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ORIGINAL PAPER

On properties of real selfadjoint operators

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Abstract
In spite of the important applications of real selfadjoint operators and monotone operators, very few papers have dealt in depth with the properties of such operators. In the present paper, we follow A. Rhodius to define the spectrum $\sigma_{\mathbb{F}}(T)$ and the numerical range $W_{\mathbb{F}}(T)$ of a selfadjoint operator T acting on a Hilbert space H over the real/complex field $\mathbb{F}$, and study their topological and geometrical properties which are well known in the complex case $\mathbb{F} = \mathbb{C}$. If $\mathbb{F} = \mathbb{R}$, the results are new; if $\mathbb{F} = \mathbb{C}$, the results constitute an expository body of results containing simple and short proofs of the known facts. The results are then applied to real selfadjoint operators and then to complex normal operators to sharpen their Borel functional calculi with new and shorter proofs avoiding the classical sophisticated Gelfand–Naimark theorem or the Berberian's amalgamation theory. For such a real selfadjoint or complex normal operator N, a normed functional algebra $L^{\infty}_{\mathbb{F}}(N)$ consisting of certain Borel functions defined on $\sigma_{\mathbb{F}}(N)$ is constructed which inherits the isometric properties of the continuous functional calculus $f \mapsto f(N) : C_{\mathbb{F}}(\sigma_{\mathbb{F}}(N)) \to B(H)$.

Keywords Selfadjoint operator · Rhodius spectrum · Spectral mapping theorem · Borel functional calculus · Spectral measure

Mathematics Subject Classification 47B15 · 47A10 · 47A12

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1 Introduction

The present paper adopts the definition of Rhodius [12] for the spectrum $\sigma_{\mathbb{F}}(T)$ and continues to define the numerical range $W_{\mathbb{F}}(T)$ for any bounded linear operator T acting on a Hilbert space H over the real or complex field $\mathbb{F}$; it then proves that, in case $T = T^*$, $\sigma_{\mathbb{R}}(T) \cap \{-||T||, ||T||\} \neq \emptyset$ and $\mathrm{co}(\sigma_{\mathbb{R}}(T)) = \overline{W_{\mathbb{R}}(T)} \subset [-||T||, ||T||]$. The results are well-known in case $\mathbb{F} = \mathbb{C}$ and the independent proofs of the general case given here provide novel short proofs for the known results. This obliges us to acknowledge that the paper is partly expository. The classical properties of complex selfadjoint operators are proven here with arguments not depending on the theory of complex functions such as Liouville's theorem in the proof of $\sigma(T) \neq \emptyset$. However, the results are not valid for real normal operators N as it is clear from the 2×2 real normal operator $N : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$N[1,0]^T = [0,1]^T \text{ and } N[0,1]^T = [-1,0]^T \tag{1.1}$$

that $\sigma_{\mathbb{R}}(N) = \emptyset \neq W_{\mathbb{R}}(N) = \{0\}$.

In Sect. 2, certain topological as well as geometrical properties of $\sigma_{\mathbb{F}}(T)$ and $W_{\mathbb{F}}(T)$ are proven. Let $(H, \mathbb{F})$ be as in Sect. 2 and assume $N = T + i\tau$ for some selfadjoint operators $T, \tau \in B(H)$, where τ is necessarily 0 if $\mathbb{F} = \mathbb{R}$. The results of Sect. 2 are applied to the pair $(N, \mathbb{F})$ to provide a new proof of the known Borel functional calculus for N in which the sophisticated Gelfand–Naimark theorem [9] is avoided and the Berberian's amalgamation theory [2] is shortened via a double integration method. (See (3.7)–(3.10) below.) The latter results, in turn, are used to sharpen the bounded Borel functional calculus $f \mapsto f(N) : \mathcal{A}_b(N) \to B(H)$ to an isometric Borel functional calculus $f \mapsto f(N) : L^\infty_{\mathbb{F}}(N) \to B(H)$, where $\mathcal{A}_b(N) \subset \mathbb{F}^{\mathbb{F}}$ denotes the algebra of all Borel functions bounded on $\sigma_{\mathbb{F}}(N)$ and $L^\infty_{\mathbb{F}}(N)$ is the completion of $\mathcal{A}_b(N)$ with respect to a norm to be defined in Sect. 3. Our improvement of the Borel functional calculus also includes an essential extension of the known spectral mapping theorem $f(\sigma_{\mathbb{F}}(N)) = \sigma_{\mathbb{F}}(f(N))$ for continuous functions to the one for the classes $\mathcal{A}_b(N)$ and $L^\infty_{\mathbb{F}}(N)$.

With each type of the field ($\mathbb{R}$ or $\mathbb{C}$), we may be dealing with separable or nonseparable Hilbert spaces. When the symbol $\mathbb{F}$ is used, we are dealing with a unified proof in which we ignore the type of the field as well as the cardinality of the orthonormal basis of the space. As a general rule, any result involving $\sigma_{\mathbb{F}}(T)$ or $W_{\mathbb{F}}(T)$ is a new result if $\mathbb{F} = \mathbb{R}$. Theorem 3.1 is known (see [9–11]) but the proof contains a novel proof of the Borel functional calculus as described above. No matter what the field $\mathbb{F}$ be, Theorem 3.2 as well as its proof are completely new. We mention that the polynomial spectral mapping theorem $f(\sigma_{\mathbb{C}}(N)) = \sigma_{\mathbb{C}}(f(N))$ may fail to be true if $\mathbb{C}$ is replaced by $\mathbb{R}$; this is revealed by examining the simple 2×2 normal matrix N given at the beginning of the paper; however, we prove it for any real selfadjoint operator. (See Remark 1.2 and Theorem 2.1 below.)

In Sect. 4, we study the spectral measures of real selfadjoint operators. As usual, the arguments remain valid for complex normal operators. Here, the spectral measures are applied to the above-mentioned algebras $L^\infty_{\mathbb{F}}(N)$. For more details on Borel measures, Borel functional calculus, and spectral measures, we may refer to [3–6].

Section 5 deals with (real or complex) normal operators whose domains are generated by their eigenvalues. We first have a different look at direct sums of infinitely many mutually perpendicular subspaces and study extensions of Bessel's inequality and Parseval equalities. We give a very simple definition of the direct sum of an arbitrary family of mutually perpendicular subspaces and then use it to extend the notion of functional calculus for discrete normal operators. A similar extension is obtained for the spectral measure of such operators.

Let us fix some conventions which will be used throughout the paper. In $\mathbb{R}$, we feel free to use expressions such as (a) the polar decomposition $\lambda = e^{i\theta}|\lambda|$, (b) the complex conjugate $\lambda^* = (\alpha + i\beta)^* = \alpha - i\beta$, (c) the real part $\mathfrak{Re}(\lambda)$, (d) the imaginary part $\mathfrak{Im}(\lambda)$, as well as the potentially real expressions (e) i^2, (f) $0i$, (g) $i0$, and (h) $e^{ik\pi}$, provided that they be interpreted as (a') $\lambda = \pm|\lambda|$, (b') $\lambda = \lambda^* = \alpha \pm i0 = \alpha$, (c') $\mathfrak{Re}(\lambda) = \lambda$, (d') $\mathfrak{Im}(\lambda) = 0$, (e') $i^2 = -1$, (f') $0i = 0$, (g') $i0 = 0$ and (h') $e^{ik\pi} = (-1)^k$ for $k \in \mathbb{Z}$, respectively. We will keep the notation $\bar{\gamma}$ to denote the closure of a set γ; in conformity with the notation a^* in a general C^*-algebra, the complex conjugate of a number $\lambda \in \mathbb{F}$ will be denoted by λ^*. If f is any $\mathbb{F}$-valued function, f^* denotes its complex conjugate; that is, $f^*(t) = (f(t))^* = \mathfrak{Re}(f(t)) - i\mathfrak{Im}(f(t))$. Also, if Δ is a subset of a topological linear space, then $\bar{\vee}(\Delta)$ will denote the closure of its linear span. (We define $\bar{\vee}(\emptyset) = \{0\}$.)

For an algebra $\mathcal{A}$ of $\mathbb{F}$-valued functions, we set $\mathcal{A}^* = \{f^* : f \in \mathcal{A}\}$. Then, $\mathcal{A}$ is called a $*$-algebra if $\mathcal{A} = \mathcal{A}^*$. In particular, if $\mathbb{F} = \mathbb{R}$, then $\mathcal{A} = \mathcal{A}^*$ for every $\mathcal{A}$. The notation $\mathfrak{B}(= \mathfrak{B}_L)$ stands for the σ-algebra of all Borel subsets of $\sigma_{\mathbb{F}}(L)$. Also, the notation $\mathcal{D}(L)$, $\mathcal{K}(L)$, and $\mathcal{R}(L)$ are fixed to denote the domain, the kernel, and the range of an operator L.

The present introductory section will be concluded with certain properties of the Rhodius spectrum $\sigma_{\mathbb{F}}(L)$ of a general $L \in B(\mathcal{X})$, where $\mathcal{X}$ is a Banach space over the real or complex field $\mathbb{F}$. The same will be done for the numerical range $W_{\mathbb{F}}(L)$ when L is a Hilbert space operator. Immediate properties of these sets follow the definitions. The case $\mathcal{X} = \{0\}$ causes ambiguity; even if we assume without loss of generality that $\mathcal{X} \neq \{0\}$, the zero subspaces will surprise us in the middle of the arguments. So, we may take care of the ambiguities as follows.

Definition 1.1 For $L \in B(\mathcal{X})$, define its spectrum $\sigma_{\mathbb{F}}(L)$ as follows.

Case 1. $\mathcal{X} \neq \{0\}$.

$$\sigma_{\mathbb{F}}(L) := \mathbb{F} \backslash \{\lambda \in \mathbb{F} : \ \lambda I - L \ \text{ is bijective}\}. \tag{1.2}$$

Case 2. $\mathcal{X} = \{0\}$. We have $B(\mathcal{X}) = \{0\}$ and define

$$\sigma_{\mathbb{F}}(L) = \emptyset. \tag{1.3}$$

Remark 1.1 To justify the case $\mathcal{X} = \{0\}$, note that the operator $L = 0$ is the only linear operator on $\mathcal{X}$, $0^{-1} = 0 = I \in B(\mathcal{X})$ and, in this particular case, $\sigma_{\mathbb{F}}(0) = \emptyset$. Note that if $\mathcal{X} \neq \{0\}$, then 0^{-1} does not exist and $\sigma_{\mathbb{F}}(0) = \{0\}$.

For a scholarly investigation of the definition and properties of the Rhodius spectrum and numerical range of an operator [12] used and verified in the following formulas (1.4)–(1.9), we may refer to Theorem 1.1 of Appell et al. [1]; however, although not explicitly mentioned, we believe the authors of [1] assume the underlying field is complex; otherwise, the statement and the proof of Theorem 1.1 of [1] would contain errors. The theorem claims the validity of the spectral mapping theorem in the real case for which we have a counterexample. (See Remark 1.2 below.) Another error is the application of Liouville's theorem which has no context in real analysis. Therefore, not being able to find a suitable reference, we present a proof which works for both real and complex fields. Of course, instead of the full strength of the spectral mapping theorem $f(\sigma_{\mathbb{F}}(L)) = \sigma_{\mathbb{F}}(f(L))$ claimed in Theorem 1.1 of [1], we only prove the weaker formula $f(\sigma_{\mathbb{F}}(L)) \subset \sigma_{\mathbb{F}}(f(L))$. (See 1.5).)

Theorem 1.1 *The spectrum $\sigma_{\mathbb{F}}(L)$ defined by* (1.2)–(1.3) *satisfies*

$$\overline{\sigma_{\mathbb{F}}(L)} = \sigma_{\mathbb{F}}(L) \subset \{\lambda \in \mathbb{F} : |\lambda| \leq ||L||\}, \tag{1.4}$$

and

$$f(\sigma_{\mathbb{F}}(L)) \subset \sigma_{\mathbb{F}}(f(L)) \quad \text{for all} f \in \mathbb{F}[x]. \tag{1.5}$$

The latter inclusion can be sharpened to equality, if $\mathbb{F} = \mathbb{C}$.

Proof The boundedness of $\sigma_{\mathbb{F}}(L)$ follows from the fact that $L_\lambda(\lambda I - L)x = (\lambda I - L)L_\lambda x \equiv x$, where

$$L_\lambda x := \sum_{n=0}^{\infty} \lambda^{-n-1} L^n x\,, \tag{1.6}$$

$$||L_\lambda x|| \leq \frac{1}{|\lambda| - ||L||} ||x||,\ \forall x \in \mathcal{X},\ \forall \lambda \in \mathbb{F} \text{ satisfying } |\lambda| > ||L||; \tag{1.7}$$

here, the power series is absolutely convergent. This proves that $\sigma_{\mathbb{F}}(L) \subset \{\lambda \in \mathbb{F} : |\lambda| \leq ||L||\}$. The closedness of $\sigma_{\mathbb{F}}(L)$ follows from the fact that if $\lambda_0 \in \mathbb{F} \backslash \sigma_{\mathbb{F}}(L)$, then to every λ in an (open) neighborhood of radius r centered at λ_0 there corresponds an operator R_λ satisfying $R_\lambda(\lambda I - L)x = (\lambda I - L)R_\lambda x \equiv x$ and $||R_\lambda x|| \leq \rho_\lambda ||x||$ for all $x \in \mathcal{X}$, where R_λ, ρ_λ and r are defined as follows:

$$R_\lambda x = -\sum_{n=0}^{\infty} (\lambda - \lambda_0)^n (L - \lambda_0 I)^{-n-1} x, \tag{1.8}$$

$$\rho_\lambda = \frac{1}{r - |\lambda - \lambda_0|}, \quad \text{and } r = ||(L - \lambda_0 I)^{-1}||^{-1} \tag{1.9}$$

Again, here, the power series is absolutely convergent. In particular, $\sigma_{\mathbb{F}}(L)$ is a compact subset of the interval/disk $\{\lambda \in \mathbb{F} : |\lambda| \leq ||L||\}$ in $\mathbb{F}$. This proves (1.4). To prove (1.5), let $\mu \in f(\sigma_{\mathbb{F}}(L))$. Then $\mu = f(\lambda)$ for some $\lambda \in \sigma_{\mathbb{F}}(L)$. Hence,

$\mu - f(t) = (\lambda - t)g(t)$ for some $g \in \mathbb{F}[x]$. Since $\lambda I - L$ is not invertible, it follows that $\mu I - f(L) = (\lambda I - L)g(L)$ is not invertible and that $\mu \in \sigma_{\mathbb{F}}(f(L))$; i.e., $f(\sigma_{\mathbb{F}}(L)) \subset \sigma_{\mathbb{F}}(f(L))$. To show the desired converse, let $\mu \in \sigma_{\mathbb{C}}(f(L))$ and factorize $\mu I - f(L)$ as

$$(\mu I - f(L)) = (\lambda_1 I - L)(\lambda_2 I - L) \dots (\lambda_k I - L) \text{ for some } \lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{C};$$

note that at least one of the factors should be singular, say $\lambda_i \in \sigma_{\mathbb{C}}(L)$. Thus $\mu = f(\lambda_i) \in f(\sigma_{\mathbb{C}}(L))$. □

Remark 1.2 Let $N : \mathbb{R}^2 \to \mathbb{R}^2$ be as in (1.1). $[\sigma_{\mathbb{R}}(N)]^2 = \emptyset \neq \{-1\} = \sigma_{\mathbb{R}}(N^2)$. Thus, the inclusion in (1.5) cannot be sharpened, and the equality 1.14 in Theorem 1.1 of [1] is not true in the real case.

Corollary 1.1 *For a general operator $L \in B(\mathcal{X})$ and a point $\lambda_0 \notin \sigma_{\mathbb{F}}(L)$,*

$$||L|| \geq \max|\sigma_{\mathbb{F}}(L)| := \max\{|\lambda| : \lambda \in \sigma_{\mathbb{F}}(L)\} \text{ and}$$
$$||(L - \lambda_0 I)^{-1}|| \geq 1/\text{dist}(\lambda_0, \sigma_{\mathbb{F}}(L)).$$

Corollary 1.2 *If U is a unitary operator on a Hilbert space H, then $\sigma(U) \subset \{z \in \mathbb{F} : |z| = 1\}$.*

Another concept needed in the spectral theory is the notion of the numerical range $W_{\mathbb{F}}(L)$ for any Hilbert space operator $L \in B(H)$:

$$W_{\mathbb{F}}(L) := \{\langle Lx, x\rangle : x \in H, ||x|| = 1\}. \tag{1.10}$$

The following theorem is well known in the complex case; the proof here is readjusted to cover the real case, too. We found no author to be particularly interested in the real case, though, we believe there must be some proof somewhere. Hence, for reader's convenience, we present the following short unified proof which works for both $\mathbb{R}$ and $\mathbb{C}$.

Theorem 1.2 *The numerical range of a Hilbert space operator $L \in B(H)$ is a convex subset of $\mathbb{F}$ when the latter is identified as one of the Euclidean spaces $\mathbb{R}$ or $\mathbb{R}^2$.*

Proof Let $u, v \in H$ be any two linearly independent unit vectors such that $\langle Lu, u\rangle = \lambda \neq \langle Lv, v\rangle = \mu$. We must show that the open line segment joining λ and μ lies in $W_{\mathbb{F}}(L)$. Replace L by $(\mu - \lambda)^{-1}(L - \lambda I)$ to assume without loss of generality that $\lambda = 0$ and $\mu = 1$. Let $L = A + B$ be the decomposition of L into its (obviously) selfadjoint part $A = (L + L^*)/2$ and its skew selfadjoint part $B = L - A$; the latter in the sense that $\underline{\langle Bx, y\rangle} = -\langle x, By\rangle$. Let $x = zv + u$ for $z \in \mathbb{F}$. Then, $||x|| > 0$, $\langle Bu, v\rangle = -\langle u, Bv\rangle = -\langle Bv, u\rangle$, and

$$\begin{aligned}\langle Lx, x\rangle =&\langle L(zv+u), zv+u\rangle\\ =&|z|^2 + z\langle Av, u\rangle + \bar{z}\langle Au, v\rangle + z\langle Bv, u\rangle + \bar{z}\langle Bu, v\rangle\\ =&|z|^2 + 2\,\Re e(z\langle Av, u\rangle) + 2i\,\Im m(z\langle Bv, u\rangle).\end{aligned}$$

Consider the polar decomposition $\langle Bv, u\rangle) = e^{i\theta}|\langle Bv, u\rangle)|$. (If $\mathbb{F} = \mathbb{R}$, then $\theta = 0$ or π.) Let $z = te^{-i\theta}$ ($t \in \mathbb{R}$), let $b = 2\,\Re e(e^{-i\theta}\langle Av, u\rangle)$, let $b' = 2\,\Re e(e^{-i\theta}\langle v, u\rangle)$ and let

$$\phi(t) = \frac{t^2 + bt}{t^2 + b't + 1} = \frac{\langle Lx, x\rangle}{||x||^2},\ \forall\, t \in \mathbb{R}.$$

Then, $\phi(t)$ is a continuous rational function traveling in $W_{\mathbb{F}}(L)$ and covering the line segment $(0, 1)$ as t runs in $(0, \infty)$. □

2 Polynomial spectral mapping theorem

In this section, we prove the spectral mapping theorem for real or complex selfadjoint operators and, as a consequence, we show that the spectrum of any selfadjoint operator is nonempty. Of course, this is well known in the complex case and is proven here for the first time for real case. We first state a lemma whose proof in the complex case is well known and works for the real case, too.

Lemma 2.1 *Let $N \in B(H)$ be a real or complex normal operator. Then $\sigma_{\mathbb{F}}(N)$ consists of approximate point spectra. Moreover, $\sigma_{\mathbb{F}}(N) \subset \overline{W_{\mathbb{F}}(N)}$. In particular, if $T = T^*$, then $\sigma_{\mathbb{F}}(T) \subset \overline{W_{\mathbb{F}}(T)} \subset \mathbb{R}$.*

Theorem 2.1 *If $T \in B(H)$ is a real or complex selfadjoint operator, then*

$$\sigma_{\mathbb{F}}(f(T)) = f(\sigma_{\mathbb{F}}(T)) \subset f(\mathbb{R}),\ \forall\, f \in \mathbb{F}[x].$$

Proof Let $f_m \in \mathbb{F}[x]$ be a prime monic polynomial of order m and define $S_m = f_m(T)$. Then, f_m is necessarily of the form

$$f_m(t) = (t-\alpha)^m - (i\beta)^m,\ t \in \mathbb{F},\ \alpha, \beta \in \mathbb{R};\ m = 1, 2. \tag{2.1}$$

Note that if $\mathbb{F} = \mathbb{C}$, then $m = 1$; also, if $\mathbb{F} = \mathbb{R}$, then either $m = 2$ in which case $-(i\beta)^2 = \beta^2 > 0$ or $m = 1$ in which case $\beta = 0$.

In general, if $u \in H$ is any unit vector, then

$$\begin{aligned}||S_1u||^2 =&||S_1^*u||^2 = \langle (T-\alpha I - i\beta I)u, (T - \alpha I - i\beta I)u\rangle\\ =&\langle (T - \alpha I + i\beta I)(T - \alpha I - i\beta I)u, u\rangle = \langle [(T-\alpha I)^2 + \beta^2 I]u, u\rangle\\ =&||(T-\alpha I)u||^2 + \beta^2 \ge \beta^2,\end{aligned}$$

and

$$||S_2u|| =||S_2^*u|| = \langle[(T-\alpha I)^2+\beta^2)]u,u\rangle = \langle(T-\alpha I)^2u,u\rangle + \langle\beta^2u,u\rangle$$
$$=||(T-\alpha I)u||^2+\beta^2 \geq \beta^2.$$

It follows that 0 is not in the approximate point spectrum of S_m if $\beta \neq 0$. Therefore, S_m is invertible if $\beta \neq 0$. Now, consider the prime factorization $\mu - f = g_1g_2\dots g_k$ for *some* $\mu \in \mathbb{F}$ and *some* $f \in \mathbb{F}[x]$, where each g_i is a prime polynomial of the form f_m defined in (2.1). If $\mu I - f(T)$ is singular, at least one of the factors $g_i(T)$ is singular. We assume without loss of generality that $i = k = 1, g_1(t) = f_m(t)$ and $f_m(T)$ is singular. Then, $\beta = 0$ and, hence, $m \neq 2$. It follows that $g_1(T) = T - \alpha I$ and, thus, $\alpha \in \sigma_{\mathbb{F}}(T)$. Therefore, $\mu - f(\alpha) = 0$ or, equivalently, $\mu = f(\alpha) \in f(\sigma_{\mathbb{F}}(T))$. This, together with (1.5), completes the proof of the theorem. □

The next theorem shows that, regardless of the type of the underlying field $\mathbb{F}$, the spectrum of a selfadjoint operator on a nonzero Hilbert space is nonempty. Here, the extreme points of the numerical range play an important role. To prove the main result of this section, we recall the following simple lemma from elementary calculus.

Lemma 2.2 *Assume* $a, b, c \in \mathbb{R}$ *and* $at^2 + bt + c$ *does not change sign as the real number* t *traverses* $(-\infty, +\infty)$. *Then, the discriminant* $\Delta := b^2 - 4ac \leq 0$.

The next theorem is the first which is not applicable to the case of real normal operators. Throughout the remainder of the paper, T stands for a selfadjoint operator on H.

Theorem 2.2 *Let* $T = T^* \in B(H)$. *Then,*

$$\emptyset \neq \{\text{extreme points of } \overline{W_{\mathbb{F}}(T)}\} \cap \{-||T||, ||T||\} \subset \sigma_{\mathbb{F}}(T) \subset \overline{W_{\mathbb{F}}(T)}; \quad (2.2)$$

in particular, we show that, if λ *is an extreme point of* $\overline{W_{\mathbb{F}}(T)}$, *then*

$$\lim_n (\lambda I - T)\xi_n = 0 \text{ whenever } \lim_n \langle T\xi_n, \xi_n\rangle = \lambda \text{ and } ||\xi_n|| = 1 (n = 1, 2, 3, \cdots). \quad (2.3)$$

Moreover,

$$||N^*N|| = ||N||^2 \text{ whenever } N \in B(H),\ N^*N = NN^*; \quad (2.4)$$

$$||f(T)|| = ||f||_{\sigma_{\mathbb{F}}(T)} := \max\{|f(t)| : t \in \sigma_{\mathbb{F}}(T)\} \text{ for all } f \in \mathbb{F}[x]. \quad (2.5)$$

Proof The inclusion $\sigma_{\mathbb{F}}(T) \subset \overline{W_{\mathbb{F}}(T)}$ follows from Lemma 2.1. Now, let $\lambda \in \mathbb{F}$ be an extreme point of $\overline{W_{\mathbb{F}}(T)}$ and choose a finite or infinite sequence of unit vectors $\{\xi_n\}_n$ such that $\lim_n \langle T\xi_n, \xi_n\rangle = \lambda$. Then, λ is either the supremum or the infimum of the set. Assume without loss of generality that $\lambda = 0 = \sup(\overline{W_{\mathbb{F}}(T)})$ and $W_{\mathbb{F}}(T) \subset (-\infty, 0]$. Thus, $\{\xi_n\}_n$ is a sequence of unit vectors such that $\lim_n \langle T\xi_n, \xi_n\rangle = 0$. (Certainly, such a sequence exists.) We must show that $\lim_n T\xi_n = 0$. If not, there exists a subsequence $\{\psi_n\}_n$ of $\{\xi_n\}_n$ and a positive number ϵ such that $||T\psi_n|| \geq \epsilon$ for all $n \in \mathbb{N}$. Then

$$0 \geq \limsup_n \langle T(\psi_n + tT\psi_n), \psi_n + tT\psi_n\rangle$$
$$= \limsup_n (t^2\langle T^2\psi_n, T\psi_n\rangle + 2t||T\psi_n||^2).$$

Since the sequences $\{\langle T^2\psi_n, T\psi_n\rangle\}_n$ and $\{2||T\psi_n||^2\}_n$ are bounded, one can further reduce to a subsequence to assume without loss of generality that $\lim_n\langle T^2\psi_n, T\psi_n\rangle = a$ and by a repeated such reduction that $\lim_n 2||T\psi_n||^2 = b \geq \epsilon^2$ for some $a, b \in \mathbb{R}$. Thus, $at^2 + bt \leq 0$ on $\mathbb{R}$ and, with the notation of Lemma 2.2, $0 < \epsilon^2 \leq b^2 = b^2 - 4ac = \Delta \leq 0$; a contradiction. Therefore, $\lim_n ||T\xi_n|| = 0$ and $0 \in \sigma_{\mathbb{F}}(T)$. Going back to a general extreme point λ, it follows that $\sigma_{\mathbb{F}}(T)$ contains the infimum and the supremum of $W_{\mathbb{F}}(T)$ and, hence, the spectrum of T is nonempty. Consequently, $||T||^2 \geq ||T^2|| \geq \sup W_{\mathbb{F}}(T^2) \in \sigma_{\mathbb{F}}(T^2)$. (Note that $W_{\mathbb{F}}(T^2) \subset [0, ||T^2||]$.)

To complete the proof of (2.2)–(2.3), it remains to show that $\sigma_{\mathbb{F}}(T) \cap \{-||T||, ||T||\} \neq \emptyset$. Let x_n be a sequence of unit vectors such that $||Tx_n|| \to ||T||$ as $n \to \infty$. Then $\langle T^2x_n, x_n\rangle = ||Tx_n||^2 \to ||T||^2$ as $n \to \infty$ which implies that $||T||^2 = ||T^2|| = \sup W_{\mathbb{F}}(T^2) = \max \sigma_{\mathbb{F}}(T^2)$. Now, Theorem 2.1 implies that $||T||^2 \in (\sigma_{\mathbb{F}}(T))^2$ and, hence, $\sigma_{\mathbb{F}}(T)$ contains $||T||$ and/or $-||T||$.

For (2.4), note that N^*N is a (real or complex) selfadjoint operator and, hence,

$$||N||^2 = \sup_{||x||=1} ||Nx||^2 = \sup_{||x||=1} \langle N^*Nx, x\rangle = ||N^*N||.$$

Finally, to prove (2.5), let $f(x) \equiv \sum_{j=0}^n (a_j + ib_j)x^j$ and define $u(x) = \sum_{j=0}^n a_jx^j$ and $v(x) = \sum_{j=0}^n b_jx^j$. Then, for all $t \in \sigma_{\mathbb{F}}(T)(\subset \mathbb{R})$, $f(t) = u(t) + iv(t)$, $|f(t)|^2 = u(t)^2 + v(t)^2$ and u, v are polynomials with real coefficients. It follows that

$$||f(T)||^2 = ||[f(T)]^*f(T)|| = ||(u^2 + v^2)(T)|| = \sup\{|s| \ : \ s \in \sigma_{\mathbb{F}}((u^2 + v^2)(T))\}$$
$$= \sup\{|s| \ : \ s \in (u^2 + v^2)(\sigma_{\mathbb{F}}(T))\} = \sup\{u^2(t) + v^2(t) \ : \ t \in \sigma_{\mathbb{F}}(T)\}$$
$$= \sup\{|f(t)|^2 \ : \ t \in \sigma_{\mathbb{F}}(T)\} = ||f||^2_{\sigma_{\mathbb{F}}(T)}.$$

□

Corollary 2.1 *Let $T = T^* \in B(H)$. The following assertions are true.*

(a) $\min \sigma_{\mathbb{F}}(T) = \inf W_{\mathbb{F}}(T) \leq \sup W_{\mathbb{F}}(T) = \max \sigma_{\mathbb{F}}(T)$.
(b) *If an extreme point of $\overline{W_{\mathbb{F}}(T)}$ belongs to $W_{\mathbb{F}}(T)$, then it is an eigenvalue of T.*
(c) *If $\sigma_{\mathbb{F}}(T) \subset \{\lambda\}$, then $T = \lambda I$.*

Proof Since $\inf(W_{\mathbb{F}}(T))$ and $\sup(W_{\mathbb{F}}(T))$ are the extreme points of $\overline{W_{\mathbb{F}}(T)}$, it follows from (2.2) that $\min(\sigma_{\mathbb{F}}(T)) \leq \inf(W_{\mathbb{F}}(T)) \leq \sup(W_{\mathbb{F}}(T)) \leq \max(\sigma_{\mathbb{F}}(T))$; the desired equalities are immediate from (2.2). This proves (a); the proof of (b) follows from (2.3).

For (c), if $\sigma_{\mathbb{F}}(T) = \emptyset$, then $H = \{0\}$ and $S = 0 = \lambda I$; otherwise, $W_{\mathbb{F}}(T) = \{\lambda\}$. Let $K = \mathcal{K}(\lambda I - T)$ and write $H = K \oplus K^{\perp}$. Observe that $\sigma_{\mathbb{F}}(T|_{K^{\perp}}) \subset \{\lambda\}$ and that λ is not an eigenvalue of $T|_{K^{\perp}}$. Thus, $\mathcal{K}^{\perp} = \{0\}$. □

The following example shows that the spectral theory for real nonselfadjoint normal operators cannot go beyond Theorem 2.2.

Example 2.1 Consider the normal (unitary) operator $N = U_1 \oplus U_2 \in B(\mathbb{R} \oplus \mathbb{R}^2)$ with the following matrix representations:

$$U_1 = [1]; \text{ and } U_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then for $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ with $x_1^2 + x_2^2 + x_3^2 = 1$, we have $\langle Nx, x\rangle = x_1^2$ and, hence, $\sigma_{\mathbb{R}}(N) = \{1\} \subset W_{\mathbb{R}}(N) = [0, 1]$. However, the extreme point 0 of $W_{\mathbb{R}}(N)$ is not an eigenvalue of the (unitary) operator N. Also, note that $\max_{t\in\{1\}} |t - 1| = 0 < 1 = ||N - I||$. In fact, the 2×2 unitary part U_2 of N has an empty spectrum. Also, if $f(t) \equiv t^2$, then $f(U_2) = U_2^2 = -I$ and $\sigma_{\mathbb{R}}(f(U_2)) = \{-1\} \neq \emptyset = f(\sigma_{\mathbb{R}}(U_2))$.

3 Borel functional calculus for normal operators

In this section, we study the Borel functional calculus of normal operators of the form $N = T + i\tau$, where (T, τ) is a commuting pair of selfadjoint operators on a Hilbert space H over the field $\mathbb{F}$. In case $\tau \neq 0$, the field $\mathbb{F}$ is necessarily equal to $\mathbb{C}$; otherwise, $\mathbb{F}$ can be either $\mathbb{C}$ or $\mathbb{R}$. For a vector $x \in H$ and a normal operator $N = T + i\tau \in B(H)$, the smallest invariant subspace and the smallest reducing invariant subspace of N containing x can be, respectively, formulated as follows:

$$\begin{aligned} \mathcal{Z}(N;x) =& \bar{\vee}\{N^m x : \ m = 0, 1, 2, \cdots\} \text{ and} \\ \mathcal{Z}(N, N^*;x) :=& \mathcal{Z}(T, \tau;x) = \bar{\vee}\{N^m N^{*n} x : \ m, n = 0, 1, 2, \cdots\} \\ =& \bar{\vee}\{T^m \tau^{*n} x : \ m, n = 0, 1, 2, \cdots\}; \end{aligned}$$

i.e., the closure of the linear span of all $p(T, \tau)x$ as p runs in the collection of all bivariate polynomials with coefficients in $\mathbb{F}$. In case $\tau = 0$, we have $\mathcal{Z}(N, N^*;x) = \mathcal{Z}(T;x)$. Note that every closed invariant subspace K of $\{T, \tau\}$ is a reducing one; i.e., $N = N_1 \oplus N_2$ with respect to the direct sum $H = K \oplus K^{\perp}$, where

$$\sigma_{\mathbb{F}}(N) = \sigma_{\mathbb{F}}(N_1) \cup \sigma_{\mathbb{F}}(N_2) \text{ and } N_\gamma N_\gamma^* = N_\gamma^* N_\gamma \ (\gamma = 1, 2).$$

The following lemma is a consequence of the results of the previous section; its simple proof is left to the interested reader.

Lemma 3.1 (Continuous functional calculus for selfadjoint operators) *Let $T = T^* \in B(H)$ and let* $\mathbf{1}$ *and* $\mathbf{id}$ *be the polynomials $p(t) \equiv 1$ and $p(t) \equiv t$, respectively. Then, the polynomial functional calculus $p \mapsto p(T) : \mathbb{F}[x] \to B(H)$ is an isometric *-algebra isomorphism such that $\mathbf{1}(T) = I$, $\mathbf{id}(T) = T$. Moreover, the functional calculus can be extended to an isometric *-algebra isomorphism $f \mapsto f(T) : C_{\mathbb{F}}(\sigma_{\mathbb{F}}(T)) \to B(H)$ such that*

$$||f(T)|| = ||f||_{\sigma_{\mathbb{F}}(T)}, \ \forall f \in C_{\mathbb{F}}(\sigma_{\mathbb{F}}(T)).$$

To prepare ourselves for the main theorem, we need the following definition.

Definition 3.1 A function $f : \mathbb{R}^2 \to \mathbb{C}$ may be also regarded as a function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ by $\tilde{f}(z) = f(s(z), t(z))$, where $s(z) = (z + z^*)/2$ and $t(z) = (z - z*)/(2i)$. Now, if $N = T + i\tau$ is a normal operator, then we may write $\tilde{f}(N)$ to mean $f(T, \tau)$ provided that the latter expression is somehow defined. If no ambiguity arises, we may even drop the tilde sign to write $f(N)$ instead of $f(T, \tau)$.

Theorem 3.1 *Let $N \in B(H)$ be a real selfadjoint or a complex normal operator. Then there exist a positive Borel measure $(\mathbb{F}, \mathfrak{B}, \mu_x^N)$ and a unitary operator $U_N : L^2_{\mathbb{F}}(\mu_x^N) \to \mathcal{Z}(N, N^*;x)$ such that for all $f \in \mathbb{F}[x]$, for all $\phi \in L^2_{\mathbb{F}}(\mu_x^N)$ and for all $y \in H$,*

$$U_{T,\tau;x}f = f(N)x, \ \ ||\mu_x^N|| = \mu_x^N(\sigma_{\mathbb{F}}(N)) = ||x||^2, \ \text{and} \tag{3.1}$$

$$\langle U_{T,\tau;x}\phi, y\rangle = \int \phi(s)\{[U^*_{T,\tau;x}Py](s)\}^* d\mu_x^N(s), \tag{3.2}$$

where $P : H \to H$ is the orthogonal projection onto $\mathcal{Z}(N, N^*;x)$. Moreover, if $\mathcal{A}_b(S)$ denotes the *-algebra of all $\mathbb{F}$-valued Borel functions bounded on $\sigma_{\mathbb{F}}(S)$, then the mapping $f \mapsto f(N) : \mathcal{A}_b(N) \to B(H)$ defined by $f(N)x = U_{T,\tau;x}f$ is a *-algebra homomorphism extending the continuous functional calculus of Lemma 3.1. Moreover,

$$Qf(N) = f(N)Q, \ \text{whenever} \ QN = NQ, \ \text{for some} \ Q \in B(H). \tag{3.3}$$

Proof Assume for the moment that $\tau = 0$ and, hence, $\mathbb{F} = \mathbb{R}$. In view of Lemma 3.1, for each $x \in H$, the mapping $f \mapsto \langle f(T)x, x\rangle : C_{\mathbb{R}}(\sigma_{\mathbb{R}}(T)) \to \mathbb{R}$ defines a positive linear functional Φ_x such that $|\Phi_x(f)| = |\langle f(T)x, x\rangle| \le ||f(T)||\,||x||^2 = ||f||_{\sigma_{\mathbb{R}}(T)}\,||x||^2$. Letting $f = \mathbf{1}$ yields $|\langle f(T)x, x\rangle| = ||x||^2$ and, thus, $||\Phi_x|| = ||x||^2$. Hence, there exists a unique positive measure $(\sigma_{\mathbb{R}}(T), \mathfrak{B}, \mu_x^T)$ such that $||\mu_x^T|| = ||x||^2$ and $\langle f(T)x, x\rangle = \int f d\mu_x^T$ for all $f \in C_{\mathbb{R}}(\sigma_{\mathbb{R}}(T))$. Also, note that

$$\begin{aligned} ||f(T)x||^2 &= \langle f(T)x, f(T)x\rangle = \langle f^*(T)f(T)x, x\rangle = \langle |f|^2(T)x, x\rangle \\ &= \int |f(s)|^2 d\mu_x^T(s) = ||f||^2_{L^2(\mu_x)}, \ \forall f \in \sigma_{\mathbb{R}}(T). \end{aligned} \tag{3.4}$$

The equation (3.4) defines an isometry U_0 from a dense subset of $L^2_{\mathbb{R}}(\mu^T_x)$ onto a dense subset of $\mathcal{Z}(T;x)$ which can be extended to a unitary operator $U_{T;x}$ from $L^2_{\mathbb{R}}(\mu^T_x)$ onto $\mathcal{Z}(T;x)$ satisfying (3.1). The proof of (3.2) follows from the fact that ϕ, $U^*_{T;x}Py \in L^2_{\mathbb{R}}(\mu^T_x)$ and

$$\langle U_{T;x}\phi, y\rangle = \langle U_{T;x}\phi, Py\rangle = \langle \phi, U^*_{T;x}Py\rangle = \int \phi(s)\{[U^*_{T;x}Py](s)\}^* d\mu^T_x(s).$$

Next, let $f \in L^\infty_{\mathbb{R}}(\mu^T_x)$ and define $M_f : L^2_{\mathbb{R}}(\mu^T_x) \to L^2(\mu^T_x)$ by $M_f\phi = f\phi$. It is well known that M_f is a bounded normal operator of norm $||M_f|| = ||f||_\infty$ with adjoint $M^*_f = M_{f^*}$. This makes us ready to define the following Borel functional calculus for any real or complex selfadjoint operator T:

$$f \mapsto f(T) : \mathcal{A}_b(T) \to B(H) \text{ via } f(T)x = U_{T;x}f,\ \forall\, x \in H. \tag{3.5}$$

We show that $f(T)$ is a well-defined bounded linear operator defined on H which satisfies (3.3), $||f(T)|| \le ||f||_{\sigma_{\mathbb{R}}(T)}$, $f(T)g(T) = (fg)(T)$, and $f(T)^* = f^*(T)$ ($\forall f, g \in \mathcal{A}_b(T)$). For arbitrary $x, y \in H$ and $\alpha \in \mathbb{R}$, define $\omega = \mu^T_x + \mu^T_y + \mu^T_{\alpha x+y} + \mu^T_{Qx} + \mu^T_{g(T)x}$ and choose sequences $\{f_n\}$ and $\{g_n\}$ of continuous functions converging to f and g in $L^2_{\mathbb{R}}(\omega)$, respectively. We modify f_n and g_n to assume without loss of generality that $||f_n||_{\sigma_{\mathbb{R}}(T)} \le ||f||_{\sigma_{\mathbb{R}}(T)} + 1$ and $||g_n||_{\sigma_{\mathbb{R}}(T)} \le ||g||_{\sigma_{\mathbb{R}}(T)} + 1$ for all $n \in \mathbb{N}$. Hence, in view of the bounded convergence theorem, f_ng_n converges to fg in $L^2_{\mathbb{R}}(\omega)$. (Note that, ω is supported on $\sigma_{\mathbb{R}}(T) \subset \mathbb{R}$.) Then, for $u = x, y, \alpha x + y, Qx$, and $g(T)x$,

$$f(T)u = U_{T,u}f = \lim_n U_{T,u}f_n = \lim_n f_n(T)u. \tag{3.6}$$

Therefore, the following assertions are true.

(a) $f(T)(\alpha x + y) = \lim_n f_n(T)(\alpha x + y) = \alpha \lim_n f_n(T)x + \lim_n f_n(T)y = \alpha f(T)x + f(T)y$ which implies that $f(T)$ is a well-defined linear operator.

(b) In view of (3.6) and the commutativity property $Qf_n(T) = f_n(T)Q$, it follows that $Qf(T)x - f(T)Qx = \lim_n[Qf_n(T)x - f_n(T)Qx] = 0$, which proves (3.3).

(c) In view of (3.4),

$$||f(T)x||^2 = \lim_n ||f_n(T)x||^2 = \lim_n \int |f_n|^2 d\mu^T_x \le \lim_n ||f_n||^2_{\sigma_{\mathbb{R}}(T)}||x||^2$$
$$\le ||f||^2_{\sigma_{\mathbb{R}}(T)}||x||^2,$$

which implies that $f(T)$ is bounded.

(d) $\langle f(T)x, y\rangle = \lim_n\langle f_n(T)x, y\rangle = \lim_n\langle x, f_n(T)^*y\rangle = \lim_n\langle x, f^*_n(T)y\rangle = \langle x, f^*(T)y\rangle$ which shows that $f(T)^* = f^*(T)$.

(e) Finally,

$$\begin{aligned}||\{(fg)(T)-f(T)g(T)\}x|| &= \lim_n ||\{(f_ng_n)(T)x - f_n(T)(g(T)x)||\\ &= \lim_n ||f_n(T)\{g_n(T)x - g(T)x\}||\\ &\le \lim_n ||f_n(T)||\ ||[g_n(T)-g(T)]x||\\ &\le (||f||_{\sigma_{\mathbb{F}}(T)}+1)\lim_n [\int |g_n - g|^2 d\mu_x^T]^{1/2} = 0,\end{aligned}$$

which implies that $(fg)(T) = f(T)g(T) = g(T)f(T)$ and that $f(T)$ is normal. This completes the proof of the theorem for the case $N = T$.

We now assume $N = T + i\tau$ with $\tau \neq 0$; here, the underlying field $\mathbb{F}$ is necessarily equal to $\mathbb{C}$. Choose a fixed rectangle $\Gamma = [\bar{a}, \bar{b}] \times [\bar{c}, \bar{d}]$ such that $\sigma_{\mathbb{F}}(N) \subset \Gamma$. Let $\mathfrak{S}$ be the algebra generated by all sets of the form $[a,b)\times[c,d)\cap\Gamma$. Define $\mu_x^N : \mathfrak{S} \to [0,\infty]$ as follows:

$$\begin{aligned}\mu_x^N([a,b)\times[c,d)) &= \langle \chi_{[a,b)}(T)x, \chi_{[c,d)}(\tau)x\rangle\\ &= \int \chi_{[a,b)}(s)\{[U^*_{T;x}P\chi_{[c,d)}(\tau)x](s)\}^* d\mu_x^T(s),\end{aligned} \tag{3.7}$$

where $P : H \to H$ is the orthogonal projection onto $\mathcal{Z}(T;x)$. To extend μ_x^N to a Borel σ-additive measure, we have to show that it is σ-additive on the ring $\mathfrak{S}$. Let $[a,b)\times[c,d) = \dot{\cup}_n([a_n,b_n)\times[c_n,d_n))$. Then

$$\begin{aligned}&\mu_x^N([a,b)\times[c,d)) = \langle \chi_{[a,b)}(T)x, \chi_{[c,d)}(\tau)x\rangle\\ &= \int \chi_{[a,b)}(s)\{[U^*_{\tau;x}P\chi_{[c,d)}(\tau)x](s)\}^* d\mu_x^T(s)\\ &= \int \left(\int_c^d \sum_{n=1}^{\infty} \chi_{[a_n,b_n)}(s)\frac{\chi_{[c_n,d_n)}(t)}{d_n-c_n}\{[U^*_{\tau;x}P\chi_{[c_n,d_n)}(\tau)x](s)\}^* dt\right) d\mu_x^T(s)\\ &= \sum_{n=1}^{\infty}\int\int_c^d \chi_{[a_n,b_n)}(s)\frac{\chi_{[c_n,d_n)}(t)}{d_n-c_n}\{[U^*_{\tau;x}P\chi_{[c_n,d_n)}(\tau)x](s)\}^* dt d\mu_x^T(s)\\ &= \sum_{n=1}^{\infty}\int \chi_{[a_n,b_n)}(s)\{[U^*_{\tau;x}P\chi_{[c_n,d_n)}(\tau)x](s)\}^* d\mu_x^T(s)\\ &= \sum_{n=1}^{\infty}\mu_x^N([a_n,b_n)\times[c_n,d_n)).\end{aligned} \tag{3.8}$$

Thus, μ_x^N is a Borel positive measure on Γ and can be extended to a measure space $(\mathbb{C}, \mathfrak{B}, \mu_x^N)$ by defining $\mu_x^N(B) = \mu_x^N(B\cap\Gamma)$ for all $B \in \mathfrak{B}$.

Thus, for simple Borel functions $f = \sum_{i=1}^m c_i\chi_{E_i}$ and $g = \sum_{j=1}^n d_j\chi_{F_j}$,

$$\langle f(T)g(\tau)x, x\rangle = \langle f(T)x, g^*(\tau)x\rangle = \int f(s)g(t)d\mu_x^N(s,t). \tag{3.9}$$

It follows easily that (3.9) as well as the following equation hold for all bivariate polynomials $f(s,t)$ with coefficients in $\mathbb{C}$:

$$||f(T,\tau)x||^2 = \langle |f|^2(T,\tau)x, x\rangle = \int |f(s,t)|^2 d\mu_x^N(s,t). \tag{3.10}$$

Thus, in view of the Stone–Weierstrass Theorem, $\mathcal{Z}(N,N^*;x) = U_{T,\tau;x}L^2_{\mathbb{C}}(\mu_x^N)$ for some unitary operator $U_{T,\tau;x} : L^2_{\mathbb{C}}(\mu_x^N) \to \mathcal{Z}(N,N^*;x)$ satisfying $U_{T,\tau;x}f = f(T,\tau)x$ for all functions of the form $f(s,t) = \sum_{j,k=1}^n c_{jk}g_j(s)h_k(t)$ with $g_j \in C_{\mathbb{C}}([\bar{a},\bar{b}])$, $h_k \in C_{\mathbb{C}}([\bar{c},\bar{d}])$.

We now replace T by $N = T + i\tau$, μ_x^T by μ_x^N, and $U(T;\,x)$ by $U(T,\tau;x)$, and then mimic the arguments leading to (3.5)–(3.6) to complete the proof of the theorem in the general case. □

We now try to equip $\mathcal{A}_b(N)$ with a norm with respect to which the functional calculus $f \mapsto f(T,\tau) : \mathcal{A}_b(N) \to B(H)$ is an isometric *-algebra isomorphism. Note that $\mathcal{A}_b(N) \subset \cap_{x\in H}L^\infty(\mu_x^N)$ and this will help us in the norm construction. Let us first fix a notation to be used in the theorem.

Definition 3.2 Let $(\mathbb{F},\mathfrak{B},\mu)$ be a positive Borel measure and let $f : \mathbb{F} \to \mathbb{F}$ be measurable. The $[\mu]$essential range of f, denoted by $[\mu]$ess.range(f), is the set of all $\zeta \in \mathbb{F}$ for which $\mu(f^{-1}(\Delta)) > 0$ for every neighborhood Δ of ζ. Also, $\mathrm{supp}(\mu) = \cap\{C = \bar{C} :\ \mu(\mathbb{F}\backslash C) = 0\}$.

Remark 3.1 It is easy to see that, for every μ-measurable set γ and every μ-measurable function $f : \mathbb{F} \to \mathbb{F}$,

$$\mathrm{supp}(\mu|_\gamma) \subset [\mu]\mathrm{ess.range}(\mathbf{id}\ \cdot \chi_\gamma) \subset \mathrm{supp}(\mu|_\gamma) \cup \{0\}, \tag{3.11}$$

$$[\mu]\mathrm{ess.range}(f) = \cap\{\overline{g(\mathrm{supp}(\mu))} :\ g = f\ a.e.[\mu]\}, \tag{3.12}$$

where $(\mathbf{id} \cdot \chi_\gamma)(t) \equiv t\chi_\gamma(t)$.

The following theorem is an improvement of Theorems 4.6, 4.8, 4.11 of [7] and of Theorem 5.5.3 of [13].

Theorem 3.2 *Suppose the notation and the hypotheses of Theorem 3.1 are valid. For each* $f \in \mathcal{A}_b(N)$, *define* $\nu(f) = \sup_{x\in H} ||f||_{L^\infty(\mu_x^N)}$. *Then,* ν *is a seminorm on* $\mathcal{A}_b(N)$ *which induces a norm* $||\cdot||_N$ *on the completion* $L^\infty(N)$ *of the quotient space* $\mathcal{A}_b(N)/\{f : \nu(f) = 0\}$ *with respect to which the functional calculus* $f \mapsto f(T,\tau) : L^\infty(N) \to B(H)$ *is an isometric* **-algebra isomorphism. Moreover, the following assertions are true for all* $f, g \in L^\infty(N)$.

(i) $\sigma_{\mathbb{F}}(f(T,\tau)) = \Lambda$, *where* $\Lambda := \cup_{x\in H}\ [\mu_x^N]\mathrm{ess.range}(f)$.
(ii) $\sigma_{\mathbb{F}}(N) = \sigma_{\mathbb{F}}(T + i\tau) = \cup_{x\in H}\ \mathrm{supp}(\mu_x^N)$ *and* $\sigma_{\mathbb{F}}(f(T,\tau)) \subset \overline{f(\sigma_{\mathbb{F}}(N))}$.
(iii) $f(T,\tau) = g(T,\tau)$ *whenever* $f(s,t) = g(s,t)$ *for all* $(s,t) \in \sigma_{\mathbb{F}}(N)$.
(iv) $||f||_N = ||f||_{\sigma_{\mathbb{F}}(N)} := \max\{|f(t)| :\ t \in \sigma_{\mathbb{F}}(N)\}$ *if* $f|_{\sigma_{\mathbb{F}}(N)}$ *is continuous.*

Proof Write $||f||_x$ as a short form for $||f||_{L^\infty(\mu_x^N)}$. It is clear that $\nu(f) \geq 0$, $\nu(cf) = \sup_x ||cf||_x = |c| \sup_x ||f||_x = |c|\nu(f)$, and

$$\nu(f+g) = \sup_x ||f+g||_x \leq \sup_x ||f||_x + \sup_x ||g||_x = \nu(f) + \nu(g),$$

for all $c \in \mathbb{F}$, for all $x \in H$ and for all $f, g \in L^\infty(N)$. Thus ν is a seminorm and induces a norm $||\cdot||_N$ on $L^\infty(N)$. Also, with the same notation, it follows from (3.10) that

$$||f(T,\tau)x||^2 = ||U_{T,\tau;x}f||^2 = \int |f|^2 d\mu_x^N \leq ||f||_x^2 ||x||^2$$

which implies that $||f(T,\tau)|| \leq ||f||_{_N}$. Next, choose $x \in H$ such that $||f||_{L^\infty(\mu_x)} > ||f||_{_N} - \epsilon/2$ for a given $\epsilon > 0$. Let Δ be a Borel set of measure $m = \mu_x^N(\Delta) > 0$ on which $|f(t)| > ||f||_{L^\infty(\mu_x)} - \epsilon/2 > ||f||_{_N} - \epsilon$, a.e.$[\mu_x^N]$. Define $g = m^{-1/2}\chi_{_\Delta}$. Then

$$||g(T,\tau)x||^2 = \int |g(s,t)|^2 d\mu_x^N(s,t) = 1,$$

and

$$\begin{aligned} ||f(T,\tau)g(T,\tau)x||^2 &= \int |f(s,t)g(s,t)|^2 d\mu_x^N(s,t) \\ &\geq (||f||_{_N} - \epsilon)^2 \int |g(s,t)|^2 d\mu_x^N(s,t) = (||f||_{_N} - \epsilon)^2. \end{aligned}$$

Thus, $||f(T,\tau)|| \geq ||f||_{_N} - \epsilon$ and, since $\epsilon > 0$ is arbitrary, it follows that $||f(T,\tau)|| = ||f||_{_N}$. It is now easy to verify that the functional calculus $f \mapsto f(T,\tau) : L^\infty(N) \to B(H)$ is an isometric *-algebra isomorphism.

To prove (i), we first show that Λ is closed. By (3.12), the set $\Lambda_x := [\mu_x^N]$ess.range(f) is closed for all $x \in H$. Fix $\lambda \in \overline{\Lambda}$. Assume no Λ_x contains λ and reach a contradiction. Let D_1 be the closed disc/interval in $\mathbb{F}$ of radius $r_1 = 1$ centered at λ. Find $x \in H$ such that $D_1 \cap \Lambda_x \neq \emptyset$. We are going to replace x by a unit vector x_1 to guarantee (3.13) holds for $n = 1$. (See below.) Define $\Gamma_1 = f^{-1}(D_1)$ and let $x_1 = U_{T,\tau;x}g_1$, where $g_1 = \mu_x^N(\Gamma_1)^{-1/2}\chi_{_{\Gamma_1}}$. Then, for all Borel subsets Ω of $\mathbb{F}$,

$$\begin{aligned} \mu_{x_1}^N(\Omega) &= \int |\chi_{_\Omega}(s,t)|^2 d\mu_{x_1}^N = ||\chi_{_\Omega}(T,\tau)x_1||^2 = ||\chi_{_\Omega}(T,\tau)U_{T,\tau;x}\chi_{_{\Gamma_1}}||^2/\mu_x^N(\Gamma_1) \\ &= ||\chi_{_{\Omega\cap\Gamma_1}}||^2_{L^2(\mu_x^N)}/\mu_x^N(\Gamma_1) = \mu_x^N(\Omega \cap \Gamma_1)/\mu_x^N(\Gamma_1), \end{aligned}$$

which implies that $||x_1||^2 = \mu_{x_1}(\mathbb{F}) = 1$, and $\Lambda_{x_1} = \Lambda_x \cap D_1 \neq \emptyset$. Since Λ_{x_1} is closed and $\lambda \notin \Lambda_{x_1}$, it follows that $\Lambda_{x_1} \subset D_1 \backslash D_2$ for some closed disc of (positive) radius $r_2 \leq r_1/2$ and centered at λ. Now, assume by induction, we have constructed a sequence of concentric closed discs $D_1, D_2, D_3, \ldots, D_{n+1}$ of respective radii $r_1, r_2, \ldots, r_k$ and an orthonormal set $\{x_1, x_2, x_3, \ldots, x_n\} \subset H$ such that, for $k = 1, 2, \ldots, n$,

$$r_{k+1} \le r_k/2, \quad x_k = \chi_{_{\Gamma_k \setminus \Gamma_{k+1}}}(T,\tau)x_k \text{ and } \Lambda_{x_k} \subset D_k \setminus D_{k+1}, \tag{3.13}$$

where $\Gamma_k = f^{-1}(D_k)$. Mimicking the construction of x_1 and D_2, yields a unit vector $x_{n+1} \in H$ and a disc D_{n+2} of radius $r_{n+1} \le r_n/2$ centered at λ such that (3.13) remains valid for $k = n+1$. We claim $\langle x_k, x_{n+1}\rangle = 0$ for $k = 1, 2, \dots, n$. In view of (3.13)

$$\begin{aligned}\langle x_k, x_{n+1}\rangle =& \langle \chi_{_{\Gamma_k \setminus \Gamma_{k+1}}}(T,\tau)x_k, \chi_{_{\Gamma_{n+1} \setminus \Gamma_{n+2}}}(T,\tau)\rangle \\ =& \langle \chi_{_{\Gamma_{n+1} \setminus \Gamma_{n+2}}}(T,\tau)\chi_{_{\Gamma_k \setminus \Gamma_{k+1}}}(T,\tau)x_k, x_{n+2}\rangle = 0, \quad \forall k \le n.\end{aligned}$$

So far, we have constructed an orthonormal sequence $x_1, x_2, x_3, \dots$ such that $\mu^N_{x_k}(f^{-1}(D_k)) \neq 0$. Define $x = \sum_{k=1}^{\infty} 2^{-k} x_k$ and observe that

$$||x|| = 1, \quad \text{and } \mu^N_x(f^{-1}(D_k \setminus D_{k+1}) = \mu^N_x(\Gamma_k \setminus \Gamma_{k+1}) > 0.$$

Thus, $D_k \setminus D_{k+1} \subset [\mu^N_x]$ess.range($f$). This implies that $\lambda \in \overline{\Lambda}_x = \Lambda_x$.

It remains to show that $\lambda \in \sigma_{\mathbb{F}}(f(T,\tau))$. Then there exists $x \in H$ such that $\lambda \in [\mu_x]$ess.range(f). For every $n \in \mathbb{N}$, let Δ_n be the disc in $\mathbb{F}$ of radius n^{-1} and center λ. Then, for all $n \in \mathbb{N}$, $\mu_x(\Gamma_n) > 0$, where $\Gamma_n := f^{-1}(\Delta_n)$. Let $g_n = \mu_x(\Gamma_n)^{-1/2}\chi_{_{\Gamma_n}}$ and $y_n = U_{T,\tau;x} g_n$ for all $n \in \mathbb{N}$. Then $||y_n|| = ||g_n||_{L^2(\mu_x)} = 1$ and

$$||(\lambda - f(T,\tau))y_n||^2 = \mu_x(\Gamma_n)^{-1} \int_{\Gamma_n} |\lambda - f(s+it)|^2 d\mu_x(s,t) \le n^{-2} \to 0 \tag{3.14}$$

Conversely, we assume $\lambda \in \sigma_{\mathbb{F}}(f(T,\tau))$ and claim $\lambda \in \Lambda$. It is sufficient to show that $\lambda \in \overline{\Lambda}$. If not, there exists an open disc Δ of radius $\delta > 0$ and center λ such that $\mu^N_x(f^{-1}(\Delta)) = 0$ for all $x \in H$. Then,

$$||(\lambda - f(T,\tau))x||^2 = \int |\lambda - f(s+it)|^2 d\mu^N_x(s,t) \ge \delta^2 ||x||^2.$$

This shows that λ is not in the approximate point spectrum of the normal operator $f(T,\tau)$. Hence, $\lambda \notin \sigma_{\mathbb{F}}(f(T,\tau))$; a contradiction.

The proof of (ii) follows from the fact that $[\mu^N_x]$ess.range(**id**) = supp(μ^N_x) and that $N = T + i\tau = \mathbf{id}(T,\tau)$. Thus, $\sigma_{\mathbb{F}}(N) = \cup_{x\in H}\text{supp}(\mu^N_x)$. The remainder of the proof of (ii) follows from Part (i) and formulas (3.11)–(3.12).

For (iii), observe that supp(μ^N_x) $\subset \sigma_{\mathbb{F}}(N)$. Letting $h = f - g$, it follows that

$$||h(T+i\tau)x||^2 = \int |h(s,t)|^2 d\mu^N_x = 0, \ \forall\, x \in H.$$

In (iv), since f is continuous, it follows that

$$||f||_N = \sup_{x\in H} ||f||_x \le \sup_{x\in H} \max\{|f(t)| \ : \ t \in \sigma_{\mathbb{F}}(N)\} \le ||f||_{\sigma_{\mathbb{F}}(N)}.$$

For the converse, choose $s \in \sigma_{\mathbb{F}}(N)$ such that $|f(s)| = \max\{|f(t)| \ : \ t \in \sigma_{\mathbb{F}}(N)\}$. In view of Part (ii), there exists $x \in H$ such that $s \in \text{supp}(\mu^N_x)$ and it follows from the continuity of f on the compact set supp(μ^N_x) that $|f(s)| \le ||f||_{L^\infty(\mu^N_x)} \le ||f||_N$. □

Corollary 3.1 *Let $\sigma_{\mathbb{F}}(N) = \sigma_1 \cup \sigma_2$ for a pair of disjoint closed sets σ_1 and σ_2. Then $N = N_1 \oplus N_2$ with $\sigma_{\mathbb{F}}(N_j) = \sigma_j$ $(j = 1, 2)$. In particular, if $\sigma_1 = \{\lambda\}$, then $N_1 = \lambda I_1$, where $I_1 = I|_{\mathcal{D}(N_1)}$.*

Proof Let χ_j be the characteristic function of the set σ_j and define $P_j = \chi_j(N)$ $(j = 1, 2)$. Obviously, each P_j is an orthogonal projection and

$$
\begin{aligned}
I = \mathbf{1}(N) &= \chi_1(N) + \chi_2(N) = P_1 + P_2,\\
P_1P_2 = P_2P_1 &= \chi_1(N)\chi_2(N) = (\chi_1\chi_2)(N) = 0.
\end{aligned}
$$

Thus, $H = H_1 \oplus H_2$ and $NH_j \subset H_j$ $(j = 1, 2)$. Hence, $N = N_1 \oplus N_2$, $\sigma_{\mathbb{F}}(N) = \sigma_{\mathbb{F}}(N_1) \cup \sigma_{\mathbb{F}}(N_2)$, and N_1, N_2 are normal operators, where $N_j = N|_{H_j}$ $(j = 1, 2)$. But

$$
\sigma(N_j) = \sigma(N\chi_j(N)|_{H_j}) \subset \{t\chi_j(t) : t \in \sigma_{\mathbb{F}}(N)\} \subset \sigma_j \cup \{0\}, \ (j = 1, 2).
$$

Assume $0 \notin \sigma_j$ and $x \in H_j$ for some $j = 1, 2$. Then,

$$
||Nx||^2 = ||N\chi_j(N)x||^2 = \int_{\sigma_j} |t|^2 d\mu_x^N \geq \operatorname{dist}(0, \sigma_j)||x||^2,
$$

which implies that 0 is not in the approximate point spectrum of N_j. Hence, $\sigma_{\mathbb{F}}(N_j) \subset \sigma_j$ and, thus, $\sigma_{\mathbb{F}}(N_j) = \sigma_j$ $(j = 1, 2)$.

In particular, if $\sigma_1 = \{\lambda\}$, then

$$
||(\lambda I - N)\chi_{\{\lambda\}}(N)x||^2 = \int_{\{\lambda\}} |\lambda - s - it|^2 d\mu_x^N = 0.
$$

Thus, $N_1 = \lambda I_1$. □

We conclude this section with a simple observation about the polar decomposition of an operator $S \in B(H)$ acting on a real (or complex) Hilbert space H. It is obvious that S^*S is a (real or complex) positive selfadjoint operator and, hence, has a positive square root $(S^*S)^{1/2}$. As usual, we denote this square root by $|S|$. Also, it is easy to verify that $\langle |S|x, |S|x\rangle = \langle Sx, Sx\rangle$ which implies that the mapping $|S|x \mapsto Sx$ is a partial isometry from the range of $|S|$ onto the range of S satisfying $S = V|S|$.

Now, if $N \in B(H)$ is a real or complex normal operator, then $\mathcal{K}(N) = \mathcal{K}(N^*) = \mathcal{K}(|N|)$, $\overline{\mathcal{R}(N)} = \overline{\mathcal{R}(N^*)} = \overline{\mathcal{R}(|N|)}$, $H = \mathcal{K}(N) \oplus \overline{\mathcal{R}(N)}$ and, accordingly,

$$
N = \begin{bmatrix} 0 & 0 \\ 0 & N_1 \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & |N_1| \end{bmatrix},
$$

where $N_1 : \overline{\mathcal{R}(N)} \to \overline{\mathcal{R}(N)}$ is an injective normal operator, $W : \mathcal{K}(N) \to \mathcal{K}(N)$ is an arbitrary operator, and $V : \overline{\mathcal{R}(N)} \to \overline{\mathcal{R}(N)}$ is a unique unitary operator. Letting W be any unitary operator, one can define a unitary operator

$$U = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} = W \oplus V$$

to establish the polar decomposition $N = U|N|$ of N with $U^*U = UU^* = I$, regardless of the type of the underlying field $\mathbb{R}$ or $\mathbb{C}$.

At this end, assuming $N = T + i\tau$ is a real selfadjoint or a complex normal operator, we want to demonstrate the operators U and N as functions of N. Define $u(s,t) = (s+it)(s^2+t^2)^{-1/2}$ if $s^2+t^2 \neq 0$ and $u(0,0) = 1$. Also, define $h(s,t) = (s^2+t^2)^{1/2}$. It follows that $s+it = u(s,t)h(s,t)$ and, hence, $N = u(T,\tau)h(T,\tau)$, where

$$\begin{aligned} ||u(T,\tau)x||^2 &= \int |u|^2 d\mu_x^N = ||x||^2, \\ ||u(T,\tau)^*x||^2 &= ||u^*(T,\tau)x||^2 = \int |u^*|^2 d\mu_x^N = ||x||^2, \\ h(T,\tau)^* &= h^*(T,\tau) = h(T,\tau), \quad \text{and} \\ \langle h(T,\tau)x, x\rangle &= \int h d\mu_x^N \geq 0, \quad \forall x \in H. \end{aligned}$$

Therefore, $u(T,\tau)$ is a unitary operator and $h(T,\tau)$ is a positive selfadjoint operator which together establish the polar decomposition of N. Also, note that $u = w + v$ with $w = \chi_{\{(0,0)\}}$, $v = (s+it)(s^2+t^2)^{-1/2}\chi_{\mathbb{F}\setminus\{(0,0)\}}(s,t)$ and $wv = 0$. This re-establishes the direct sum $U = W \oplus V$ with the unique part $V = v(T,\tau)|_{\mathcal{K}(N)^\perp}$ and the choice $W = w(T,\tau)|_{\mathcal{K}(N)} = I_{\mathcal{K}(N)}$.

Here, we make a very important observation about the relation between the spectrum of U and that of $|N|$; i.e., given a Borel partition $\mathbb{T} = \eta \cup \omega$ of the torus $\mathbb{T} := \{z \in \mathbb{F} : |z| = 1\}$, the space H is decomposed into the direct sum $H = \chi_\eta(U)H \oplus \chi_\omega(U)H$ according to which U, N and $|N|$ are decomposed as $U = U_\eta \oplus U_\omega$, $N = N_\eta \oplus N_\omega$ and $|N| = |N_\eta| \oplus |N_\omega|$. Moreover, $\sigma_{\mathbb{F}}(N_\eta)$ is included in the sector of the unit disc consisting of radii joining 0 to the points on $\bar{\eta}$. (Note that if η contains a portion of $\sigma_{\mathbb{F}}(W)\setminus\sigma_{\mathbb{F}}(V)$, then its counterpart in $\sigma_{\mathbb{F}}(N)$ is the singleton $\{(0,0)\}$.)

4 Spectral measure

Once a Borel functional calculus is established, one can immediately construct a so-called *Borel spectral measure* satisfying (4.1)–(4.4) of the next theorem. We review the theory just to see how our improving Theorem 3.2 affects the spectral measure theorems.

Theorem 4.1 *Let $N = T + i\tau \in B(H)$ be a real selfadjoint operator or a complex normal operator, where $T = (N+N^*)/2$, $i\tau = (N-N^*)/2$ and H is a Hilbert space with the underlying field $\mathbb{F}$. (If $\tau \neq 0$, then $\mathbb{F}$ is necessarily equal to $\mathbb{C}$.) Define $E : \mathcal{B} \to B(H)$ by $E(\gamma) = \chi_\gamma(N)$, where $\mathfrak{B}$ is the σ-algebra of all Borel subsets of $\mathbb{F}$.*

The following assertions are true for all $\sigma, \gamma, \gamma_1, \gamma_2, \gamma_3, \ldots \in \mathfrak{B}$ *whenever* $\gamma_m \cap \gamma_n = \emptyset$ *for* $m \neq n$.

$$E(\gamma) = E(\gamma)^2 = E(\gamma)^*. \tag{4.1}$$

$$E(\mathbb{F}) = I,\ E(\gamma)N = NE(\gamma). \tag{4.2}$$

$$||E(\dot{\cup}_{n=1}^{\infty}\gamma_n)x||^2 = \sum_{n=1}^{\infty} ||E(\gamma_n)x||^2,\ \forall x \in H. \tag{4.3}$$

$$\sigma_{\mathbb{F}}(N_\gamma) \subset \bar{\gamma} \cap \sigma_{\mathbb{F}}(N), \tag{4.4}$$

where $N_\gamma = N|_{E(\gamma)H}$.

Proof Assertions (4.1)–(4.2) follow from Theorem 3.1 and the definition of E. Assertion (4.3) is an immediate consequence of $\mu_x^N(\gamma) = ||E(\gamma)x||^2$. For (4.4), assume without loss of generality that $E(\mathbb{F}\backslash\gamma)H \neq \{0\}$ and observe that $N = N_\gamma \oplus \tilde{N}_\gamma$ with respect to $H = E(\gamma)H \oplus E(\mathbb{F}\backslash\gamma)H$. Then $N_\gamma = N\chi_\gamma(N)|_{E(\gamma)H}$ and, hence, $\sigma_{\mathbb{F}}(N_\gamma) \subset \sigma_{\mathbb{F}}(N\chi_\gamma(N))$. (Note that both N_γ and $N\chi_\gamma(N)$ are normal.) Since $N\chi_\gamma(N) = N_\gamma \oplus 0$, it follows from Remark 3.1 that

$$\begin{aligned}\sigma_{\mathbb{F}}(N_\gamma) \cup \{0\} =& \sigma_{\mathbb{F}}(N_\gamma \oplus 0) = \sigma_{\mathbb{F}}(N\chi_\gamma(N)) = \overline{\cup_{x\in H}[\mu_x^N]\text{ess.range}(\mathbf{id}\cdot\chi_\gamma)} \\ =& \overline{\{\cup_x \text{supp}(\mu_x^N|_\gamma)\}} \cup \{0\} \subset (\bar{\gamma} \cap \sigma_{\mathbb{F}}(N)) \cup \{0\}.\end{aligned}$$

Assume $0 \in \sigma_{\mathbb{F}}(N_\gamma)$. Then, there exists a sequence of unit vectors $x_n = E(\gamma)x_n$ such that $N\chi_\gamma(N)x_n = N_\gamma x_n \to 0$. Hence, $1 = ||x_n||^2 = \int_\gamma d\mu_{x_n}^N$ and

$$\text{dist}(0, \text{supp}(\mu_{x_n}^N))^2 \leq \int_\gamma |s+it|^2 d\mu_{x_n}^N = ||Nx_n||^2 \to 0.$$

This implies that $0 \in \overline{\cup_{x\in H}\text{supp}(\mu_x^N|_\gamma)} \subset \bar{\gamma} \cap \sigma_{\mathbb{F}}(N)$ and, thus, (4.4) is proven. □

Definition 4.1 Let H be a real or complex Hilbert space and let $\mathfrak{B}$ denote the σ-algebra of all Borel subsets of the (underlying) field $\mathbb{F}$. Any triple $(\mathbb{F}, \mathfrak{B}, E)$ is called a Borel spectral measure for a real selfadjoint operator or a complex normal operator $N \in B(H)$ if $E : \mathfrak{B} \to B(H)$ satisfies (4.1)–(4.4).

Theorem 4.2 *If* $(\mathbb{F}, \mathfrak{B}, E_j)$ *is a Borel spectral measure for the real selfadjoint operator or the complex normal operator* N $(j = 1, 2)$, *then* $E_1 = E_2$.

Proof We can assume without loss of generality that E_1 is equal to the spectral measure E of Theorem 4.1. Also, observe that, for $j = 1, 2$,

$$\begin{aligned}&4\langle E_j(\gamma)x, y\rangle := \\ &\langle E_j(\gamma)(x+y), x+y\rangle - \langle E_j(\gamma)(x-y), x-y\rangle \\ &+ i\delta_{\mathbb{F},\mathbb{C}}[\langle E_j(\gamma)(x+iy), x+iy\rangle - \langle E_j(\gamma)(x-iy), x-iy\rangle].\end{aligned} \tag{4.5}$$

(Here, $\delta_{\cdot,\cdot}$ denotes the Kronecker δ.) Letting $M \in \mathbb{N}$ and choosing $\gamma_n = \emptyset$ ($\forall n > M$), it follows from (4.3) and (4.5) that

$$E_j(\emptyset) = 0 \ \text{ and } \ E_j(\cup_{n=1}^{M}\gamma_n) = \sum_{n=1}^{M} E_j(\gamma_n). \tag{4.6}$$

In view of (4.1), every projection $E_j(\gamma)$ is an orthogonal projection and, hence, it follows from [8] (page 18) that $E_j(\gamma_h)E_j(\gamma_k) = E_j(\gamma_k)E_j(\gamma_h) = 0$ whenever $h \neq k$.

Now, define $\nu_{j,x}(\gamma) = ||E_j(\gamma)x||^2$. It follows from (4.3) and (4.6) that $(\mathbb{F}, \mathfrak{B}, \nu_{j,x})$ is a positive σ-additive measure ($j = 1, 2$). Thus, to show $E_1 = E_2$, it is sufficient to show that $\nu_{1,x}(\gamma) = \nu_{2,x}(\gamma)$ for all closed sets γ. Note that, in view of (4.2) and Theorem 3.1, $E_h(\gamma)E_k(\eta) = E_k(\gamma)E_h(\eta)$ and $E_h(\gamma)N = NE_h(\gamma)$ $\forall h, k = 1, 2$ and $\forall \gamma, \eta \in \mathfrak{B}$. Thus, for every $\gamma \in \mathfrak{B}$,

$$H = H_{j,\gamma} \oplus H_{j,\mathbb{F}\setminus\gamma}, \quad \text{where } \ H_{j,\cdot} = E_j(\cdot)H.$$

We claim $H_{1,\gamma} = H_{2,\gamma}$ for every closed set γ. Write $\mathbb{F}\setminus\gamma = \cup_{n=1}^{\infty}\eta_n$ for some monotone increasing sequence of open sets η_n satisfying $\bar{\eta}_n \cap \gamma = \emptyset$. Let $x \in H_{1\gamma}$ and write $y_n = E_2(\eta_n)x$. Then $||E_2(\mathbb{F}\setminus\gamma)x||^2 = \lim_n ||y_n||^2$. We prove $y_n = 0$. Fix n and let $\mathcal{M} := \mathcal{Z}(T, \tau; y_n)$. It follows from the commutativity of $N, E_1(\cdot)$ and $E_2(\cdot)$ that $\mathcal{M} \subset E_2(\eta_n)H \cap E_1(\gamma)H$ and, hence, $\sigma_{\mathbb{F}}(T|_{\mathcal{M}}) \subset \gamma \cap \bar{\eta}_n = \emptyset$. Thus $\mathcal{M} = \{0\}$ and, therefore, $E_1(\gamma)H \subset E_2(\gamma)H$. By a similar argument, $E_2(\gamma)H \subset E_1(\gamma)H$ which implies that $E_1(\gamma) = E_2(\gamma)$ for all closed sets γ. Since $E_j(\gamma) + E_j(\mathbb{F}\setminus\gamma) = E_j(\mathbb{F}) = I$, it follows that $E_1(\eta) = E_2(\eta)$ for all open sets η. Thus, the two positive measures $\nu_{1,x}$ and $\nu_{2,x}$ are identical for all $x \in H$. It follows from (4.5) that E_1 and E_2 are equal in the weak sense; i.e., $\langle E_1(\gamma)x, y\rangle = \langle E_2(\gamma)x, y\rangle$ for all $x, y \in H$. □

Corollary 4.1 *For the spectral measure $(\mathbb{F}, \mathfrak{B}, E)$ of a real selfadjoint or complex normal operator N, the following assertions are true.*

1. *For every $x \in H$, the set function $\mu_x^N(\gamma) = ||E(\gamma)x||^2$ is a positive σ-additive measure and*

$$||f(T, \tau)x||^2 = \int_{\sigma_{\mathbb{F}}(N)} |f(s, t)|^2 d\mu_x^N(s, t), \ \forall\, x \in H, \ \forall f \in L^\infty(N).$$

2. *For every $x, y \in H$, the set function $\mu_{x,y}^N(\gamma) = \langle E(\gamma)x, y\rangle$ is a complex σ-additive measure and*

$$\langle f(T,\tau)x,y\rangle = \int_{\sigma_{\mathbb{F}}(N)} f(s,t)d\mu^N_{x,y}(s,t),\ \forall\, x,y \in H,\ \forall f \in L^\infty(N).$$

5 Discrete normal operators

In the present section, we assume $N \in B(H)$ is a real selfadjoint or a complex normal operator; we further assume the collection of its eigenvectors span H. In case H is separable, then it follows from the Gram–Schmidt process that there exists a finite or countable orthonormal basis $\{e_n\}_{n\in\mathbb{J}}$ consisting of eigenvectors of N such that $N = \sum_{n\in\mathbb{J}} \lambda_n e_n \otimes e_n$; more generally, if $p \in \mathbb{F}[x]$, then $p(N) = \sum_{n\in\mathbb{J}} p(\lambda_n) e_n \otimes e_n$. In fact, if $f : \mathbb{F} \to \mathbb{F}$ is a bounded function, one can define $f(N) = \sum_{n\in\mathbb{J}} f(\lambda_n) e_n \otimes e_n \in B(H)$. With no need of axiom of choice or Hausdorff maximality principle, we can partition $\mathbb{J}$ as $\mathbb{J} = \cup_{\lambda\in\Lambda} J_\lambda$ such that

$$H = \oplus_{\lambda\in\Lambda} \mathcal{K}(\lambda I - N),\ f(N) = \oplus_{\lambda\in\Lambda} f(\lambda) I_\lambda. \tag{5.1}$$

In (5.1), one may either extend Λ to $\mathbb{F}$ by taking $\mathcal{K}(\lambda I - N) = \{0\}$ for the redundant λ's or assume H is nonseparable and N has uncountably many eigenvalues. In either case, there is a need for the definition of uncountable direct sums and extended Bessel inequalities or extended Parseval equalities. The following definition is surprisingly simple. Recall that an orthogonal projection $P \in B(H)$ is characterized as $P = P^* = P^2$; such an orthogonal projection decomposes the space as the orthogonal direct sum $H = PH \oplus (I - P)H$. To avoid any kind of confusion between orthogonality term used for matrices or projection, we use the term perpendicularity for linear subspaces. Also, recall that the notation $\oplus$ in this paper is solely used for orthogonal direct sum; i.e., the expression $\oplus_{\alpha\in\Lambda} M_\alpha$ necessarily implies that M_α and M_β are mutually perpendicular (for all $\alpha \neq \beta$ in Λ).

However, we acknowlege that until Theorem 5.2, the results are known for which we refer to Shirbisheh [13].

Definition 5.1 A subspace M of H is said to be the orthogonal direct sum of a family $\{M_\alpha\}_{\alpha\in\Lambda}$ of mutually perpendicular subspaces M_α, and write $M = \oplus_{\alpha\in\Lambda} M_\alpha$, if M is the smallest closed subspace containing $\cup_{\alpha\in\Lambda} M_\alpha$.

To justify our definition of orthogonal direct sum, we prove the following theorem which extends Bessel's inequality and Parseval's equality for families of mutually perpendicular closed subspaces. We could not persuade ourselves to skip the proof; instead, we avail ourselves of acknowledging that it is somehow known to the experts. Also, we have to mention that the terms extended Bessel's inequality and extended Parseval's equality are new.

Theorem 5.1 *Let $\{M_\alpha : \alpha \in \Lambda\}$ be a family of mutually perpendicular subspaces of a Hilbert space H over the field $\mathbb{F}$ and let $P_\alpha \in B(H)$ be the orthogonal projection onto M_α. Let $M = \oplus_{\alpha\in\Lambda} M_\alpha$ with the corresponding orthogonal projection $P \in B(H)$.*

Then the following extended Parseval's equality and extended Bessel's inequality hold:

$$\sup\{\sum_{\alpha\in L} ||P_\alpha x||^2 \ : \ L \text{ finite } \subset \Lambda\} = ||Px||^2 \le ||x||^2 \ \forall\, x \in H.$$

Also, the set $\Lambda_x := \{\alpha \in \Lambda \ : \ P_\alpha x \neq 0\}$ *has an identification with* $\mathbb{N}$ *or some finite initial segment* $\{1, 2, \dots, n\}$, *independent of which the following conditions* (a)–(c) *hold*.

(a) *The Bessel inequality and the Parseval equality can be modified as follows*:

$$\sum_{\alpha\in\Lambda_x} ||P_\alpha x||^2 = ||Px||^2 \le ||x||^2 \quad (\forall x \in H).$$

(b) *The series* $Px = \sum_{\alpha\in\Lambda_x} P_\alpha x$ *is convergent in norm* $(\forall x \in H)$.
(c) *The numerical series* $\langle Px, y\rangle = \sum_{\alpha\in\Lambda_x} \langle P_\alpha x, y\rangle$ *is convergent* $(\forall x, y \in H)$.

Proof Fix $x \in H$. For each $n \in \mathbb{N}$, let $\Lambda_{x,n} = \{\alpha \in \Lambda \ : \ ||P_\alpha x|| \ge 1/n\}$. Then, for any finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ of $\Lambda_{x,n}$,

$$m/n^2 \le \sum_{k=1}^{m} ||P_{\alpha_k} x||^2 = \sum_{k=1}^{m} \langle P_{\alpha_k} x, x\rangle = \langle (P_{\alpha_1} + \dots + P_{\alpha_m})x, x\rangle \le ||x||^2. \quad (5.2)$$

Thus, $m \le n^2 ||x||^2$, which shows that $\Lambda_{x,n}$ is finite and, hence, $\Lambda_x = \cup_{n\in\mathbb{N}} \Lambda_{x,n}$ is countable. For the sake of uniformity, we assume without loss of generality that $\{M_\alpha\}_{\alpha\in\Lambda}$ contains infinitely many copies of $\{0\}$; this enables us to identify each Λ_x with $\mathbb{N}$ by supplementing it with enough copies of $\{0\}$, if necessary. Now, let L be any finite subset of Λ. Using the argument leading to (5.2) yields $\sum_{\alpha\in L} ||P_\alpha x||^2 \le ||x||^2$, which establishes the extended Bessel's inequality as well as its discrete form in (a). Note that the positive series in (a) is absolutely convergent and, hence, does not depend on the order of Λ_x.

To prove extended Parseval equality, observe that

$$|| \sum_{k=m+1}^{n} P_k x||^2 = \langle \sum_{k=m+1}^{n} P_k x, x\rangle = \sum_{k=m+1}^{n} \langle P_k x, x\rangle$$
$$= \sum_{k=m+1}^{n} ||P_k x||^2 \to 0 \quad \text{as} \quad m, n \to \infty.$$

Therefore, there exists $\xi \in M$ such that the series $\sum_{n=1}^{\infty} P_n x$ and $\sum_{n=1}^{\infty} ||P_n x||^2$ converge to ξ and $||\xi||^2$, respectively. Also, the arguments leading to (5.2) reveal that Parseval equality and conclusions (b)-(c) hold if x is replaced by $\xi = P\xi$. To complete the proof, we claim $\xi = Px$ or, equivalently, $x - \xi \perp M$. This, in turn, is equivalent with showing that $x - \xi \perp M_\beta$ for all $\beta \in \Lambda_x$. Fix $\beta \in \Lambda$ and choose arbitrary $y = P_\beta y \in M_\beta$. Assume without loss of generality that $\beta \in \Lambda_x$ and observe that

$$\begin{aligned}\langle x-\xi, P_\beta y\rangle =&\langle P_\beta(x-\xi), y\rangle = \langle P_\beta x, y\rangle - \langle P_\beta \sum_{n=1}^{\infty} P_n x, y\rangle \\ &=\langle P_\beta x, y\rangle - \langle \sum_{n=1}^{\infty} P_\beta P_n x, y\rangle = \langle P_\beta x, y\rangle - \langle P_\beta x, y\rangle = 0.\end{aligned}$$

Thus, $x-\xi \perp M_\beta$ and, by linearity and continuity of the inner product in the first component, $x-\xi \perp M$. □

The following corollary provides a list of what we expect from an orthogonal direct sum.

Corollary 5.1 *With the notation of the theorem, let* $x_\alpha \in M_\alpha$ *for all* $\alpha \in \Lambda$ *and let* $x \in \bigoplus_{\alpha\in\Lambda} M_\alpha$. *The following assertions are equivalent.*

(i) $x_\alpha = P_\alpha x$ *for all* $\alpha \in \Lambda$.
(ii) $x_\alpha = P_\alpha x$ *for all* $\alpha \in \Lambda$ *and* $\sup\{\sum_{\alpha\in F} ||x_\alpha||^2 \,:\, F \text{ finite subset of } \Lambda\} = ||x||^2$.
(iii) *The set* $\Lambda_x := \{\alpha \in \Lambda \,:\, x_\alpha \neq 0\}$ *is countable and* $x = \sum_{\alpha\in\Lambda_x} x_\alpha$.
(iv) *The vector* x *is uniquely represented as* $x = \sum_{\alpha\in\Lambda} x_\alpha$ *in the sense that for all* $\epsilon > 0$, *there exists a finite subset* F *of* Λ *such that* $||x - \sum_{\alpha\in L} x_\alpha|| < \epsilon$ *whenever* $F \subset L \subset \Lambda$ *and* $\mathrm{card}(L) < \infty$.

The next corollary is an extension of the Pythagorean Theorem.

Corollary 5.2 (Extended Pythagorean Theorem) *Let* $\{M_\alpha\}_{\alpha\in\Lambda}$ *be a family of closed linear subspaces of the Hilbert space* H *with the underlying field* $\mathbb{F}$. *Let* P_α *denote the orthogonal projection onto* M_α $(\alpha \in \Lambda)$. *Then, the following assertions are equivalent.*

(a) $M_\alpha \perp M_\beta$ *for all distinct* $\alpha, \beta \in \Lambda$.
(b) $P_\alpha P_\beta = 0$ *for all distinct* $\alpha, \beta \in \Lambda$.
(c) $P_\alpha + P_\beta$ *is an orthogonal projection for all distinct* $\alpha, \beta \in \Lambda$.
(d) $||x_\alpha + x_\beta||^2 = ||x_\alpha||^2 + ||x_\beta||^2$ *whenever* $x_\alpha \in M_\alpha$, $x_\beta \in M_\beta$ *and* $\alpha \neq \beta$.

Proof The equivalence of (a) and (b) follows from the simple verification of $\langle P_\alpha P_\beta x, y\rangle$ for all $x, y \in H$. The equivalence of (b) and (c) is verified on page 18 of [8]. The proof of (a) $\Rightarrow$ (d) is a straightforward computation. For the converse, we must show that $\langle x_\alpha, x_\beta\rangle = 0$ whenever $\alpha \neq \beta \in \Lambda$, $x_\alpha \in M_\alpha$ and $x_\beta \in M_\beta$. Assume $\langle x_\alpha, x_\beta\rangle \neq 0$ to reach a contradiction. Let $z = \langle x_\beta, x_\alpha\rangle / |\langle x_\alpha, x_\beta\rangle|$. Then

$$||zx_\alpha||^2 + ||x_\beta||^2 = ||zx_\alpha + x_\beta||^2 = ||zx_\alpha||^2 + ||x_\beta||^2 + 2\,\Re e(z\langle x_\alpha, x_\beta\rangle)$$

and, hence, $0 = 2\Re e|\langle x_\alpha, x_\beta\rangle| = 2|\langle x_\alpha, x_\beta\rangle| \neq 0$; a contradiction. □

We are now ready to state our main theorem about the functional calculus of discrete normal operators. For the definitions of Borel functional calculus $f \mapsto f(N)$ and Borel spectral measure E we refer to Theorems 3.1 and 4.1. Also, we fix the notation I_λ for the identity operator on the subspace $\mathcal{K}(\lambda I - N)$.

Theorem 5.2 (Discrete spectral measure) *Let $N \in B(H)$ be a real or complex normal operator whose domain H is the smallest closed subspace containing the collection of all eigenvectors of N corresponding to the eigenvalues $\sigma_p(N) \subset \sigma_{\mathbb{F}}(N)$. Let $B(\sigma_p(N))$ denote the collection of all functions $f : \mathbb{F} \to \mathbb{F}$ such that $f|_{\sigma_p(N)}$ is bounded. Then $L^\infty(N) \subset B(\sigma_p(N))$ and*

$$||f||_N = ||f||_{p,N}, \quad \text{where } ||f||_{p,N} := \sup\{|f(\lambda)| : \lambda \in \sigma_p(N)\}. \tag{5.3}$$

Moreover, the functional calculus $f \mapsto f(N) : B(\sigma_p(N) \to B(H)$ and the spectral measure $(\mathbb{F}, 2^{\mathbb{F}}, \tilde{E})$ are, respectively, isometric extensions of the Borel functional calculus $f \mapsto f(N) : L^\infty(N) \to B(H)$ and the Borel spectral measure $(\mathbb{F}, \mathfrak{B}, E)$, where

$$f(N) = \oplus_{\lambda \in \sigma_p(N)} f(\lambda) I_\lambda, \quad \forall f \in B(\sigma_p(N)), \tag{5.4}$$

$$\tilde{E}(\gamma) = \oplus_{\lambda \in \{\sigma_p(N) \cap \gamma\}} I_\lambda, \quad \forall \gamma \in 2^{\mathbb{F}}. \tag{5.5}$$

Proof Choose arbitrary $f \in L^\infty(N)$ and $\epsilon > 0$. There exists $x \in H$ such that $||f||_N < ||f||_{L^\infty(\mu_x^N)} + \epsilon$. Let x_λ be the projection of x into $\mathcal{K}(\lambda I - T)$. Then, for all $\gamma \in \mathfrak{B}$,

$$\mu_x^N(\gamma) = ||\chi_\gamma(N)x||^2 = \sum_{\lambda \in \sigma_p(N)} ||\chi_\gamma(N)x_\lambda||^2 = \sum_{\lambda \in \gamma} ||x_\lambda||^2 = \sum_{\lambda \in \gamma} \mu_x^N(\{\lambda\}). \tag{5.6}$$

Choose $\lambda \in \sigma_p(N)$ such that $||f||_{L^\infty(\mu_x^N)} < |f(\lambda)| + \epsilon$. It follows that $||f||_N < |f(\lambda)| + 2\epsilon < ||f||_{\sigma_p(N)} + 2\epsilon$. Letting $\epsilon \to 0$ yields $||f||_N \le ||f||_{p,N}$.

For the converse, choose arbitrary $\lambda \in \sigma_p(N)$ and let $Nx = \lambda x$ for some unit vector $x \in H$. Then $\mu_x^N(\{\lambda\}) \neq 0$ and, hence, $|f(\lambda)| \le ||f||_{L^\infty(\mu_x^N)} \le ||f||_N$. Thus, $||f||_{p,N} \le ||f||_N$ which proves $L^\infty(N) \subset B(\sigma_p(N))$ and completes the proof of (5.3).

To prove (5.4), let $f \in L^\infty(N)$ and apply (5.6) to conclude that

$$\begin{aligned} ||f(N)x||^2 &= \int |f|^2 d\mu_x^N = \sum_{\lambda \in \Lambda_x} \int_{\{\lambda\}} |f(\lambda)|^2 d\mu_x^N \\ &= \sum_{\lambda \in \Lambda_x} ||f(\lambda) I_\lambda x|| = \sum_{\lambda \in \sigma_p(N)} ||f(\lambda) I_\lambda x||. \end{aligned}$$

Now, in view of (4.5), the assertion (5.4) follows. It is now easy to see that the functional calculus (5.4) is a well-defined isometric $*$-algebra isomorphism which extends the Borel functional calculus.

For (5.5), it follows from (5.4) that

$$E(\gamma) = \chi_\gamma(N) = \bigoplus_{\lambda \in \sigma_p(N)} \chi_\gamma(\lambda) I_\lambda$$
$$= \bigoplus_{\lambda \in \{\sigma_p(N) \cap \gamma\}} \chi_\gamma(\lambda) I_\lambda = \tilde{E}(\gamma)$$

Therefore, $\tilde{E}$ is an extension of E. It is easy to see that $(\mathbb{F}, 2^{\mathbb{F}}, \tilde{E})$ is a spectral measure. □

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