ct^{-(n+1)/2}e^{-c|x-y|^2/t}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0}. \end{aligned}$$

Similar to I_1, for the term I_2, we can obtain

$$I_2 \leq Ct^{-(n+1)/2}e^{-c|x-y|^2/t}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}.$$

It follows from (i) of Lemma 2 that

$$\frac{\sqrt{t}}{\rho(x)} \leq C\left(1+\frac{|x-y|}{\sqrt{t}}\frac{\sqrt{t}}{\rho(y)}\right)^{l_0}\frac{\sqrt{t}}{\rho(y)} \leq C_\varepsilon e^{\varepsilon|x-y|^2/t}\frac{\sqrt{t}}{\rho(y)},$$

where $\varepsilon > 0$ is an arbitrary small constant. Hence

$$I_2 \leq Ct^{-(n+1)/2}e^{-c|x-y|^2/t}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0}.$$

□

Lemma 9 [7, *Proposition* 2.17] *Let* $0 < \delta < \min\{1, \delta_0\}$. *For every* $C' > 0$ *there exists a constant* C *such that for every* $z, x, y \in \mathbb{R}^n$, $|y-z| \leq |x-y|/4$, $|y-z| \leq C'\rho(x)$ *we have*

$$\left|\left(K_t^{\mathcal{L}}(x,y) - K_t(x-y)\right) - \left(K_t^{\mathcal{L}}(x,z) - K_t(x-z)\right)\right| \leq C\left(\frac{|y-z|}{\rho(y)}\right)^{\delta} t^{-n/2}e^{-c|x-y|^2/t}.$$

Lemma 10 *Suppose that* $V \in B_q$ *for some* $q > n$. *Let* $\delta_1 = 1 - n/q$ *and* $0 < \delta' < \delta_1$. *For every* $C' > 0$ *there exists a constant* C *such that for every* $u, x, y \in \mathbb{R}^n$, $|u| \leq |x-y|/4$, $|u| \leq C'\rho(x)$ *we have*

$$\begin{aligned}&\left|\left(\nabla_x K_t^{\mathcal{L}}(x,y) - \nabla_x K_t(x-y)\right) - \left(\nabla_x K_t^{\mathcal{L}}(x+u,y) - \nabla_x K_t(x+u-y)\right)\right| \\ &\quad \leq Ct^{-(n+1)/2}e^{-c|x-y|^2/t}\left(\frac{|u|}{\rho(y)}\right)^{\delta'}.\end{aligned}$$

Proof We prove this lemma by the same argument as Lemma 8. It is enough to verify that

$$\begin{aligned}&\left|\left(\nabla_x K_t^{\mathcal{L}}(x,y) - \nabla_x K_t(x-y)\right) - \left(\nabla_x K_t^{\mathcal{L}}(x+u,y) - \nabla_x K_t(x+u-y)\right)\right| \\ &\quad \leq C_\epsilon t^{-(n+1)/2}e^{\epsilon|x-y|^2/t}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1},\end{aligned} \tag{9}$$

where $\epsilon > 0$ is an arbitrary small constant. In fact, under the condition $|u| < |x-y|/4$, it is easy to see that $|x-y| \sim |x+u-y|$. We can deduce from Lemma 7 that

$$
\begin{aligned}
&\left|\left(\nabla_x K_t^{\mathcal{L}}(x,y)-\nabla_x K_t(x-y)\right)-\left(\nabla_x K_t^{\mathcal{L}}(x+u,y)-\nabla_x K_t(x+u-y)\right)\right| \\
&\quad \le \left|\nabla_x K_t^{\mathcal{L}}(x,y)\right|+\left|\nabla_x K_t(x-y)\right|+\left|\nabla_x K_t^{\mathcal{L}}(x+u,y)\right|+\left|\nabla_x K_t(x+u-y)\right| \\
&\quad \le Ct^{-(n+1)/2}e^{-c|x-y|^2/t}.
\end{aligned} \tag{10}
$$

Then, for $\delta' \in (0,\delta_1)$, it follows from (9) and (10) that

$$
\begin{aligned}
&\left|\left(\nabla_x K_t^{\mathcal{L}}(x,y)-\nabla_x K_t(x-y)\right)-\left(\nabla_x K_t^{\mathcal{L}}(x+u,y)-\nabla_x K_t(x+u-y)\right)\right| \\
&\quad \le C\left\{t^{-(n+1)/2}e^{\epsilon|x-y|^2/t}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1}\right\}^{\delta'/\delta_1}\left\{t^{-(n+1)/2}e^{-c|x-y|^2/t}\right\}^{1-\delta'/\delta_1},
\end{aligned}
$$

which gives the desired estimate.

Now we prove (9). Since the case for $|u| \ge \rho(y)$ is trivial, we may assume $|u| < \rho(y)$. If $t \le 2|u|^2$, the required estimate follows from Lemma 8. Hence we consider the case $t > 2|u|^2$ only. Recall that for the classical heat kernel $K_t(\cdot)$, it holds

$$
\left|\nabla_x^2 K_t(x)\right| \le Ct^{-n/2-1}e^{-c|x|^2/t}.
$$

A direct computation gives

$$
\left|\nabla_x K_t(x+u)-\nabla_x K_t(x)\right| \le C|u|t^{-n/2-1}, \tag{11}
$$

and for $|u| \le |x|/2$,

$$
\left|\nabla_x K_t(x+u)-\nabla_x K_t(x)\right| \le C|u|t^{-n/2-1}e^{-c|x|^2/t}. \tag{12}
$$

Similar to (8), we split

$$
\left|\left(\nabla_x K_t^{\mathcal{L}}(x,y)-\nabla_x K_t(x-y)\right)-\left(\nabla_x K_t^{\mathcal{L}}(x+u,y)-\nabla_x K_t(x+u-y)\right)\right| \le J_1+J_2,
$$

where

$$
\begin{cases}
J_1 := \displaystyle\int_0^{t/2}\int_{\mathbb{R}^n}\left|\nabla_x K_{t-s}(w-(x+u))-\nabla_x K_{t-s}(w-x)\right|V(w)K_s^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s; \\
J_2 := \displaystyle\int_0^{t/2}\int_{\mathbb{R}^n}\left|\nabla_x K_s(w-(x+u))-\nabla_x K_s(w-x)\right|V(w)K_{t-s}^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s.
\end{cases}
$$

For J_1, if $t < 2\rho(y)^2$, using Lemma 3 and (11), we get

$$\begin{aligned} J_1 &\le C|u|t^{-n/2-1}\int_0^{t/2}\int_{\mathbb{R}^n} V(w)s^{-n/2}e^{-c|y-w|^2/s}\mathrm{d}w\mathrm{d}s \\ &\le C|u|t^{-n/2-1}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0} \\ &\le Ct^{-(n+1)/2}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1}. \end{aligned}$$

If $t \ge 2\rho(y)^2$, applying Lemmas 3 and 5, we have

$$\begin{aligned} J_1 &\le C|u|t^{-n/2-1}\int_0^{t/2}\int_{\mathbb{R}^n} V(w)s^{-n/2}e^{-c|y-w|^2/s}\left(1+\frac{\sqrt{s}}{\rho(y)}\right)^{-N}\mathrm{d}w\mathrm{d}s \\ &\le C|u|t^{-n/2-1}\left\{\int_0^{\rho(y)^2}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(y)}\right)^{\delta_0}ds+\int_{\rho(y)^2}^{t/2}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(y)}\right)^{l_0-N}\mathrm{d}s\right\} \\ &\le C|u|t^{-n/2-1} \\ &\le Ct^{-(n+1)/2}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1}, \end{aligned}$$

where N is chosen large enough satisfying $N > l_0$.

To estimate J_2, we use Lemma 5 and write $J_2 \le C(J_{2,1}+J_{2,2}+J_{2,3})$, where

$$\begin{cases} J_{2,1} := t^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_0^{|u|^2}\int_{\mathbb{R}^n}\left|\nabla_x K_s(w-(x+u))-\nabla_x K_s(w-x)\right|V(w)\mathrm{d}w\mathrm{d}s; \\ J_{2,2} := t^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_{|u|^2}^{t/2}\int_{|w-x|<2|u|}\left|\nabla_x K_s(w-(x+u))-\nabla_x K_s(w-x)\right|V(w)\mathrm{d}w\mathrm{d}s; \\ J_{2,3} := t^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_{|u|^2}^{t/2}\int_{|w-x|\ge 2|u|}\left|\nabla_x K_s(w-(x+u))-\nabla_x K_s(w-x)\right|V(w)\mathrm{d}w\mathrm{d}s. \end{cases}$$

Notice that $\rho(x+u)\sim\rho(x)$ as $|u|\le\rho(x)$. It holds

$$\begin{aligned} J_{2,1} &\le Ct^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_0^{|u|^2}\frac{1}{s^{3/2}}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\mathrm{d}s \\ &\le Ct^{-n/2}|u|^{-1}\left(\frac{|u|}{\rho(y)}\right)^{-1}\left(\frac{\sqrt{t}}{|u|}\right)^{-1}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N+1}\left(\frac{|u|}{\rho(y)}\right)^{\delta_0}\left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0} \\ &\le Ct^{-(n+1)/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N+1}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1}\left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0}, \end{aligned}$$

where in the last inequality we have used the fact that $\delta_0 = 2 - n/q$. By Lemmas 1 and 2, we apply (11) to get

$$\begin{aligned}
J_{2,2} &\le Ct^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_{|u|^2}^{t/2}\int_{|w-x|<2|u|}|u|s^{-n/2-1}V(w)\mathrm{d}w\mathrm{d}s\\
&\le Ct^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_{|u|^2}^{t/2}|u|^{n-1}s^{-n/2-1}\left(\frac{|u|}{\rho(x)}\right)^{\delta_0}\mathrm{d}s\\
&\le Ct^{-n/2}|u|^{-1}\left(\frac{|u|}{\rho(y)}\right)^{\delta_0}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0}\\
&\le Ct^{-(n+1)/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N+1}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1}\left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0}.
\end{aligned}$$

For $J_{2,3}$, if $t \le 2\rho(x)^2$, it can be deduced from Lemma 3 and (12) that

$$\begin{aligned}
J_{2,3} &\le Ct^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_{|u|^2}^{t/2}\int_{|w-x|\ge 2|u|}|u|s^{-n/2-1}e^{-c|w-x|^2/s}V(w)\mathrm{d}w\mathrm{d}s\\
&\le Ct^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}|u|\int_{|u|^2}^{t/2}\frac{1}{s^2}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\mathrm{d}s\\
&\le Ct^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}|u|^{-1}\left(\frac{|u|}{\rho(y)}\right)^{\delta_0}\left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0}\\
&\le Ct^{-(n+1)/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N+1}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1}\left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0}.
\end{aligned}$$

If $t > 2\rho(x)^2$, then

$$\begin{aligned}
J_{2,3} \le & Ct^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_{|u|^2}^{\rho(x)^2}\int_{|w-x|\ge 2|u|} \\
& \left|\nabla_x K_s(w-(x+u)) - \nabla_x K_s(w-x)\right| V(w)\mathrm{d}w\mathrm{d}s \\
& + Ct^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_{\rho(x)^2}^{t/2}\int_{|w-x|\ge 2|u|} |u| s^{-n/2-1} e^{-c|w-x|^2/s} V(w)\mathrm{d}w\mathrm{d}s \\
\le & Ct^{-(n+1)/2}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N+1}\left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0} \\
& + Ct^{-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_{\rho(x)^2}^{t/2}\frac{|u|}{s}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{l_0}\mathrm{d}s \\
\le & Ct^{-(n+1)/2}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N+1}\left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0} \\
& + Ct^{-(n+1)/2}\frac{|u|}{\rho(y)}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N-1+l_0}\left(\frac{\rho(y)}{\rho(x)}\right)^{l_0},
\end{aligned} \tag{13}$$

where in (13) we have used the estimate obtained for $t \le 2\rho(x)^2$ and Lemma 3 for $s \ge \rho(x)^2$.

By Lemma 2,

$$\frac{\rho(y)}{\rho(x)} \le C\left(1+\frac{|x-y|}{\sqrt{t}}\frac{\sqrt{t}}{\rho(y)}\right)^{m_0} \le C_\epsilon e^{\epsilon|x-y|^2/t}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{m_0},$$

where $\epsilon > 0$ is an arbitrary small constant. Choosing N large enough in the estimates of $J_{2,1}, J_{2,2}$ and $J_{2,3}$, we obtain (9) and hence Lemma 10 is proved. □

We can obtain the following estimates, which generalize [4, Lemmas 3.7 and 3.8]. We also refer to [13, (57)] for the case $m = 1$ in the setting of Heisenberg groups.

Proposition 1

(i) *There exist constants $C, c > 0$ such that*

$$\left|t^m\partial_t^m K_t^{\mathcal{L}}(x,y) - t^m\partial_t^m K_t(x-y)\right| \le Ct^{-n/2}e^{-c|x-y|^2/t}\min\left\{\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}, \left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0}\right\}.$$

(ii) *Let* $0 < \delta < \min\{1, \delta_0\}$*. For every* $C' > 0$ *there exist constants* C *and* c *such that for every* $z, x, y \in \mathbb{R}^n, |y-z| \le |x-y|/4, |y-z| \le C'\rho(x)$ *we have*

$$\left|\left(t^m\partial_t^m K_t^{\mathcal{L}}(x,y) - t^m\partial_t^m K_t(x-y)\right) - \left(t^m\partial_t^m K_t^{\mathcal{L}}(x,z) - t^m\partial_t^m K_t(x-z)\right)\right|$$
$$\le C\left(\frac{|y-z|}{\rho(y)}\right)^{\delta} t^{-n/2}e^{-c|x-y|^2/t}.$$

Proof For $t>0$ and $m \in \mathbb{Z}_+$, define $Q_{t,m}^{\mathcal{L}}(x,y) := t^m\partial_t^m K_t^{\mathcal{L}}(x,y)$ and $Q_{t,m}(x-y) := t^m\partial_t^m K_t(x-y)$. The proof of (i) is similar to [18, Lemmas 4.10 and 4.11], so we omit the details.

For (ii), by (7), we get

$$\begin{aligned}&\left(K_t(x+u-y) - K_t^{\mathcal{L}}(x+u,y)\right) - \left(K_t(x-y) - K_t^{\mathcal{L}}(x,y)\right)\\ &= \int_0^{t/2}\int_{\mathbb{R}^n}\left(K_{t-s}(w-(x+u)) - K_{t-s}(w-x)\right)V(w)K_s^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s\\ &\quad + \int_0^{t/2}\int_{\mathbb{R}^n}\left(K_s(w-(x+u)) - K_s(w-x)\right)V(w)K_{t-s}^{\mathcal{L}}(w,y)\mathrm{d}w\mathrm{d}s.\end{aligned}$$

Similar to [18, Proposition 4.8], we can use a direct calculus to deduce:

(1) If m is even with $m \ge 2$, there exists a sequence of coefficients $\{C_{m,j}\}_{m\ge 2, 2\le j\le m/2}$ such that

$$\begin{aligned}&t^m\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left\{\left(K_t(x+u-y) - K_t^{\mathcal{L}}(x+u,y)\right) - \left(K_t(x-y) - K_t^{\mathcal{L}}(x,y)\right)\right\}\\ &= \frac{m+1}{2}(E_1+E_2) + \sum_{j=2}^{m/2}(C_{m-1,j-1} + C_{m-1,j})\left(E_{3,1}^j + E_{3,2}^j\right) + E_4 + E_5.\end{aligned} \tag{14}$$

(2) If m is odd with $m \ge 3$, there exists a sequence of coefficients $\{C_{m,j}\}_{m\ge 3, 2\le j\le [m/2]}$ such that

$$\begin{aligned}&t^m\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left\{\left(K_t(x+u-y) - K_t^{\mathcal{L}}(x+u,y)\right) - \left(K_t(x-y) - K_t^{\mathcal{L}}(x,y)\right)\right\}\\ &= \frac{m+1}{2}(E_1+E_2) + E_4 - E_5 + \sum_{j=2}^{[m/2]}(C_{m-1,j-1} + C_{m-1,j})\left(E_{3,1}^j + E_{3,2}^j\right)\\ &\quad + 2C_{m,[m/2]}\frac{\mathrm{d}^{[m/2]}}{\mathrm{d}t^{[m/2]}}\left(K_{t/2}(w-(x+u)) - K_{t/2}(w-x)\right)V(w)\frac{\mathrm{d}^{[m/2]}}{\mathrm{d}t^{[m/2]}}K_{t/2}^{\mathcal{L}}(w,y).\end{aligned} \tag{15}$$

Here in the above (14) and (15),

$$\begin{cases} E_1 := \int_{\mathbb{R}^n} t\left\{ t^{m-1}\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\big(K_{t/2}(w-x-u)-K_{t/2}(w-x)\big)\right\}V(w)K^{\mathcal{L}}_{t/2}(w,y)\mathrm{d}w;\\ E_2 := \int_{\mathbb{R}^n} t\big(K_{t/2}(w-x-u))-K_{t/2}(w-x)\big)V(w)\left(t^{m-1}\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}K^{\mathcal{L}}_{t/2}(w,y)\right)\mathrm{d}w;\\ E^j_{3,1} := \int_{\mathbb{R}^n} t\left\{ t^{m-j}\frac{\mathrm{d}^{m-j}}{\mathrm{d}t^{m-j}}\big(K_{t/2}(w-x-u)-K_{t/2}(w-x)\big)\right\}V(w)\left(t^{j-1}\frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}}K^{\mathcal{L}}_{t/2}(w,y)\right)\mathrm{d}w;\\ E^j_{3,2} := \int_{\mathbb{R}^n} t\left\{ t^{j-1}\frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}}\big(K_{t/2}(w-x-u)-K_{t/2}(w-x)\big)\right\}V(w)\left(t^{m-j}\frac{\mathrm{d}^{m-j}}{\mathrm{d}t^{m-j}}K^{\mathcal{L}}_{t/2}(w,y)\right)\mathrm{d}w;\\ E_4 := \int_0^{t/2}\int_{\mathbb{R}^n}\left(t^m\frac{\mathrm{d}^m}{\mathrm{d}t^m}(K_{t-s}(w-x-u)-K_{t-s}(w-x)\right)V(w)K^{\mathcal{L}}_s(w,y)\mathrm{d}w\mathrm{d}s;\\ E_5 := \int_0^{t/2}\int_{\mathbb{R}^n}\big(K_s(w-x-u)-K_s(w,x)\big)V(w)\left(t^m\frac{\mathrm{d}^m}{\mathrm{d}t^m}K^{\mathcal{L}}_{t-s}(w,y)\right)\mathrm{d}w\mathrm{d}s. \end{cases}$$

Below, for the sake of simplicity, we only estimate E_1, E_4, E_5. The estimations for E_2, $E^j_{3,1}$, $E^j_{3,2}$ are similar, and so we omit the details. By the mean value theorem, we know that there exist constants C, c such that

$$\big|Q_{t,m}(x+u)-Q_{t,m}(x)\big| \le \begin{cases} C|u|t^{-(n+1)/2}, & \forall\, x,u\in\mathbb{R}^n,\ t\in(0,\infty);\\ C|u|t^{-(n+1)/2}e^{-c|x|^2/t}, & |u|\le |x|/2,\ t>0. \end{cases} \tag{16}$$

We divide E_1 as $E_1 \le E_{1,1}+E_{1,2}$, where

$$\begin{cases} E_{1,1} := t\int_{|w-x|<2u}\Big|Q_{t/2,m-1}(w-(x+u))-Q_{t/2,m-1}(w-x)\Big|V(w)K^{\mathcal{L}}_{t/2}(w,y)\mathrm{d}w;\\ E_{1,2} := t\int_{|w-x|\ge 2u}\Big|Q_{t/2,m-1}(w-(x+u))-Q_{t/2,m-1}(w-x)\Big|V(w)K^{\mathcal{L}}_{t/2}(w,y)\mathrm{d}w. \end{cases}$$

If $t<2\rho(y)^2$, for $E_{1,1}$, By (16), Lemma 3 (i) and Lemma 5, we obtain

$$\begin{aligned} E_{1,1} &\le Ct^{1-n/2}\int_{|w-x|<2|u|}|u|t^{-(n+1)/2}V(w)e^{-c|w-y|^2/t}\mathrm{d}w\\ &\le Ct^{1-n/2}|u|\frac{1}{t^{3/2}}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0}\\ &= Ct^{-n/2}\frac{|u|}{\sqrt{t}}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0} \le Ct^{-n/2}\left(\frac{|u|}{\rho(y)}\right)^{\delta}. \end{aligned}$$

Similar to $E_{1,1}$, by (16) again, we get

$$
\begin{aligned}
E_{1,2} &\leq Ct^{1-n/2}\int_{|w-x|\geq 2|u|}|u|t^{-(n+1)/2}e^{-c|w-y|^2/t}V(w)\mathrm{d}w\\
&\leq Ct^{1-n/2}\frac{|u|}{t^{3/2}}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0}\\
&\leq Ct^{-n/2}\left(\frac{|u|}{\rho(y)}\right)^{\delta}.
\end{aligned}
$$

If $t > 2\rho(y)^2$, for $E_{1,1}$, by (16), Lemma 3 (ii) and Lemma 5, we obtain

$$
\begin{aligned}
E_{1,1} &\leq Ct^{1-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\int_{|w-x|<2|u|}|u|t^{-(n+1)/2}V(w)e^{-c|w-y|^2/t}\mathrm{d}w\\
&\leq Ct^{1-n/2}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}|u|\frac{1}{t^{3/2}}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{l_0}\\
&\leq Ct^{-n/2}\left(\frac{|u|}{\rho(y)}\right)^{\delta}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N+l_0}.
\end{aligned}
$$

Similar to $E_{1,1}$, we can also choose N large enough such that

$$
E_{1,2} \leq Ct^{-n/2}\left(\frac{|u|}{\rho(y)}\right)^{\delta}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-N+l_0}.
$$

For E_4 and E_5, similar to [13, Lemma 10], by Lemma 5 and (16), we can obtain

$$
|E_4+E_5| \leq C_\epsilon t^{-n/2}e^{\epsilon|x-y|^2/t}\left(\frac{|u|}{\rho(y)}\right)^{\delta}, \tag{17}
$$

where $\epsilon > 0$ is an arbitrary small constant. Hence Proposition 1 is proved. □

3.2 Fractional heat kernels associated with $\mathcal{L}$

In the following, we will derive some regularity estimates for the fractional heat kernels related with $\mathcal{L}$. For $\alpha \in (0,1)$, the fractional power of $\mathcal{L}$, denoted by $\mathcal{L}^\alpha$, is defined as

$$
\mathcal{L}^\alpha := \frac{1}{\Gamma(-\alpha)}\int_0^\infty\left(e^{-t\sqrt{\mathcal{L}}}f(x)-f(x)\right)\frac{\mathrm{d}t}{t^{1+2\alpha}} \quad \forall f \in L^2(\mathbb{R}^n). \tag{18}
$$

We use the subordinative formula to express the integral kernel $K^{\mathcal{L}}_{\alpha,t}(\cdot,\cdot)$ of $e^{-t\mathcal{L}^\alpha}$ as (cf. [10])

$$K_{\alpha,t}^{\mathcal{L}}(x,y)=\int_0^\infty \eta_t^\alpha(s)K_s^{\mathcal{L}}(x,y)\mathrm{d}s,$$

where $\eta_t^\alpha(\cdot)$ satisfies

$$\begin{cases}\eta_t^\alpha(s)=1/t^{1/\alpha}\eta_1^\alpha(s/t^{1/\alpha});\\ \eta_t^\alpha(s)\le t/s^{1+\alpha}\quad \forall\, s,t>0;\\ \displaystyle\int_0^\infty s^{-r}\eta_1^\alpha(s)\mathrm{d}s<\infty,\quad r>0;\\ \eta_t^\alpha(s)\simeq t/s^{1+\alpha}\ \ \forall\, s\ge t^{1/\alpha}>0.\end{cases}\tag{19}$$

By the subordinative formula (4) and Lemma 5, Li et al. [12] proved the following estimates for $K_{\alpha,t}^{\mathcal{L}}(\cdot,\cdot)$.

Proposition 2 [12, *Propositions* 3.1 *and* 3.2] *Let* $0<\alpha<1$.

(i) *For any* $N>0$, *there exists a constant* $C_N>0$ *such that*

$$\left|K_{\alpha,t}^{\mathcal{L}}(x,y)\right|\le\frac{C_Nt}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-N}.$$

(ii) *Let* $0<\delta\le\min\{1,\delta_0\}$. *For any* $N>0$, *there exists a constant* $C_N>0$ *such that for all* $|h|\le\sqrt{t^{1/\alpha}}$,

$$\begin{aligned}&\left|K_{\alpha,t}^{\mathcal{L}}(x+h,y)-K_{\alpha,t}^{\mathcal{L}}(x,y)\right|\\ &\le C_N\left(\frac{|h|}{\sqrt{t^{1/\alpha}}}\right)^{\delta}\frac{t}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-N}.\end{aligned}$$

For the kernels $\widetilde{D}_{\alpha,t}^{\mathcal{L}}(\cdot,\cdot)$ and $D_{\alpha,t}^{\mathcal{L},m}(\cdot,\cdot), m\in\mathbb{Z}_+, t>0$, defined by (5), the following regularity estimates were obtained by Li et al. [12].

Proposition 3 [12, *Proposition* 3.3] *Let* $0<\alpha<1$.

(i) *For every* N, *there is a constant* $C_N>0$ *such that*

$$\left|D_{\alpha,t}^{\mathcal{L},m}(x,y)\right|\le\frac{C_Nt}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-N}.$$

(ii) *Let* $0<\delta<\min\{2\alpha,\delta_0,1\}$. *For every* $N>0$, *there exists a constant* $C_N>0$ *such that, for all* $|h|<\sqrt{t^{1/\alpha}}$,

$$\left|D_{\alpha,t}^{\mathcal{L},m}(x+h,y)-D_{\alpha,t}^{\mathcal{L},m}(x,y)\right|$$
$$\le C_N\left(\frac{h}{\sqrt{t^{1/\alpha}}}\right)^{\delta}\frac{t}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-N}.$$

(iii) *There exists a constant* $C_N>0$ *such that*

$$\left|\int_{\mathbb{R}^n}D_{\alpha,t}^{\mathcal{L},m}(x,y)dy\right|\le\frac{C_N(\sqrt{t^{1/\alpha}}/\rho(x))^{\delta}}{(1+\sqrt{t^{1/\alpha}}/\rho(x))^{N}}.$$

Proposition 4 [12, *Propositions* 3.6, 3.9 *and* 3.10] *Suppose that and* $V\in B_q$ *for some* $q>n$.

(i) *Let* $\alpha\in(0,1)$. *For every* N, *there is a constant* $C_N>0$ *such that*

$$\left|\widetilde{D}_{\alpha,t}^{\mathcal{L}}(x,y)\right|\le\frac{C_Nt}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-N}.$$

(ii) *Let* $\alpha\in(0,1)$ *and* $\delta_1=1-n/q$. *For every* $N>0$, *there exists a constant* $C_N>0$ *such that for all* $|h|<\sqrt{t^{1/\alpha}}$,

$$\left|\widetilde{D}_{\alpha,t}^{\mathcal{L}}(x+h,y)-\widetilde{D}_{\alpha,t}^{\mathcal{L}}(x,y)\right|$$
$$\le C_N\left(\frac{h}{\sqrt{t^{1/\alpha}}}\right)^{\delta_1}\frac{t}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-N}.$$

(iii) *Let* $\alpha\in(0,1/2-n/2q)$. *There exists a constant* $C_N>0$ *such that*

$$\left|\int_{\mathbb{R}^n}\widetilde{D}_{\alpha,t}^{\mathcal{L}}(x,y)\mathrm{d}y\right|\le C_N\min\left\{\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{1+2\alpha},\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{-N}\right\}.$$

To establish the BMO_L^{γ}-boundedness of operators via $T1$ type theorem, we need the following propositions.

Proposition 5 *There exists a constant* $C>0$ *such that*

$$\left|K_{\alpha,t}^{\mathcal{L}}(x,y)-K_{\alpha,t}(x-y)\right|\le\begin{cases}C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_0}\frac{t}{(|x-y|+\sqrt{t^{1/\alpha}})^{n+2\alpha}}, & \sqrt{t^{1/\alpha}}\le|x-y|;\\ C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}\frac{t}{(|x-y|+\sqrt{t^{1/\alpha}})^{n+2\alpha}}, & \sqrt{t^{1/\alpha}}\ge|x-y|.\end{cases}$$

Proof By the subordinative formula (4) and Lemma 6, we obtain

$$\left|K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right| \le \int_0^\infty \eta_t^\alpha(s)\left|K^{\mathcal{L}}_s(x,y)-K_s(x-y)\right|\mathrm{d}s$$
$$\le C\int_0^\infty \eta_t^\alpha(s)\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0} s^{-n/2}e^{-|x-y|^2/s}\mathrm{d}s.$$

On the one hand, letting $s=t^{1/\alpha}u$, we can get

$$\begin{aligned}
&\left|K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right|\\
&\quad\le C\int_0^\infty \frac{t}{s^{1+\alpha}}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0} s^{-n/2}e^{-|x-y|^2/s}\mathrm{d}s\\
&\quad\le C\int_0^\infty \frac{t}{(t^{1/\alpha}u)^{1+\alpha}}\left(\frac{\sqrt{t^{1/\alpha}u}}{\rho(x)}\right)^{\delta_0} (t^{1/\alpha}u)^{-n/2}e^{-c|x-y|^2/(t^{1/\alpha}u)}t^{1/\alpha}\mathrm{d}u\\
&\quad\le C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0} t^{-n/(2\alpha)}\int_0^\infty u^{-1-\alpha-n/2+\delta_0/2}e^{-|x-y|^2/(t^{1/\alpha}u)}\mathrm{d}u.
\end{aligned}$$

Applying the change of variables: $|x-y|^2/(t^{1/\alpha}u)=r^2$, we deduce that

$$\begin{aligned}
&\left|K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right|\\
&\quad\le C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0} t^{-n/(2\alpha)}\int_0^\infty \left(\frac{|x-y|^2}{t^{1/\alpha}r^2}\right)^{-1-\alpha+\delta_0/2-n/2} e^{-r^2}\frac{|x-y|^2}{t^{1/\alpha}r^3}\mathrm{d}r\\
&\quad\le C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0} |x-y|^{-2\alpha+\delta_0-n}t^{1-\delta_0/(2\alpha)}\int_0^\infty e^{-r^2}r^{2\alpha-\delta_0+n+1}\mathrm{d}r\\
&\quad\le C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0} \frac{t^{1-\delta_0/(2\alpha)}}{|x-y|^{2\alpha+n-\delta_0}}.
\end{aligned}$$

On the other hand, taking $\tau=s/t^{1/\alpha}$, we obtain

$$\begin{aligned}\left|K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right| \leq& C\int_0^\infty \left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0} s^{-n/2}t^{-1/\alpha}\eta_1^\alpha(s/t^{1/\alpha})\mathrm{d}s\\ \leq& C\int_0^\infty \left(\frac{\sqrt{t^{1/\alpha}\tau}}{\rho(x)}\right)^{\delta_0}(t^{1/\alpha}\tau)^{-n/2}\eta_1^\alpha(\tau)\mathrm{d}\tau\\ \leq& C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0} t^{-n/2\alpha}\int_0^\infty \eta_1^\alpha(\tau)\tau^{\delta_0/2-n/2}\mathrm{d}\tau\\ \leq& C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0} t^{-n/2\alpha}.\end{aligned}$$

If $\sqrt{t^{1/\alpha}} \leq |x-y|$, then

$$\begin{aligned}\left|K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right| \leq& C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_0}\frac{t}{|x-y|^{2\alpha+n}}\\ \leq& C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_0}\frac{t}{(|x-y|+\sqrt{t^{1/\alpha}})^{2\alpha+n}}.\end{aligned}$$

If $\sqrt{t^{1/\alpha}} > |x-y|$, we can see that

$$\begin{aligned}\left|K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right| \leq& C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}\frac{t}{t^{n/(2\alpha)+1}}\\ \leq& C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}\frac{t}{(|x-y|+\sqrt{t^{1/\alpha}})^{2\alpha+n}}.\end{aligned}$$

□

Let $\widetilde{D}_{\alpha,t}(\cdot) = t^{1/(2\alpha)}\nabla_x e^{-t(-\Delta)^\alpha}(\cdot)$. Similar to the proof of Proposition 5, by (4) and Lemma 8, we have

Proposition 6 *There exists a constant $C > 0$ such that*

$$\left|\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)-\widetilde{D}_{\alpha,t}(x-y)\right| \leq \begin{cases} C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_0}\frac{t}{(|x-y|+\sqrt{t^{1/\alpha}})^{n+2\alpha}}, & \sqrt{t^{1/\alpha}} \leq |x-y|;\\ C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}\frac{t}{(|x-y|+\sqrt{t^{1/\alpha}})^{n+2\alpha}}, & \sqrt{t^{1/\alpha}} \geq |x-y|.\end{cases}$$

Let $D^m_{\alpha,t}(\cdot) = t^m\partial^m_t e^{-t(-\Delta)^\alpha}(\cdot)$. We have

Proposition 7 *There exists a constant $C > 0$ such that*

$$\left|D_{\alpha,t}^{\mathcal{L},m}(x,y) - D_{\alpha,t}^{m}(x-y)\right| \leq \begin{cases} C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_0} \frac{t}{(|x-y|+\sqrt{t^{1/\alpha}})^{n+2\alpha}}, & \sqrt{t^{1/\alpha}} \leq |x-y|; \\ C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0} \frac{t}{(|x-y|+\sqrt{t^{1/\alpha}})^{n+2\alpha}}, & \sqrt{t^{1/\alpha}} \geq |x-y|. \end{cases}$$

Proof The proposition can be proved by Proposition 1 and (4). Since the argument is similar to that of Proposition 5, the details is omitted. □

Proposition 8 *Let $0 < \delta < \min\{2\alpha, \delta_0\}$. For every $C' > 0$ there exists a constant C such that for every $z, x, y \in \mathbb{R}^n$, $|y - z| \leq |x - y|/4$, $|y - z| \leq C'\rho(y)$ we have*

$$\begin{aligned} &\left|\left(K_{\alpha,t}^{\mathcal{L}}(x,y) - K_{\alpha,t}(x-y)\right) - \left(K_{\alpha,t}^{\mathcal{L}}(x,z) - K_{\alpha,t}(x-z)\right)\right| \\ &\leq C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta} \frac{t}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}. \end{aligned}$$

Proof By the subordinative formula (4), we can use Lemma 9 to deduce

$$\begin{aligned} &\left|\left(K_{\alpha,t}^{\mathcal{L}}(x,y) - K_{\alpha,t}(x-y)\right) - \left(K_{\alpha,t}^{\mathcal{L}}(x,z) - K_{\alpha,t}(x-z)\right)\right| \\ &\leq \int_0^\infty \eta_t^\alpha(s)\left|\left(K_s^{\mathcal{L}}(x,y) - K_s(x-y)\right) - \left(K_s^{\mathcal{L}}(x,z) - K_s(x-z)\right)\right| \mathrm{d}s \\ &\leq C\int_0^\infty \eta_t^\alpha(s)\left(\frac{|y-z|}{\rho(x)}\right)^{\delta} s^{-n/2}e^{-|x-y|^2/s}\mathrm{d}s. \end{aligned}$$

On the one hand, taking $s = t^{1/\alpha}u$, we obtain

$$\begin{aligned} &\left|\left(K_{\alpha,t}^{\mathcal{L}}(x,y) - K_{\alpha,t}(x-y)\right) - \left(K_{\alpha,t}^{\mathcal{L}}(x,z) - K_{\alpha,t}(x-z)\right)\right| \\ &\leq \int_0^\infty \frac{Ct}{s^{1+\alpha}}\left(\frac{|y-z|}{\rho(x)}\right)^{\delta} s^{-n/2}e^{-|x-y|^2/s}\mathrm{d}s \\ &\leq C\int_0^\infty \frac{t}{(t^{1/\alpha}u)^{1+\alpha}}\left(\frac{|y-z|}{\rho(x)}\right)^{\delta}(t^{1/\alpha}u)^{-n/2}e^{-|x-y|^2/(t^{1/\alpha}u)}t^{1/\alpha}\mathrm{d}u \\ &\leq C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta} t^{-n/(2\alpha)}\int_0^\infty e^{-|x-y|^2/(t^{1/\alpha}u)}u^{-1-\alpha-n/2}\mathrm{d}u. \end{aligned}$$

Let $\frac{|x-y|^2}{(t^{1/\alpha}u)} = r$. We can see that

$$\begin{aligned}
&\left|\left(K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right)-\left(K^{\mathcal{L}}_{\alpha,t}(x,z)-K_{\alpha,t}(x-z)\right)\right|\\
&\quad\le C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta} t^{-n/(2\alpha)}\int_0^\infty e^{-r}\left(\frac{|x-y|^2}{t^{1/\alpha}r}\right)^{-1-\alpha-n/2}\frac{|x-y|^2}{t^{1/\alpha}r^2}\mathrm{d}r\\
&\quad\le C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta}\frac{t}{|x-y|^{2\alpha+n}}.
\end{aligned}$$

On the other hand, letting $\frac{s}{t^{1/\alpha}}=\tau$, we obtain

$$\begin{aligned}
&\left|\left(K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right)-\left(K^{\mathcal{L}}_{\alpha,t}(x,z)-K_{\alpha,t}(x-z)\right)\right|\\
&\quad\le \int_0^\infty \frac{1}{t^{1/\alpha}}\eta_1^\alpha(s/t^{1/\alpha})\left(\frac{|y-z|}{\rho(x)}\right)^{\delta} s^{-n/2}\mathrm{d}s\\
&\quad\le C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta}\int_0^\infty \eta_1^\alpha(\tau)(t^{1/\alpha}\tau)^{-n/2}\mathrm{d}\tau\\
&\quad\le C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta} t^{-n/(2\alpha)}.
\end{aligned}$$

Case 1: $\sqrt{t^{1/\alpha}}\le|x-y|$. We obtain

$$\begin{aligned}
&\left|\left(K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right)-\left(K^{\mathcal{L}}_{\alpha,t}(x,z)-K_{\alpha,t}(x-z)\right)\right|\\
&\quad\le C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta}\frac{t}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}.
\end{aligned}$$

Case 2: $\sqrt{t^{1/\alpha}}>|x-y|$. We can see that

$$\begin{aligned}
&\left|\left(K^{\mathcal{L}}_{\alpha,t}(x,y)-K_{\alpha,t}(x-y)\right)-\left(K^{\mathcal{L}}_{\alpha,t}(x,z)-K_{\alpha,t}(x-z)\right)\right|\\
&\quad\le C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta}\frac{t}{t^{n/(2\alpha)+1}}\le C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta}\frac{t}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}.
\end{aligned}$$

□

Proposition 9 *Let* $0<\delta<\min\{2\alpha,\delta_0\}$. *For every* $C'>0$ *there exists a constant* C *such that for every* $z,x,y\in\mathbb{R}^n$, $|y-z|\le|x-y|/4$, $|y-z|\le C'\rho(y)$ *we have*

$$\begin{aligned}
&\left|\left(D^{\mathcal{L},m}_{\alpha,t}(x,y)-D^m_{\alpha,t}(x-y)\right)-\left(D^{\mathcal{L},m}_{\alpha,t}(x,z)-D^m_{\alpha,t}(x-z)\right)\right|\\
&\quad\le C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta}\frac{t}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}.
\end{aligned}$$

Proof The proof is similar to that of Proposition 8, so we omit the details. □

Proposition 10 *Suppose that $V \in B_q$ for some $q > n$. Let $0 < \delta' < \delta_1 := 1 - n/q$. For every $C' > 0$ there exists a constant C such that for every $z, x, y \in \mathbb{R}^n$, $|y - z| \le |x - y|/4, |y - z| \le C'\rho(y)$ we have*

$$\left|\left(\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y) - \widetilde{D}_{\alpha,t}(x-y)\right) - \left(\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,z) - \widetilde{D}_{\alpha,t}(x-z)\right)\right| \le C\left(\frac{|y-z|}{\rho(x)}\right)^{\delta'} \frac{t}{(\sqrt{t^{1/\alpha}} + |x-y|)^{n+2\alpha}}.$$

Proof The proof is similar to that of Proposition 8, so we omit the details. □

4 BMO$^{\gamma}_{\mathcal{L}}$-boundedness via *T*1 theorem

4.1 Maximal operators for fractional heat semigroups

Definition 4 Let $0 < \gamma \le 1$. The Campanato type space $\mathrm{BMO}^{\gamma}_{\mathcal{L},L^\infty((0,\infty),\mathrm{d}t)}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f satisfying

$$\|f\|_{\mathrm{BMO}^{\gamma}_{\mathcal{L},L^\infty((0,\infty),\mathrm{d}t)}} := \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|^{1+\gamma/n}} \int_B \|f(x,t) - f(B,V)\|_{L^\infty((0,\infty),\mathrm{d}t)} \mathrm{d}x \right\} < \infty.$$

To prove that $M^{\alpha}_{\mathcal{L}}$ is bounded from $\mathrm{BMO}^{\gamma}_{\mathcal{L}}(\mathbb{R}^n)$, $0 < \gamma < \min\{2\alpha, \delta_0, 1\}$, into itself, we give a vector-valued interpretation of the operator and apply Lemma 4. Indeed, it is clear that $M^{\alpha}_{\mathcal{L}} f = \|e^{-t\mathcal{L}^\alpha} f\|_{L^\infty((0,\infty),\mathrm{d}t)}$. Hence, it is enough to show that the operator $\Lambda(f) := \{e^{-t\mathcal{L}^\alpha} f\}_{t>0}$ is bounded from $\mathrm{BMO}^{\gamma}_{\mathcal{L}}$ into $\mathrm{BMO}^{\gamma}_{\mathcal{L},L^\infty((0,\infty),\mathrm{d}t)}$.

By the spectral theorem, Λ is bounded from $L^2(\mathbb{R}^n)$ into $L^2_{L^\infty((0,\infty),\mathrm{d}t)}(\mathbb{R}^n)$. The desired result can be then deduced from the following theorem.

Theorem 3 *Assume that the potential $V \in B_q$ with $q > n/2$. Let $x, y, z \in \mathbb{R}^n$.*

(i) *For any $N > 0$, there exists a constant C_N such that*

$$\left\|K^{\mathcal{L}}_{\alpha,t}(x,y)\right\|_{L^\infty((0,\infty),\mathrm{d}t)} \le \frac{C_N}{|x-y|^n}\left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-N}.$$

(ii) *For $|x - y| > 2|y - z|$ and any $0 < \delta < \min\{2\alpha, \delta_0\}$, there exists a constant $C > 0$ such that*

$$\begin{aligned} &\left\|K^{\mathcal{L}}_{\alpha,t}(x,y)-K^{\mathcal{L}}_{\alpha,t}(x,z)\right\|_{L^\infty((0,\infty),\mathrm{d}t)}+\left\|K^{\mathcal{L}}_{\alpha,t}(y,x)-K^{\mathcal{L}}_{\alpha,t}(z,x)\right\|_{L^\infty((0,\infty),\mathrm{d}t)} \\ &\quad\le \frac{C|y-z|^\delta}{|x-y|^{n+\delta}}. \end{aligned} \tag{20}$$

(iii) *There exists a constant C such that for every ball $B=B(x,r)$ with $0<r\le\rho(x)/2$,*

$$\log\left(\frac{\rho(x)}{r}\right)\frac{1}{|B|}\int_B\left\|e^{-t\mathcal{L}^\alpha}1(y)-(e^{-t\mathcal{L}^\alpha}1)_B\right\|_{L^\infty((0,\infty),\mathrm{d}t)}\mathrm{d}y\le C,$$

and, if $\gamma<\min\{2\alpha,1,\delta_0\}$ then

$$\left(\frac{\rho(x)}{r}\right)^\gamma\frac{1}{|B|}\int_B\left\|e^{-t\mathcal{L}^\alpha}1(y)-(e^{-t\mathcal{L}^\alpha}1)_B\right\|_{L^\infty((0,\infty),\mathrm{d}t)}\mathrm{d}y\le C.$$

Proof For (i), from (i) of Proposition 2, we can get

$$\left|K^{\mathcal{L}}_{\alpha,t}(x,y)\right|\le C_N\min\left\{\frac{t^{1+N/\alpha}}{|x-y|^{n+2\alpha+2N}},\ t^{-n/(2\alpha)}\right\}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-N}.$$

If $\sqrt{t^{1/\alpha}}>|x-y|$, then

$$\begin{aligned}\left|K^{\mathcal{L}}_{\alpha,t}(x,y)\right|&\le\frac{C_N}{(\sqrt{t^{1/\alpha}})^n}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}\\ &\le\frac{C}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}.\end{aligned}$$

If $\sqrt{t^{1/\alpha}}\le|x-y|$, we obtain

$$\begin{aligned}\left|K^{\mathcal{L}}_{\alpha,t}(x,y)\right|&\le\frac{C_Nt^{1+N/\alpha}}{|x-y|^{n+2\alpha+2N}}\left(\frac{\sqrt{t^{1/\alpha}}}{|x-y|}\right)^{-N}\left(\frac{|x-y|}{\sqrt{t^{1/\alpha}}}+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}\\ &\le\frac{C_N(\sqrt{t^{1/\alpha}})^{2\alpha+N}}{|x-y|^{n+2\alpha+N}}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}\\ &\le\frac{C_N}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}.\end{aligned}$$

For (ii), from (ii) of Proposition 2, we obtain

$$\left|K^{\mathcal{L}}_{\alpha,t}(x,y)-K^{\mathcal{L}}_{\alpha,t}(x,z)\right|\le C_N\min\left\{\frac{t^{1+N/\alpha}|y-z|^\delta}{|x-y|^{2\alpha+n+2N+\delta}},\ t^{-n/(2\alpha)}\right\}\left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}}\right)^\delta.$$

If $\sqrt{t^{1/\alpha}} > |x-y|$, then

$$\left|K_{\alpha,t}^{\mathcal{L}}(x,y)-K_{\alpha,t}^{\mathcal{L}}(x,z)\right| \leq \frac{C}{|x-y|^n}\left(\frac{|y-z|}{|x-y|}\right)^{\delta} \leq \frac{C|y-z|^{\delta}}{|x-y|^{n+\delta}}.$$

If $|x-y| > \sqrt{t^{1/\alpha}}$, we can also get

$$\left|K_{\alpha,t}^{\mathcal{L}}(x,y)-K_{\alpha,t}^{\mathcal{L}}(x,z)\right| \leq \frac{C|x-y|^{2\alpha+2N}}{|x-y|^{2\alpha+n+2N+\delta}}|y-z|^{\delta} \leq \frac{C|y-z|^{\delta}}{|x-y|^{n+\delta}}.$$

The symmetry of the kernel $K_{\alpha,t}^{L}(\cdot,\cdot)$ gives the conclusion of (ii).

For (iii), letting $B=B(x,r)$ with $0<r\leq \rho(x)/2$, the triangle inequality gives

$$\left\|e^{-t\mathcal{L}^{\alpha}}1(y)-(e^{-t\mathcal{L}^{\alpha}}1)_B\right\|_{L^{\infty}((0,\infty),\mathrm{d}t)} \leq \frac{1}{|B|}\int_B \left\|e^{-t\mathcal{L}^{\alpha}}1(y)-e^{-t\mathcal{L}^{\alpha}}1(z)\right\|_{L^{\infty}((0,\infty),\mathrm{d}t)}\mathrm{d}z.$$

We estimate $\|e^{-t\mathcal{L}^{\alpha}}1(y)-e^{-t\mathcal{L}^{\alpha}}1(z)\|_{L^{\infty}((0,\infty),\mathrm{d}t)}$. Because $y,z\in B$, $\rho(y)\sim\rho(z)\sim\rho(x)$. By Proposition 5, we split $|e^{-t\mathcal{L}^{\alpha}}1(y)-e^{-t\mathcal{L}^{\alpha}}1(z)|\leq S_1+S_2$, where

$$\begin{cases} S_1 := \int_{\mathbb{R}^n}\left|K_{\alpha,t}^{\mathcal{L}}(y,w)-K_{\alpha,t}(y,w)\right|\mathrm{d}w; \\ S_2 := \int_{\mathbb{R}^n}\left|K_{\alpha,t}^{\mathcal{L}}(z,w)-K_{\alpha,t}(z,w)\right|\mathrm{d}w. \end{cases}$$

For S_1, if $|y-w|\leq\sqrt{t^{1/\alpha}}$, we obtain

$$S_1 \leq C\int_{\mathbb{R}^n}\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}\frac{t}{(\sqrt{t^{1/\alpha}}+|y-w|)^{n+2\alpha}}\mathrm{d}w \leq C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}.$$

If $|y-w|>\sqrt{t^{1/\alpha}}$, we can see that

$$\begin{aligned} S_1 &\leq C\int_{\mathbb{R}^n}\left(\frac{|y-w|}{\rho(x)}\right)^{\delta_0}\frac{t}{(\sqrt{t^{1/\alpha}}+|y-w|)^{n+2\alpha}}\mathrm{d}w \\ &\leq \frac{C}{\rho(x)^{\delta_0}}\int_0^{\infty}\frac{|y-w|^{\delta_0+n-1}t}{(|y-w|+\sqrt{t^{1/\alpha}})^{n+2\alpha}}\mathrm{d}|y-w| \\ &\leq C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}\int_0^{\infty}u^{\delta_0+n-1}(1+u)^{-n-2\alpha}\mathrm{d}u \\ &\leq C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}, \end{aligned}$$

where we have chosen $\delta_0<2\alpha$ since $\delta_0=2-n/q$, $q>n/2$. The proof of the term S_2 is similar to that of the term S_1, so we omit it. Then we can get

$$\left|e^{-t\mathcal{L}^\alpha}1(y)-e^{-t\mathcal{L}^\alpha}1(z)\right|\le C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0},$$

which shows that if $\sqrt{t^{1/\alpha}}\le 2r$,

$$\left|e^{-t\mathcal{L}^\alpha}1(y)-e^{-t\mathcal{L}^\alpha}1(z)\right|\le C\left(\frac{r}{\rho(x)}\right)^{\delta_0}.$$

If $\sqrt{t^{1/\alpha}}>2r$, then $|y-z|\le 2r<\sqrt{t^{1/\alpha}}$. Hence, Proposition 2 implies that for $0<\delta<\delta_0$,

$$\begin{aligned}\left|e^{-t\mathcal{L}^\alpha}1(y)-e^{-t\mathcal{L}^\alpha}1(z)\right|&\le\int_{\mathbb{R}^n}\left|K^{\mathcal{L}}_{\alpha,t}(y,w)-K^{\mathcal{L}}_{\alpha,t}(z,w)\right|\mathrm{d}w\\&\le C\int_{\mathbb{R}^n}\left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}}\right)^{\delta}\frac{t}{(\sqrt{t^{1/\alpha}}+|y-w|)^{n+2\alpha}}\mathrm{d}w\le C\left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}}\right)^{\delta}\le C\left(\frac{r}{\sqrt{t^{1/\alpha}}}\right)^{\delta}.\end{aligned}\tag{21}$$

Therefore, if $\sqrt{t^{1/\alpha}}>\rho(x)$, (21) gives

$$|e^{-t\mathcal{L}^\alpha}1(y)-e^{-t\mathcal{L}^\alpha}1(z)|\le C\left(\frac{r}{\rho(x)}\right)^{\delta}.$$

When $2r<\sqrt{t^{1/\alpha}}<\rho(x)$, we have $|e^{-t\mathcal{L}^\alpha}1(y)-e^{-t\mathcal{L}^\alpha}1(z)|=\mathrm{I}+\mathrm{II}+\mathrm{III}$, where

$$\begin{cases}\mathrm{I}:=\displaystyle\int_{|w-y|>c\rho(y)>4|y-z|}\left|\Big(K^{\mathcal{L}}_{\alpha,t}(y,w)-K_{\alpha,t}(y,w)\Big)-\Big(K^{\mathcal{L}}_{\alpha,t}(z,w)-K_{\alpha,t}(z,w)\Big)\right|\mathrm{d}w;\\ \mathrm{II}:=\displaystyle\int_{4|y-z|<|w-y|<c\rho(y)}\left|\Big(K^{\mathcal{L}}_{\alpha,t}(y,w)-K_{\alpha,t}(y,w)\Big)-\Big(K^{\mathcal{L}}_{\alpha,t}(z,w)-K_{\alpha,t}(z,w)\Big)\right|\mathrm{d}w;\\ \mathrm{III}:=\displaystyle\int_{|w-y|<4|y-z|}\left|\Big(K^{\mathcal{L}}_{\alpha,t}(y,w)-K_{\alpha,t}(y,w)\Big)-\Big(K^{\mathcal{L}}_{\alpha,t}(z,w)-K_{\alpha,t}(z,w)\Big)\right|\mathrm{d}w.\end{cases}$$

Notice that the estimate (20) is valid for the classical fractional heat kernel. For I, by (20), we can get

$$\mathrm{I}\le C\int_{|w-y|>c\rho(y)>4|y-z|}\frac{|y-z|^{\delta}}{|w-y|^{n+\delta}}\mathrm{d}w\le C\left(\frac{r}{\rho(x)}\right)^{\delta}.$$

For II, we apply Proposition 8 and the fact that $\rho(w)\sim\rho(y)$ in the region of integration to deduce that

$$\mathrm{II}\le C|y-z|^{\delta}\int_{4|y-z|<|w-y|<c\rho(y)}\frac{t\mathrm{d}w}{\rho(w)^{\delta}(\sqrt{t^{1/\alpha}}+|w-y|)^{n+2\alpha}}\le C\left(\frac{r}{\rho(x)}\right)^{\delta}.$$

For III, since $|y-z|\le 2r<\sqrt{t^{1/\alpha}}$, we have $|w-y|<C\sqrt{t^{1/\alpha}}$. For $n-\delta_0>0$, by Proposition 5, we obtain

$$\begin{aligned}\mathrm{III} &\le C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}\left(\int_{|w-y|<4|y-z|}\frac{t\mathrm{d}w}{(\sqrt{t^{1/\alpha}}+|w-y|)^{n+2\alpha}}\right.\\
&\quad\left.+\int_{|w-z|\le 5|y-z|}\frac{t\mathrm{d}w}{(\sqrt{t^{1/\alpha}}+|w-z|)^{n+2\alpha}}\right)\\
&\le C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}\int_0^{5|y-z|/\sqrt{t^{1/\alpha}}}u^{n-1}\mathrm{d}u \le C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}\right)^{\delta_0}\left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}}\right)^n\\
&\le \frac{Cr^n}{\rho(x)^{\delta_0}(\sqrt{t^{1/\alpha}})^{n-\delta_0}} \le \frac{Cr^n}{\rho(x)^{\delta_0}r^{n-\delta_0}} = C\left(\frac{r}{\rho(x)}\right)^{\delta_0}.\end{aligned}$$

Thus, when $2r < \sqrt{t^{1/\alpha}} < \rho(x)$,

$$\left|e^{-t\mathcal{L}^\alpha}1(y) - e^{-t\mathcal{L}^\alpha}1(z)\right| \le C\left(\frac{r}{\rho(x)}\right)^{\delta}.$$

Combining the above estimates, we can get

$$\left\|e^{-t\mathcal{L}^\alpha}1(y) - e^{-t\mathcal{L}^\alpha}1(z)\right\|_{L^\infty((0,\infty),\mathrm{d}t)} \le C\left(\frac{r}{\rho(x)}\right)^{\delta}. \tag{22}$$

Therefore, it holds

$$\begin{aligned}&\log\left(\frac{\rho(x)}{r}\right)\frac{1}{|B|}\int_B\left\|e^{-t\mathcal{L}^\alpha}1(y) - (e^{-t\mathcal{L}^\alpha}1)_B\right\|_{L^\infty((0,\infty),\mathrm{d}t)}\mathrm{d}y\\
&\le C\left(\frac{r}{\rho(x)}\right)^{\delta}\log\left(\frac{\rho(x)}{r}\right) \le C,\end{aligned}$$

which is the first conclusion of (iii).

For the second estimate of (iii), take $\delta \in (\gamma, \min\{2\alpha, 2-n/q\})$. By (22), we have

$$\left(\frac{\rho(x)}{r}\right)^{\gamma}\frac{1}{|B|}\int_B\left\|e^{-t\mathcal{L}^\alpha}1(y) - (e^{-t\mathcal{L}^\alpha}1)_B\right\|_{L^\infty((0,\infty),\mathrm{d}t)}\mathrm{d}y \le C\left(\frac{r}{\rho(x)}\right)^{\delta-\gamma} \le C.$$

□

4.2 Boundedness of the Littlewood–Paley *g*-function $g_\alpha^{\mathcal{L}}$

Similar to Sect. 4.1, we introduce the following function space:

Definition 5 Let $0 < \gamma \le 1$. The Campanato type space $\mathrm{BMO}^{\gamma}_{\mathcal{L},L^2((0,\infty),\mathrm{d}t/t)}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f satisfying

$$\|f\|_{\mathrm{BMO}^{\gamma}_{\mathcal{L},L^2((0,\infty),\mathrm{d}t)}} := \sup_{B\subset\mathbb{R}^n}\left\{\frac{1}{|B|^{1+\gamma/n}}\int_B \|f(x,t)-f(B,V)\|_{L^2((0,\infty),\mathrm{d}t/t)}\mathrm{d}x\right\}<\infty.$$

The functional calculus and the spectral theorem imply that $g^{\mathcal{L}}_{\alpha}$ is an isometry on $L^2(\mathbb{R}^n)$. As before, to get the boundedness of $g^{\mathcal{L}}_{\alpha}$ on $\mathrm{BMO}^{\gamma}_{\mathcal{L}}(\mathbb{R}^n)$, $0<\gamma<\min\{2\alpha,\delta_0,1\}$, it is sufficient to prove the following result.

Theorem 4 *Assume that the potential $V\in B_q$ with $q>n/2$. Let $x,y,z\in\mathbb{R}^n$ and $N>0$.*

(*i*) *For any $N>0$, there exists a constant C_N such that*

$$\left\|D^{\mathcal{L},m}_{\alpha,t}(x,y)\right\|_{L^2((0,\infty),\frac{\mathrm{d}t}{t})}\le\frac{C_N}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}.$$

(*ii*) *If $|x-y|>2|y-z|$ and $0<\delta<\min\{2\alpha,\delta_0,1\}$, there exists a constant C such that*

$$\begin{aligned}&\left\|D^{\mathcal{L},m}_{\alpha,t}(x,y)-D^{\mathcal{L},m}_{\alpha,t}(x,z)\right\|_{L^2((0,\infty),\frac{\mathrm{d}t}{t})}+\left\|D^{\mathcal{L},m}_{\alpha,t}(y,x)-D^{\mathcal{L},m}_{\alpha,t}(z,x)\right\|_{L^2\left((0,\infty),\frac{\mathrm{d}t}{t}\right)}\\&\quad\le C\frac{|y-z|^{\delta}}{|x-y|^{n+\delta}}.\end{aligned}\tag{23}$$

(iii) *There exists a constant C such that for every ball $B=B(x_0,r)$ with $0<r\le\rho(x)/2$,*

$$\log\left(\frac{\rho(x)}{r}\right)\frac{1}{|B|}\int_B\left\|t^m\partial_t^m e^{-t\mathcal{L}^{\alpha}}1(y)-(t^m\partial_t^m e^{-t\mathcal{L}^{\alpha}}1)_B\right\|_{L^2\left((0,\infty),\frac{\mathrm{d}t}{t}\right)}\mathrm{d}y\le C,$$

and, if $\gamma<\min\{2\alpha,\delta_0,1\}$, then

$$\left(\frac{\rho(x)}{r}\right)^{\gamma}\frac{1}{|B|}\int_B\left\|t^m\partial_t^m e^{-t\mathcal{L}^{\alpha}}1(y)-(t^m\partial_t^m e^{-t\mathcal{L}^{\alpha}}1)_B\right\|_{L^2\left((0,\infty),\frac{\mathrm{d}t}{t}\right)}\mathrm{d}y\le C.$$

Proof For (i), from Proposition 3, we have

$$\left|D^{\mathcal{L},m}_{\alpha,t}(x,y)\right|\le C_N\min\left\{\frac{t^{1+N/\alpha}}{|x-y|^{n+2\alpha+2N}},\ t^{-n/(2\alpha)}\right\}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-N}.$$

If $\sqrt{t^{1/\alpha}} \leq |x-y|$, we obtain

$$\begin{aligned}
&\left\| D^{\mathcal{L},m}_{\alpha,t}(x,y) \cdot \chi_{\{t^{1/2\alpha} \leq |x-y|\}} \right\|^2_{L^2\left((0,\infty),\frac{dt}{t}\right)} \\
&\leq C_N \int_0^{|x-y|^{2\alpha}} \frac{t^{2+2N/\alpha}}{|x-y|^{2n+4\alpha+4N}} \left(1 + \frac{\sqrt{t^{1/\alpha}}}{\rho(x)} + \frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-2N} \frac{dt}{t} \\
&\leq C_N \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-2N} \int_0^{|x-y|^{2\alpha}} \frac{(\sqrt{t^{1/\alpha}})^{4\alpha+4N}}{|x-y|^{2n+4\alpha+4N}} \left(\frac{\sqrt{t^{1/\alpha}}}{|x-y|}\right)^{-2N} \frac{dt}{t} \\
&\leq \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-2N} \frac{C_N}{|x-y|^{2n}} \int_0^{|x-y|^{2\alpha}} \left(\frac{\sqrt{t^{1/\alpha}}}{|x-y|}\right)^{4\alpha+2N} \frac{dt}{t}.
\end{aligned}$$

Let $\sqrt{t^{1/\alpha}}/|x-y| = u$. We can see that

$$\begin{aligned}
&\left\| D^{\mathcal{L},m}_{\alpha,t}(x,y) \cdot \chi_{\{t^{1/2\alpha} \leq |x-y|\}} \right\|^2_{L^2\left((0,\infty),\frac{dt}{t}\right)} \\
&\leq \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-2N} \frac{C_N}{|x-y|^{2n}} \int_0^1 u^{4\alpha+2N-1} du \\
&\leq \frac{C_N}{|x-y|^{2n}} \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-2N}.
\end{aligned}$$

If $\sqrt{t^{1/\alpha}} \geq |x-y|$, we can get

$$\begin{aligned}
&\left\| D^{\mathcal{L},m}_{\alpha,t}(x,y) \cdot \chi_{\{t^{1/2\alpha} \geq |x-y|\}} \right\|^2_{L^2\left((0,\infty),\frac{dt}{t}\right)} \\
&\leq C_N \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-2N} \int_{|x-y|^{2\alpha}}^{\infty} t^{-n/\alpha-1} dt \\
&\leq \frac{C_N}{|x-y|^{2n}} \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-2N}.
\end{aligned}$$

For (ii), by Proposition 3, we have

$$\begin{aligned}
\left\| D^{\mathcal{L},m}_{\alpha,t}(x,y) - D^{\mathcal{L},m}_{\alpha,t}(x,z) \right\|^2_{L^2\left((0,\infty),\frac{dt}{t}\right)} &\leq \int_0^\infty \frac{Ct}{(\sqrt{t^{1/\alpha}} + |x-y|)^{2n+4\alpha}} \left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}}\right)^{2\delta} dt \\
&\leq C|y-z|^{2\delta} \int_0^\infty \frac{(\sqrt{t^{1/\alpha}})^{2\alpha-2\delta}}{(\sqrt{t^{1/\alpha}} + |x-y|)^{2n+4\alpha}} dt.
\end{aligned}$$

Let $\sqrt{t^{1/\alpha}}/|x-y| = u$. We obtain

$$\begin{aligned}\left\|D^{\mathcal{L},m}_{\alpha,t}(x,y) - D^{\mathcal{L},m}_{\alpha,t}(x,z)\right\|^2_{L^2\left((0,\infty),\frac{dt}{t}\right)} &\le C|y-z|^{2\delta}|x-y|^{-2n-2\delta}\int_0^\infty \frac{u^{4\alpha-2\delta-1}}{(1+u)^{2n+4\alpha}}du\\ &\le \frac{C|y-z|^{2\delta}}{|x-y|^{2n+2\delta}}.\end{aligned}$$

The symmetry of the kernel $D^{L,m}_{\alpha,t}(\cdot,\cdot)$ gives the conclusion of (ii).

For (iii), let us fix $y, z \in B = B(x_0, r)$, $0 < r \le \rho(x_0)/2$. Similar to Theorem 3, we must handle

$$\left\|t^m\partial_t^m e^{-t\mathcal{L}^\alpha}1(y) - t^m\partial_t^m e^{-t\mathcal{L}^\alpha}1(z)\right\|_{L^2\left((0,\infty),\frac{dt}{t}\right)}.$$

We can write

$$\left\|t^m\partial_t^m e^{-t\mathcal{L}^\alpha}1(y) - t^m\partial_t^m e^{-t\mathcal{L}^\alpha}1(z)\right\|^2_{L^2\left((0,\infty),\frac{dt}{t}\right)} = M_1 + M_2 + M_3,$$

where

$$\begin{cases} M_1 := \displaystyle\int_0^{(2r)^{2\alpha}} \left|\int_{\mathbb{R}^n}\left(D^{\mathcal{L},m}_{\alpha,t}(x,y) - D^{\mathcal{L},m}_{\alpha,t}(x,z)\right)dx\right|^2\frac{dt}{t};\\ M_2 := \displaystyle\int_{(2r)^{2\alpha}}^{\rho(x_0)^{2\alpha}} \left|\int_{\mathbb{R}^n}\left(D^{\mathcal{L},m}_{\alpha,t}(x,y) - D^{\mathcal{L},m}_{\alpha,t}(x,z)\right)dx\right|^2\frac{dt}{t};\\ M_3 := \displaystyle\int_{\rho(x_0)^{2\alpha}}^{\infty} \left|\int_{\mathbb{R}^n}\left(D^{\mathcal{L},m}_{\alpha,t}(x,y) - D^{\mathcal{L},m}_{\alpha,t}(x,z)\right)dx\right|^2\frac{dt}{t}.\end{cases}$$

Since $y, z \in B \subset B(x_0, \rho(x_0))$, it follows that $\rho(y) \sim \rho(x_0) \sim \rho(z)$. By Proposition 3 (iii),

$$M_1 \le C\int_0^{(2r)^{2\alpha}} \frac{(\sqrt{t^{1/\alpha}}/\rho(x_0))^{2\delta}}{(1+\sqrt{t^{1/\alpha}}/\rho(x_0))^{2N}}\frac{dt}{t} \le C\int_0^{(2r)^{2\alpha}}\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x_0)}\right)^{2\delta}\frac{dt}{t} = C\left(\frac{r}{\rho(x_0)}\right)^{2\delta}.$$

Also, by Proposition 3(ii),

$$\begin{aligned}M_3 &\le C\int_{\rho(x_0)^{2\alpha}}^\infty \left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}}\right)^{2\delta}\left|\int_{\mathbb{R}^n}\frac{t}{(\sqrt{t^{1/\alpha}}+|x-y|)^{n+2\alpha}}dx\right|^2\frac{dt}{t}\\ &\le C\int_{\rho(x_0)^{2\alpha}}^\infty \left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}}\right)^{2\delta}\frac{dt}{t} \le C\left(\frac{r}{\rho(x_0)}\right)^{2\delta}.\end{aligned}$$

It remains to estimate the term M_2. In this case, $|y-z| \le 2r \le \sqrt{t^{1/\alpha}} \le \rho(x_0)$. Then we can use the methods in Theorem 3 to obtain

$$M_2 = \int_{(2r)^{2\alpha}}^{\rho(x_0)^{2\alpha}} \left|M_{2,1} + M_{2,2} + M_{2,3}\right|^2 \frac{\mathrm{d}t}{t},$$

where

$$\begin{cases} M_{2,1} := \int_{|x-y|>c\rho(y)>4|y-z|} \left(D_{\alpha,t}^{\mathcal{L},m}(x,y) - D_{\alpha,t}^{m}(x-y)\right) - \left(D_{\alpha,t}^{\mathcal{L},m}(x,z) - D_{\alpha,t}^{m}(x-z)\right)\mathrm{d}x; \\ M_{2,2} := \int_{4|y-z|<|x-y|<c\rho(y)} \left(D_{\alpha,t}^{\mathcal{L},m}(x,y) - D_{\alpha,t}^{m}(x-y)\right) - \left(D_{\alpha,t}^{\mathcal{L},m}(x,z) - D_{\alpha,t}^{m}(x-z)\right)\mathrm{d}x; \\ M_{2,3} := \int_{|x-y|<4|y-z|} \left(D_{\alpha,t}^{\mathcal{L},m}(x,y) - D_{\alpha,t}^{m}(x-y)\right) - \left(D_{\alpha,t}^{\mathcal{L},m}(x,z) - D_{\alpha,t}^{m}(x-z)\right)\mathrm{d}x. \end{cases}$$

For $M_{2,1}$, similar to prove (23), we can also get

$$\left|D_{\alpha,t}^{\mathcal{L},m}(x,y) - D_{\alpha,t}^{\mathcal{L},m}(x,z)\right| \le \frac{C|y-z|^\delta}{|x-y|^{n+\delta}},$$

which is valid to $D_{\alpha,t}^{m}(\cdot)$. So we obtain

$$|M_{2,1}| \le C \int_{|x-y|>c\rho(y)>4|y-z|} \frac{|y-z|^\delta}{|x-y|^{n+\delta}}\mathrm{d}x \le C\left(\frac{r}{\rho(x_0)}\right)^\delta.$$

For $M_{2,2}$, by Proposition 9 and the fact that $\rho(x) \sim \rho(y)$ in the region of integration.

$$|M_{2,2}| \le C|y-z|^\delta \int_{4|y-z|<|x-y|<c\rho(y)} \frac{t}{\rho(x)^\delta(\sqrt{t^{1/\alpha}} + |x-y|)^{n+2\alpha}}\mathrm{d}x \le C\left(\frac{r}{\rho(x_0)}\right)^\delta.$$

For $M_{2,3}$, since $|y-z| \le 2r < \sqrt{t^{1/\alpha}}$, we have $|x-y| < C\sqrt{t^{1/\alpha}}$. For $n - \delta_0 > 0$, by Proposition 7, we obtain

$$\begin{aligned} |M_{2,3}| \le & C\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x_0)}\right)^{\delta_0} \left(\int_{|x-y|<4|y-z|} \frac{t\mathrm{d}x}{(\sqrt{t^{1/\alpha}} + |x-y|)^{n+2\alpha}}\right. \\ & \left. + \int_{|x-z|\le 5|y-z|} \frac{t\mathrm{d}x}{(\sqrt{t^{1/\alpha}} + |x-z|)^{n+2\alpha}}\right) \\ \le & \frac{Cr^n}{\rho(x_0)^{\delta_0}(\sqrt{t^{1/\alpha}})^{n-\delta_0}} \le C\left(\frac{r}{\rho(x_0)}\right)^{\delta_0}. \end{aligned}$$

The estimates for $M_{2,i}$, $i = 1, 2, 3$, imply that

$$M_2 \le \int_{(2r)^{2\alpha}}^{\rho(x_0)^{2\alpha}} \left(\frac{r}{\rho(x_0)}\right)^{2\delta} \frac{\mathrm{d}t}{t} = C\left(\frac{r}{\rho(x_0)}\right)^{2\delta} \log\left(\frac{\rho(x_0)}{r}\right).$$

Finally, we can get

$$\left\| t^m \partial_t^m e^{-t\mathcal{L}^\alpha} 1(y) - t^m \partial_t^m e^{-t\mathcal{L}^\alpha} 1(z) \right\|_{L^2\left((0,\infty), \frac{dt}{t}\right)} \leq C \left(\frac{r}{\rho(x_0)} \right)^\delta \left(\log(\frac{\rho(x_0)}{r}) \right)^{1/2}.$$

Thus (iii) readily follows. □

4.3 Boundedness of Littlewood–Paley *g*-function $\widetilde{g}_\alpha^{\mathcal{L}}$

By the L^2-boundedness of Riesz transforms $\nabla_x \mathcal{L}^{-1/2}$, we can see that

$$\|\widetilde{g}_\alpha^{\mathcal{L}} f\|_{L^2}^2 \leq C \int_0^\infty \left(\int_{\mathbb{R}^n} |t^{1/2\alpha} \mathcal{L}^{1/2} e^{-t\mathcal{L}^\alpha} f(x)|^2 dx \right) \frac{dt}{t}.$$

Then by the spectral theorem, we know that $\widetilde{g}_\alpha^{\mathcal{L}}$ is bounded from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

Theorem 5 *Assume that the potential* $V \in B_q$ *with* $q > n$. *Let* $x, y, z \in \mathbb{R}^n$.

(i) *For any* $N > 0$, *there exists a constant* C_N *such that*

$$\left\| \widetilde{D}_{\alpha,t}^{\mathcal{L}}(x, y) \right\|_{L^2\left((0,\infty), \frac{dt}{t}\right)} \leq \frac{C_N}{|x - y|^n} \left(1 + \frac{|x - y|}{\rho(x)} + \frac{|x - y|}{\rho(y)} \right)^{-N};$$

(ii) *Let* $|x - y| > 2|y - z|$ *and* $0 < \delta < \min\{2\alpha, \delta_1, 1\}$. *There exists a constant* C *such that*

$$\left\| \widetilde{D}_{\alpha,t}^{\mathcal{L}}(x, y) - \widetilde{D}_{\alpha,t}^{\mathcal{L}}(x, z) \right\|_{L^2\left((0,\infty), \frac{dt}{t}\right)} + \left\| \widetilde{D}_{\alpha,t}^{\mathcal{L}}(y, x) - \widetilde{D}_{\alpha,t}^{\mathcal{L}}(z, x) \right\|_{L^2\left((0,\infty), \frac{dt}{t}\right)} \leq C \frac{|y - z|^\delta}{|x - y|^{n+\delta}};$$

(iii) *There exists a constant* C *such that for every ball* $B = B(x_0, r)$ *with* $0 < r \leq \rho(x)/2$,

$$\log\left(\frac{\rho(x)}{r} \right) \frac{1}{|B|} \int_B \left\| t^{1/(2\alpha)} \nabla_x e^{-t\mathcal{L}^\alpha} 1(y) - (t^{1/(2\alpha)} \nabla_x e^{-t\mathcal{L}^\alpha} 1)_B \right\|_{L^2\left((0,\infty), \frac{dt}{t}\right)} dy \leq C,$$

and, if $\gamma < \min\{2\alpha, \delta_1, 1\}$ *then*

$$\left(\frac{\rho(x)}{r}\right)^{\gamma}\frac{1}{|B|}\int_B \left\| t^{1/(2\alpha)}\nabla_x e^{-t\mathcal{L}^{\alpha}}1(y) - (t^{1/(2\alpha)}\nabla_x e^{-t\mathcal{L}^{\alpha}}1)_B \right\|_{L^2\left((0,\infty),\frac{\mathrm{d}t}{t}\right)}\mathrm{d}y \le C.$$

Proof For (i), from Proposition 4, we have

$$\left|\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)\right| \le C_N \min\left\{\frac{t^{1+N/\alpha+1/(2\alpha)}}{|x-y|^{n+2\alpha+2N+1}},\ t^{-n/(2\alpha)}\right\}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-N}.$$

If $\sqrt{t^{1/\alpha}} \le |x-y|$, we obtain

$$\begin{aligned}
&\left\|\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)\cdot\chi_{\{t^{1/2\alpha}\le|x-y|\}}\right\|^2_{L^2\left((0,\infty),\frac{\mathrm{d}t}{t}\right)}\\
&\le C_N\int_0^{|x-y|^{2\alpha}}\frac{t^{2+2N/\alpha+1/\alpha}}{|x-y|^{2n+4\alpha+4N+2}}\left(1+\frac{\sqrt{t^{1/\alpha}}}{\rho(x)}+\frac{\sqrt{t^{1/\alpha}}}{\rho(y)}\right)^{-2N}\frac{\mathrm{d}t}{t}\\
&\le C_N\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-2N}\int_0^{|x-y|^{2\alpha}}\frac{(\sqrt{t^{1/\alpha}})^{4\alpha+4N+2}}{|x-y|^{2n+4\alpha+4N+2}}\left(\frac{\sqrt{t^{1/\alpha}}}{|x-y|}\right)^{-2N}\frac{\mathrm{d}t}{t}\\
&\le \left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-2N}\frac{C_N}{|x-y|^{2n}}\int_0^{|x-y|^{2\alpha}}\left(\frac{\sqrt{t^{1/\alpha}}}{|x-y|}\right)^{4\alpha+2N+2}\frac{\mathrm{d}t}{t}.
\end{aligned}$$

Let $\sqrt{t^{1/\alpha}}/|x-y| = u$. We can see that

$$\begin{aligned}
&\left\|\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)\cdot\chi_{\{t^{1/2\alpha}\le|x-y|\}}\right\|^2_{L^2\left((0,\infty),\frac{\mathrm{d}t}{t}\right)}\\
&\le \left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-2N}\frac{C_N}{|x-y|^{2n}}\int_0^1 u^{4\alpha+2N+1}\mathrm{d}u\\
&\le \frac{C_N}{|x-y|^{2n}}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-2N}.
\end{aligned}$$

If $\sqrt{t^{1/\alpha}} \ge |x-y|$, we can get

$$\begin{aligned}
&\left\|\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)\cdot\chi_{\{t^{1/2\alpha}\ge|x-y|\}}\right\|^2_{L^2\left((0,\infty),\frac{\mathrm{d}t}{t}\right)}\\
&\le C_N\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-2N}\int_{|x-y|^{2\alpha}}^{\infty} t^{-n/\alpha-1}\mathrm{d}t\\
&\le \frac{C_N}{|x-y|^{2n}}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-2N},
\end{aligned}$$

which proves (i).

For (ii), by Proposition 4, we have

$$\left\|\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)-\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,z)\right\|^2_{L^2\left((0,\infty),\frac{dt}{t}\right)} \leq \int_0^\infty \frac{Ct}{(\sqrt{t^{1/\alpha}}+|x-y|)^{2n+4\alpha}}\left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}}\right)^{2\delta}dt$$
$$\leq C|y-z|^{2\delta}\int_0^\infty \frac{(\sqrt{t^{1/\alpha}})^{2\alpha-2\delta}}{(\sqrt{t^{1/\alpha}}+|x-y|)^{2n+4\alpha}}dt.$$

Let $\sqrt{t^{1/\alpha}}/|x-y|=u$. We obtain

$$\left\|\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)-\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,z)\right\|^2_{L^2\left((0,\infty),\frac{dt}{t}\right)} \leq C|y-z|^{2\delta}|x-y|^{-2n-2\delta}\int_0^\infty \frac{u^{4\alpha-2\delta-1}}{(1+u)^{2n+4\alpha}}du$$
$$\leq \frac{C|y-z|^{2\delta}}{|x-y|^{2n+2\delta}}.$$

The symmetry of the kernel $D^{\mathcal{L}}_{\alpha,t}(\cdot,\cdot)$ gives the desired conclusion of (ii).

For (iii), fix $y,z\in B=B(x_0,r)$ with $0<r\leq\rho(x_0)/2$. Similar to Theorem 4, we need to deal with the term

$$\left\|t^{1/(2\alpha)}\nabla_x e^{-t\mathcal{L}^\alpha}1(y)-t^{1/(2\alpha)}\nabla_x e^{-t\mathcal{L}^\alpha}1(z)\right\|_{L^2\left((0,\infty),\frac{dt}{t}\right)}$$

first. Write

$$\left\|t^{1/(2\alpha)}\nabla_x e^{-t\mathcal{L}^\alpha}1(y)-t^{1/(2\alpha)}\nabla_x e^{-t\mathcal{L}^\alpha}1(z)\right\|^2_{L^2\left((0,\infty),\frac{dt}{t}\right)} = G_1+G_2+G_3,$$

where

$$\begin{cases} G_1 := \int_0^{(2r)^{2\alpha}}\left|\int_{\mathbb{R}^n}\left(\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)-\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,z)\right)dx\right|^2\frac{dt}{t};\\ G_2 := \int_{(2r)^{2\alpha}}^{\rho(x_0)^{2\alpha}}\left|\int_{\mathbb{R}^n}\left(\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)-\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,z)\right)dx\right|^2\frac{dt}{t};\\ G_3 := \int_{\rho(x_0)^{2\alpha}}^{\infty}\left|\int_{\mathbb{R}^n}\left(\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,y)-\widetilde{D}^{\mathcal{L}}_{\alpha,t}(x,z)\right)dx\right|^2\frac{dt}{t}. \end{cases}$$

Since $y,z\in B\subset B(x_0,\rho(x_0))$, then $\rho(y)\sim\rho(x_0)\sim\rho(z)$. It follows from Proposition 4 (iii) that

$$G_1\leq C\int_0^{(2r)^{2\alpha}}(\sqrt{t^{1/\alpha}}/\rho(x_0))^{1+2\alpha}\frac{dt}{t}\leq C\int_0^{(2s)^{2\alpha}}\left(\frac{\sqrt{t^{1/\alpha}}}{\rho(x_0)}\right)^{2\delta}\frac{dt}{t}=C\left(\frac{r}{\rho(x_0)}\right)^{2\delta}.$$

Also, we apply Proposition 4 (ii) to deduce that

$$G_3 \leq C \int_{\rho(x_0)^{2\alpha}}^{\infty} \left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}} \right)^{2\delta} \left| \int_{\mathbb{R}^n} \frac{t}{(\sqrt{t^{1/\alpha}} + |x-y|)^{n+2\alpha}} \mathrm{d}x \right|^2 \frac{\mathrm{d}t}{t}$$
$$\leq C \int_{\rho(x_0)^{2\alpha}}^{\infty} \left(\frac{|y-z|}{\sqrt{t^{1/\alpha}}} \right)^{2\delta} \frac{\mathrm{d}t}{t} \leq C \left(\frac{r}{\rho(x_0)} \right)^{2\delta}.$$

Then for G_2, following the procedure of the treatment for M_2 in Theorem 4, we obtain

$$G_2 \leq \int_{(2r)^{2\alpha}}^{\rho(x_0)^{2\alpha}} \left(\frac{r}{\rho(x_0)} \right)^{2\delta} \frac{\mathrm{d}t}{t} = C \left(\frac{r}{\rho(x_0)} \right)^{2\delta} \log \left(\frac{\rho(x_0)}{r} \right).$$

From the above estimates, we can get

$$\left\| t^{1/(2\alpha)} \nabla_x e^{-t\mathcal{L}^\alpha} 1(y) - t^{1/(2\alpha)} \nabla_x e^{-t\mathcal{L}^\alpha} 1(z) \right\|^2_{L^2\left((0,\infty), \frac{\mathrm{d}t}{t}\right)} \leq C \left(\frac{r}{\rho(x_0)} \right)^{\delta} \left(\log \left(\frac{\rho(x_0)}{r} \right) \right)^{1/2}.$$

Thus (iii) readily follows. □

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