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ORIGINAL PAPER

Dual spaces for variable martingale Lorentz–Hardy spaces

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Abstract

Let $H_{p(\cdot),q}$ be the variable Lorentz–Hardy martingale spaces. In this paper, we give a new atomic decomposition for these spaces via simple L_r-atoms ($1 < r \leq \infty$). Using this atomic decomposition, we consider the dual spaces of variable Lorentz-Hardy spaces $H_{p(\cdot),q}$ for the case $0 < p(\cdot) \leq 1$, $0 < q \leq 1$, and $0 < p(\cdot) < 2$, $1 < q < \infty$ respectively, and prove that they are equivalent to the *BMO* spaces with variable exponent. Furthermore, we also obtain several John-Nirenberg theorems based on the dual results.

Keywords Variable martingales · Martingale Hardy spaces · Dualities · John–Nirenberg theorems

Mathematics Subject Classification Primary 60G42 · 60G46 · Secondary 42B30 · 42C10

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1 Introduction

In this paper, we focus on the dual space of Hardy spaces in martingale setting. A martingale analogue of $H_1 - BMO$ duality can be found in [9]. For dyadic martingales, Herz [13] proved the dual space of H_p $(0 < p < 1)$. In 1990, Weisz [28] characterized the dual space of H_p $(0 < p \leq 1)$ for general martingales via atomic decomposition. Recently, these results were extended to more general cases. Jiao et al. [14] got the atomic decomposition for martingale Lorentz–Hardy spaces $H_{p,q}$. Later, Jiao et al. [17] extended the atomic decomposition in [14] and investigated the dual space of $H_{p,q}$. Miyamoto et al. [24] studied the atomic decomposition of martingale Orlicz–Hardy space H_Φ and proved the dual of it. The weak type martingale Hardy spaces were also studied by several authors, see for instance [16, 31].

Recently, motivated by the development of harmonic analysis based on variable Lebesgue spaces (see e.g. [5] and references therein), people began to study martingales associated with variable exponents. In particular, Aoyama [1] established the Doob maximal inequality when $p(\cdot)$ is $\mathcal{F}_n$-measurable for all $n \geq 0$. Shortly later, Nakai and Sadasue [25] showed that Aoyama's assumption is not necessary for the Doob maximal inequality. In [19] (see also [12]), with additional assumption that $\mathcal{F}_n$ is atomic σ-algebra, Jiao et al. introduced a new condition on $p(\cdot)$ to ensure that Doob maximal operator is bounded on $L_{p(\cdot)}(\Omega)$. Xie et al. [33] proved several martingale inequalities in Musielak-Orlicz spaces. Jiao et al. [15] did a systematic study of variable martingale Lorentz-Hardy spaces $H_{p(\cdot),q}$. Actually, the authors in [15] constructed atomic decomposition for Hardy spaces and gave applications to Fourier analysis.

In the present paper, we continue to study the variable martingale Lorentz-Hardy spaces $H_{p(\cdot),q}$. Our first aim is to show the dual space of $H_{p(\cdot),q}$. The main tool we use here is atomic decomposition of $H_{p(\cdot),q}$. Recall that only ∞-atoms works for the atomic decomposition in [15]. As we will see, by Lemma 3.6, we can construct atomic decomposition via r-atoms $(r < \infty)$ in the sense of simple atoms (see Theorem 3.5). The proof is given in Sect. 3. In Sect. 4, as applications of this kind of atomic decomposition, we establish the dual space of $H_{p(\cdot),q}$.

Our second objective is to prove John-Nirenberg theorem associated with variable exponent. Consider martingales with respect to a non-decreasing stochastic basic $(\mathcal{F}_n)_{n\geq 0}$. Let $\mathcal{T}$ be the set of all stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. The well known (classical) John–Nirenberg theorem says that if the the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ is regular, then

$$BMO_p = BMO_1, \quad 1 \leq p < \infty, \tag{1}$$

where $\|f\|_{BMO_p} = \|f\|_{BMO_{L_p}}$ defined below. We refer the reader to [9] for the above fact (1). This result was generalized by Yi et al. [35]: if the stochastic basis is regular and E is a rearrangement invariant Banach function space (see e.g. [2]), then

$$BMO_E = BMO_1, \tag{2}$$

where

$$\|f\|_{BMO_E} = \sup_{\tau\in\mathcal{T}} \frac{\|(f - f^{\tau})\chi_{\{\tau<\infty\}}\|_E}{\|\chi_{\{\tau<\infty\}}\|_E}.$$

In this paper, we introduce variable Lipschitz space $BMO_E(\alpha(\cdot))$ and show that (Theorem 5.10)

$$BMO_E(\alpha(\cdot)) = BMO_1(\alpha(\cdot))$$

where $\alpha(\cdot)$ satisfies (5). This result is just (2) when $\alpha(\cdot) \equiv 0$. We also have John-Nirenberg theorem for generalized BMO martingale spaces associated with variable exponent. Recall that Jiao et al. introduced the generalized BMO martingale spaces $BMO_{r,q}(\alpha)$ $(r, q \geq 1,\ \alpha \geq 0)$ and proved that $BMO_{r,q}(\alpha)$ is the dual space of $H_{p,q}$ $(0 < p \leq 1$ and $\alpha = 1/p - 1)$. Jiao et al. [17, Theorem 1.2] obtained that if the stochastic basis is regular, then

$$BMO_{r,q}(\alpha) = BMO_{1,q}(\alpha). \tag{3}$$

The variable exponent version of (3) is presented in Theorem 5.14. The proof is given in Sect. 5.

Throughout this paper, the integer set and nonnegative integer set are denoted by $\mathbb{Z}$ and $\mathbb{N}$, respectively. We denote by C a positive constant, which can vary from line to line. The symbol $A \lesssim B$ stands for the inequality $A \leq CB$. If we write $A \approx B$, then it mean $A \lesssim B \lesssim A$.

2 Preliminaries

2.1 Variable Lebesgue spaces $L_{p(\cdot)}$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A measurable function $p(\cdot) : \Omega \to (0, \infty)$ is called a variable exponent. For a measurable set $A \subset \Omega$, we denote

$$p_-(A) := \operatorname*{ess\,inf}_{x\in A} p(x), \quad p_+(A) := \operatorname*{ess\,sup}_{x\in A} p(x)$$

and for convenience

$$p_- := p_-(\Omega), \quad p_+ := p_+(\Omega).$$

Denote by $\mathcal{P}(\Omega)$ the collection of all variable exponents $p(\cdot)$ such that $0 < p_- \leq p_+ < \infty$. The variable Lebesgue space $L_{p(\cdot)} = L_{p(\cdot)}(\Omega)$ is the collection of all measurable functions f defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for some $\lambda > 0$,

$$\rho(f/\lambda) = \int_{\Omega} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} d\mathbb{P} < \infty.$$

This becomes a quasi-Banach function space when it is equipped with the quasi-norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \rho(f/\lambda) \le 1\}.$$

For any $f \in L_{p(\cdot)}$, we have $\rho(f) \le 1$ if and only if $\|f\|_{p(\cdot)} \le 1$; see [6, Theorem 1.3]. In the sequel, we always use the symbol

$$\underline{p} = \min\{p_-, 1\}.$$

Throughout the paper, the variable exponent $p'(\cdot)$ is defined pointwise by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \Omega.$$

For $p(\cdot) \in \mathcal{P}(\Omega)$, it is clear that $p'(x) \in \mathbb{R} \cup \{\infty\} \setminus \{0\}$ for any $x \in \Omega$. We present some basic properties here (see [26]):

1. $\|f\|_{p(\cdot)} \ge 0$; $\|f\|_{p(\cdot)} = 0 \Leftrightarrow f \equiv 0$.
2. $\|cf\|_{p(\cdot)} = |c| \cdot \|f\|_{p(\cdot)}$ for $c \in \mathbb{C}$.
3. for $0 < b \le \underline{p}$, we have

$$\|f + g\|^b_{p(\cdot)} \le \|f\|^b_{p(\cdot)} + \|g\|^b_{p(\cdot)}. \tag{4}$$

Lemma 2.1 ([5, Corollary 2.28]) *Let $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}(\Omega)$ satisfy*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$

Then there exists a constant C such that for all $f \in L_{q(\cdot)}$ and $g \in L_{r(\cdot)}$, we have $fg \in L_{p(\cdot)}$ and

$$\|fg\|_{p(\cdot)} \le C\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

Furthermore, we have the following reverse Minkowski inequality. It was stated without a proof in [34, Remark 2.4] for $p_+ < 1$. We give a detailed proof here.

Lemma 2.2 *Let $p(\cdot) \in \mathcal{P}(\Omega)$. If $p_+ \le 1$, we have, for positive functions $f, g \in L_{p(\cdot)}$,*

$$\|f\|_{p(\cdot)} + \|g\|_{p(\cdot)} \le \|f + g\|_{p(\cdot)}.$$

Proof Take positive functions $f, g \in L_{p(\cdot)}$. For arbitrary small positive number $\varepsilon > 0$, set $\lambda_f = \|f\|_{p(\cdot)} - \varepsilon$ and $\lambda_g = \|g\|_{p(\cdot)} - \varepsilon$. Note that, as mentioned before, $\rho(f) > 1$ if and only if $\|f\|_{p(\cdot)} > 1$. Then, by concavity, we have

$$\int_\Omega \left(\frac{f(x) + g(x)}{\lambda_f + \lambda_g}\right)^{p(x)} d\mathbb{P} \ge \frac{\lambda_f}{\lambda_f + \lambda_g} \int_\Omega \left(\frac{f(x)}{\lambda_f}\right)^{p(x)} d\mathbb{P} + \frac{\lambda_g}{\lambda_f + \lambda_g} \int_\Omega \left(\frac{g(x)}{\lambda_g}\right)^{p(x)} d\mathbb{P} > 1,$$

which implies

$$\|f+g\|_{p(\cdot)} > \lambda_f + \lambda_g = \|f\|_{p(\cdot)} + \|g\|_g - 2\varepsilon.$$

Taking $\varepsilon \to 0$, we get the desired result. □

2.2 Variable Lorentz spaces $L_{p(\cdot),q}$

In this section, we recall the definition of Lorentz spaces $L_{p(\cdot),q}(\Omega)$ with variable exponents $p(\cdot) \in \mathcal{P}(\Omega)$ and $0 < q \le \infty$ is a constant. For more information about general cases $L_{p(\cdot),q(\cdot)}(\Omega)$, we refer the reader to [21]. Following [21] (see also [36]), we introduce the definition below.

Definition 2.3 Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $0 < q \le \infty$. Then $L_{p(\cdot),q}(\Omega)$ is the collection of all measurable functions f such that

$$\|f\|_{L_{p(\cdot),q}} := \begin{cases} \left(\int_0^\infty \lambda^q \|\chi_{\{|f|>\lambda\}}\|_{p(\cdot)}^q \frac{d\lambda}{\lambda}\right)^{1/q}, & q < \infty, \\ \sup_\lambda \lambda \|\chi_{\{|f|>\lambda\}}\|_{p(\cdot)}, & q = \infty \end{cases}$$

is finite.

Next we introduce a closed subspace of $L_{p(\cdot),\infty}$.

Definition 2.4 Let $p(\cdot) \in \mathcal{P}(\Omega)$. We define $\mathscr{L}_{p(\cdot),\infty}(\Omega)$ as the set of measurable functions f such that

$$\lim_{n\to\infty} \|f\chi_{A_n}\|_{L_{p(\cdot),\infty}} = 0$$

for every sequence $(A_n)_{n\ge 0}$ satisfying $\mathbb{P}(A_n) \to 0$ as $n \to \infty$.

It follows from the dominated convergence theorem for $\mathscr{L}_{p(\cdot),\infty}(\Omega)$ (see Lemma 2.13 in Jiao et al. [15]) that the simple functions are dense in $\mathscr{L}_{p(\cdot),\infty}(\Omega)$.

2.3 Variable martingale Hardy spaces

In this section, we introduce some standard notations from martingale theory. We refer to the books [9, 23, 29] for the theory of classical martingale space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let the subalgebras $(\mathcal{F}_n)_{n\ge 0}$ be increasing such that $\mathcal{F} = \sigma(\cup_{n\ge 0}\mathcal{F}_n)$, and let $\mathbb{E}_n$ denote the conditional expectation operator relative to $\mathcal{F}_n$. A sequence of measurable functions $f = (f_n)_{n\ge 0} \subset L_1(\Omega)$ is called a martingale with respect to $(\mathcal{F}_n)_{n\ge 0}$ if $\mathbb{E}_n(f_{n+1}) = f_n$ for every $n \ge 0$. For a martingale $f = (f_n)_{n\ge 0}$,

$$d_n f = f_n - f_{n-1}, \quad n \ge 0,$$

denote the martingale difference. If in addition $f_n \in L_{p(\cdot)}$ for any $n \geq 0$, then f is called an $L_{p(\cdot)}$-martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. In this case, we set

$$\|f\|_{p(\cdot)} = \sup_{n\geq 0} \|f_n\|_{p(\cdot)}.$$

If $\|f\|_{p(\cdot)} < \infty$, f is called a bounded $L_{p(\cdot)}$-martingale and it is denoted by $f \in L_{p(\cdot)}$. For a martingale relative to $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_n)_{n\geq 0})$, we define the maximal function, the square function and the conditional square function of f, respectively, as follows ($f_{-1} = 0$):

$$M_m(f) = \sup_{0\leq n\leq m} |f_n|, \quad M(f) = \sup_{n\geq 0} |f_n|;$$

$$S_m(f) = \left(\sum_{n=0}^{m} |d_n f|^2\right)^{1/2}, \quad S(f) = \left(\sum_{n=0}^{\infty} |d_n f|^2\right)^{1/2};$$

$$s_m(f) = \left(\sum_{n=0}^{m} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}}, \quad s(f) = \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}}.$$

Denote by Λ the collection of all sequences $(\lambda_n)_{n\geq 0}$ of non-decreasing, non-negative and adapted functions with $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $0 < q \leq \infty$.

Similarly, the variable martingale Lorentz-Hardy spaces associated with variable Lorentz spaces $L_{p(\cdot),q}$ are defined as follows:

$$H^M_{p(\cdot),q} = \{f = (f_n)_{n\geq 0} : \|f\|_{H^M_{p(\cdot),q}} = \|M(f)\|_{L_{p(\cdot),q}} < \infty\};$$

$$H^S_{p(\cdot),q} = \{f = (f_n)_{n\geq 0} : \|f\|_{H^S_{p(\cdot),q}} = \|S(f)\|_{L_{p(\cdot),q}} < \infty\};$$

$$H^s_{p(\cdot),q} = \{f = (f_n)_{n\geq 0} : \|f\|_{H^s_{p(\cdot),q}} = \|s(f)\|_{L_{p(\cdot),q}} < \infty\};$$

$$Q_{p(\cdot),q} = \{f = (f_n)_{n\geq 0} : \exists (\lambda_n)_{n\geq 0} \in \Lambda, \text{ s.t. } S_n(f) \leq \lambda_{n-1}, \lambda_\infty \in L_{p(\cdot),q}\},$$
$$\|f\|_{Q_{p(\cdot),q}} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{L_{p(\cdot),q}};$$

$$P_{p(\cdot),q} = \{f = (f_n)_{n\geq 0} : \exists (\lambda_n)_{n\geq 0} \in \Lambda, \text{ s.t. } |f_n| \leq \lambda_{n-1}, \lambda_\infty \in L_{p(\cdot),q}\},$$
$$\|f\|_{P_{p(\cdot),q}} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{L_{p(\cdot),q}}.$$

We define $\mathscr{H}^M_{p(\cdot),\infty}$ as the space of all martingales such that $M(f) \in \mathscr{L}_{p(\cdot),\infty}$. Analogously, we can define $\mathscr{H}^S_{p(\cdot),\infty}$ and $\mathscr{H}^s_{p(\cdot),\infty}$, respectively.

Remark 2.5 If $p(\cdot) = p$ is a constant, then the above definitions of variable Hardy spaces go back to the classical definitions stated in [9] and [29].

2.4 The Doob maximal operator

We need some more notations. Recall that $B \in \mathcal{F}_n$ is called an atom, if for any $A \subset B$ with $A \in \mathcal{F}_n$ satisfying $\mathbb{P}(A) < \mathbb{P}(B)$, we have $\mathbb{P}(A) = 0$. In the theory of variable spaces, we usually use the log-Hölder continuity of $p(\cdot)$. In the sequel of this paper, we will always suppose that every σ-algebra $\mathcal{F}_n$ is generated by countably many atoms. We denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$ for each $n \geq 0$. Instead of the log-Hölder continuity, we suppose that there exists an absolute constant $K_{p(\cdot)} \geq 1$ depending only on $p(\cdot)$ such that

$$\mathbb{P}(A)^{p_-(A)-p_+(A)} \leq K_{p(\cdot)}, \quad \forall A \in \bigcup_n A(\mathcal{F}_n). \tag{5}$$

Note that in this paper, under condition (5), we also mean that every σ-algebra $\mathcal{F}_n$ is generated by countably many atoms.

It is clear that for $f \in L_1(\Omega)$

$$\mathbb{E}_n(f) = \sum_{A \in A(\mathcal{F}_n)} \left(\frac{1}{\mathbb{P}(A)} \int_A f(x) d\mathbb{P} \right) \chi_A, \quad n \in \mathbb{N}.$$

We now recall the definition of regularity. The stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ is said to be regular, if for $n \geq 0$ and $A \in \mathcal{F}_n$, there exists $B \in \mathcal{F}_{n-1}$ such that $A \subset B$ and $P(B) \leq RP(A)$, where R is a positive constant independent of n. A martingale is said to be regular if it is adapted to a regular σ-algebra sequence. This implies that there exists a constant $R > 0$ such that

$$f_n \leq Rf_{n-1} \tag{6}$$

for all non-negative martingales $(f_n)_{n\geq 0}$ adapted to the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$. We refer the reader to [23, Chapter 7] for more details.

The following results are taken from [12] and [19].

Lemma 2.6 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (5). Then, for any atom $B \in \cup_n A(\mathcal{F}_n)$,*

$$\mathbb{P}(B)^{1/p_-(B)} \approx \mathbb{P}(B)^{1/p(x)} \approx \mathbb{P}(B)^{1/p_+(B)} \approx \|\chi_B\|_{p(\cdot)}, \quad \forall x \in B.$$

Lemma 2.7 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (5) with $p_- \geq 1$. Then, for any atom $B \in \cup_n A(\mathcal{F}_n)$*
,

$$\|\chi_B\|_{r(\cdot)} \approx \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{q(\cdot)},$$

where

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \quad x \in \Omega.$$

Theorem 2.8 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (5) and $1 < p_- \leq p_+ < \infty$. Then, there is a positive constant $C_{p(\cdot)}$ such that*

$$\|M(f)\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)}.$$

3 Atomic decomposition via simple atoms

In this section, we consider the atomic characterizations of variable Lorentz-Hardy spaces. Recall that, without any restriction, $H^s_{p(\cdot),q}$ has atomic decomposition via $(1, p(\cdot), \infty)$-atoms (see [15]). In this section, we show that, under the assumption that the filtration $(\mathcal{F}_n)_n$ is generated by countably many atoms and $p(\cdot)$ satisfies (5), then $H^s_{p(\cdot),q}$ has atomic decomposition via simple $(1, p(\cdot), r)$-atoms with $\max\{p_+, 1\} < r \le \infty$. To be able the prove the duality results later, we need this new atomic decomposition. We will use it not only for $r = \infty$ but also for $r < \infty$. The results later cannot be proved with the atomic decomposition obtained in [15]. We begin this section with the definition of the simple atoms (see [30] for the classical definition).

Definition 3.1 Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $1 < r \le \infty$. A measurable function a is called a simple $(1, p(\cdot), r)$-atom (briefly $(s, 1, p(\cdot), r)$-atom) if there exist $j \in \mathbb{N}$, $I \in A(\mathcal{F}_j)$ such that

(1) the support of a is contained in I,
(2) $\|s(a)\|_r \le \frac{\|\chi_I\|_r}{\|\chi_I\|_{p(\cdot)}}$,
(3) $\mathbb{E}_j(a) = 0$.

If $s(a)$ in (2) is replaced by $S(a)$ (or $M(a)$), then the function a is called $(s, 2, p(\cdot), r)$ -atom (or $(s, 3, p(\cdot), r)$-atom).

The result below is a simple but useful observation.

Proposition 3.2 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $1 < r \le \infty$. If a is an $(s, i, p(\cdot), r)$-atom $(i = 1, 2, 3)$ associated with $I \in A(\mathcal{F}_j)$ for some $j \in \mathbb{N}$, then*

$$s(a)\chi_I = s(a), \quad S(a)\chi_I = S(a) \quad \text{and} \quad M(a)\chi_I = M(a).$$

Proof Observe that $\mathbb{E}_m(a) = 0$ for $m \le j$. Hence, for each $m \in \mathbb{N}$, $\mathbb{E}_m(a)\chi_I = \mathbb{E}_m(a)$. From this,

$$M(a)\chi_I = \sup_{m\ge 0} \mathbb{E}_m(a)\chi_I = \sup_{m\ge 0} \mathbb{E}_m(a) = M(a).$$

Also,

$$\begin{aligned} s^2(a) &= \sum_{m=0}^{\infty} \mathbb{E}_{m-1}|d_m a|^2 = \sum_{m=j+1}^{\infty} \mathbb{E}_{m-1}|d_m a|^2 \\ &= \chi_I \sum_{m=j+1}^{\infty} \mathbb{E}_{m-1}|d_m a|^2 = s^2(a)\chi_I. \end{aligned}$$

This means $s(a)\chi_I = s(a)$. In a similar way, we have

$$S(a)\chi_I = S(a).$$

□

We introduce the definition of atomic Hardy spaces.

Definition 3.3 Let $p(\cdot) \in \mathcal{P}(\Omega)$, $0 < q \le \infty$ and $1 < r \le \infty$. Assume that $d = 1, 2$ or 3. The atomic Hardy space $H^{\mathrm{sat,d,r}}_{p(\cdot),q}$ is defined as the space of all martingales $f = (f_n)_{n\ge 0}$ such that

$$f_n = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{n-1} \sum_{i} \mu_{k,j,i} a_n^{k,j,i} \quad \text{a.e. }, \quad n \in \mathbb{N}, \tag{7}$$

where $(a_{k,j,i})_{k\in\mathbb{Z},j,i\in\mathbb{N}}$ is a sequence of $(s, d, p(\cdot), r)$-atoms associated with $(I_{k,j,i})_{k,j,i} \subset A(\mathcal{F}_j)$, which are disjoint for fixed k, and $\mu_{k,j,i} = 3 \cdot 2^k \|\chi_{I_{k,j,i}}\|_{p(\cdot)}$. For $f \in H^{\mathrm{sat,d,r}}_{p(\cdot),q}$, define

$$\|f\|_{H^{\mathrm{sat,d,r}}_{p(\cdot),q}} = \inf \left(\sum_{k\in\mathbb{Z}} \left\| \sum_{j=0}^{\infty} \sum_{i} \frac{\mu_{k,j,i}\chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{p(\cdot)}} \right\|_{p(\cdot)}^{q} \right)^{1/q},$$

where the infimum is taken over all the decompositions of the form (7).

Remark 3.4 From the above definition, since $\mu_{k,j,i} = 3 \cdot 2^k \|\chi_{I_{k,j,i}}\|_{p(\cdot)}$, we have

$$\|f\|_{H^{\mathrm{sat,d,r}}_{p(\cdot),q}} \approx \inf \left(\sum_{k\in\mathbb{Z}} 2^{kq} \left\| \sum_{j=0}^{\infty} \sum_{i} \chi_{I_{k,j,i}} \right\|_{p(\cdot)}^{q} \right)^{1/q},$$

where the infimum is the same as above.

We state the main result of this section. The atomic decomposition via simple $(1, p(\cdot), r)$-atoms $(r < \infty)$ are much more complicated than the atomic decomposition via $(1, p(\cdot), \infty)$-atoms proved in [15, Theorem 3.9].

Theorem 3.5 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (5) and $\max\{p_+, 1\} < r \le \infty$. Then*

$$H^s_{p(\cdot),q} = H^{\mathrm{sat,1,r}}_{p(\cdot),q}, \quad 0 < q \le \infty$$

with equivalent quasi-norms.

Before going further, we show the next lemma. Let $T : X \to Y$ be a sublinear operator, where X is a martingale space and Y is a function space.

Lemma 3.6 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (5) and $\max\{p_+, 1\} < r < \infty$. Take $0 < \varepsilon < p$ and $L \in (1, \frac{r}{p_+} \wedge \frac{1}{\varepsilon})$. If for a sublinear operator T and all $(s, d, p(\cdot), r)$-atoms $a^{k,\bar{j},\bar{i}}$ $(d = 1, 2, 3)$,*

$$\|T(a^{k,j,i})\|_r \lesssim \frac{\|\chi_{I_{k,j,i}}\|_r}{\|\chi_{I_{k,j,i}}\|_{p(\cdot)}},$$

then

$$Z := \left\| \sum_{j=0}^{\infty} \sum_i \left[\|\chi_{I_{k,j,i}}\|_{p(\cdot)} T(a^{k,j,i}) \chi_{I_{k,j,i}} \right]^{L\varepsilon} \right\|_{p(\cdot)/\varepsilon} \lesssim \left\| \sum_{j=0}^{\infty} \sum_i \chi_{I_{k,j,i}} \right\|_{p(\cdot)/\varepsilon}.$$

Proof According to the duality $(L_{\frac{p(\cdot)}{\varepsilon}})^* = L_{(\frac{p(\cdot)}{\varepsilon})'}$ (see e.g. [5, Theorem 2.80]), we can choose a positive function $g \in L_{(\frac{p(\cdot)}{\varepsilon})'}$ with $\|g\|_{L_{(\frac{p(\cdot)}{\varepsilon})'}} \le 1$ such that

$$Z = \int_\Omega \sum_{j=0}^{\infty} \sum_i \left[\|\chi_{I_{k,j,i}}\|_{p(\cdot)} T(a^{k,j,i}) \chi_{I_{k,j,i}} \right]^{L\varepsilon} g d\mathbb{P}.$$

Applying Hölder's inequality (here, note that $L\varepsilon < 1 < r$), we obtain that

$$\begin{aligned}
Z &\le \sum_{j=0}^{\infty} \sum_i \|\chi_{I_{k,j,i}}\|_{p(\cdot)}^{L\varepsilon} \|T(a^{k,j,i})^{L\varepsilon}\|_{\frac{r}{L\varepsilon}} \|\chi_{I_{k,j,i}} g\|_{(\frac{r}{L\varepsilon})'} \\
&\lesssim \sum_{j=0}^{\infty} \sum_i \|\chi_{I_{k,j,i}}\|_r^{L\varepsilon} \left(\int_{I_{k,j,i}} g^{(\frac{r}{L\varepsilon})'} \right)^{1/(\frac{r}{L\varepsilon})'} \\
&= \sum_{j=0}^{\infty} \sum_i \int_\Omega \chi_{I_{k,j,i}} d\mathbb{P} \left(\frac{1}{\mathbb{P}(I_{k,j,i})} \int_{I_{k,j,i}} g^{(\frac{r}{L\varepsilon})'} \right)^{1/(\frac{r}{L\varepsilon})'} \\
&\le \sum_{j=0}^{\infty} \sum_i \int_\Omega \chi_{I_{k,j,i}} [M(g^{(\frac{r}{L\varepsilon})'})]^{1/(\frac{r}{L\varepsilon})'} d\mathbb{P} \\
&\le \left\| \sum_{j=0}^{\infty} \sum_i \chi_{I_{k,j,i}} \right\|_{p(\cdot)/\varepsilon} \|[M(g^{(\frac{r}{L\varepsilon})'})]^{1/(\frac{r}{L\varepsilon})'}\|_{(p(\cdot)/\varepsilon)'}.
\end{aligned}$$

The "$\lesssim$" above is due to the definition of the operator T. Since $L < \frac{r}{p_+}$, we deduce that

$$\left(\frac{r}{L\varepsilon}\right)' < (p(\cdot)/\varepsilon)'.$$

Noting that $\varepsilon < \underline{p}$, hence $((p(\cdot)/\varepsilon)')_+ < \infty$. Using the maximal inequality (Theorem 2.8), we have

$$\|[M(g^{(\frac{r}{L\varepsilon})'})]^{1/(\frac{r}{L\varepsilon})'}\|_{(p(\cdot)/\varepsilon)'} \lesssim \|g\|_{(p(\cdot)/\varepsilon)'} \le 1,$$

which completes the proof. □

Now we are in a position to prove the main result of this section.

Proof of Theorem 3.5 Let us consider the following stopping times for all $k \in \mathbb{Z}$,

$$\tau_k = \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}.$$

The sequence of these stopping times is obviously non-decreasing. For each stopping time τ, denote $f_n^\tau = f_{n\wedge\tau}$, where $n \wedge \tau = \min(n, \tau)$. Hence

$$f_n = \sum_{k\in\mathbb{Z}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}).$$

Note that, for fixed k, j, there exist disjoint atoms $(I_{k,j,i})_i \subset \mathcal{F}_j$ such that

$$\bigcup_i I_{k,j,i} = \{\tau_k = j\} \in \mathcal{F}_j.$$

Then, it is easy to see that

$$f_n = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{n-1} \sum_i \chi_{I_{k,j,i}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}).$$

Let

$$\mu_k = 3 \cdot 2^k \left\| \chi_{I_{k,j,i}} \right\|_{p(\cdot)} \quad \text{and} \quad a_n^k = \chi_{I_{k,j,i}} \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

Observe that

$$\begin{aligned} f_n^{\tau_{k+1}} &= \sum_{m=0}^{n-1} f_m \chi_{\{\tau_{k+1}=m\}} + f_n \chi_{\{\tau_{k+1}\geq n\}} \\ &= \sum_{m=0}^{n-1} f_m (\chi_{\{\tau_{k+1}\geq m\}} - \chi_{\{\tau_{k+1}\geq m+1\}}) + f_n \chi_{\{\tau_{k+1}\geq n\}} \\ &= \sum_{m=0}^{n} (f_m - f_{m-1}) \chi_{\{\tau_{k+1}\geq m\}} = \sum_{m=0}^{n} d_m f \chi_{\{\tau_{k+1}\geq m\}}. \end{aligned}$$

Then we conclude that

$$\begin{aligned} \chi_{I_{k,j,i}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}) &= \chi_{I_{k,j,i}} \sum_{m=0}^{n} d_m f \chi_{\{\tau_{k+1}\geq m>\tau_k\}} \\ &= \chi_{I_{k,j,i}} \sum_{m=j+1}^{n} d_m f \chi_{\{\tau_{k+1}\geq m>\tau_k\}}, \end{aligned}$$

where the last estimate is due to $I_{k,j,i} \subset \{\tau_k = j\}$. Consequently,

$$\mathbb{E}_j(a_n^{k,j,i}) = 0, \qquad \int_{I_{k,j,i}} a_n^{k,j,i} = 0$$

and, for fixed k, j, i, $(a_n^{k,j,i})_{n\geq 0}$ is a martingale. By the definition of τ_k, we obtain that

$$s((a_n^{k,j,i})_n) \leq \frac{1}{\|\chi_{I_{k,j,i}}\|_{p(\cdot)}}.$$

Thus $(a_n^{k,j,i})_n$ is an L_2-bounded martingale and so there exists $a^{k,j,i} \in L_2$ such that

$$\mathbb{E}_n(a^{k,j,i}) = a_n^{k,j,i} \quad \text{and} \quad s(a^{k,j,i}) \leq \frac{1}{\|\chi_{I_{k,j,i}}\|_{p(\cdot)}}.$$

We conclude that $a^{k,j,i}$ is a $(s, 1, p(\cdot), \infty)$-atom according to the above estimates. Note that for any fixed $k \in \mathbb{Z}$,

$$\sum_{j=0}^{\infty}\sum_i \chi_{I_{k,j,i}} = \chi_{\{\tau_k<\infty\}}.$$

Hence,

$$f_n = \sum_{k\in\mathbb{Z}}\sum_{j=0}^{n-1}\sum_i \mu_{k,j,i}a_n^{k,j,i} \quad \text{a.e.}, \quad n \in \mathbb{N}.$$

Since every $(s, 1, p(\cdot), \infty)$-atom is a $(s, 1, p(\cdot), r)$-atom, it follows that

$$\|f\|_{H^{\mathrm{sat},1,r}_{p(\cdot),q}} \leq \|f\|_{H^{\mathrm{sat},1,\infty}_{p(\cdot),q}} \lesssim \|f\|_{H^s_{p(\cdot),q}},$$

where the second inequality is from Theorem 3.9 in [15].

Now we prove the converse part of the theorem. Assume that f has the decomposition 7. For the case $r = \infty$, the result can be referred to Theorem 3.9 in [15]. We focus on the cases $r < \infty, q < \infty$ and $r < \infty, q = \infty$. For any $k_0 \in \mathbb{Z}$, set

$$f = \sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty}\sum_i \mu_{k,j,i}a^{k,j,i} = F_1 + F_2,$$

where

$$F_1 = \sum_{k=-\infty}^{k_0-1}\sum_{j=0}^{\infty}\sum_i \mu_{k,j,i}a^{k,j,i}, \quad F_2 = f - F_1.$$

We note that F_2 can be handled in the same way as in [15, Theorem 3.9], we can prove similarly that

$$\|F_2\|_{H^s_{p(\cdot),q}} = \|s(F_2)\|_{L_{p(\cdot),q}} \lesssim \|f\|_{H^{\mathrm{sat},1,r}_{p(\cdot),q}}, \qquad r < \infty, 0 < q \leq \infty.$$

Case 1: In this step suppose that $r < \infty$ and $q < \infty$. Firstly, we need to estimate $\|s(F_1)\|_{L_{p(\cdot),q}}$. Assume that $0 < \varepsilon < \underline{p}$. We choose $L \in (1, \frac{1}{\varepsilon})$ such that $L < r/p_+$. By Hölder's inequality for $\frac{1}{L} + \frac{1}{L'} = 1$, we have

$$
\begin{aligned}
s(F_1) &\le \sum_{k=-\infty}^{k_0-1} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} s(a^{k,j,i}) \chi_{I_{k,j,i}} \\
&\le \Big(\sum_{k=-\infty}^{k_0-1} 2^{k\ell L'} \Big)^{1/L'} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{-k\ell L} \left[\sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} s(a^{k,j,i}) \chi_{I_{k,j,i}} \right]^L \right\}^{1/L} \\
&= \Big(\frac{2^{k_0 \ell L'}}{2^{\ell L'} - 1} \Big)^{1/L'} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{-k\ell L} \left[\sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} s(a^{k,j,i}) \chi_{I_{k,j,i}} \right]^L \right\}^{1/L},
\end{aligned}
$$

where ℓ is a constant such that $0 < \ell < 1 - 1/L$. Then, by (4), we have that

$$
\begin{aligned}
\|\chi_{\{s(F_1) > 2^{k_0}\}}\|_{p(\cdot)} &\le \left\| \frac{s(F_1)^L}{2^{k_0 L}} \right\|_{p(\cdot)} \\
&\lesssim 2^{k_0 L(\ell-1)} \left\| \sum_{k=-\infty}^{k_0-1} 2^{-k\ell L} \left[\sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} s(a^{k,j,i}) \chi_{I_{k,j,i}} \right]^L \right\|_{p(\cdot)} \\
&\lesssim 2^{k_0 L(\ell-1)} \left\| \sum_{k=-\infty}^{k_0-1} 2^{(1-\ell)kL\varepsilon} \sum_{j=0}^{\infty} \sum_i \left[\|\chi_{I_{k,j,i}}\|_{p(\cdot)} s(a^{k,j,i}) \chi_{I_{k,j,i}} \right]^{L\varepsilon} \right\|_{\frac{p(\cdot)}{\varepsilon}}^{\frac{1}{\varepsilon}} \\
&\lesssim 2^{k_0 L(\ell-1)} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{(1-\ell)kL\varepsilon} \left\| \sum_{j=0}^{\infty} \sum_i \left[\|\chi_{I_{k,j,i}}\|_{p(\cdot)} s(a^{k,j,i}) \chi_{I_{k,j,i}} \right]^{L\varepsilon} \right\|_{\frac{p(\cdot)}{\varepsilon}} \right\}^{\frac{1}{\varepsilon}}.
\end{aligned}
$$

It follows from Lemma 3.6 that

$$
\begin{aligned}
\|\chi_{\{s(F_1) > 2^{k_0}\}}\|_{p(\cdot)} &\lesssim 2^{k_0 L(\ell-1)} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{(1-\ell)kL\varepsilon} \left\| \sum_{j=0}^{\infty} \sum_i \chi_{I_{k,j,i}} \right\|_{\frac{p(\cdot)}{\varepsilon}} \right\}^{\frac{1}{\varepsilon}} \\
&= 2^{k_0 L(\ell-1)} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{(1-\ell)kL\varepsilon} \left\| \sum_{j=0}^{\infty} \sum_i \chi_{I_{k,j,i}} \right\|_{p(\cdot)}^{\varepsilon} \right\}^{\frac{1}{\varepsilon}},
\end{aligned} \tag{8}
$$

where the first "=" is because that $I_{k,j,i}$ are disjoint for fixed k. To continue the estimation, we set

$$
\delta = \frac{(1-\ell)L + 1}{2} > 1.
$$

So we get $(1-\ell)L-\delta>0$. Using again Hölder's inequality for $\frac{q-\varepsilon}{q}+\frac{\varepsilon}{q}=1$, we obtain

$$\begin{aligned}\|\chi_{\{s(F_1)>2^{k_0}\}}\|_{p(\cdot)} &\le 2^{k_0L(\ell-1)}\left(\sum_{k=-\infty}^{k_0-1}2^{k((1-\ell)L-\delta)\varepsilon\frac{q}{q-\varepsilon}}\right)^{\frac{q-\varepsilon}{\varepsilon q}}\\ &\quad\times\left(\sum_{k=-\infty}^{k_0-1}2^{k\delta q}\left\|\sum_{j=0}^{\infty}\sum_i\chi_{I_{k,j,i}}\right\|_{p(\cdot)}^q\right)^{1/q}\\ &\lesssim 2^{-k_0\delta}\left(\sum_{k=-\infty}^{k_0-1}2^{k\delta q}\left\|\sum_{j=0}^{\infty}\sum_i\chi_{I_{k,j,i}}\right\|_{p(\cdot)}^q\right)^{1/q}.\end{aligned}\tag{9}$$

Consequently,

$$\begin{aligned}\sum_{k_0=-\infty}^{\infty}2^{k_0q}\|\chi_{\{s(F_1)>2^{k_0}\}}\|_{p(\cdot)}^q &\lesssim \sum_{k_0=-\infty}^{\infty}2^{k_0(1-\delta)q}\sum_{k=-\infty}^{k_0-1}2^{k\delta q}\left\|\sum_{j=0}^{\infty}\sum_i\chi_{I_{k,j,i}}\right\|_{p(\cdot)}^q\\ &=\sum_{k=-\infty}^{\infty}2^{k\delta q}\left\|\sum_{j=0}^{\infty}\sum_i\chi_{I_{k,j,i}}\right\|_{p(\cdot)}^q\sum_{k_0=k+1}^{\infty}2^{k_0(1-\delta)q}\\ &=\frac{2^{(1-\delta)q}}{1-2^{(1-\delta)q}}\sum_{k=-\infty}^{\infty}2^{kq}\left\|\sum_{j=0}^{\infty}\sum_i\chi_{I_{k,j,i}}\right\|_{p(\cdot)}^q,\end{aligned}$$

where the last "=" is because $1-\delta<0$. This implies that

$$\|F_1\|_{H^s_{p(\cdot),q}}=\|s(F_1)\|_{L_{p(\cdot),q}}\lesssim\|f\|_{H^{\mathrm{sat},1,r}_{p(\cdot),q}}.$$

Case 2: Suppose that $r<\infty$ and $q=\infty$. Using Lemma 3.6 and (8), we conclude

$$\begin{aligned}&\|\chi_{\{s(F_1)>2^{k_0}\}}\|_{p(\cdot)}\\ &\lesssim 2^{k_0L(\ell-1)}\left(\sum_{k=-\infty}^{k_0-1}2^{-k\ell L\varepsilon}2^{kL\varepsilon}2^{-k\varepsilon}2^{k\varepsilon}\left\|\sum_{j=0}^{\infty}\sum_i\chi_{I_{k,j,i}}\right\|_{p(\cdot)}^{\varepsilon}\right)^{1/\varepsilon}\\ &\le\left(\sup_{k\in\mathbb{Z}}2^k\left\|\sum_{j=0}^{\infty}\sum_i\chi_{I_{k,j,i}}\right\|_{p(\cdot)}\right)2^{k_0L(\ell-1)}\left(\sum_{k=-\infty}^{k_0-1}2^{k\varepsilon(L(1-\ell)-1)}\right)^{1/\varepsilon}\\ &\lesssim 2^{-k_0}\|f\|_{H^{\mathrm{sat},1,r}_{p(\cdot),\infty}},\end{aligned}$$

where the last inequality is because of $(1-\ell)L-1>0$. Consequently,

$$\|F_1\|_{H^s_{p(\cdot),\infty}}=\|s(F_1)\|_{L_{p(\cdot),\infty}}\lesssim\|f\|_{H^{\mathrm{sat},1,r}_{p(\cdot),\infty}}.$$

The proof is complete. □

We present the following result without proof because it is similar to the one of Theorem 3.5. Note that for the $(s, d, p(\cdot), \infty)$-atomic characterizations, we do not need to assume that $p(\cdot)$ satisfies (5).

Theorem 3.7 *Let* $p(\cdot) \in \mathcal{P}(\Omega)$ *and* $0 < q \le \infty$. *Then*

$$Q_{p(\cdot),q} = H^{\mathrm{sat},2,\infty}_{p(\cdot),q}, \quad P_{p(\cdot),q} = H^{\mathrm{sat},3,\infty}_{p(\cdot),q}$$

with equivalent quasi-norms.

If $\{\mathcal{F}_n\}_{n\ge0}$ is regular, then three kinds of simple atoms are equivalent. Then, we can get the following corollary by [15, Theorem 4.11], Theorem 3.5 and Theorem 3.7.

Corollary 3.8 *Let* $p(\cdot) \in \mathcal{P}(\Omega)$ *satisfy* (5), $0 < q \le \infty$ *and let* $\max\{p_+, 1\} < r \le \infty$. *If* $\{\mathcal{F}_n\}_{n\ge0}$ *is regular, then*

$$H^s_{p(\cdot),q} = H^S_{p(\cdot),q} = H^M_{p(\cdot),q} = Q_{p(\cdot),q} = P_{p(\cdot),q} = H^{\mathrm{sat},d,r}_{p(\cdot),q}, \quad d = 1, 2, 3$$

with equivalent quasi-norms.

4 The dual spaces of Lorentz–Hardy spaces

In this section, we study the dual spaces of Lorentz–Hardy spaces $H_{p(\cdot),q}$. We consider the problem according to the range of q.

4.1 The dual of $H_{p(\cdot),q}$, $0 < q \le 1$

Definition 4.1 Let $\alpha(\cdot) + 1 \in \mathcal{P}(\Omega)$ and $1 < r < \infty$. Define $BMO_r(\alpha(\cdot))$ as the space of functions $f \in L_r$ for which

$$\|f\|_{BMO_r(\alpha(\cdot))} = \sup_{n\ge0} \sup_{I\in A(\mathcal{F}_n)} \left\|\chi_I\right\|^{-1}_{\frac{1}{\alpha(\cdot)+1}} \left\|\chi_I\right\|_{r/(r-1)} \|(f - f_n)\chi_I\|_r$$

is finite. For $r = 1$, we define $BMO_1(\alpha(\cdot))$ with the norm

$$\|f\|_{BMO_1(\alpha(\cdot))} = \sup_{n\ge0} \sup_{I\in A(\mathcal{F}_n)} \left\|\chi_I\right\|^{-1}_{\frac{1}{\alpha(\cdot)+1}} \|(f - f_n)\chi_I\|_1.$$

Remark 4.2 If $\alpha(\cdot) = 0$, then this definition goes back to classical martingale BMO space. If $\alpha(\cdot) = \alpha_0 > 0$ is a constant, then this definition becomes the classical martingale Lipschitz space. We refer the reader to [29] for details.

Proposition 4.3 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ with $0 < p_+ \le 1$, $0 < q \le 1$ and $1 < r \le \infty$. Let $f = (f_n)_{n\ge 0} \in H^{\mathrm{sat},d,r}_{p(\cdot),q}$, $d = 1, 2, 3$. Then f has a decomposition as in (7), and moreover,*

$$\sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \lesssim \|f\|_{H^{\mathrm{sat},d,r}_{p(\cdot),q}},$$

where $\mu_{k,j,i} = 3 \cdot 2^k \|\chi_{I_{k,j,i}}\|_{p(\cdot)}$ and $(I_{k,j,i})_{k,i} \subset A(\mathcal{F}_j)$ are as in Definition 3.3.

Proof By Lemma 2.2, we have

$$\sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \lesssim \sum_{k\in\mathbb{Z}} 2^k \sum_{j=0}^{\infty} \sum_{i} \|\chi_{I_{k,j,i}}\|_{p(\cdot)} \le \sum_{k\in\mathbb{Z}} 2^k \|\sum_{j=0}^{\infty} \sum_{i} \chi_{I_{k,j,i}}\|_{p(\cdot)}$$
$$\lesssim \|f\|_{H^{\mathrm{sat},d,r}_{p(\cdot),1}} \le \|f\|_{H^{\mathrm{sat},d,r}_{p(\cdot),q}}.$$

□

Theorem 4.4 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (5), $0 < p_+ \le 1$ and $0 < q \le 1$. Then*

$$\left(H^s_{p(\cdot),q}\right)^* = BMO_2(\alpha(\cdot)), \quad \alpha(\cdot) = 1/p(\cdot) - 1.$$

Proof Let $\varphi \in BMO_2(\alpha(\cdot)) \subset L_2$. Define

$$l_\varphi(f) = \mathbb{E}(f\varphi), \quad \forall f \in L_2.$$

We claim that l_φ is a bounded linear functional on $H^s_{p(\cdot),q}$. Note that L_2 can be embedded continuously in $H^s_{p(\cdot),q}$, namely,

$$\|f\|_{H^s_{p(\cdot),q}} = \|s(f)\|_{L_{p(\cdot),q}} \lesssim \|s(f)\|_2, \quad \forall f \in L_2$$

because of [21, Theorem 3.3(i, iv)] and $0 < p_+ \le 1$. It follows from Theorem 3.5 that for each $f \in L_2$

$$f = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} a^{k,j,i}$$

and the convergence holds also in the L_2-norm, where $a^{k,j,i}$ is an $(s, 1, p(\cdot), \infty)$-atom and $\mu_{k,j,i} = 3 \cdot 2^k \|\chi_{I_{k,j,i}}\|_{p(\cdot)}$. Hence

$$l_\varphi(f) = \mathbb{E}(f\varphi) = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \mathbb{E}(a^{k,j,i}\varphi).$$

By the definition of an atom, then

$$\mathbb{E}(a^{k,j,i}\varphi) = \mathbb{E}((a^{k,j,i} - \mathbb{E}_j(a^{k,j,i}))\varphi) = \mathbb{E}(a^{k,j,i}(\varphi - \varphi_j)),$$

where $\varphi_j = \mathbb{E}_j(\varphi)$. Thus, using Hölder's inequality we conclude that

$$\begin{aligned}|l_\varphi(f)| &\le \sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty}\sum_i \mu_{k,j,i}\left|\int_\Omega a^{k,j,i}(\varphi-\varphi_j)d\mathbb{P}\right| \\ &\le \sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty}\sum_i \mu_{k,j,i}\|a^{k,j,i}\|_2\|(\varphi-\varphi_j)\chi_{I_{k,j,i}}\|_2 \\ &\le \sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty}\sum_i \mu_{k,j,i}\frac{\mathbb{P}(I_{k,j,i})^{\frac{1}{2}}}{\left\|\chi_{I_{k,j,i}}\right\|_{p(\cdot)}}\|(\varphi-\varphi_j)\chi_{I_{k,j,i}}\|_2 \\ &\lesssim \sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty}\sum_i \mu_{k,j,i}\|\varphi\|_{BMO_2(\alpha(\cdot))}.\end{aligned}$$

Since $0 < q \le 1$, we obtain from Proposition 4.3 and Theorem 3.5 that

$$|l_\varphi(f)| \lesssim \|f\|_{H^{\mathrm{sat},1,\infty}_{p(\cdot),q}}\|\varphi\|_{BMO_2(\alpha(\cdot))} \lesssim \|f\|_{H^s_{p(\cdot),q}}\|\varphi\|_{BMO_2(\alpha(\cdot))}.$$

By Remark 3.12 of [15], we know that L_2 is dense in $H^s_{p(\cdot),q}$. Consequently, l_φ can be uniquely extended to a linear functional on $H^s_{p(\cdot),q}$.

Conversely, let l be an arbitrary bounded linear functional on $H^s_{p(\cdot),q}$. We will show that there exists $\varphi \in BMO_2(\alpha(\cdot))$ such that $l = l_\varphi$ and

$$\|\varphi\|_{BMO_2(\alpha(\cdot))} \lesssim \|l\|.$$

Indeed, since L_2 can be embedded continuously to $H^s_{p(\cdot),q}$, there exists $\varphi \in L_2$ such that

$$l(f) = \mathbb{E}(f\varphi), \quad \forall f \in L_2.$$

For $I \in A(\mathcal{F}_j)$, we set

$$g = \frac{(\varphi-\varphi_j)\chi_I}{\|(\varphi-\varphi_j)\chi_I\|_2\|\chi_I\|_{\frac{1}{\alpha(\cdot)+1}}\|\chi_I\|_2^{-1}}.$$

Then the function g is a $(s, 1, p(\cdot), 2)$-atom. It follows from Theorem 3.5 that $g \in H^s_{p(\cdot),q}$ and

$$\|g\|_{H^s_{p(\cdot),q}} \lesssim \|g\|_{H^{\mathrm{sat},1,2}_{p(\cdot),q}} \lesssim 1.$$

Finally, we obtain

$$\|l\| \gtrsim l(g) = \mathbb{E}\big(g(\varphi-\varphi_j)\big) = \|\chi_I\|^{-1}_{\frac{1}{\alpha(\cdot)+1}}\|\chi_I\|_2\|(\varphi-\varphi_j)\chi_I\|_2$$

and $\|\varphi\|_{BMO_2(\alpha(\cdot))} \lesssim \|l\|$. □

4.2 The dual of $H_{p(\cdot),q}$, $1 < q < \infty$

Strongly motivated by [17, 18] and [32], in the present paper, we introduce the following generalized martingale spaces associated with variable exponents.

Definition 4.5 Let $1 \le r < \infty, 0 < q \le \infty$ and $\alpha(\cdot) + 1 \in \mathcal{P}(\Omega)$. The generalized martingale space $BMO_{r,q}(\alpha(\cdot))$ is defined by

$$BMO_{r,q}(\alpha(\cdot)) = \left\{ f \in L_r : \|f\|_{BMO_{r,q}(\alpha)} < \infty \right\},$$

where

$$\|f\|_{BMO_{r,q}(\alpha(\cdot))} = \sup \frac{\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_i 2^k \mathbb{P}(I_{k,j,i})^{1-\frac{1}{r}} \|(f - f_j)\chi_{I_{k,j,i}}\|_r}{\left(\sum_{k\in\mathbb{Z}} 2^{kq} \| \sum_{j\in\mathbb{N}} \sum_i \chi_{I_{k,j,i}} \|^q_{\frac{1}{\alpha(\cdot)+1}} \right)^{1/q}}$$

and the supremum is taken over all sequence of atoms $\{I_{k,j,i}\}_{k\in\mathbb{Z},j\in\mathbb{N},i}$ such that that $I_{k,j,i}$ are disjoint if k is fixed, $I_{k,j,i}$ belong to $\mathcal{F}_j$ and

$$\left\{ 2^k \left\| \sum_{j\in\mathbb{N}} \sum_i \chi_{I_{k,j,i}} \right\|_{\frac{1}{\alpha(\cdot)+1}} \right\}_k \in \ell_q.$$

$BMO_{r,\infty}(\alpha(\cdot))$ can be similarly defined.

First of all, $BMO_r(\alpha(\cdot))$ and $BMO_{r,q}(\alpha(\cdot))$ have the following connection.

Proposition 4.6 *Let* $1 \le r < \infty, 0 < q \le \infty$, $\alpha(\cdot) \ge 0$ *and* $\alpha(\cdot) + 1 \in \mathcal{P}(\Omega)$. *Then*

$$\|f\|_{BMO_r(\alpha(\cdot))} \le \|f\|_{BMO_{r,q}(\alpha(\cdot))}.$$

If in addition $0 < q \le 1$, *then* $BMO_r(\alpha(\cdot)) \sim BMO_{r,q}(\alpha(\cdot))$.

Proof If we take the supremum in the definition of $BMO_{r,q}(\alpha(\cdot))$ only for one atom, then we get back the $BMO_r(\alpha(\cdot))$-norm, so the first inequality is shown. On the other hand, if $0 < q \le 1$, then

$$\begin{aligned}
\|f\|_{BMO_{r,q}(\alpha(\cdot))} &\le \sup \frac{\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_i 2^k \|\chi_{I_{k,j,i}}\|_{\frac{1}{\alpha(\cdot)+1}} \|f\|_{BMO_{r,q}(\alpha(\cdot))}}{\left(\sum_{k\in\mathbb{Z}} 2^{kq} \| \sum_{j\in\mathbb{N}} \sum_i \chi_{I_{k,j,i}} \|^q_{\frac{1}{\alpha(\cdot)+1}} \right)^{1/q}} \\
&\le \sup \frac{\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_i 2^k \|\chi_{I_{k,j,i}}\|_{\frac{1}{\alpha(\cdot)+1}} \|f\|_{BMO_{r,q}(\alpha(\cdot))}}{\sum_{k\in\mathbb{Z}} 2^k \| \sum_{j\in\mathbb{N}} \sum_i \chi_{I_{k,j,i}} \|_{\frac{1}{\alpha(\cdot)+1}}} \\
&\le \|f\|_{BMO_{r,q}(\alpha(\cdot))},
\end{aligned}$$

because of Lemma 2.2. □

Theorem 4.7 *Let* $p(\cdot) \in \mathcal{P}(\Omega)$ *satisfy* (5), $0 < p_+ < 2$ *and* $1 < q < \infty$. *Then we have*

$$\left(H^s_{p(\cdot),q}\right)^* = BMO_{2,q}(\alpha(\cdot)), \quad \alpha(\cdot) = \frac{1}{p(\cdot)} - 1,$$

with equivalent norms.

Proof Let $\varphi \in BMO_{2,q}(\alpha(\cdot)) \subset L_2$. We define the functional as

$$l_\varphi(f) = \mathbb{E}(fg), \quad \forall f \in L_2.$$

Using Theorem 3.5 and similar argument used in Theorem 4.4, we have

$$\begin{aligned} |l_\varphi(f)| &\le \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \left| \int_\Omega a^{k,j,i}(\varphi - \varphi_j) d\mathbb{P} \right| \\ &\lesssim \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i 2^k \mathbb{P}(I_{k,j,i})^{\frac{1}{2}} \|(\varphi - \varphi_j)\chi_{I_{k,j,i}}\|_2. \end{aligned}$$

It follows from the definition of $\|\cdot\|_{BMO_{2,q}(\alpha(\cdot))}$ and Theorem 3.5 that

$$\begin{aligned} |l_\varphi(f)| &\lesssim \left(\sum_{k\in\mathbb{Z}} 2^{kq} \left\| \sum_{j\in\mathbb{N}} \sum_i \chi_{I_{k,j,i}} \right\|_{p(\cdot)}^q \right)^{1/q} \|g\|_{BMO_{2,q}(\alpha(\cdot))} \\ &\lesssim \|f\|_{H^s_{p(\cdot),q}} \|g\|_{BMO_{2,q}(\alpha(\cdot))}. \end{aligned}$$

Since L_2 is dense in $H^s_{p(\cdot),q}$ (see [15, Remark 3.12]), the functional l_g can be uniquely extended to a continuous functional on $H^s_{p(\cdot),q}$.

Conversely, let $l \in (H^s_{p(\cdot),q})^*$. Since $L_2 \subset H^s_{p(\cdot),q}$, there exists $\varphi \in L_2$ such that

$$l(f) = \mathbb{E}(f\varphi) \quad \forall f \in L_2.$$

Let $\{I_{k,j,i}\}_{k\in\mathbb{Z},j\in\mathbb{N},i}$ be an arbitrary sequence of atoms such that $I_{k,j,i}$ are disjoint if k is fixed, $I_{k,j,i}$ belong to $\mathcal{F}_j$ and

$$\left\{ 2^k \left\| \sum_{j\in\mathbb{N}} \sum_i \chi_{I_{k,j,i}} \right\|_{\frac{1}{\alpha(\cdot)+1}} \right\}_k \in \ell_q.$$

We set

$$h_{k,j,i} = \frac{(\varphi - \varphi_j)\chi_{I_{k,j,i}} \|\chi_{I_{k,j,i}}\|_2}{\|(\varphi - \varphi_j)\chi_{I_{k,j,i}}\|_2 \|\chi_{I_{k,j,i}}\|_{p(\cdot)}}.$$

It is obvious that $h_{k,j,i}$ is a $(s, 1, p(\cdot), 2)$-atom. By Theorem 3.5, we find that

$$f = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_{i} 2^k \|\chi_{I_{k,j,i}}\|_{p(\cdot)} h_{k,j,i} \in H^s_{p(\cdot),q},$$

and

$$\|f\|_{H^s_{p(\cdot),q}} \lesssim \left(\sum_{k\in\mathbb{Z}} 2^{kq} \left\| \sum_{j\in\mathbb{N}} \sum_{i} \chi_{I_{k,j,i}} \right\|^q_{p(\cdot)} \right)^{\frac{1}{q}}. \tag{10}$$

Now we have the following estimate:

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_{i} 2^k \mathbb{P}(I_{k,j,i})^{\frac{1}{2}} \|(\varphi - \varphi_j)\chi_{I_{k,j,i}}\|_2 \\ &\quad = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_{i} 2^k \|\chi_{I_{k,j,i}}\|_{p(\cdot)} \mathbb{E}(h_{k,j,i}(\varphi - \varphi_j)) \\ &\quad = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_{i} 2^k \|\chi_{I_{k,j,i}}\|_{p(\cdot)} \mathbb{E}(h_{k,j,i}\varphi) \\ &\quad = \mathbb{E}(f\varphi) = l(f) \le \|f\|_{H^s_{p(\cdot),q}} \|l\|. \end{aligned}$$

Thus, applying (10) and the definition of $\|\cdot\|_{BMO_{2,q}(\alpha(\cdot))}$ we obtain

$$\|\varphi\|_{BMO_{2,q}(\alpha(\cdot))} \lesssim \|l\|.$$

The proof is complete. □

4.3 The case $q = \infty$

This case is different from the case $q < \infty$ due to the well known fact that L_p is not dense in $L_{p,\infty}$ $(0 < p < \infty)$. We refer to [31, p. 143] or [11, Remark 1.4.14] for this fact. In order to describe the duality, we define

$$\mathscr{H}^s_{p(\cdot),\infty} = \{f = (f_n)_{n\ge 0} : s(f) \in \mathscr{L}_{p(\cdot),\infty}\}.$$

It is not hard to check that $\mathscr{H}^s_{p(\cdot),\infty}$ is a closed subspace of $H^s_{p(\cdot),\infty}$. Similarly, we can define $\mathscr{H}^M_{p(\cdot),\infty}$ and $\mathscr{H}^S_{p(\cdot),\infty}$ which are closed subspaces of $H^M_{p(\cdot),\infty}$ and $H^S_{p(\cdot),\infty}$, respectively.

According to [15, Remark 3.12], we know that L_2 is dense in $\mathscr{H}^s_{p(\cdot),\infty}$. On the lines of the proof of Theorem 4.7, we can get the result below by using Theorem 3.5. We omit the proof.

Theorem 4.8 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (5) and $0 < p_+ < 2$. Then*

$$\left(\mathscr{H}_{p(\cdot),\infty}\right)^* = BMO_{2,\infty}(\alpha(\cdot)), \quad \alpha(\cdot) = \frac{1}{p(\cdot)} - 1$$

with equivalent norms.

Remark 4.9 The dual space of weak Hardy space was first studied in harmonic analysis, see [8]. In martingale setting, we refer the reader to [31].

Besides Proposition 4.3, one of the key points of the proofs of Theorems 4.4, 4.7 and 4.8 is the fact that L_2 can be embedded continuously in $H^s_{p(\cdot),q}$. So we cannot expect to characterize the dual spaces in a similar way for a wider range of p_+. It is an unknown question, how we can characterize the duals for other p_+.

5 John–Nirenberg theorems

In this section, we investigate John–Nirenberg theorems. We divide this section into two subsections.

5.1 Atomic decomposition for *E*-atoms

In this subsection, we give the atomic decomposition for $H_{p(\cdot),q}$ by using $(s, 1, p(\cdot), E)$ -atoms, where E is a rearrangement invariant Banach function space. Let $E(\Omega)$ be a rearrangement invariant Banach function space over $(\Omega, \mathcal{F}, \mathbb{P})$. We refer to [2, Chapters 1 and 2] for the definitions of Banach function spaces and rearrangement invariant Banach function spaces. In this section, we always suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic.

Let E be a rearrangement invariant Banach function space over Ω. According to the Luxemburg representation theorem (see for instance [2, Page 62]), there exists a rearrangement invariant $\widehat{E}$ over (0, 1) equipped with the norm $\|\cdot\|_{\widehat{E}}$ such that

$$\|f\|_E = \|\mu(\cdot,f)\|_{\widehat{E}}, \quad \forall f \in E, \tag{11}$$

where $\mu(\cdot,f)$ is the non-increasing rearrangement function of f defined by

$$\mu(t,f) = \inf_{s>0}\{s : \mathbb{P}(|f| > s) \le t\}, \quad t > 0.$$

We call $(\widehat{E}, \|\cdot\|_{\widehat{E}})$ the Luxemburg representation space of $(E, \|\cdot\|_E)$.

We need the Boyd indices of E introduced by Boyd [3]. Define the dilation operator D_s $(0 < s < \infty)$ acting on the space of measurable functions on (0, 1) by $D_sf(t) = f(t/s)$, if $0 < t < \min(1, s)$; $D_sf(t) = 0$, if $s < t < 1$. Let $(\widehat{E}, \|\cdot\|_{\widehat{E}})$ be the the Luxemburg representation space of E. The upper Boyd index and the lower Boyd index of E are respectively defined by

$$q_E := \inf_{s>1} \frac{\log s}{\log \|D_s\|}$$

and

$$p_E := \sup_{0<s<1} \frac{\log s}{\log \|D_s\|},$$

where $\|D_s\|$ is the operator norm on $\widehat{E}$. Note that for any rearrangement invariant Banach function space E,

$$1 \le p_E \le q_E \le \infty.$$

The associate space E' of E is defined by

$$E' = \{f : \|f\|_{E'} < \infty, \}$$

where

$$\|f\|_{E'} = \sup_{g \in E, \|g\|_E \le 1} \int_\Omega |fg| \, d\mathbb{P}.$$

A rearrangement invariant Banach function E has a Fatou norm if and only if E embeds isometrically into its second Köthe dual $E'' = (E')'$. We shall need the following duality for Boyd indices (see [22, Theorem II.4.11]). If E is a rearrangement invariant Banach function space with Fatou norm, then

$$\frac{1}{p_E} + \frac{1}{q_{E'}} = 1, \quad \frac{1}{p_{E'}} + \frac{1}{q_E} = 1. \tag{12}$$

Note that the spaces L_p $(1 \le p \le \infty)$ are rearrangement invariant Banach function spaces with Fatou norms.

We also need some basic lemmas which can be found in [2].

Lemma 5.1 *Let E be a Banach function space with associated space E'. If $f \in E$ and $g \in E'$, then fg is integrable and*

$$\left| \int_\Omega fg d\mathbb{P} \right| \le \|f\|_E \|g\|_{E'}.$$

Note that we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic in this section. The following result is referred to Theorems 5.2 and 2.7 in [2, Chapter 2].

Lemma 5.2 *Let E be a rearrangement invariant space, and E' be its associated space. Then, for all set $B \in \mathcal{F}$, we have*

$$\|\chi_B\|_1 = \|\chi_B\|_E \|\chi_B\|_{E'}.$$

The following Doob maximal inequality in rearrangement invariant Banach function space was studied in [27]. Here we give a simple proof.

Lemma 5.3 *If* $1 < p_E \le q_E \le \infty$, *then*

$$\|M(f)\|_E \le C\|f\|_E.$$

Proof As proved in [23, Theorem 3.6.3] that

$$\mu(t, M(f)) \le \frac{1}{t}\int_0^t \mu(s,f)ds.$$

It follows from $1 < p_E \le q_E \le \infty$ and Theorem 5.15 in [2, Chapter 3] that,

$$\|\frac{1}{t}\int_0^t \mu(s,f)ds\|_{\widehat{E}} \le C_E\|\mu(\cdot,f)\|_{\widehat{E}},$$

where $\widehat{E}$ is as in (11). By (11), we have

$$\|M(f)\|_E = \|\mu(\cdot, Mf)\|_{\widehat{E}} \le \|\frac{1}{t}\int_0^t \mu(s,f)ds\|_{\widehat{E}} \le C_E\|f\|_E.$$

□

We introduce the definition of $(s, 1, p(\cdot), E)$-atoms.

Definition 5.4 Let $p(\cdot) \in \mathcal{P}(\Omega)$, and let $(E, \|\cdot\|_E)$ be a rearrangement invariant Banach function space. Replacing (2) in Definition 3.1 by

$$\|s(a)\|_E \le \frac{\|\chi_I\|_E}{\|\chi_I\|_{p(\cdot)}},$$

we get the definition of $(s, 1, p(\cdot), E)$-atoms.

The following lemma plays a similar role as Lemma 3.6.

Lemma 5.5 *Let* $p(\cdot) \in \mathcal{P}(\Omega)$ *satisfy* (5), $p_+ < 1$ *and* E *be a rearrangement invariant Banach function space. Take* $0 < \varepsilon < \underline{p}$ *and* $L \in (1, \frac{1}{p_+} \wedge \frac{1}{\varepsilon})$. *If* $a^{k,j,i}$ *is a* $(s, 1, p(\cdot), E)$-*atom for every* k, j, i *associated with* $I_{k,j,i} \in A(\mathcal{F}_j)$, *then we have*

$$Z := \left\| \sum_{j=0}^{\infty}\sum_i \left[\|\chi_{I_{k,j,i}}\|_{p(\cdot)} s(a^{k,j,i})\chi_{I_{k,j,i}}\right]^{L\varepsilon} \right\|_{p(\cdot)/\varepsilon} \lesssim \left\| \sum_{j=0}^{\infty}\sum_i \chi_{I_{k,j,i}} \right\|_{p(\cdot)/\varepsilon}.$$

Proof According to the duality $(L_{\frac{p(\cdot)}{\varepsilon}})^* = L_{(\frac{p(\cdot)}{\varepsilon})'}$ (see e.g. [5, Theorem 2.80]), we hoose a positive function $g \in L_{(\frac{p(\cdot)}{\varepsilon})'}$ with $\|g\|_{(\frac{p(\cdot)}{\varepsilon})'} \le 1$ such that

$$Z = \int_\Omega \sum_{j=0}^{\infty}\sum_i \left[\|\chi_{I_{k,j,i}}\|_{p(\cdot)} s(a^{k,j,i})\chi_{I_{k,j,i}}\right]^{L\varepsilon} g d\mathbb{P}.$$

Note that the support of $s(a^{k,j,i})$ is $I_{k,j,i}$ (see Proposition 3.2). Then, applying Lemmas 5.1 and 5.2, we have

$$\begin{aligned}\|s(a^{k,j,i})\|_1 &= \|s(a^{k,j,i})\chi_{I_{k,j,i}}\|_1 \le \|s(a^{k,j,i})\|_E \|\chi_{I_{k,j,i}}\|_{E'} \\ &\le \frac{\|\chi_{I_{k,j,i}}\|_E}{\|\chi_{I_{k,j,i}}\|_{p(\cdot)}} \|\chi_{I_{k,j,i}}\|_{E'} = \frac{\|\chi_{I_{k,j,i}}\|_1}{\|\chi_{I_{k,j,i}}\|_{p(\cdot)}}.\end{aligned}$$

Hence, by Hölder's inequality, we obtain

$$\begin{aligned}Z &\le \sum_{j=0}^{\infty}\sum_i \|\chi_{I_{k,j,i}}\|_{p(\cdot)}^{L\varepsilon} \|s(a^{k,j,i})^{L\varepsilon}\|_{\frac{1}{L\varepsilon}} \|\chi_{I_{k,j,i}} g\|_{(\frac{1}{L\varepsilon})'} \\ &= \sum_{j=0}^{\infty}\sum_i \|\chi_{I_{k,j,i}}\|_{p(\cdot)}^{L\varepsilon} \|s(a^{k,j,i})\|_1^{L\varepsilon} \|\chi_{I_{k,j,i}} g\|_{(\frac{1}{L\varepsilon})'} \\ &\le \sum_{j=0}^{\infty}\sum_i \|\chi_{I_{k,j,i}}\|_1^{L\varepsilon} \left(\int_{I_{k,j,i}} g^{(\frac{1}{L\varepsilon})'}\right)^{1/(\frac{1}{L\varepsilon})'}.\end{aligned}$$

Thus, we find

$$\begin{aligned}Z &= \sum_{j=0}^{\infty}\sum_i \int_\Omega \chi_{I_{k,j,i}} d\mathbb{P} \left(\frac{1}{\mathbb{P}(I_{k,j,i})}\int_{I_{k,j,i}} g^{(\frac{1}{L\varepsilon})'}\right)^{1/(\frac{1}{L\varepsilon})'} \\ &\le \sum_{j=0}^{\infty}\sum_i \int_\Omega \chi_{I_{k,j,i}} [M(g^{(\frac{1}{L\varepsilon})'})]^{1/(\frac{1}{L\varepsilon})'} d\mathbb{P} \\ &\le \left\|\sum_{j=0}^{\infty}\sum_i \chi_{I_{k,j,i}}\right\|_{p(\cdot)/\varepsilon} \|[M(g^{(\frac{1}{L\varepsilon})'})]^{1/(\frac{1}{L\varepsilon})'}\|_{(p(\cdot)/\varepsilon)'} \\ &\le \left\|\sum_{j=0}^{\infty}\sum_i \chi_{I_{k,j,i}}\right\|_{p(\cdot)/\varepsilon} \|g\|_{(p(\cdot)/\varepsilon)'} \le \left\|\sum_{j=0}^{\infty}\sum_i \chi_{I_{k,j,i}}\right\|_{p(\cdot)/\varepsilon},\end{aligned}$$

which completes the proof. □

Applying the above lemma, we improve Theorem 3.5 to the result below. The proof is omitted.

Theorem 5.6 *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (5) with $0 < p_+ < 1$, $0 < q \le \infty$, and let E be a rearrangement invariant Banach function space. Then*

$$H^s_{p(\cdot),q} = H^{\mathrm{sat},1,E}_{p(\cdot),q}$$

with equivalent quasi-norms.

Remark 5.7 Let $(\mathcal{F}_n)_{n\geq 0}$ be regular. According to Corollary 3.8, $H^s_{p(\cdot),q} = H^M_{p(\cdot),q}$. We also can prove $(s, 1, p(\cdot), E)$-atomic decomposition for $H^M_{p(\cdot),q}$ when $p(\cdot) \in \mathcal{P}(\Omega)$ satisfying condition (5) and $p_+ < 1$.

We refer the reader to [10, Theorem 4.10] for the fact that $H_1(\mathbb{R}^n)$ does not have such atomic decomposition when $E = L_1$ in classical harmonic analysis.

5.2 $BMO_E(\alpha(\cdot))$ spaces with variable exponent

We first present the definition of $BMO_E(\alpha(\cdot))$.

Definition 5.8 Let $\alpha(\cdot) + 1 \in \mathcal{P}(\Omega)$ and E be a Banach function space with associate space E'. Define $BMO_E(\alpha(\cdot))$ as the space of functions $f \in E$ for which

$$\|f\|_{BMO_E(\alpha(\cdot))} = \sup_{n\geq 0} \sup_{I\in A(\mathcal{F}_n)} \left\|\chi_I\right\|^{-1}_{\frac{1}{\alpha(\cdot)+1}} \left\|\chi_I\right\|_{E'} \|(f - f_n)\chi_I\|_E$$

is finite.

Lemma 5.9 *Let* $p(\cdot) \in \mathcal{P}(\Omega)$ *satisfy* (5), $0 < p_+ < 1$ *and* $0 < q \leq 1$. *Let* E *be a rearrangement invariant space with Fatou norm such that* $1 \leq p_E \leq q_E < \infty$. *If* $\{\mathcal{F}_n\}_{n\geq 0}$ *is regular, then*

$$(H^M_{p(\cdot),q})^* = BMO_E(\alpha(\cdot)), \quad \alpha(\cdot) = \frac{1}{p(\cdot)} - 1$$

with equivalent norms.

Proof Let $\varphi \in BMO_E(\alpha(\cdot))$. It follows from Lemma 5.1 that

$$\|\varphi\|_{BMO_1(\alpha(\cdot))} \leq \|\varphi\|_{BMO_E(\alpha(\cdot))}.$$

Then $\varphi \in BMO_1(\alpha(\cdot)) \subset L_1$. Define the functional as

$$l_\varphi(f) = \mathbb{E}(f\varphi), \quad \forall f \in L_\infty.$$

Similar to the proof of Theorem 4.4, one can easily apply Corollary 3.8 and 4.3 to get

$$|l_\varphi(f)| \lesssim \|\varphi\|_{BMO_1(\alpha(\cdot))}\|f\|_{H^M_{p(\cdot),q}}.$$

On the other hand, since L_∞ is dense in $H^M_{p(\cdot),q}$, l_φ can be uniquely extended to a continuous functional on $H^M_{p(\cdot),q}$.

Conversely, let $l \in (H^M_{p(\cdot),q})^*$. Since $L_2 \subset H^M_{p(\cdot),q}$, there exists $\varphi \in L_2$ such that

$$l(f) = \mathbb{E}(f\varphi), \qquad f \in L_2.$$

We still need to show $\varphi \in BMO_E(\alpha(\cdot))$. It follows from Theorem 4.4 that $\|\varphi\|_{BMO_2(\alpha(\cdot))} \le \|l\|$. Hence, to show $\varphi \in BMO_E(\alpha(\cdot))$, it suffices to prove $\|\varphi\|_{BMO_E(\alpha(\cdot))} \lesssim \|\varphi\|_{BMO_2(\alpha(\cdot))}$.

By duality, for each n and $I \in A(\mathcal{F})$, there exists $h \in E'$ with $\|h\|_{E'} \le 1$ such that

$$\|(\varphi - \varphi_n)\chi_I\|_E \le 2\left|\int_I (\varphi - \varphi_n) h d\mathbb{P}\right|.$$

Define

$$a = \frac{\|\chi_I\|_{E'}(h - h_n)\chi_I}{2c_0\|\chi_I\|_{p(\cdot)}}, \quad \alpha(\cdot) = \frac{1}{p(\cdot)} - 1,$$

where c_0 is the constant in the Doob maximal inequality given in Lemma 5.3. According to (12), it is obvious that $1 < p_{E'} \le q_{E'} \le \infty$. Then, by Lemma 5.3, we have

$$\|M(a)\|_{E'} \le c_0\|a\|_{E'} \le \frac{\|\chi_I\|_{E'}}{\|\chi_I\|_{p(\cdot)}}.$$

So a is an $(s, 3, p(\cdot), E')$-atom. Thus, by Theorems 5.6 and Corollary 3.8,

$$(h - h_n)\chi_I = \frac{2c_0\|\chi_I\|_{p(\cdot)}}{\|\chi_I\|_{E'}} a \in H^M_{p(\cdot),q}$$

with

$$\|(h - h_n)\chi_I\|_{H^M_{p(\cdot),q}} \le \frac{2c_0\|\chi_I\|_{p(\cdot)}}{\|\chi_I\|_{E'}}.$$

Since $(\mathcal{F}_n)_{n\ge 0}$ is regular, we have

$$\begin{aligned}
\frac{\|\chi_I\|_{E'}\|(\varphi - \varphi_n)\chi_I\|_E}{\|\chi_I\|_{\frac{1}{\alpha(\cdot)+1}}} &\le \frac{2\|\chi_I\|_{E'}\left|\int_I (\varphi - \varphi_n) h d\mathbb{P}\right|}{\|\chi_I\|_{\frac{1}{\alpha(\cdot)+1}}} \\
&= \frac{2\|\chi_I\|_{E'}\left|\int_I \varphi (h - h_n) d\mathbb{P}\right|}{\|\chi_I\|_{\frac{1}{\alpha(\cdot)+1}}} \\
&\le \frac{2\|\chi_I\|_{E'}\|\varphi\|_{BMO_2(\alpha(\cdot))}\|(h - h_n)\chi_I\|_{H^M_{p(\cdot),q}}}{\|\chi_I\|_{\frac{1}{\alpha(\cdot)+1}}} \\
&\le 2c_0\|\varphi\|_{BMO_2(\alpha(\cdot))},
\end{aligned}$$

where the second "$\le$" is due to $(H^M_{p(\cdot),q})^* = (H^s_{p(\cdot),q})^* = BMO_2(\alpha(\cdot))$ (by Corollary 3.8). Consequently, we obtain

$$\|f\|_{BMO_E(\alpha(\cdot))} \lesssim \|f\|_{BMO_2(\alpha(\cdot))},$$

which completes the proof. □

As a consequence of the above result, we have the following John–Nirenberg inequality.

Theorem 5.10 *Let* $\alpha(\cdot)+1\in\mathcal{P}(\Omega)$ *satisfy* (5) *and* $0<\alpha_-\le\alpha_+<\infty$. *Let* E *be a rearrangement invariant Banach function space with Fatou norm such that* $1\le p_E\le q_E<\infty$. *If* $(\mathcal{F}_n)_{n\ge0}$ *is regular, then*

$$BMO_E(\alpha(\cdot))=BMO_1(\alpha(\cdot))$$

with equivalent norms.

5.3 $BMO_{E,q}(\alpha(\cdot))$ spaces with variable exponent

Definition 5.11 Let E be a rearrangement invariant Banach function space, and let $0<q\le\infty$ and $\alpha(\cdot)+1\in\mathcal{P}(\Omega)$. The generalized martingale space $BMO_{E,q}(\alpha(\cdot))$ is defined by

$$BMO_{E,q}(\alpha(\cdot))=\left\{f\in E:\|f\|_{BMO_{E,q}(\alpha)}<\infty\right\},$$

where

$$\|f\|_{BMO_{E,q}(\alpha(\cdot))}=\sup\frac{\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\sum_i 2^k\|\chi_{I_{k,j,i}}\|_{E'}\|(f-f_j)\chi_{I_{k,j,i}}\|_E}{\left(\sum_{k\in\mathbb{Z}}2^{kq}\|\sum_{j\in\mathbb{N}}\sum_i\chi_{I_{k,j,i}}\|^q_{\frac{1}{\alpha(\cdot)+1}}\right)^{1/q}}$$

and the supremum is taken over all sequence of atoms $\{I_{k,j,i}\}_{k\in\mathbb{Z},j\in\mathbb{N},i}$ such that that $I_{k,j,i}$ are disjoint if k is fixed, $I_{k,j,i}$ belong to $\mathcal{F}_j$ and

$$\left\{2^k\left\|\sum_{j\in\mathbb{N}}\sum_i\chi_{I_{k,j,i}}\right\|_{\frac{1}{\alpha(\cdot)+1}}\right\}_k\in\ell_q.$$

$BMO_{r,\infty}(\alpha(\cdot))$ can be similarly defined.

Similarly to Proposition 4.6, we can show the following result.

Proposition 5.12 *Let* E *be a rearrangement invariant Banach function space*, $0<q\le\infty$, $\alpha(\cdot)\ge0$ *and* $\alpha(\cdot)+1\in\mathcal{P}(\Omega)$. *Then*

$$\|f\|_{BMO_E(\alpha(\cdot))}\le\|f\|_{BMO_{E,q}(\alpha(\cdot))}.$$

If in addition $0<q\le1$, *then* $BMO_E(\alpha(\cdot))\sim BMO_{E,q}(\alpha(\cdot))$.

We establish the following lemma.

Lemma 5.13 *Let* $p(\cdot) \in \mathcal{P}(\Omega)$ *satisfy* (5), $0 < p_+ < 1$ *and* $1 < q < \infty$. *Let* E *be a rearrangement Banach function space with Fatou norm such that* $1 \le p_E \le q_E < \infty$. *If* $\{\mathcal{F}_n\}_{n\ge 0}$ *is regular, then*

$$(H^M_{p(\cdot),q})^* = BMO_{E,q}(\alpha(\cdot)), \quad \alpha(\cdot) = \frac{1}{p(\cdot)} - 1,$$

with equivalent norms.

Proof It follows from Lemma 5.1 that $\|g\|_{BMO_{1,q}(\alpha(\cdot))} \le \|g\|_{BMO_{E,q}(\alpha(\cdot))}$ for every $g \in BMO_{E,q}(\alpha(\cdot))$. Let $\varphi \in BMO_{E,q}(\alpha(\cdot))$. Then $\varphi \in BMO_{1,q}(\alpha(\cdot)) \subset L_1$ We define the functional as

$$l_\varphi(f) = \mathbb{E}(f\varphi), \quad \forall f \in L_\infty.$$

It follows from the inclusion $L_\infty \subset H^M_{p(\cdot),q}$ and Corollary 3.8 that

$$f = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} a^{k,j,i} \quad \forall f \in L_\infty$$

with $\mu_{k,j,i} = 3 \cdot 2^k \|\chi_{I_{l,j,i}}\|_{p(\cdot)}$ and $a^{k,j,i}$'s are $(s, 3, p(\cdot), \infty)$-atoms associated with $(I_{k,j,i})_{k,i} \subset A(\mathcal{F}_j)$. By the Definition 3.1(3), $\mathbb{E}(a^{k,j,i}\varphi) = \mathbb{E}(a^{k,j,i}(\varphi - \varphi_j))$ always holds for every k, j, i, where $\varphi_j = \mathbb{E}_j(\varphi)$. Thus, we find that

$$\begin{aligned} |l_\varphi(f)| &\le \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \left| \int_\Omega a^{k,j,i}(\varphi - \varphi_j) d\mathbb{P} \right| \\ &\le \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \|a^{k,j,i}\|_\infty \|(\varphi - \varphi_j)\chi_{I_{k,j,i}}\|_1 \\ &\lesssim \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i 2^k \|(\varphi - \varphi_j)\chi_{I_{k,j,i}}\|_1. \end{aligned}$$

It follows from the definition of $\|\cdot\|_{BMO_{1,q}(\alpha(\cdot))}$ and Corollary 3.8 that

$$\begin{aligned} |l_\varphi(f)| &\lesssim \left(\sum_{k\in\mathbb{Z}} 2^{kq} \left\| \sum_{j\in\mathbb{N}} \sum_i \chi_{I_{k,j,i}} \right\|_{p(\cdot)}^q \right)^{1/q} \|\varphi\|_{BMO_{1,q}(\alpha(\cdot))} \\ &\lesssim \|f\|_{H^M_{p(\cdot),q}} \|\varphi\|_{BMO_{E,q}(\alpha(\cdot))}. \end{aligned}$$

Since L_∞ is dense in $H^M_{p(\cdot),q}$ (see [15, Remark 3.12]), the functional l_g can be uniquely extended to a continuous functional on $H^M_{p(\cdot),q}$.

Conversely, suppose that $l \in (H^M_{p(\cdot),q})^*$. Since $L_2 \subset H^M_{p(\cdot),q}$, there exists $\varphi \in L_2 \subset L_1$ such that

$$l(f) = \mathbb{E}(f\varphi), \qquad f \in L_2.$$

We still need to show $\varphi \in BMO_{E,q}(\alpha(\cdot))$. According to Theorem 4.7, $\|\varphi\|_{BMO_{2,q}(\alpha(\cdot))} \lesssim \|l\|$. Hence, it is sufficient to show $\|\varphi\|_{BMO_{E,q}(\alpha(\cdot))} \lesssim \|\varphi\|_{BMO_{2,q}(\alpha(\cdot))}$.

Let $\{I_{k,j,i}\}_{k\in\mathbb{Z},j\in\mathbb{N},i}$ be an arbitrary atom sequence such that that $I_{k,j,i}$ are disjoint if k is fixed, $I_{k,j,i}$ belong to $\mathcal{F}_j$ if k, j are fixed, and

$$\left\{2^k \left\| \sum_{j\in\mathbb{N}} \sum_i \chi_{I_{k,j,i}} \right\|_{p(\cdot)} \right\}_k \in \ell_q.$$

We shall estimate the following term

$$\mathrm{A} := \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_i 2^k \|\chi_{I_{k,j,i}}\|_{E'} \|(g-g_j)\chi_{I_{k,j,i}}\|_E.$$

By duality, for every k, j, i, we can find $h^{k,j,i} \in E'$ such that

$$\begin{aligned} \|(g-g_j)\chi_{I_{k,j,i}}\|_E &\le 2\int (g-g_j)\chi_{I_{k,j,i}} h^{k,j,i} d\mathbb{P} \\ &= 2\int (h^{k,j,i} - \mathbb{E}_j(h^{k,j,i}))\chi_{I_{k,j,i}} g d\mathbb{P} \end{aligned}$$

Similar to the proof of Theorem 5.10, define

$$a^{k,j,i} = \frac{\|\chi_{I_{k,j,i}}\|_{E'} (h^{k,j,i} - \mathbb{E}_j(h^{k,j,i}))\chi_{I_{k,j,i}}}{2c_0 \|\chi_{I_{k,j,i}}\|_{p(\cdot)}}, \quad \alpha(\cdot) = \frac{1}{p(\cdot)} - 1,$$

where c_0 is the constant in the Doob maximal inequality in Lemma 5.3. Then each $a^{k,j,i}$ is an $(s, 3, p(\cdot), E)$-atom. By Corollary 3.8, we find that

$$f = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_i 2^k \|\chi_{I_{k,j,i}}\|_{p(\cdot)} a^{k,j,i} \in H^M_{p(\cdot),q}$$

and

$$\|f\|_{H^M_{p(\cdot),q}} \lesssim \left(\sum_{k\in\mathbb{Z}} 2^{kq} \left\| \sum_{j\in\mathbb{N}} \sum_i \chi_{I_{k,j,i}} \right\|^q_{p(\cdot)} \right)^{\frac{1}{q}}.$$

Thus, combining the above argument and Theorem 4.7, we have

$$\begin{aligned} \mathrm{A} &:= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_i 2^k \|\chi_{I_{k,j,i}}\|_{E'} \|(g-g_j)\chi_{I_{k,j,i}}\|_E \\ &\le 2 \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \sum_i 2^k \|\chi_{I_{k,j,i}}\|_{E'} \int (h^{k,j,i} - \mathbb{E}_j(h^{k,j,i}))\chi_{I_{k,j,i}} g d\mathbb{P} \\ &\lesssim \|g\|_{BMO_{2,q}(\alpha(\cdot))} \|f\|_{H^M_{p(\cdot),q}}, \end{aligned}$$

which implies that

$$\|\varphi\|_{BMO_{E,q}(\alpha(\cdot))} \lesssim \|\varphi\|_{BMO_{2,q}(\alpha(\cdot))}$$

The proof is complete. □

Similar to Lemma 5.13, one also can prove that

$$(\mathscr{H}^{M}_{p(\cdot),\infty})^{*} = BMO_{E,\infty}(\alpha(\cdot)).$$

Then the following John-Nirenberg theorem is an immediate consequence of the combination of Lemma 5.13.

Theorem 5.14 *Let $\alpha(\cdot)+1 \in \mathcal{P}(\Omega)$ satisfy (5), $1 < q \leq \infty$ and $0 < \alpha_{-} \leq \alpha_{+} < \infty$. Let E be a rearrangement invariant space with Fatou norm such that $1 \leq p_E \leq q_E < \infty$. If $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then*

$$BMO_{E,q}(\alpha(\cdot)) = BMO_{2,q}(\alpha(\cdot))$$

with equivalent norms.

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