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ORIGINAL PAPER

Ranks of commutators for a class of truncated Toeplitz operators

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Abstract
We consider truncated Toeplitz operators acting on infinite dimensional model spaces. We then describe the kernels and ranks of commutators of truncated Toeplitz operators with symbols induced by certain inner functions. Our results generalize recent results of Chen et al. [Oper Matrices (to appear)] to infinite dimensional model spaces.

Keywords Truncated Toeplitz operator · Model space · Rank

Mathematics Subject Classification 47B35 · 32A37

1 Introduction

Let $\mathbb{D}$ be the unit disk in the complex place and $\mathbb{T}$ be the unit circle. We let H^2 be the classical Hardy space on $\mathbb{D}$ which can be identified with a closed subspace of L^2. Here, $L^p := L^p(\mathbb{T}, \sigma)$ denotes the usual Lebesgue space on $\mathbb{T}$ where σ is the normalized Lebesgue measure on $\mathbb{T}$. A function $\theta \in H^2$ is said to be *inner* if $|\theta| = 1$ a.e. on $\mathbb{T}$. To each non-constant inner function θ, we associate the model space $\mathcal{H}_\theta$ defined by

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$$\mathcal{H}_\theta = H^2 \ominus \theta H^2$$

which is a nontrivial invariant subspace for the backward shift operator on H^2. Let P_θ be the Hilbert space orthogonal projection from L^2 to $\mathcal{H}_\theta$. Given a function $\varphi \in L^\infty$, the *truncated Toeplitz operator* (briefly, TTO) A_φ with symbol φ is defined on $\mathcal{H}_\theta$ by

$$A_\varphi f = P_\theta(\varphi f)$$

for functions $f \in \mathcal{H}_\theta$. Then A_φ is a bounded linear operator on $\mathcal{H}_\theta$ and $A_\varphi^* = A_{\overline{\varphi}}$; see [9] for details and related facts.

In a recent paper [4], the rank of a commutator of two TTOs has been studied and it has been shown that for any $\varphi, \psi \in L^\infty$ and inner θ, the rank of $[A_\varphi, A_\psi]$ must be even on $\mathcal{H}_\theta$ if it has finite rank. Here, $[S, T] := ST - TS$ denotes the commutator of two bounded linear operators S and T on a Hilbert space. Conversely, it has been also studied that whether there is a commutator of two TTOs with a given even rank exactly on a model space. At the same paper, it was proved that this is true on model spaces corresponding to monomials by showing the self-commutator $[A_{z^N}, A_{z^N}^*]$ has rank exactly $2N$ on $\mathcal{H}_{(-z)^n}$ when $2N \le n$; see Proposition 7 of [4].

Motivated by this result, one might naturally ask whether the same is true on any infinite dimensional model spaces corresponding to general inner functions. More generally, one can naturally ask the following question.

Question. *For two inner functions θ and η, what is the rank of the self-commutator $[A_\eta, A_\eta^*]$ on $\mathcal{H}_\theta$?*

Recently, this question has been studied for finite dimensional model spaces corresponding to finite Blaschke products. To introduce the result, we need some notations. Given $\lambda \in \mathbb{D}$, let

$$b_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}, \quad z \in \mathbb{D}$$

be the Möbius transformation of $\mathbb{D}$. Given finite points $\lambda_1, \dots, \lambda_N$ in $\mathbb{D}$, the inner function of the form $B = b_{\lambda_1} \cdots b_{\lambda_N}$ is called a finite Blaschke product of order N and we then write $\operatorname{ord} B = N$. Also, $\operatorname{rank} T$ and $\ker T$ denote the rank and kernel respectively of a bounded linear operator T on a Hilbert space.

In a recent paper [3], TTOs with finite Blaschke product symbol acting on several types of model spaces corresponding to finite Blaschke products have been considered. Then, the ranks of their self-commutators have been obtained as shown in Theorem 1.1 below and, as an immediate consequence, the result in [4] mentioned above has been extended to finite dimensional model spaces.

Theorem 1.1 *The following statements hold.*

(a) *Let θ and η be any finite Blaschke products of order N and L respectively. If $2L \le N$, then* $\operatorname{rank} [A_\eta, A_\eta^*] = 2L$ *on $\mathcal{H}_\theta$.*

(b) *Let η and ζ be any two finite Blaschke products and $\theta = \eta\zeta$. Then* $\operatorname{rank}[A_\eta, A_\eta^*] = 2\min\{\operatorname{ord}\eta, \operatorname{ord}\zeta\}$ *on* $\mathcal{H}_\theta$.

The aim of this paper is to generalize Theorem 1.1 to infinite dimensional model spaces corresponding to general inner functions. As is well known, since $[A_\eta, A_\eta^*]$ is self-adjoint for any TTO A_η defined on the model space $\mathcal{H}_\theta$, we have

$$\ker[A_\eta, A_\eta^*] = \mathcal{H}_\theta \ominus [A_\eta, A_\eta^*]\mathcal{H}_\theta.$$

Thus, when we study the rank of $[A_\eta, A_\eta^*]$, the kernel description will be important for our purpose.

We first consider TTOs with general inner symbol acting on a model space corresponding to general inner function. We then show that the kernel of the self-commutator of such a TTO is closely related to that of certain classical Toeplitz operator. Specially, if the symbol is a finite Blaschke product, we obtain the rank of the self-commutator as shown in the following our first result which generalizes Theorem 1.1(a). Given $\psi \in L^\infty$, recall that the usual Toeplitz operator T_ψ with symbol ψ is defined on H^2 by

$$T_\psi f = P(\psi f)$$

for $f \in H^2$ where P is the orthogonal projection from L^2 onto H^2; see Sect. 2 for details.

Theorem 1.2 *Let θ and η be two inner functions. If A_η is the TTO defined on $\mathcal{H}_\theta$, then the following statements hold.*

(a) $\ker[A_\eta, A_\eta^*] \cap \ker T_{\bar{\theta}\eta} = \ker[A_\eta, A_\eta^*] \cap \eta H^2 = \eta \ker T_{\bar{\theta}\eta^2}$.
(b) *If η is a finite Blaschke product and* $2\operatorname{ord}\eta \le \dim \mathcal{H}_\theta$, *then*

$$\ker[A_\eta, A_\eta^*] = \eta \ker T_{\bar{\theta}\eta^2} \quad \text{and} \quad \operatorname{rank}[A_\eta, A_\eta^*] = 2\operatorname{ord}\eta.$$

This result also exhibits the close connection of kernel of $[A_\eta, A_\eta^*]$ with the multipliers between certain model spaces; see Corollary 3.2.

Next, we consider two TTOs with general inner symbols acting on the model space corresponding to the inner function which is the product of two symbols. We describe the kernels of the self-commutators of such TTOs and then characterize when the self-commutator has finite rank. Our result shows that the self-commutators of two TTOs are the same and such a self-commutator has finite rank only in a trivial case as shown in the following which will generalize Theorem 1.1(b).

Theorem 1.3 *Let η and ζ be inner functions and put $\theta = \eta\zeta$. If A_η and A_ζ are TTOs defined on $\mathcal{H}_\theta$, then the following statements hold.*

(a) $[A_\eta, A_\eta^*] = [A_\zeta, A_\zeta^*] = P_\theta - P_\eta - P_\zeta$ *on* $\mathcal{H}_\theta$ *and*

$$\ker[A_\eta, A_\eta^*] = \ker[A_\zeta, A_\zeta^*] = (\mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta) \oplus (\mathcal{H}_\zeta \cap \eta\mathcal{H}_\zeta).$$

(b) $[A_\eta, A_\eta^*]$ *has finite rank on* $\mathcal{H}_\theta$ *if and only if either* η *or* ζ *is a finite Blaschke product.*

(c) *If* η *is a finite Blaschke product and* $\operatorname{ord}\eta \le \dim \mathcal{H}_\zeta$, *then*

$$\ker[A_\eta, A_\eta^*] = \mathcal{H}_\zeta \cap \eta\mathcal{H}_\zeta, \quad [A_\eta, A_\eta^*]\mathcal{H}_\theta = P_\zeta\mathcal{H}_\eta \oplus \zeta\mathcal{H}_\eta$$

and $\operatorname{rank}[A_\eta, A_\eta^*] = 2\operatorname{ord}\eta$.

In Sect. 2, we collect some basic results which will be needed in our proofs. In Sect. 3, we will prove Theorems 1.2 and 1.3 except the rank formula in Theorem 1.2(b). In the proofs, we use complete different arguments from those used in the proof of Theorem 1.1. We will use the characterizations of finite rank Hankel products in [1, 8]. As a special case of Theorem 1.3, we give an example that the kernel of $[A_\eta, A_\eta^*]$ can be $\eta \ker T_{\bar\theta\eta^2}$ even when η is not a finite Blaschke product; see Corollary 3.4. As another application of Theorem 1.3, we characterize finite rank products of two projections; see Corollary 3.5.

In Sect. 4, we would like to generalize the rank formula in Theorem 1.2(b) for $[A_\eta, A_\zeta^*]$ on $\mathcal{H}_\theta$ when η and ζ both are finite Blaschke products satisfying $\operatorname{ord}\eta + \operatorname{ord}\zeta \le \dim \mathcal{H}_\theta$; see Theorem 4.1. As a special case, the rank formula in Theorem 1.2(b) is then obtained.

2 Preliminaries

Given $\psi \in L^\infty$, we recall the classical Toeplitz operator T_ψ with symbol ψ defined on H^2 by $T_\psi f = P(\psi f)$ for $f \in H^2$ where P is the orthogonal projection from L^2 onto H^2 which can be given by

$$Pg(w) = \int_{\mathbb{T}} \frac{g(\zeta)}{1 - w\bar\zeta}\, d\sigma(\zeta), \quad w \in \mathbb{D}$$

for functions $g \in L^2$. Note that $T_\psi^* = T_{\bar\psi}$. For an inner function θ and $\varphi \in H^\infty$, the set of all bounded analytic functions on $\mathbb{D}$, it is easy to check that $T_\varphi^*\mathcal{H}_\theta \subset \mathcal{H}_\theta$ and hence

$$A_\varphi^* f = T_\varphi^* f \tag{2.1}$$

for functions $f \in \mathcal{H}_\theta$. Also, for θ inner and $\varphi, \psi \in H^\infty$, we note $A_\varphi A_\psi = A_{\varphi\psi}$ on $\mathcal{H}_\theta$.

Given an inner function θ, it is well known that the orthogonal projection P_θ admits the following integral representation

$$P_\theta f(w) = \int_{\mathbb{T}} f(\zeta)\frac{1 - \theta(w)\overline{\theta(\zeta)}}{1 - w\bar\zeta}\, d\sigma(\zeta), \quad w \in \mathbb{D}$$

and hence

$$P_\theta f = Pf - \theta P(\bar{\theta} f) \tag{2.2}$$

for functions $f \in L^2$. In particular, we have

$$T^*_\theta f = P(\bar{\theta} f) = \frac{f - P_\theta f}{\theta} \tag{2.3}$$

for every $f \in H^2$. See Chapter 5 of [7] for details and related facts.

We have the following explicit description for the model space of a finite Blaschke product; see Corollary 5.18 of [7] for details. For an integer $k \geq 0$, we let $\mathcal{P}_k$ denote the set of all analytic polynomials with degree less than or equal to k. Also, we write $\deg p$ as the degree of an analytic polynomial p.

Lemma 2.1 *Let $a_1, \ldots, a_N$ be finite points in $\mathbb{D}$ and put $B = b_{a_1} \cdots b_{a_N}$. Then we have*

$$\mathcal{H}_B = \frac{1}{\prod_{n=1}^{N}(1 - \overline{a_n} z)} \mathcal{P}_{N-1}.$$

Particularly, we have $\dim \mathcal{H}_B = N$.

The following kernel description of a certain commutator will be very useful. The special case of $\zeta = \eta$ has been proved in Lemma 3 of [3].

Lemma 2.2 *Let η, ζ, and θ be inner functions and $f \in \mathcal{H}_\theta$. Suppose A_η and A_ζ are TTOs defined on $\mathcal{H}_\theta$. Then $f \in \ker\,[A_\eta, A^*_\zeta]$ if and only if*

$$\eta f - \eta P_\zeta f - P_\theta(\eta f) + P_\zeta P_\theta(\eta f) \in \zeta\theta H^2.$$

In particular, when $\zeta = \eta$, then $f \in \ker\,[A_\eta, A^*_\eta]$ if and only if

$$\eta f - \eta P_\eta f - P_\theta(\eta f) + P_\eta P_\theta(\eta f) \in \eta\theta H^2.$$

Proof By (2.1) and (2.3), we have

$$A_\eta A^*_\zeta f = A_\eta T^*_\zeta f = A_\eta \frac{f - P_\zeta f}{\zeta} = P_\theta\left(\eta \frac{f - P_\zeta f}{\zeta}\right)$$

and similarly

$$A^*_\zeta A_\eta f = A^*_\zeta P_\theta(\eta f) = T^*_\zeta P_\theta(\eta f) = \frac{P_\theta(\eta f) - P_\zeta P_\theta(\eta f)}{\zeta}$$

for functions $f \in \mathcal{H}_\theta$. Thus, $f \in \ker\,[A_\eta, A^*_\eta]$ if and only if

$$\eta \frac{f - P_\zeta f}{\zeta} - \frac{P_\theta(\eta f) - P_\zeta P_\theta(\eta f)}{\zeta} \in \theta H^2,$$

which induces the desired conclusions. The proof is complete.

3 The proofs of Theorems 1.2 and 1.3

In this section, we will prove our main theorems. We start with the following simple lemma which will be useful in our proofs. We will use the notation

$$\|g\| = \left(\int_{\mathbb{T}} |g|^2 \mathrm{d}\sigma \right)^{1/2}$$

for functions $g \in L^2$ in what follows.

Lemma 3.1 *Let θ, η be two inner functions and $f \in \mathcal{H}_\theta$. If $P_\eta f \in \theta H^2$, then we have $P_\eta f = 0$.*

Proof Write $P_\eta f = \theta g$ for some $g \in H^2$. Using (2.2), we then get $f - \theta g = \eta P(\bar{\eta} f)$. Since $f \in \mathcal{H}_\theta$, we note $f \perp \theta g$ and then

$$\|f\|^2 + \|g\|^2 = \|f - \theta g\|^2 = \|\eta P(\bar{\eta} f)\|^2 \leq \|f\|^2,$$

which yields $\|g\| = 0$ and then $g = 0$. Hence $P_\eta f = 0$ as desired. The proof is complete.

We are ready to prove our first main theorem.

Proof of Theorem 1.2 We first prove (a). Let $f \in \ker [A_\eta, A_\eta^*] \cap \ker T_{\bar{\theta}\eta}$. Notice that $f \in \ker T_{\bar{\theta}\eta}$ if and only if $\eta f \in \mathcal{H}_\theta$. Since $P_\eta(\eta f) = 0$, Lemma 2.2 shows $\eta P_\eta f \in \eta\theta H^2$, which implies $P_\eta f \in \theta H^2$. By Lemma 3.1, we see $P_\eta f = 0$ and hence $f \in \eta H^2$. On the other hand, one can easily see that for $\Phi \in H^2$,

$$\Phi \in \eta \ker T_{\bar{\theta}\eta^2} \iff \eta \Phi \in \mathcal{H}_\theta \quad \text{and} \quad \Phi \in \eta H^2 \tag{3.1}$$

holds. By the observation above together with Lemma 2.2 again, we get

$$\eta \ker T_{\bar{\theta}\eta^2} = \ker [A_\eta, A_\eta^*] \cap \ker T_{\bar{\theta}\eta} \subset \ker[A_\eta, A_\eta^*] \cap \eta H^2.$$

Now, to complete the proof, let $f \in \ker[A_\eta, A_\eta^*] \cap \eta H^2$ and show $f \in \ker T_{\bar{\theta}\eta}$. It suffices to show $\eta f \in \mathcal{H}_\theta$ as mentioned above. Since $f \in \eta H^2$, we note $P_\eta f = 0$ and then Lemma 2.2 gives

$$\eta f - P_\theta(\eta f) + P_\eta P_\theta(\eta f) \in \eta\theta H^2. \tag{3.2}$$

Since $\eta f - P_\theta(\eta f) \in \theta H^2$ and $\eta\theta H^2 \subset \theta H^2$, the above shows that

$$P_\eta P_\theta(\eta f) \in \theta H^2.$$

Now, by Lemma 3.1 (replace f by $P_\theta(\eta f)$), we see $P_\eta P_\theta(\eta f) = 0$ and hence $\eta f - P_\theta(\eta f) \in \eta\theta H^2$ by (3.2). Clearly, since

$$\eta f - P_\theta(\eta f) \perp \eta\theta H^2,$$

we have $\eta f - P_\theta(\eta f) = 0$ and hence $\eta f = P_\theta(\eta f) \in \mathcal{H}_\theta$ as desired.

Now we prove (b). Write $\eta = b_{a_1} \cdots b_{a_N}$ where $a_1, \dots, a_N \in \mathbb{D}$. Let $f \in \ker[A_\eta, A_\eta^*]$. Then Lemma 2.2 shows that $\eta P_\eta f - P_\eta P_\theta(\eta f) \in \theta H^2$. On the other hand, by Lemma 2.1, we may write

$$P_\eta f = \frac{\sum_{j=0}^{N-1} c_j z^j}{\prod_{n=1}^{N}(1 - \overline{a}_n z)}, \quad P_\eta P_\theta(\eta f) = \frac{\sum_{j=0}^{N-1} d_j z^j}{\prod_{n=1}^{N}(1 - \overline{a}_n z)}$$

for some constants c_j and d_j. Thus

$$\eta P_\eta f - P_\eta P_\theta(\eta f) = \frac{p}{\prod_{n=1}^{N}(1 - \overline{a}_n z)^2}$$

where

$$p(z) = \prod_{n=1}^{N}(a_n - z) \sum_{j=0}^{N-1} c_j z^j - \prod_{n=1}^{N}(1 - \overline{a}_n z) \sum_{j=0}^{N-1} d_j z^j$$

for $z \in \mathbb{D}$. Since $\eta P_\eta f - P_\eta P_\theta(\eta f) \in \theta H^2$, we also have $p \in \theta H^2$. Note that

$$\deg p \le 2N - 1 < \dim \mathcal{H}_\theta.$$

Since $p/\theta \in H^2$, we have $p = 0$ and then $\eta P_\eta f = P_\eta P_\theta(\eta f)$. Because $\eta P_\eta f \perp P_\eta P_\theta(\eta f)$, we see $\eta P_\eta f = 0$, which implies $f \in \eta H^2$. Therefore, we have

$$\ker[A_\eta, A_\eta^*] \subset \ker[A_\eta, A_\eta^*] \cap \eta H^2,$$

and the first part of (b) follows immediately from (a).

The rank equality in the second part of (b) is the special case of Theorem 4.1, so we finish the proof.

For two inner functions u and v, let $\mathcal{M}(u, v)$ be the set of all multipliers from $\mathcal{H}_u$ into $\mathcal{H}_v$, that is,

$$\mathcal{M}(u, v) = \{\phi \in \mathrm{Hol}(\mathbb{D}) : \phi\mathcal{H}_u \subset \mathcal{H}_v\}.$$

In Theorem 3.1 of [6], it is proved that $\phi \in \mathcal{M}(u, v) \cap H^\infty$ if and only if $\phi \in \ker T_{\overline{zv}u} \cap H^\infty$. Therefore, as an immediate application of our first main result, we have the following.

Corollary 3.2 *Let θ and η be two inner functions and A_η be the TTO defined on $\mathcal{H}_\theta$. Then we have*

$$\bar{\eta}\ker[A_\eta, A_\eta^*] \cap H^\infty = \mathcal{M}(z\eta^2, \theta) \cap H^\infty.$$

We also have the following corollary by using the same proof as done in Corollary 8 of [3].

Corollary 3.3 *Let* $\lambda \in \mathbb{D}\backslash\{0\}$. *Let* B, θ, η *and* ζ *be inner functions for which* $\eta = B \cdot b_\lambda \circ (\theta\zeta)$ *or* $B\eta = b_\lambda \circ (\theta\zeta)$. *If* B *is a finite Blaschke product with* $2\,\mathrm{ord}B \le \dim \mathcal{H}_\theta$, *then* $\mathrm{rank}[A_\eta, A_\eta^*] = 2\,\mathrm{ord}B$ *on* $\mathcal{H}_\theta$.

The above says that even for η with $\dim \mathcal{H}_\eta = \infty$, the self-commutator of A_η defined on $\mathcal{H}_\theta$ may have finite rank. Moreover, when $\eta = B \cdot b_\lambda \circ (\theta\zeta)$ for some $\lambda \in \mathbb{D}$ as in the above, the proof shows $[A_\eta, A_\eta^*] = |\lambda|^2 \cdot [A_B, A_B^*]$, so it follows from Theorem 1.2 that

$$\ker[A_\eta, A_\eta^*] = \ker[A_B, A_B^*] = B \ker T_{\bar{\theta}B^2}.$$

We remark in passing that even for not finite Blaschke product η, the kernel of the self-commutator of A_η defined on $\mathcal{H}_\theta$ may satisfy

$$\ker[A_\eta, A_\eta^*] = \eta \ker T_{\bar{\theta}\eta^2};$$

see Corollary 3.4.

We also remark that for any two inner functions θ and η, if A_η is a TTO on $\mathcal{H}_\theta$, then Theorem 1.2 tells that $\eta \ker T_{\bar{\theta}\eta^2} \subset \ker [A_\eta, A_\eta^*]$, so generally we have

$$\mathrm{rank}\,[A_\eta, A_\eta^*] \le \dim \mathcal{H}_\theta \ominus \eta \ker T_{\bar{\theta}\eta^2}.$$

An interesting question arises: When is the right hand side above finite? We have the answer as follows.

Firstly, if

$$\dim \mathcal{H}_\theta \ominus \eta \ker T_{\bar{\theta}\eta^2} = 0,$$

then $\eta \ker T_{\bar{\theta}\eta^2} = \mathcal{H}_\theta$, which together with (3.1) yields $\eta\mathcal{H}_\theta = \mathcal{H}_\theta$. So $\mathcal{H}_\theta \perp \mathcal{H}_\eta$. Notice that $\mathcal{H}_\eta$ and $\mathcal{H}_\theta$ both are backward shift invariant, hence by Corollary 2.5 of [2] we have either $\mathcal{H}_\eta = \{0\}$ or $\mathcal{H}_\theta = \{0\}$. Therefore, if θ is nonconstant, we have η is a unimodular constant, a trivial case.

Secondly, suppose

$$\dim \mathcal{H}_\theta \ominus \eta \ker T_{\bar{\theta}\eta^2} < \infty.$$

For $f \in \mathcal{H}_\theta$, note that $\eta f \in \mathcal{H}_\theta$ implies $\eta f \perp \theta H^2$. Since $\eta f \perp \eta\theta H^2$, we have

$$\eta f \perp \theta H^2 \ominus \eta\theta H^2 = \theta \mathcal{H}_\eta$$

and hence $f \perp T_{\bar{\eta}\theta}\mathcal{H}_\eta$. Note that $T_{\bar{\eta}\theta}\mathcal{H}_\eta \subset \mathcal{H}_\theta$. This observation shows that for $f \in \mathcal{H}_\theta$, $\eta f \in \mathcal{H}_\theta$ if and only if $f \in \mathcal{H}_\theta \ominus T_{\bar{\eta}\theta}\mathcal{H}_\eta$. Also, for $f \in \mathcal{H}_\theta$, observing that $f \in \eta H^2$ implies $f \perp P_\theta \mathcal{H}_\eta$, we see that for $f \in \mathcal{H}_\theta$, $f \in \eta H^2$ if and only if $f \in \mathcal{H}_\theta \ominus P_\theta \mathcal{H}_\eta$. Now, by (3.1) and observations above, we see that

$$\eta \ker T_{\bar{\theta}\eta^2} = \mathcal{H}_\theta \ominus (T_{\bar{\eta}\theta}\mathcal{H}_\eta \cup P_\theta \mathcal{H}_\eta).$$

It induces that

$$\dim[\operatorname{span}(T_{\bar{\eta}\theta}\mathcal{H}_\eta \cup P_\theta \mathcal{H}_\eta)] < \infty.$$

So $\dim T_{\bar{\eta}\theta}\mathcal{H}_\eta < \infty$ and $\dim P_\theta \mathcal{H}_\eta < \infty$. Since $P_\theta \mathcal{H}_\eta = P_\theta P_\eta H^2$, we see that $\dim P_\theta \mathcal{H}_\eta < \infty$ if and only if either θ or η is a finite Blaschke product; see Corollary 3.5 below. Conversely, it is easy to see that this case implies $\dim T_{\bar{\eta}\theta}\mathcal{H}_\eta < \infty$.

To prove the second main result, we introduce some known results on Hankel operators. For $f \in L^\infty$, the (little) Hankel operator $H_f : H^2 \to H^2$ with symbol f is defined by

$$H_f h = PU(fh), \quad h \in H^2$$

where $U\varphi(z) = \bar{z}\widetilde{\varphi}(z)$ and $\widetilde{\varphi}(z) = \varphi(\bar{z})$. For $f, g \in L^\infty$, it is easy to see that Toeplitz and Hankel operators are related as follows;

$$T_{fg} - T_f T_g = H_{\widetilde{f}} H_g, \quad H_{fg} = H_f T_g + T_{\widetilde{f}} H_g. \tag{3.3}$$

In particular, if $f \in H^\infty$, since $H_f = 0$, we have

$$T_{\widetilde{f}} H_g = H_{fg} = H_g T_f. \tag{3.4}$$

Kronecker's theorem says that H_f has finite rank if and only if there is a nonzero analytic polynomial p such that $pf \in H^\infty$. Also, the Axler–Chang–Sarason theorem says that $H_f H_g$ has finite rank if and only if one of H_f and H_g has finite rank. See [1, 8] for details.

For two inner functions u and v, we have

$$\mathcal{H}_{uv} = \mathcal{H}_u \oplus u\mathcal{H}_v; \tag{3.5}$$

see Lemma 5.10 of [7] for example. For a closed subspace M of H^2, we use notation P_M for the projection from L^2 onto M.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 We first prove (a). Using (2.1) and (2.2), we have

$$A_\eta A_\eta^* f = A_\eta T_\eta^* f = P_\theta(\eta P(\bar{\eta} f))$$

and

$$A_\eta^* A_\eta f = T_\eta^* P_\theta(\eta f) = T_\eta^*(\eta f - \theta P(\bar{\theta}\eta f)) = f - \zeta P(\bar{\zeta} f)$$

for functions $f \in \mathcal{H}_\theta$. Since $-f + \eta P(\overline{\eta} f) + \zeta P(\overline{\zeta} f)$ is orthogonal to θH^2 and $P(\bar{\theta} f) = 0$ for every $f \in \mathcal{H}_\theta$, we obtain by (2.2) again

$$\begin{aligned}[A_\eta, A_\eta^*]f &= P_\theta\big(-f + \eta P(\overline{\eta}f) + \zeta P(\overline{\zeta}f)\big) \\ &= -f + \eta P(\overline{\eta}f) + \zeta P(\overline{\zeta}f) \\ &= (P_\theta - P_\eta - P_\zeta)f\end{aligned} \tag{3.6}$$

for all $f \in \mathcal{H}_\theta$. Noting (3.6) is symmetric with respect to η and ζ, we have the first part of (a). To prove the second part of (a), we let $f \in \ker[A_\eta, A_\eta^*]$. By (3.6) and (3.5), we see

$$P_\zeta f = f - P_\eta f \in \mathcal{H}_\theta \ominus \mathcal{H}_\eta = \eta\mathcal{H}_\zeta,$$

which implies $P_\zeta f \in \mathcal{H}_\zeta \cap \eta\mathcal{H}_\zeta$. Similarly we can see $P_\eta f \in \mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta$. It follows from (3.6) again that

$$f = P_\zeta f + P_\eta f \in (\mathcal{H}_\zeta \cap \eta\mathcal{H}_\zeta) \oplus (\mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta)$$

and hence $\ker[A_\eta, A_\eta^*] \subset (\mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta) \oplus (\mathcal{H}_\zeta \cap \eta\mathcal{H}_\zeta)$ holds. Also, using (3.6) again, we can easily see that the reverse inclusion also holds. Thus the second part of (a) follows.

Now we prove (b). First suppose $[A_\eta, A_\eta^*]$ has finite rank on $\mathcal{H}_\theta$. Note $P_\theta - P_\eta - P_\zeta = 0$ on θH^2. Hence, by (3.6) and (2.2), we have

$$\begin{aligned}[A_\eta, A_\eta^*]P_\theta &= P_\theta - P_\eta - P_\zeta \\ &= (I - T_\theta T_{\overline{\theta}}) - (I - T_\eta T_{\overline{\eta}}) - (I - T_\zeta T_{\overline{\zeta}}) \\ &= (T_{\theta\overline{\theta}} - T_\theta T_{\overline{\theta}}) - (T_{\eta\overline{\eta}} - T_\eta T_{\overline{\eta}}) - (T_{\zeta\overline{\zeta}} - T_\zeta T_{\overline{\zeta}}) \\ &= H_{\widetilde{\theta}} H_{\overline{\theta}} - H_{\widetilde{\eta}} H_{\overline{\eta}} - H_{\widetilde{\zeta}} H_{\overline{\zeta}}\end{aligned}$$

on H^2 and hence

$$H := H_{\widetilde{\theta}} H_{\overline{\theta}} - H_{\widetilde{\eta}} H_{\overline{\eta}} - H_{\widetilde{\zeta}} H_{\overline{\zeta}}$$

has finite rank on H^2. To complete the proof, assume η is not a finite Blaschke product. Note that none of $H_{\widetilde{\theta}}$, $H_{\overline{\eta}}$ and $H_{\widetilde{\eta}}$ has finite rank by Kronecker's theorem. Also, we see from (3.4) that

$$HT_\zeta = H_{\widetilde{\theta}} H_{\overline{\eta}} - H_{\widetilde{\eta}} H_{\overline{\eta}\zeta}$$

has finite rank on H^2, which implies that $H_{\overline{\eta}\zeta}$ doesn't have finite rank by the Axler–Chang–Sarason theorem. Thus, by Theorem 3.1 in [5], there exist nonzero analytic polynomials p, q such that $p\widetilde{\theta} + q\widetilde{\eta} \in H^\infty$, which implies $\widetilde{\eta}(p\widetilde{\zeta} + q) \in H^\infty$, so $p\widetilde{\zeta} + q \in \overline{\widetilde{\eta}} H^\infty \subset H^2$.

In particular, we have $p\widetilde{\zeta} \in H^2$ and then $p \in \overline{\widetilde{\zeta}} H^2$, which means ζ is a finite Blaschke product, as desired. Since the converse implication is clear by Theorem 1.2 and (a), we complete the proof of (b).

Finally we prove (c). For the kernel identity, it suffices to show $\mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta = \{0\}$ by (a). Suppose there exists a nonzero function f such that $f \in \mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta$. By Lemma 2.1, the number of zeros of f (counting multiplicities) on $\mathbb{D}$ is less than or equal to $\dim \mathcal{H}_\eta - 1$. Since $f = \zeta f_1$ for some $f_1 \in \mathcal{H}_\eta$, the number of zeros of f (counting multiplicities) on $\mathbb{D}$ is greater than or equal to $\dim \mathcal{H}_\zeta$, which is a contradiction.

For the range of $[A_\eta, A_\eta^*]$, first notice that $\mathcal{H}_\theta = \mathcal{H}_\eta \oplus \eta\mathcal{H}_\zeta$ by (3.5) and $A_\eta A_\eta^* f = 0$ for $f \in \mathcal{H}_\eta$. Since

$$\eta H^2 = \eta(\mathcal{H}_\zeta \oplus \zeta H^2) = \eta\mathcal{H}_\zeta \oplus \theta H^2,$$

we have $P_\theta(\eta h) = \eta P_\zeta h$ for every $h \in H^2$. Hence

$$A_\eta^* A_\eta f = A_\eta^* P_\theta(\eta f) = A_\eta^* \eta P_\zeta f = P_\zeta f$$

and $[A_\eta, A_\eta^*]f = -P_\zeta f$ for every $f \in \mathcal{H}_\eta$. So $[A_\eta, A_\eta^*]\mathcal{H}_\eta = P_\zeta \mathcal{H}_\eta$. On the other hand, since $A_\eta A_\eta^* \eta g = \eta g$ and

$$A_\eta^* A_\eta \eta g = A_\eta^* \eta P_\zeta \eta g = P_\zeta \eta g,$$

we have

$$[A_\eta, A_\eta^*]\eta g = \eta g - P_\zeta \eta g = P_{\zeta\mathcal{H}_\eta} \eta g$$

for every $g \in \mathcal{H}_\zeta$ and hence $[A_\eta, A_\eta^*]\eta\mathcal{H}_\zeta = P_{\zeta\mathcal{H}_\eta}\eta\mathcal{H}_\zeta$. Now, we show that the map $P_{\zeta\mathcal{H}_\eta} : \eta\mathcal{H}_\zeta \to \zeta\mathcal{H}_\eta$ is onto. Suppose there is $g \in \mathcal{H}_\eta$ such that $P_{\zeta\mathcal{H}_\eta}\eta\mathcal{H}_\zeta \perp \zeta g$. Then $\eta\mathcal{H}_\zeta \perp \zeta g$, so $\zeta g \in \mathcal{H}_\eta$. If $g \neq 0$, Lemma 2.1 implies that the inner factor φ of ζg is a finite Blaschke product (which might be constant) with $\dim \mathcal{H}_\varphi \leq \dim \mathcal{H}_\eta - 1$. But, we have $\dim \mathcal{H}_\zeta \leq \dim \mathcal{H}_\varphi$ because φ is the inner factor of ζg. This contradiction gives $g = 0$ and $P_{\zeta\mathcal{H}_\eta}\eta\mathcal{H}_\zeta = \zeta\mathcal{H}_\eta$ holds. It follows that

$$\begin{aligned} [A_\eta, A_\eta^*]\mathcal{H}_\theta &= [A_\eta, A_\eta^*](\mathcal{H}_\eta \oplus \eta\mathcal{H}_\zeta) \\ &= P_\zeta\mathcal{H}_\eta \oplus P_{\zeta\mathcal{H}_\eta}\eta\mathcal{H}_\zeta \\ &= P_\zeta\mathcal{H}_\eta \oplus \zeta\mathcal{H}_\eta, \end{aligned}$$

as desired. Finally, since

$$\{f \in \mathcal{H}_\eta : P_\zeta f = 0\} = \mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta = \{0\},$$

we have $\dim P_\zeta\mathcal{H}_\eta = \dim \mathcal{H}_\eta$. Hence

$$\operatorname{rank}[A_\eta, A_\eta^*] = \dim P_\zeta\mathcal{H}_\eta + \dim \zeta\mathcal{H}_\eta = 2 \operatorname{ord} \eta,$$

which completes the proof of (c). The proof is complete.

As a special case of Theorem 1.3, we have the following corollary which provides an example as mentioned at the remark just after Corollary 3.3.

Corollary 3.4 *Let η and ζ be two inner functions and $\theta = \eta^2\zeta$. If A_η is a TTO defined on $\mathcal{H}_\theta$, then*

$$\ker[A_\eta, A_\eta^*] = \eta\mathcal{H}_\zeta = \eta \ker T_{\bar\theta\eta^2}$$

and

$$\mathcal{H}_\theta \ominus \ker[A_\eta, A_\eta^*] = \mathcal{H}_\eta \oplus \eta\zeta\mathcal{H}_\eta.$$

Moreover, $[A_\eta, A_\eta^*]$ *has finite rank only when* η *is a finite Blaschke product and* $\operatorname{rank}[A_\eta, A_\eta^*] = 2 \operatorname{ord} \eta$.

Proof By Theorem 1.3(a), we have

$$\ker[A_\eta, A_\eta^*] = (\mathcal{H}_\eta \cap \eta\zeta\mathcal{H}_\eta) \oplus (\mathcal{H}_{\eta\zeta} \cap \eta\mathcal{H}_{\eta\zeta}).$$

Since $\mathcal{H}_\eta \perp \eta\zeta\mathcal{H}_\eta$, we see $\mathcal{H}_\eta \cap \eta\zeta\mathcal{H}_\eta = \{0\}$. By (3.5) we have

$$\mathcal{H}_{\eta\zeta} = \mathcal{H}_\eta \oplus \eta\mathcal{H}_\zeta, \quad \eta\mathcal{H}_{\eta\zeta} = \eta\mathcal{H}_\zeta \oplus \eta\zeta\mathcal{H}_\eta,$$

which gives that $\mathcal{H}_{\eta\zeta} \cap \eta\mathcal{H}_{\eta\zeta} = \eta\mathcal{H}_\zeta$. It is clear that $\eta \ker T_{\bar{\theta}\eta^2} = \eta\mathcal{H}_\zeta$ because $\theta = \eta^2\zeta$ and $\ker T_{\bar{\zeta}} = \mathcal{H}_\zeta$. Thus the equality of $\ker[A_\eta, A_\eta^*]$ holds. For the second equality of $\mathcal{H}_\theta \ominus \ker[A_\eta, A_\eta^*]$, it quickly follows from the obtained equality of kernel and the fact

$$\mathcal{H}_\theta = \mathcal{H}_\eta \oplus \eta\mathcal{H}_{\eta\zeta} = \mathcal{H}_\eta \oplus \eta\mathcal{H}_\zeta \oplus \eta\zeta\mathcal{H}_\eta.$$

Now, we see easily that

$$\dim \mathcal{H}_\theta \ominus \ker[A_\eta, A_\eta^*] = 2 \dim \mathcal{H}_\eta.$$

Therefore, $[A_\eta, A_\eta^*]$ has finite rank if and only if $\dim \mathcal{H}_\eta < \infty$, that is, η is a finite Blaschke product. We complete the proof.

Recall that $P_\eta P_\zeta = 0$ if and only if one of η and ζ is a constant. Indeed, note $P_\eta P_\zeta = 0$ if and only if $\mathcal{H}_\eta \perp \mathcal{H}_\zeta$. Also, since $\mathcal{H}_\eta$ and $\mathcal{H}_\zeta$ are backward shift invariant, we note that $\mathcal{H}_\eta \perp \mathcal{H}_\zeta$ if and only if either $\mathcal{H}_\eta = \{0\}$ or $\mathcal{H}_\zeta = \{0\}$ by Corollary 2.5 of [2], which gives the desired fact.

More generally, as an application of Theorem 1.3, we prove that a product of two orthogonal projections onto model spaces can be of finite rank in a trivial case as shown in the following corollary. We provide independent two proofs.

Corollary 3.5 *Let* η *and* ζ *be two inner functions. Then* $P_\eta P_\zeta$ *has finite rank on* H^2 *if and only if one of* η *and* ζ *is a finite Blaschke product.*

Proof Put $\theta = \eta\zeta$, consider TTOs A_η and A_ζ defined on $\mathcal{H}_\theta$. By (3.5) and Theorem 1.3(a), we can easily see that

$$\mathcal{H}_\theta \ominus \ker[A_\eta, A_\eta^*] = [\mathcal{H}_\zeta \ominus (\mathcal{H}_\zeta \cap \eta\mathcal{H}_\zeta)] \oplus [\zeta\mathcal{H}_\eta \ominus (\mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta)].$$

Also, by routine verifications, we have

$$\mathcal{H}_\zeta \ominus (\mathcal{H}_\zeta \cap \eta\mathcal{H}_\zeta) = P_\zeta\mathcal{H}_\eta$$

and

$$\zeta\mathcal{H}_\eta \ominus (\mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta) = P_{\zeta\mathcal{H}_\eta}\eta\mathcal{H}_\zeta.$$

It follows that

$$\mathcal{H}_\theta \ominus \ker[A_\eta, A_\eta^*] = P_\zeta\mathcal{H}_\eta \oplus P_{\zeta\mathcal{H}_\eta}\eta\mathcal{H}_\zeta, \tag{3.7}$$

which is what we have obtained in the proof of Theorem 1.3(c) by using a different method. To complete the proof, define $C_\eta(f) = \eta\bar{z}\bar{f}$ for $f \in \mathcal{H}_\eta$, then one can see that $C_\eta^2 = I$ on $\mathcal{H}_\eta$ and C_η is bijective on $\mathcal{H}_\eta$. Put

$$M := \bar{\zeta}\mathcal{H}_\eta \cap \mathcal{H}_\eta = \{g \in \mathcal{H}_\eta : \quad \zeta g \in \mathcal{H}_\eta\}.$$

For $g \in M$, we have $\zeta g \in \mathcal{H}_\eta$, and

$$C_\eta(\zeta g) = \eta z\overline{\zeta g} = \bar{\zeta}C_\eta(g).$$

From this we have $C_\eta(\zeta M) \subset M$ and $C_\eta(M) \subset \zeta M$, which gives that

$$\zeta M = C_\eta^2(\zeta M) \subset C_\eta(M) \subset \zeta M,$$

to yield that $C_\eta M = \zeta M$. Hence we have

$$\begin{aligned}\dim \mathcal{H}_\eta \ominus M &= \dim C_\eta(\mathcal{H}_\eta \ominus M)\\ &= \dim C_\eta\mathcal{H}_\eta \ominus C_\eta M\\ &= \dim \mathcal{H}_\eta \ominus \zeta M\\ &= \dim \mathcal{H}_\eta \ominus (\mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta)\\ &= \dim P_\eta\mathcal{H}_\zeta.\end{aligned}$$

It follows that

$$\begin{aligned}\dim P_{\zeta\mathcal{H}_\eta}\eta\mathcal{H}_\zeta &= \dim \zeta\mathcal{H}_\eta \ominus (\mathcal{H}_\eta \cap \zeta\mathcal{H}_\eta)\\ &= \dim \mathcal{H}_\eta \ominus M\\ &= \dim P_\eta\mathcal{H}_\zeta.\end{aligned} \tag{3.8}$$

Notice that

$$\dim P_\eta\mathcal{H}_\zeta = \dim P_\eta P_\zeta H^2 = \dim P_\zeta P_\eta H^2 = \dim P_\zeta\mathcal{H}_\eta.$$

Now, combining (3.7) with (3.8), we obtain

$$\begin{aligned}\dim \mathcal{H}_\theta \ominus \ker[A_\eta, A_\eta^*] &= \dim P_\zeta\mathcal{H}_\eta + \dim P_\eta\mathcal{H}_\zeta\\ &= 2\dim P_\eta P_\zeta H^2.\end{aligned}$$

Therefore, by Theorem 1.3(b), we see that $P_\eta P_\zeta$ has finite rank on H^2 if and only if one of η and ζ is a finite Blaschke product, as desired. The proof is complete.

We remark in passing that the proof above shows $\operatorname{rank}[A_\eta, A_\eta^*] = 2\operatorname{rank} P_\eta P_\zeta$. In the following, we give a short and independent proof of Corollary 3.5.

Another proof of Corollary 3.5 First note that $P_\theta = H_{\bar\theta}^* H_{\bar\theta}$ for every inner functions θ. Let F be the set of all finite rank operators. Then we note

$$\begin{aligned} P_\eta P_\zeta = H_{\bar\eta}^* H_{\bar\eta} H_{\bar\zeta}^* H_{\bar\zeta} \in F &\Longleftrightarrow H_{\bar\eta}^* H_{\bar\eta} H_{\bar\zeta}^* \in F \\ &\Longleftrightarrow H_{\bar\zeta} H_{\bar\eta}^* H_{\bar\eta} \in F \\ &\Longleftrightarrow H_{\bar\zeta} H_{\bar\eta}^* \in F, \end{aligned}$$

where we repeatedly use the fact that the ranges of AB^*B and AB^* are the same for operators A and B for which AB^*B or AB^* has finite rank. Thus, by the Axler–Chang–Sarason theorem and Kronecker's theorem, we conclude that $P_\eta P_\zeta$ has finite rank if and only if either η or ζ is a finite Blaschke product.

4 A more general case for rank formula

In this section, we generalize the rank formula in Theorem 1.2(b) to commutators of a TTO with inner symbol and an adjoint of a TTO with inner symbol as shown in the following.

Theorem 4.1 *Let η, ζ be two finite Blaschke products and θ an inner function satisfying* $\operatorname{ord}\eta + \operatorname{ord}\zeta \le \dim \mathcal{H}_\theta$. *If A_η and A_ζ are TTOs defined on $\mathcal{H}_\theta$, then $[A_\eta, A_\zeta^*]$ has finite rank and*

$$\operatorname{rank}[A_\eta, A_\zeta^*] = 2\min\{\operatorname{ord}\eta, \operatorname{ord}\zeta\}.$$

Proof Since the ranks of $[A_\eta, A_\zeta^*]$ and $[A_\zeta, A_\eta^*]$ are the same, we may assume $\operatorname{ord}\zeta \le \operatorname{ord}\eta$ without loss of generality. For $f \in \ker[A_\eta, A_\zeta^*]$, we have by Lemma 2.2

$$\eta f - \eta P_\zeta f - P_\theta(\eta f) + P_\zeta P_\theta(\eta f) \in \zeta\theta H^2. \tag{4.1}$$

Since $\eta f - P_\theta(\eta f) \in \theta H^2$, the above implies

$$\eta P_\zeta f - P_\zeta P_\theta(\eta f) \in \theta H^2. \tag{4.2}$$

Write $\eta = b_{\alpha_1} \cdots b_{\alpha_N}$ and $\zeta = b_{\beta_1} \cdots b_{\beta_L}$ where $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_L \in \mathbb{D}$ and $L \le N$. Then, by a similar argument as in the proof of (b) in Theorem 1.2, we may get

$$\eta P_\zeta f = P_\zeta P_\theta(\eta f), \tag{4.3}$$

which forces (4.1) to become

$$\eta f - P_\theta(\eta f) \in \zeta\theta H^2 \subset \zeta H^2.$$

Hence

$$P_\zeta(\eta f) - P_\zeta P_\theta(\eta f) = P_\zeta(\eta f - P_\theta(\eta f)) = 0$$

and then (4.3) shows

$$\eta P_\zeta f = P_\zeta(\eta f).$$

Now, Lemma 2.1 tells that the right hand side of the above is a rational function with at most $L-1$ zeros, which is less than N, the number of zeros of η. Hence we get $P_\zeta f = 0$ and so $f \in \zeta H^2$, which implies $P_\zeta(\eta f) = 0$ and so $P_\zeta P_\theta(\eta f) = 0$. To sum up, we have obtained that

$$\ker[A_\eta, A_\zeta^*] = \{f \in \mathcal{H}_\theta : f \in \zeta H^2, \eta f - P_\theta(\eta f) \in \zeta\theta H^2\}.$$

Note that $\eta f - P_\theta(\eta f) \in \zeta\theta H^2$ implies $P(\bar\theta\eta f) \in \zeta H^2$ or $P(\bar\theta\eta f) \perp \mathcal{H}_\zeta$, which is equivalent to $f \in \mathcal{H}_\theta \ominus P_\theta(\theta\bar\eta\mathcal{H}_\zeta)$. Also, for $f \in \mathcal{H}_\theta$, observing that $f \in \zeta H^2$ implies $f \perp P_\theta\mathcal{H}_\zeta$, we see that for $f \in \mathcal{H}_\theta$, $f \in \zeta H^2$ if and only if $f \in \mathcal{H}_\theta \ominus P_\theta\mathcal{H}_\zeta$. Therefore, we get

$$\mathcal{H}_\theta \ominus \ker[A_\eta, A_\zeta^*] = \overline{\text{span}}\big(P_\theta(\theta\bar\eta\mathcal{H}_\zeta) \cup P_\theta\mathcal{H}_\zeta\big),$$

where $\overline{\text{span}}(E)$ is the closure of the linear span of a set E. Now we only need to show that

$$\dim[\text{span}(P_\theta(\theta\bar\eta\mathcal{H}_\zeta) \cup P_\theta\mathcal{H}_\zeta)] = 2L.$$

For this end, let $f, g \in \mathcal{H}_\zeta$ and suppose

$$P_\theta(\theta\bar\eta f + g) = 0.$$

It suffices to show $f = g = 0$. Note that the above shows that

$$\theta\bar\eta f + g \in \theta H^2 \oplus \overline{zH^2}$$

or

$$\theta f + \eta g \in \eta\theta H^2 \oplus \eta\overline{zH^2}$$

where $\overline{zH^2} = \{\overline{zf} : f \in H^2\}$. Since

$$\eta\overline{zH^2} = \eta\overline{z\mathcal{H}_\eta} \oplus \eta\overline{z\eta H^2} = \mathcal{H}_\eta \oplus \overline{zH^2},$$

it follows that

$$\theta f + \eta g \in \eta\theta H^2 \oplus \mathcal{H}_\eta.$$

Since $\eta g \perp \mathcal{H}_\eta$ and $I - P_\eta$ is an orthogonal projection onto ηH^2, the above implies that

$$(I - P_\eta)(\theta f) + \eta g \in \eta\theta H^2.$$

Using the fact that $(I - P_\eta)f = \eta P(\bar{\eta}f)$, one can see that the above gives

$$P(\bar{\eta}\theta f) + g \in \theta H^2. \tag{4.4}$$

Since

$$P(\bar{\eta}\theta f) = T_\eta^*(\theta f) = \frac{\theta f - P_\eta(\theta f)}{\eta}$$

by (2.3), then (4.4) implies

$$\frac{\theta f - P_\eta(\theta f)}{\eta} + g \in \theta H^2$$

and thus

$$\theta f - P_\eta(\theta f) + \eta g \in \eta\theta H^2.$$

Hence there is $h \in H^2$ for which

$$P_\eta(\theta f) - \eta g = \theta(f + \eta h).$$

Now, by the same argument as in the proof of the first paragraph, we have

$$P_\eta(\theta f) = \eta g \quad \text{and} \quad f = -\eta h.$$

It is clear that the first equality of the above gives that $g = 0$. Since $\operatorname{ord}\zeta \le \operatorname{ord}\eta$, the second equality of the above together with Lemma 2.1 yields $f = 0$, as desired. Therefore, we finally get that

$$\operatorname{rank}[A_\eta, A_\zeta^*] = \dim \mathcal{H}_\theta \ominus \ker[A_\eta, A_\zeta^*] = 2L,$$

as desired. The proof is complete.

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