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Strongly irreducible factorization of quaternionic operators and Riesz decomposition theorem

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Abstract

Let $\mathcal{H}$ be a right quaternionic Hilbert space and let T be a bounded quaternionic normal operator on $\mathcal{H}$. In this article, we show that T can be factorized in a strongly irreducible sense, that is, for any $\delta > 0$ there exist a compact operator K with the norm $\|K\| < \delta$, a partial isometry W and a strongly irreducible operator S on $\mathcal{H}$ such that

$$T = (W + K)S.$$

We illustrate our result with an example. In addition, we discuss the quaternionic version of the Riesz decomposition theorem and obtain a consequence that if the S-spectrum of a bounded (need not be normal) quaternionic operator is disconnected by a pair of disjoint axially symmetric closed subsets, then the operator is strongly reducible.

Keywords Axially symmetric set · Quaternionic Hilbert space · S-specturm · Strongly irreducible operator · Riesz decomposition theorem

Mathematics Subject Classification 47A68 · 47A15 · 47B99

1 Introduction

According to the Frobenius theorem for real division algebras, the real algebra of Hamilton quaternions is the only finite-dimensional associative division algebra that contains $\mathbb{R}$ and $\mathbb{C}$ as proper real subalgebras [7]. The theory of matrices over the real algebra of quaternions has been well developed in the literature parallel to the

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case of real and complex matrices (see [1, 8, 22, 23] and references therein). For example, some of the fundamental results such as Schur's canonical form and Jordan canonical form are extended to matrices with quaternion entries [23]. In particular, the Jordan canonical form infers that every square matrix over quaternions can be reduced under similarity to a direct sum of Jordan blocks. Equivalently, the Jordan canonical form determines its complete similarity invariants and establishes the structure of matrices over quaternions. Another significant aspect in this direction is the diagonalization, through which every matrix can be factorized in terms of its restriction to eigenspaces. As it is well known that every quaternionic normal matrix is diagonalizable, it is worth mentioning the result due to Weigmann [22] that a quaternion matrix is normal if and only if its adjoint is given by a polynomial of the given matrix with real coefficients. In this article, we concentrate on the class of normal operators on the right quaternionic Hilbert spaces and their factorization in a strongly irreducible sense. For this, we adopt the notion of strong irreducibility proposed by Gilfeather [10] and Jiang [14] to the class of quaternionic operators in order to replace the notion of Jordan block for infinite-dimensional right quaternionic Hilbert spaces.

On the other hand, the study of quaternionic normal operators gained much attention in the form of spectral theory for quantum theories. Also various versions of quaternionic functional calculus has been developed (see [2–5, 9, 17, 18] for details). In fact, it is important to note that the spectral theorem for bounded and unbounded operators based on the notion called "S-spectrum" is proved by Alpay, Colombo and Kimsey in [3]. The multiplication form of the spectral theorem is the immediate consequence of the result proved in [3]. Here the notion of S-spectrum is one of the crucial objects in the theory of quaternionic operators. In the book, [4] by Colombo, Gantner and Kimsey, the discovery of S-spectrum is briefly explained and gave a systematic foundation of quaternionic spectral theory based on the S-spectrum, and present the theory of slice hyperholomorphic functions which will be used in the treatment of quaternionic operator theory.

As far as the factorization of quaternionic operators concern, the well known polar decomposition theorem shows that every bounded or densely defined closed quaternionic operator can be decomposed as a product of a partial isometry and a positive operator (see [9, 19]). In fact, the authors of [19] gave a necessary and sufficient condition for any arbitrary factorization of densely defined closed quaternionic operator to be the polar decomposition. One of the crucial observations is the positive operator that appears in the polar decomposition is the modulus of the given quaternionic operator which has several reducing subspaces as in the complex case [6]. In summary, the polar decomposition establishes the factorization of the quaternionic operator as a product of a partial isometry and an operator having several reducing subspaces. At this point, a natural question that arises from previous observation is the following: "**whether a given quaternionic operator be decomposed as a product of a partial isometry and an operator with no reducing subspace?**" We answer this question for quaternionic normal operators by proving a factorization in a strongly irreducible sense, by means of decomposing the given operator as a product of a sufficiently small compact perturbation of a partial isometry and a strongly irreducible quaternionic operator. Our result is the quaternionic version (for normal

operators) of the result proved recently in [21] by Tian, Cao, Ji and Li. In which, the authors employed the properties of Cowen–Douglas operators related to complex geometry on complex Hilbert spaces. Also, see [15] for similar factorization in the case of finite-dimensional Hilbert spaces.

We organize this article into four sections. In the second section, we give some basic definitions and results that are useful for proving our assertions. In the third section, we prove the factorization of quaternionic normal operators in a strongly irreducible sense. The final section is dedicated to the Riesz decomposition theorem for quternionic operators and provide a sufficient condition for strong irreducibility.

2 Preliminaries

An Irish mathematician Sir William Rowan Hamilton [7] described this system of numbers known as "quaternions" as an extension of complex numbers. A quaternion is of the form:

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

where $q_\ell \in \mathbb{R}$ for $\ell = 0, 1, 2, 3$ and i, j, k are the fundamental quaternion units satisfying,

$$i^2 = j^2 = k^2 = -1 = i \cdot j \cdot k. \tag{1}$$

The collection of all quaternions denoted by $\mathbb{H}$ is a non-commutative real division algebra equipped with the usual vector space operations addition and scalar multiplication defined as in the complex field $\mathbb{C}$, and with the ring multiplication given by Eq. (1). For every $q \in \mathbb{H}$, the real and the imaginary part of 'q' is defined as, $\mathrm{re}(q) = q_0$ and $\mathrm{im}(q) = q_1 i + q_2 j + q_3 k$ respectively. Then the conjugate and the modulus of q is given respectively by

$$\bar{q} = q_0 - (q_1 i + q_2 j + q_3 k) \text{ and } |q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

The set of all imaginary unit quaternions in $\mathbb{H}$ denoted by $\mathbb{S}$ and it is defined as

$$\mathbb{S} := \{q \in \mathbb{H} : \bar{q} = -q \ \& \ |q| = 1\} = \{q \in \mathbb{H} : q^2 = -1\}.$$

Note that $\mathbb{S}$ is a 2-dimensional sphere in $\mathbb{R}^4$. Let $m \in \mathbb{S}$. A real subalgebra of $\mathbb{H}$ generated by $\{1, m\}$ is called a *slice* of $\mathbb{H}$ and it is denoted by $\mathbb{C}_m$. Clearly, $\mathbb{C}_m \cap \mathbb{C}_n = \mathbb{R}$ for all $m \neq \pm n \in \mathbb{S}$. For every non real $q \in \mathbb{H}$, there is a unique $m_q := \frac{\mathrm{im}(q)}{|\mathrm{im}(q)|} \in \mathbb{S}$ such that $q = \mathrm{re}(q) + m_q |\mathrm{im}(q)| \in \mathbb{C}_{m_q}$. It follows that $\mathbb{H} = \bigcup_{m \in \mathbb{S}} \mathbb{C}_m$. There is an equivalence relation on $\mathbb{H}$ given by

$$p \sim q \Leftrightarrow p = s^{-1} q s, \text{ for some } s \in \mathbb{H} \setminus \{0\}.$$

For every $q \in \mathbb{H}$, the equivalence class of q is expressed in terms of its real and imaginary parts as follows:

$$[q] = \big\{ p \in \mathbb{H} : \ \text{re}(p) = \text{re}(q) \text{ and } |\text{im}(p)| = |\text{im}(q)| \big\}. \tag{2}$$

Definition 1 [9] Let S be a non-empty subset of $\mathbb{C}$.

1. If S is invariant under complex conjugation, then the *circularization* Ω_S of S is defined by

$$\Omega_S = \big\{ \alpha + m\beta : \ \alpha, \beta \in \mathbb{R}, \alpha + i\beta \in S, m \in \mathbb{S} \big\}.$$

2. a subset K of $\mathbb{H}$ is called *circular* or *axially symmetric*, if $K = \Omega_S$, for some $S \subseteq \mathbb{C}$ which is invariant under complex conjugation.

Note that by Definition 1 and Eq. (2), we can express Ω_S as the union of the equivalence class of its elements,

$$\Omega_S = \bigcup_{z \in S} [z].$$

It follows that the closure of Ω_S is the circularization of the closure of S. That is,

$$\overline{\Omega_S} = \Omega_{\overline{S}}. \tag{3}$$

The properties described above are useful for later sections. A more detailed discussion about quaternions can be found in [1, 5, 9] and [23]. Now we turn our discussion to some basic definitions and known results from the theory of quaternionic Hilbert spaces.

Definition 2 [9, Sect. 2.2] An *inner product* on a right $\mathbb{H}$-module $\mathcal{H}$ is a map

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{H}$$

satisfy the following properties,

1. *Positivity*: $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{H}$. In particular,

$$\langle x, x \rangle = 0 \ \text{ if and only if } \ x = 0.$$

2. *Right linearity*: $\langle x, yq + z \rangle = \langle x, y \rangle \, q + \langle x, z \rangle$, for all $x, y, z \in \mathcal{H}, q \in \mathbb{H}$.
3. *Quaternionic hermiticity*: $\langle x, y \rangle = \overline{\langle y, x \rangle}$, for all $x, y \in \mathcal{H}$.

The pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is called quaternionic pre-Hilbert space. Moreover, $\mathcal{H}$ is said to be

(i) *Quaternionic Hilbert space*, if $\mathcal{H}$ is complete with respect to the norm induced from the inner product $\langle \cdot, \cdot \rangle$, which is defined by

$$\|x\| = \sqrt{\langle x, x \rangle}, \text{ for all } x \in \mathcal{H}.$$

(ii) *Separable*, if $\mathcal{H}$ has a countable dense subset.

Furthermore, for any subset $\mathcal{N}$ of $\mathcal{H}$, the span of $\mathcal{N}$ is defined as

$$Span \, \mathcal{N} := \left\{ \sum_{\ell=1}^{n} x_\ell q_\ell; \, x_\ell \in \mathcal{N}, n \in \mathbb{N} \right\}.$$

The orthogonal complement of a subspace $\mathcal{M}$ of $\mathcal{H}$ is,

$$\mathcal{M}^{\perp} = \left\{ x \in \mathcal{H}; \langle x, y \rangle = 0, \text{ for every } y \in \mathcal{M} \right\}.$$

Note 1 The inner product $\langle \cdot, \cdot \rangle$ defined on $\mathcal{H}$ satisfies the Cauchy–Schwarz inequality:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle, \text{ for all } x, y \in \mathcal{H}.$$

Definition 3 [9] Let $\left(\mathcal{H}, \langle \cdot, \cdot \rangle \right)$ be a quaternionic Hilbert space. A subset $\mathcal{N}$ of $\mathcal{H}$ with the property that

$$\langle z, z' \rangle = \begin{cases} 1, & \text{if } z = z' \\ 0, & \text{if } z \neq z' \end{cases}$$

is said to be *Hilbert basis* if for every $x, y \in \mathcal{H}$, the series $\sum_{z \in \mathcal{N}} \langle x, z \rangle \langle z, y \rangle$ converges absolutely and it holds:

$$\langle x, y \rangle = \sum_{z \in \mathcal{N}} \langle x, z \rangle \langle z, y \rangle.$$

Equivalentely, $\overline{Span} \, \mathcal{N} = \mathcal{H}$.

Remark 1 By [9, Proposition 2.6], every quaternionic Hilbert space $\mathcal{H}$ has a Hilbert basis $\mathcal{N}$. For every $x \in \mathcal{H}$, it is uniquely decomposed as,

$$x = \sum_{z \in \mathcal{N}} z \langle z, x \rangle.$$

Example 1 [17] Let μ be a Lebesgue measure on [0, 1] and the set of all $\mathbb{H}$-valued square integrable μ-measurable functions on [0, 1] is defined as,

$$L^2 \left([0, 1]; \mathbb{H}; \mu \right) = \left\{ f : [0, 1] \to \mathbb{H} \, ; \int_0^1 |f(x)|^2 \, d\mu(x) < \infty \right\}.$$

It is a right quaternionic Hilbert space with respect to the inner product given by

$$\langle f, g \rangle = \int_0^1 \overline{f(x)} g(x) \, d\mu(x), \text{ for all } f, g \in L^2\big([0, 1]; \mathbb{H}; \mu\big).$$

Let us fix $m \in \mathbb{S}$. If we define $e_r^{(m)}(x) = \exp\{2\pi m r x\}$, *for all* $x \in [0, 1]$ and $r \in \mathbb{Z}$. Then the set $\mathcal{N} = \{e_r^{(m)} : r \in \mathbb{Z}\}$ is an orthornormal system in $L^2\big([0, 1]; \mathbb{H}; \mu\big)$. Further, by the Stone-Weierstrass theorem $\mathcal{N}$ is a Hilbert basis for $L^2\big([0, 1]; \mathbb{H}; \mu\big)$. Since $\mathcal{N}$ is countable, we conclude that $L^2\big([0, 1]; \mathbb{H}; \mu\big)$ is a separable quaternionic Hilbert space.

Definition 4 Let $\mathcal{H}$ be a quaternionic Hilbert space and $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$, where $\mathcal{D}(T)$ denote the domain of T, which is a right linear subspace of $\mathcal{H}$. Then T is said to be *right* $\mathbb{H}$- linear or *quaternionic operator*, if

$$T(x + yq) = T(x) + T(y) \, q, \text{ for all } x \in \mathcal{D}(T).$$

The operator T is said to be densely defined, if $\overline{\mathcal{D}(T)} = \mathcal{H}$. Moreover, it is said to be *closed*, if the graph $\mathcal{G}(T) := \big\{(x, Tx) : x \in \mathcal{D}(T)\big\}$ is a closed right linear subspace of $\mathcal{H} \times \mathcal{H}$. By the closed graph theorem, the operator T is bounded or continuous if and only if $\mathcal{D}(T) = \mathcal{H}$ and T is closed.

We denote the class of all bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$ and it is a real Banach algebra with respect to the operator norm defined by

$$\|T\| = \sup \big\{\|Tx\| : x \in \mathcal{H}, \|x\| \le 1\big\}.$$

For every $T \in \mathcal{B}(\mathcal{H})$, by the quaternionic version of the Riesz representation theorem [9, Theorem 2.8], there exists a unique operator denoted by $T^* \in \mathcal{B}(\mathcal{H})$, called the adjoint of T satisfying,

$$\langle x, Ty \rangle = \langle T^*x, y \rangle, \text{ for all } x, y \in \mathcal{H}.$$

If $T \in \mathcal{B}(H)$, then the null space of T is defined by $N(T) = \{x \in \mathcal{H} : Tx = 0\}$ and the range space of T is defined by $R(T) = \{Tx : x \in \mathcal{H}\}$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be invariant subspace of T, if

$$T(x) \in \mathcal{M}, \text{ for every } x \in \mathcal{M}.$$

Furthermore, $\mathcal{M}$ is said to be reducing subspace of T if $\mathcal{M}$ is an invariant subspace of both T and T^*.

Definition 5 Let $T \in \mathcal{B}(\mathcal{H})$. Then T is said to be

1. *Self-adjoint*, if $T^* = T$,
2. *Anti self-adjoint*, if $T^* = -T$,
3. *Normal* if $T^*T = TT^*$,
4. *Positive*, if $T^* = T$ and $\langle x, Tx \rangle \ge 0$ for all $x \in \mathcal{H}$,
5. *Orthogonal projection*, if $T^* = T$ and $T^2 = T$,
6. *Partial isometry*, if $\|Tx\| = \|x\|$, for all $x \in N(T)^\perp$,

7. *Unitary*, if $T^*T = TT^* = I$.

Suppose that $T \in \mathcal{B}(\mathcal{H})$ is positive, then by [9, Theorem 2.18], there exists a unique positive operator $S \in \mathcal{B}(\mathcal{H})$ such that $S^2 = T$. Such an operator S is called the positive square root of T and it is dentoed by $S := T^{\frac{1}{2}}$. In fact, for every $T \in \mathcal{B}(\mathcal{H})$, the modulus $|T|$ is defined as the positive square root of T^*T, that is, $|T| := (T^*T)^{\frac{1}{2}}$.

We know from the well known Cartesian decomposition that every bounded normal operator on a complex Hilbert space can be decomposed uniquely as $A + iB$, where A, B are bounded self-adjoint operators. There is a quaternionic analog of this result due to Teichmüller [20] in which the role of 'i' is replaced by an anti self-adjoint unitary operator. Also see [9] for a detailed discussion.

Theorem 1 [9, *Theorem* 5.9] *Let* $T \in \mathcal{B}(\mathcal{H})$ *be normal. Then there exists an anti self-adjoint unitary operator* $J \in \mathcal{B}(\mathcal{H})$ *such that* $TJ = JT$, $T^*J = JT^*$ *and*

$$T = \frac{1}{2}(T + T^*) + \frac{1}{2} J|T - T^*|.$$

Here J *is uniquely determined by* T *on* $N(T - T^*)^{\perp}$. *Moreover, the operators* $(T + T^*)$, $|T - T^*|$ *and* J *commute mutually.*

The notion of the spectrum in the case of quaternionic operator theory has not been settled until the S-spectrum is introduced in 2006 [5]. Since then, research on quaternionic spectral theory has developed rapidly. Now we recall these definitions from the book [4].

Definition 6 [4, *Definition* 9.2, 9.2.4] Let $T : \mathcal{D}(T) \to \mathcal{H}$ be densely defined and $\Delta_q(T) : \mathcal{D}(T^2) \to \mathcal{H}$ is given by

$$\Delta_q(T) := T^2 - 2 \operatorname{re}(q)T + |q|^2 I.$$

The S-resolvent set of T is defined as follows:

$$\rho_S(T) = \big\{ q \in \mathbb{H} : N(\Delta_q(T)) = \{0\}, \overline{R(\Delta_q(T))} = \mathcal{H} \text{ and } \Delta_q(T)^{-1} \in \mathcal{B}(\mathcal{H}) \big\}.$$

The S-spectrum of T is defined as

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T).$$

Note that the S-spectrum $\sigma_S(T)$ is a nonempty compact subset of $\mathbb{H}$. Further, the S-spectrum is divided into three disjoint sets:

(i) The point S-spectrum of T as, $\sigma_{pS}(T) = \big\{ q \in \mathbb{H} : N(\Delta_q(T)) \neq \{0\} \big\}$.
(ii) The residual S-spectrum of T as,

$$\sigma_{rS}(T) = \big\{ q \in \mathbb{H} : N(\Delta_q(T)) = \{0\}, \overline{R(\Delta_q(T))} \neq \mathcal{H} \big\}.$$

(iii) The continuous S-spectrum of T as,

$$\sigma_{cs}(T) = \left\{ q \in \mathbb{H} : \ N(\Delta_q(T)) = \{0\}, \ \overline{R(\Delta_q(T))} = \mathcal{H}, \ \Delta_q(T)^{-1} \notin \mathcal{B}(\mathcal{H}) \right\}.$$

Now we recall the notion of slice Hilbert space associated to the given quaternionic Hilbert space $\mathcal{H}$, anti self-adjoint unitary operator $J \in \mathcal{B}(\mathcal{H})$ and $m \in \mathbb{S}$.

Definition 7 [9, Definition 3.6] Let $m \in \mathbb{S}$ and $J \in \mathcal{B}(\mathcal{H})$ be an anti self-adjoint unitary operator. These subsets $\mathcal{H}_{\pm}^{Jm}$ of $\mathcal{H}$ associated with J and m are defined by setting

$$\mathcal{H}_{\pm}^{Jm} := \{ x \in \mathcal{H} \ : \ J(x) = \pm \, x \cdot m \}.$$

Remark 2 For each $x \in \mathcal{H}$ and $m \in \mathbb{S}$, define $x_{\pm} := \frac{1}{2}(x \mp Jx \cdot m)$, then

$$J(x_{\pm}) = \frac{1}{2}\left(Jx \mp J^2 x \cdot m\right) = \frac{1}{2}\left(Jx \pm x \cdot m\right) = \pm \, x_{\pm} \cdot m. \tag{4}$$

It implies that $x_{\pm} \in \mathcal{H}_{\pm}^{Jm}$. Since $x \mapsto \pm \, x \cdot m$ and $x \mapsto Jx$ are continuous, then $\mathcal{H}_{\pm}^{Jm}$ is a non-trivial closed subsets of $\mathcal{H}$. In fact, we see that the inner product on $\mathcal{H}$ restricted to $\mathcal{H}_{\pm}^{Jm}$ is $\mathbb{C}_m$-valued as follows: let $\alpha + m\beta \in \mathbb{C}_m$ and $u, v \in \mathcal{H}_{\pm}^{Jm}$ for $m \in \mathbb{S}$, then

$$\begin{aligned}
\langle u, v \rangle (\alpha \pm m\beta) &= \alpha \langle u, v \rangle \pm \beta \langle u, v \cdot m \rangle \\
&= \alpha \langle u, v \rangle \pm \beta \langle u, Jv \rangle \\
&= \alpha \langle u, v \rangle \mp \beta \langle Ju, v \rangle \quad (\text{since } J^* = -J) \\
&= \alpha \langle u, v \rangle \mp \beta \langle u \cdot m, v \rangle \\
&= (\alpha \pm m\beta) \langle u, v \rangle \quad (\text{since } \overline{m} = -m).
\end{aligned}$$

Then by being a closed subspace of $\mathcal{H}$, we conclude that $\mathcal{H}_{\pm}^{Jm}$ is a Hilbert space over the field $\mathbb{C}_m$. These Hilbert spaces $\mathcal{H}_{\pm}^{Jm}$ are known as slice Hilbert spaces. As a $\mathbb{C}_m$-Hilbert space $\mathcal{H}$ has the following decomposition [9, Lemma 3.10]

$$\mathcal{H} = \mathcal{H}_{+}^{Jm} \oplus \mathcal{H}_{-}^{Jm}, \text{ for every } m \in \mathbb{S}. \tag{5}$$

Furthermore, if $\mathcal{N}$ is a Hilbert basis of $\mathcal{H}_{+}^{Jm}$, then $\mathcal{N} \cdot n = \{ z \cdot n : \ z \in \mathcal{N} \}$, where $n \in \mathbb{S}$ such that $mn = -nm$, is a Hilbert basis of $\mathcal{H}_{-}^{Jm}$. From Equation (5), it follows that $\mathcal{N}$ is also Hilbert basis of $\mathcal{H}$.

We denote the class of all bounded $\mathbb{C}_m$- linear operators on $\mathcal{H}_{+}^{Jm}$ by $\mathcal{B}(\mathcal{H}_{+}^{Jm})$. The following proposition develop a technique to extend $\mathbb{C}_m$- linear operator on $\mathcal{H}_{+}^{Jm}$ (for any $m \in \mathbb{S}$) to the quaternionic operator on $\mathcal{H}$.

Proposition 1 [9] *Let* $J \in \mathcal{B}(\mathcal{H})$ *be anti self-adjoint unitary and* $m \in \mathbb{S}$. *If* $T \in \mathcal{B}(\mathcal{H}_+^{Jm})$, *then there exist a unique quaternionic operator* $\widetilde{T} \in \mathcal{B}(\mathcal{H})$ *such that* $\widetilde{T}(x) = T(x)$, *for all* $x \in \mathcal{H}_+^{Jm}$. *The following additional facts hold.*

1. $\|\widetilde{T}\| = \|T\|$.
2. $J\widetilde{T} = \widetilde{T}J$.
3. $(\widetilde{T})^* = \widetilde{T^*}$.
4. *If* $S \in \mathcal{B}(\mathcal{H}_+^{Jm})$, *then* $\widetilde{ST} = \widetilde{S}\widetilde{T}$
5. *If* T *is invertible, then* $\widetilde{T}$ *is invertible and the inverse is given by*

$$(\widetilde{T})^{-1} = \widetilde{T^{-1}}.$$

On the other hand, let $V \in \mathcal{B}(\mathcal{H})$, *then* $V = \widetilde{U}$ *for some* $U \in \mathcal{B}(\mathcal{H}_+^{Jm})$ *if and only if* $JV = VJ$.

Precisely, the extension $\widetilde{T}$ of the operator T is defined as,

$$\begin{aligned} \widetilde{T}(x) = \widetilde{T}(x_+ + x_-) &= \widetilde{T}(x_+) + \widetilde{T}(x_-) \\ &= \widetilde{T}(x_+) - \widetilde{T}(x_- \cdot n) \cdot n \\ &= T(x_+) - T(x_- \cdot n) \cdot n, \end{aligned}$$

for all $x = x_+ + x_- \in \mathcal{H}_+^{Jm} \oplus \mathcal{H}_-^{Jm}$.

Note 2 In the case of normal operator $T \in \mathcal{B}(\mathcal{H})$, there exists an anti self-adjoint unitary operator $J \in \mathcal{B}(\mathcal{H})$ commutes with T by Theorem 1. Then by Proposition 1, there is a complex linear operator, we denote it by $T_+ \in \mathcal{B}(\mathcal{H}_+^{Ji})$ such that $T = \widetilde{T}_+$.

3 Factorization in a strongly irreducible sense

One of the fundamental results in the direction of factorizing quaternionic operators is the well known polar decomposition theorem [9, 19]. It states that if T is bounded or densely defined closed operator on a right quaternionic Hilbert space $\mathcal{H}$, then there exists a unique partial isometry $W_0 \in \mathcal{B}(\mathcal{H})$ satisfying

$$T = W_0|T| \text{ and } N(T) = N(W_0). \tag{6}$$

Recently, the authors of [19] obtained a necessary and sufficient condition for any arbitrary decomposition to coincide with the polar decomposition given in Equation (6). We recall the result here.

Theorem 2 [19, *Theorem* 5.13] *Let* T *be a bounded or densely defined closed operator defined on a right quaternionic Hilbert space* $\mathcal{H}$. *If* $W \in \mathcal{B}(\mathcal{H})$ *is a partial isometry satisfying* $T = W|T|$, *then* $W = W_0$ *if and only if either* $N(T) = \{0\}$ *or* $R(T)^\perp = \{0\}$, *where* W_0 *is the partial isometry satisfying Equation* (6).

In this section, first, we adopt the notion of strong irreducibility [10, 13] to the class of bounded quaternionic operators, and prove a relation between strong irreducibility and the point S-spectrum. Later, we prove a factorization of quaternionic normal operators in a strongly irreducible sense, by means of replacing the partial isometry W by a desirably small compact perturbation of W and $|T|$ is replaced by the strongly irreducible operator. It is a quaternionic extension (for normal operators) of the result proved in [21].

Definition 8 Let $T \in \mathcal{B}(\mathcal{H})$. Then T is said to be

1. *Irreducible*, if there does not exist a nontrivial orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ (i.e., $P \neq 0$ and $P \neq I$) such that $PT = TP$. Otherwise, T is called reducible.
2. *Strongly irreducible*, if there does not exist a nontrivial idempotent $E \in \mathcal{B}(\mathcal{H})$ (i.e., $E \neq 0$ and $E \neq I$) such that $TE = ET$. Otherwise, T is called strongly reducible.

It is clear from Definition 8 that every strongly irreducible operator is irreducible. Similar to the classical setup, the class of strongly irreducible quaternionic operators is closed under similarity invariance. To describe strong irreducibility or irreducibility of quaternionic normal operators, we show that it is enough to deal with the corresponding complex linear operator defined on slice Hilbert space.

Lemma 1 *Let $J \in \mathcal{B}(\mathcal{H})$ be anti self-adjoint unitary operator and $S \in \mathcal{B}(\mathcal{H}_+^{Ji})$, then*

1. *$\widetilde{S}$ is irreducible if and only if S is irreducible.*
2. *$\widetilde{S}$ is strongly irreducible if and only if S is strongly irreducible.*

Proof Proof of (1) : Suppose that $\widetilde{S}$ is irreducible, then we show that S is irreducible. If there is an orthogonal projection $0 \neq P \in \mathcal{B}(H_+^{Ji})$ such that $SP = PS$, then by Proposition 1, we see that

$$(\widetilde{P})^* = \widetilde{P^*} = \widetilde{P} \text{ and } (\widetilde{P})^2 = \widetilde{P^2} = \widetilde{P}.$$

Further,

$$\widetilde{SP} = \widetilde{SP} = \widetilde{PS} = \widetilde{PS}.$$

This shows that $0 \neq \widetilde{P} \in \mathcal{B}(\mathcal{H})$ is an othogonal projection which commutes with $\widetilde{S}$. Since $\widetilde{S}$ is irreducible, we conclude that $\widetilde{P} = I$, the identity operator on $\mathcal{H}$. Thus P is an identity operator on $\mathcal{H}_+^{Ji}$ and hence S is irreducible. Now we prove contrapositive statement. Suppose that $\{0\} \neq \mathcal{M} \subsetneq \mathcal{H}$ is a reducing subspace for $\widetilde{S}$. Define

$$\mathcal{M}_+^{Ji} := \{x \in \mathcal{M} \,:\, Jx = x \cdot i\} = \mathcal{M} \cap \mathcal{H}_+^{Ji}.$$

For $x \in \mathcal{M}_+^{Ji}$, we have

$$JS(x) = J\widetilde{S}(x) = \widetilde{S}J(x) = \widetilde{S}(x \cdot i) = \widetilde{S}(x) \cdot i = S(x) \cdot i$$

and

$$JS^*(x) = J\widetilde{S^*}(x) = \widetilde{S^*}J(x) = \widetilde{S^*}(x \cdot i) = \widetilde{S^*}(x) \cdot i = S^*(x) \cdot i.$$

This implies that $\mathcal{M}_+^{Ji}$ is a reducing subspace of S. It is enough to show that $\mathcal{M}_+^{Ji}$ is a non-trivial proper subspace of $\mathcal{H}_+^{Ji}$.

Claim: $\{0\} \neq \mathcal{M}_+^{Ji} \subsetneq \mathcal{H}_+^{Ji}$.

First, we assume that $\mathcal{M}_+^{Ji} = \{0\}$. In this case, $\mathcal{M}_-^{Ji} = \{0\}$ since $x \in \mathcal{M}_-^{Ji}$ if and only if $x \cdot j \in \mathcal{M}_+^{Ji}$, where $i \cdot j = -j \cdot i$. Therefore, $M = M_+^{Ji} \oplus M_-^{Ji} = \{0\}$. This is a contradiction to the fact that $\mathcal{M}$ is a non trivial subspace of $\mathcal{H}$. Hence $\mathcal{M}_+^{Ji} \neq \{0\}$.

Next, suppose that $\mathcal{M}_+^{Ji} = \mathcal{H}_+^{Ji}$. Let $y \in \mathcal{H}$ with $y = y_+ + y_-$, where $y_\pm \in \mathcal{H}_\pm^{Ji}$. We know that $y_- \cdot j \in \mathcal{H}_+^{Ji} = \mathcal{M}_+^{Ji}$ and so $y_- \in \mathcal{M}$. Therefore, $y \in \mathcal{M}$. This is contradiction to the fact that $\mathcal{M}$ is a proper subspace of $\mathcal{H}$. Hence $\mathcal{M}_+^{Ji} \subsetneq \mathcal{H}_+^{Ji}$ and therefore $\widetilde{S}$ is irreducible if and only if S is irreducible.

Proof of (2) : Suppose that $\widetilde{S}$ is strongly irreducible. If there is an idempotent $0 \neq E \in \mathcal{B}(\mathcal{H}_+^{Ji})$ such that $SE = ES$, then by Proposition [1] we have that $0 \neq \widetilde{E} \in \mathcal{B}(\mathcal{H})$ with $\widetilde{E}^2 = \widetilde{E}$ and

$$\widetilde{SE} = \widetilde{S}\widetilde{E} = \widetilde{E}\widetilde{S} = \widetilde{ES}.$$

Since $\widetilde{S}$ is strongly irreducible, we conclude that $\widetilde{E} = I$, the identity operator on $\mathcal{H}$. This implies that E is the identity operator on $\mathcal{H}_+^{Ji}$ and hence S is strongly irreducible.

Conversely, assume that S is strongly irreducible. If there is an idempotent say $0 \neq F \neq I$ in $\mathcal{B}(\mathcal{H})$ such that $\widetilde{S}F = F\widetilde{S}$. Since $R(F)$ is a closed subspace of $\mathcal{H}$, we have $\mathcal{H}_+^{Ji} = R(F)_+^{Ji} \oplus (R(F)_-^{Ji})^\perp$, where

$$R(F)_+^{Ji} = \{x \in R(F) : J(x) = x \cdot i\} = R(F) \cap \mathcal{H}_+^{Ji}.$$

Furthermore, if $x \in R(F)_+^{Ji}$, then

$$JS(x) = J\widetilde{S}(x) = \widetilde{S}J(x) = \widetilde{S}(x \cdot i) = \widetilde{S}(x) \cdot i = S(x) \cdot i.$$

and

$$JS^*(x) = J\widetilde{S^*}(x) = \widetilde{S^*}J(x) = \widetilde{S^*}(x \cdot i) = \widetilde{S^*}(x) \cdot i = S^*(x) \cdot i.$$

This shows that S is reducible. It is a contradiction to the fact that S is irreducible. Hence $\widetilde{S}$ is strongly irreducible. □

Now we prove the relation between point S-spectrum and strong irreduicibility. It is a quaternionic analogue of the well known result proved in [10] by F. Gilfeather.

Theorem 3 *Let $T \in \mathcal{B}(\mathcal{H})$ be normal. If the point S- spectrum of T is empty set (i.e., $\sigma_{p^S}(T) = \emptyset$), then T is strongly irreducible.*

Proof Since T is normal, then by Note 2 there is an anti self-adjoint unitary operator $J \in \mathcal{B}(\mathcal{H})$ commuting with T such that $T = \widetilde{T_+}$, where $T_+ \in \mathcal{B}(\mathcal{H}_+^{Ji})$ is normal operator. Now we show that the point spectrum $\sigma_p(T_+)$ of T_+ is empty. Suppose that $\lambda \in \sigma_p(T_+)$, then

$$T_+(x) = x \cdot \lambda, \text{ for some } x \in \mathcal{H}_+^{Ji} \setminus \{0\}.$$

This implies that $\Delta_\lambda(T)x = 0$.

Equivalently, $x \in N(\Delta_\lambda(T)) \neq \{0\}$. This shows that

$$\lambda \in \sigma_p(T_+) \iff [\lambda] \in \sigma_{p^s}(T).$$

If follows that $\sigma(T_+) = \emptyset$ (since $\sigma_{p^s}(T) = \emptyset$). Since T_+ is a bounded complex normal operator with empty point spectrum, then by [10, Theorem 2] the operator T_+ is strongly irreducible. Finally, by Lemma 1 we conclude that T is strongly irreducible. $\qquad \square$

Now we prove our main result in this section, which shows that every quaternionic normal operator can be factorized in a strongly irreducible sense.

Theorem 4 *Let $T \in \mathcal{B}(\mathcal{H})$ be normal and $\delta > 0$. Then there exist a partial isometry W, a compact operator K with $\|K\| < \delta$ and a strongly irreducible operator S in $\mathcal{B}(\mathcal{H})$ such that*

$$T = (W + K)S.$$

Proof Since T is normal, by Theorem 1 and Note 2, there exists an anti self-adjoint unitary operator $J \in \mathcal{B}(\mathcal{H})$ commuting with T such that $T = \widetilde{T_+}$, where $T_+ \in \mathcal{B}(\mathcal{H}_+^{Ji})$ is a normal operator. It is clear from [21, Theorem 1.1] that for a given $\delta > 0$, there exists a partial isometry W_+, a compact operator K_+ with $\|K_+\| < \delta$ and a strongly irreducible operator S_+ in $\mathcal{B}(\mathcal{H}_+^{Ji})$ such that

$$T_+ = (W_+ + K_+)S_+. \tag{7}$$

If we define $W := \widetilde{W_+}$, one can see that W is a partial isometry as follows: for every $x \in N(W)^\perp$, there exists $x_\pm \in N(W)^\perp \cap \mathcal{H}_\pm^{Ji}$ such that $x = x_+ + x_-$ and

$$\begin{aligned}
\|Wx\|^2 &= \|W_+(x_+) - W_+(x_- \cdot j)j\|^2 \\
&= \|W_+(x_+)\|^2 + \|W_+(x_- \cdot j)\|^2 \\
&= \|x_+\|^2 + \|x_-\|^2 \\
&= \|x\|^2.
\end{aligned}$$

Now we show that the operator defined by $K := \widetilde{K_+}$ is a quaternionic compact operator on $\mathcal{H}$. Since K_+ is a compact operator on $\mathcal{H}_+^{Ji}$, there is a sequence of finite rank operators $\{F_n : n \in \mathbb{N}\} \subset \mathcal{B}(\mathcal{H}_+^{Ji})$ converging to K_+ (uniformly) with respect to the topology induced from the operator norm. Then by (1) of Proposition 1 we see that

$$\|\widetilde{F}_n - K\| = \|\widetilde{F}_n - \widetilde{K}_+\| = \|F_n - K_+\| \longrightarrow 0, \text{ as } n \to \infty.$$

This implies that the sequence $\{\widetilde{F}_n : n \in \mathbb{N}\} \subset \mathcal{B}(\mathcal{H})$ of finite rank quaternionic operators converges to K uniformly. Thus the operator K is compact and its norm is given by

$$\|K\| = \|K_+\| < \delta.$$

Moreover, by Lemma 1, the quaternionic operator defined by $S := \widetilde{S}_+ \in \mathcal{B}(\mathcal{H})$ is strongly irreducible. Now we apply quaternionic extension to bounded complex linear operator T_+ and use its factorization given in Eq. (7), we conclude that

$$T = \widetilde{T}_+ = (\widetilde{W}_+ + \widetilde{K}_+)\widetilde{S}_+ = (W + K)S.$$

Hence the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We illustrate our result with the following example.

Example 2 Let $T : L^2\big([0, 1];\mathbb{H};\mu\big) \to L^2\big([0, 1];\mathbb{H};\mu\big)$ be defined by

$$(Tg)(x) = \begin{cases} xg(x) + \frac{1}{2} \int\limits_0^1 xy^2 g(y)\, dy, & \text{if } 0 \le x \le \frac{1}{3}; \\[2mm] \frac{1}{2} \int\limits_0^1 xy^2 g(y)\, dy, & \text{if } \frac{1}{3} \le x \le 1, \end{cases}$$

for all $g \in L^2\big([0, 1];\mathbb{H};\mu\big)$. Then the adjoint of T is given by

$$(T^*g)(x) = \begin{cases} xg(x) + \frac{1}{2} \int\limits_0^1 x^2 y g(y)\, dy, & \text{if } 0 \le x \le \frac{1}{3}; \\[2mm] \frac{1}{2} \int\limits_0^1 x^2 y g(y)\, dy, & \text{if } \frac{1}{3} \le x \le 1, \end{cases}$$

for all $g \in L^2\big([0, 1];\mathbb{H};\mu\big)$. Clearly, T is normal. Suppose that $\delta = \frac{1}{2}$. Now we factorize T in a strongly irreducible sense. Define the integral operator $K : L^2\big([0, 1];\mathbb{H};\mu\big) \to L^2\big([0, 1];\mathbb{H};\mu\big)$ by

$$(Kg)(x) = \frac{1}{2} \int\limits_0^1 xy\, g(y)dy, \text{ for all } g \in L^2\big([0, 1];\mathbb{H};\mu\big).$$

It is well known that K is a compact operator. Using Caucy-Schwarz inequality, the norm of K is computed as,

$$\|Kg\|_2 = \Big(\int\limits_0^1 |(Kg)(x)|^2 \, dx \Big)^{\frac{1}{2}} \le \Big(\int\limits_0^1 \int\limits_0^1 |xy|^2 \, |g(y)|^2 dy dx \Big)^{\frac{1}{2}}$$

$$\le \Big(\int\limits_0^1 |g(y)|^2 \, dy \Big)^{\frac{1}{2}} \Big(\int\limits_0^1 \int\limits_0^1 |xy|^2 dy dx \Big)^{\frac{1}{2}}$$

$$= \|g\|_2 \Big(\int\limits_0^1 \int\limits_0^1 x^2 y^2 dy dx \Big)^{\frac{1}{2}}$$

$$= \frac{1}{3} \|g\|_2.$$

This shows that $\|K\| \le \frac{1}{3} < \frac{1}{2}$. We recall that the class of all bounded $\mathbb{H}$-valued measurabale functions on $[0, 1]$ is denoted by $L^\infty\big([0,1];\mathbb{H};\mu\big)$. For every $f \in L^\infty\big([0,1];\mathbb{H};\mu\big)$, the multiplication operator $M_f : L^2\big([0,1];\mathbb{H};\mu\big) \to L^2\big([0,1];\mathbb{H};\mu\big)$ defined by

$$M_f(g)(x) = f(x)g(x), \text{ for all } g \in L^2\big([0,1];\mathbb{H};\mu\big)$$

is a bounded quaternionic operator with the norm $\|M_f\| = \|f\|_\infty$. The adjoint of M_f is given by $M_f^* = M_{\overline{f}}$, where $\overline{f}(x) = \overline{f(x)}$ for all $x \in [0,1]$. Let $\phi(x) = x$, for all $x \in [0,1]$ and the characteristic function

$$\chi_{[0,\frac{1}{3}]} = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{3}]; \\ 0, & \text{otherwise} . \end{cases}$$

Then by the direct verification, we get that

$$T = \Big(M_{\chi_{[0,\frac{1}{3}]}} + K \Big) M_\phi. \tag{8}$$

Note that $M_{\chi_{[0,\frac{1}{3}]}}$ is a partial isometry, and since M_ϕ is normal with $\sigma_{ps}(M_\phi) = \emptyset$, then M_ϕ is strongly irreducible by Theorem 3. Therefore, the factorization of T given in Eq. (8) is a strongly irreducible factorization.

Now we construct an example of a non-normal operator by a slight modification of the linear operator defined in Example 2 and compute its strongly irreducible factorization.

Example 3 Let us define $T : L^2\big([0,1];\mathbb{H};\mu\big) \to L^2\big([0,1];\mathbb{H};\mu\big)$ by

$$T(g)(x) = \begin{cases} xg(x) + \dfrac{i}{2} \displaystyle\int_0^x yg(y)dy, & \text{if } 0 \leq x \leq \frac{1}{3}; \\[3mm] \dfrac{i}{2} \displaystyle\int_0^x yg(y)dy, & \text{if } \frac{1}{3} < x \leq 1, \end{cases}$$

for all $g \in L^2([0,1];\mathbb{H};\mu)$ and suppose that $\delta = \frac{1}{2}$. First, we show that T is a bounded quaternionic non-normal operator. Let $g, h \in L^2([0,1];\mathbb{H};\mu)$. Then

$$\langle h, Tg \rangle$$

$$= \int_0^1 \overline{h(x)}\, (Tg)(x)\, dx$$

$$= \int_0^{\frac{1}{3}} \overline{h(x)} \left[xg(x) + \frac{j}{2} \int_0^x yg(y)\, dy \right] dx + \int_{\frac{1}{3}}^1 \overline{h(x)} \left[\frac{j}{2} \int_0^x yg(y)\, dy \right] dx$$

$$= \int_0^{\frac{1}{3}} \overline{h(x)}xg(x)\, dx + \int_0^{\frac{1}{3}} \int_0^x \overline{h(x)} \left[\frac{j}{2}yg(y) \right] dydx + \int_{\frac{1}{3}}^1 \int_0^x \overline{h(x)} \left[\frac{j}{2}yg(y) \right] dydx.$$

By Fubini's theorem, the above integral can be written as,

$$\langle h, Tg \rangle$$

$$= \int_0^{\frac{1}{3}} \overline{h(y)}\, yg(y)\, dy + \int_0^{\frac{1}{3}} \int_y^{\frac{1}{3}} \overline{h(x)} \left[\frac{j}{2}yg(y) \right] dxdy$$

$$+ \int_{\frac{1}{3}}^1 \int_{y-\frac{1}{3}}^y \overline{h(x)} \left[\frac{j}{2}yg(y) \right] dxdy$$

$$= \int_0^{\frac{1}{3}} \overline{yh(y)}\, g(y)\, dy + \int_0^{\frac{1}{3}} \left[\overline{\frac{-j}{2}y \int_y^{\frac{1}{3}} h(x)\, dx} \right] g(y)\, dy$$

$$+ \int_{\frac{1}{3}}^1 \left[\frac{-j}{2}y \int_{y-\frac{1}{3}}^y h(x)\, dx \right] g(y)\, dy.$$

Thus the adjoint of T is given by

$$
(T^*h)(y) =
\begin{cases}
yg(y) - \dfrac{j}{2}y \displaystyle\int_{y}^{\frac{1}{3}} h(x)\,dx, & \text{if } 0 \le y \le \tfrac{1}{3}; \\[2ex]
-\dfrac{j}{2}y \displaystyle\int_{y-\frac{1}{3}}^{y} h(x)\,dx, & \text{if } \tfrac{1}{3} \le y \le 1.
\end{cases}
$$

It follows that $TT^* \ne T^*T$. Now we show that T can be factorized in a strongly irreducible sense. First, we define $K : L^2\big([0,1];\mathbb{H};\mu\big) \to L^2\big([0,1];\mathbb{H};\mu\big)$ by

$$
(Kg)(x) = \frac{j}{2} \int_{0}^{x} g(t)\,dt, \ \text{for all } g \in L^2\big([0,1];\mathbb{H};\mu\big).
$$

Our aim is to show that K is a compact operator with $\|K\| < \tfrac{1}{2}$. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $L^2\big([0,1];\mathbb{H};\mu\big)$ with $\|g_n\| \le 1$, for all $n \in \mathbb{N}$. Then

$$
|(Kg_n)(x)| = \left| \frac{j}{2} \int_{0}^{x} g_n(t)\,dt \right| \le \frac{1}{2} \int_{0}^{x} |g_n(x)|\,dt \le \frac{1}{2}, \tag{9}
$$

for all $x \in [0,1]$ and $n \in \mathbb{N}$. Further, by Hölder's inequality, we get

$$
\left|Kg(x) - Kg(y)\right| = \left|\frac{j}{2}\right|\left| \int_{0}^{x} g(t)\,dt - \int_{0}^{y} g(t)\,dt \right| \le \frac{1}{2} \int_{y}^{x} |g(t)|\,dt \le \frac{1}{2}\|g\|_2 \sqrt{|x-y|}. \tag{10}
$$

It follows from Eqs. (9), (10) that the sequence $\{Kg_n\}_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous. By Arzela-Ascoli's theorem, there is a subsequence $\{g_{n_k}\}$ of $\{g_n\}_{n\in\mathbb{N}}$ such that $\{Kg_{n_k}\}$ converges uniformly. Thus K is a compact operator.

Now we compute the norm of K. First, by applying the Fubini's theorem, we get the adjoint of K as,

$$
(K^*g)(x) = \frac{-j}{2} \int_{x}^{1} g(t)\,dt, \ \text{for all } g \in L^2\big([0,1];\mathbb{H};\mu\big).
$$

So the operator K^*K is given by,

$$
(K^*Kg)(x) = \frac{-j}{2} \int_{x}^{1} \left(\frac{j}{2} \int_{0}^{t} g(s)\,ds \right) dt = \frac{1}{4} \int_{x}^{1} \int_{0}^{t} g(s)\,ds\,dt
$$

is a positive quaternionic compact operator. We know from [17, Corollary 2.13] that $L^2\big([0,1];\mathbb{C};\mu\big)$ is an associated slice Hilbert space and let $(K^*K)_+$ be the bounded complex linear operator on $L^2([0,1];\mathbb{C};\mu)$ such that $\widetilde{(K^*K)_+} = K^*K$. Then by norm of the Volterra integral operator computed as in Solution 188 of [11] and in the view of Proposition 1, we conclude that

$$\|K\| = \|K^*K\|^{\frac{1}{2}} = \|(K^*K)_+\|^{\frac{1}{2}} = \left(\frac{1}{\pi^2}\right)^{\frac{1}{2}} = \frac{1}{\pi} < \frac{1}{2}.$$

Let us take $W := M_{\chi_{[0,\frac{1}{3}]}}$ and $S = M_{\varphi}$, where $\varphi(x) = x$, for all $x \in [0, 1]$. Clearly, W is a partial isometry and since S is normal with $\sigma_{p^s}(S) = \emptyset$, it follows that S is strongly irreducible from Theorem 3. Finally, we have that

$$T = (W + K)S.$$

Now we pose the following question.

Question 1 Let $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$ be densely defined closed right $\mathbb{H}$- linear operator (need not be normal), where $\mathcal{D}(T)$ is the domain of T. Then, can T be factorized in a strongly irreducible sense?

We expect that, using the notion of quaternionic Cowen–Douglas operators related to the geometry of quaternionic Hilbert spaces developed in [12] and further suitable arguments, one might achieve the affirmative answer to the Question 1.

4 Riesz decomposition

In this section, we discuss Riesz decomposition theorem for bounded quaternionic operators on the right quaternionic Hilbert spaces and obtain a sufficient condition for strong irreducibility. We recall some definitions and known results form [2, 4, 5, 9] that are useful to establish our result.

Let U be an open subset of $\mathbb{H}$. If U is a domain in $\mathbb{H}$ such that $U \cap \mathbb{R} \neq \emptyset$ and $U \cap \mathbb{C}_m$ is domain in $\mathbb{C}_m$, for all $m \in \mathbb{S}$ then U is called slice domain or s-domain. A real differentiable function $f : U \to \mathbb{H}$ is said to be left s-regular (right s-regular), if for every $m \in \mathbb{S}$, the function f satisfy $\frac{1}{2}\left[\frac{\partial}{\partial x}f(x + my) + m\,\frac{\partial}{\partial y}f(x + my)\right] = 0$ $\left(\frac{1}{2}[\frac{\partial}{\partial x}f(x + my) + \frac{\partial}{\partial y}f(x + my)\,m] = 0\right)$. We denote the class of left and right s-regular functions on U by $\mathcal{R}^L(U)$ and $\mathcal{R}^R(U)$, respectively. One can verify that $\mathcal{R}^L(U)$ is a right $\mathbb{H}$-module and $\mathcal{R}^R(U)$ is a left $\mathbb{H}$-module.

As in the case of complex holomorphic functions, there is a Cauchy integral formula for s-regular functions (see [5, Theorem 4.5.3] for details).

4.1 The quaternionic functional calculus

Let $\mathcal{H}$ be a right quaternionic Hilbert space and let $\mathcal{N}$ be a Hilbert basis of $\mathcal{H}$. It is immediate to see that the class of all bounded right $\mathbb{H}$- linear operators denoted by $\mathcal{B}(\mathcal{H})$ is a two sided quaternionic Banach module with respect to the module actions given by

$$(q \cdot T)(x) := \sum_{z \in \mathcal{N}} z \cdot q \langle z, Tx \rangle \text{ and } (T \cdot q)(x) := \sum_{z \in \mathcal{N}} T(z) \cdot q \langle z, x \rangle,$$

for all $T \in \mathcal{B}(\mathcal{H}), q \in \mathbb{H}, x \in \mathcal{H}$. In particular, for an identity operator $I \in \mathcal{B}(\mathcal{H})$, we have

$$(q \cdot I)(x) = \sum_{z \in \mathcal{N}} z \cdot q \langle z, x \rangle = (I \cdot q)(x), \text{ for all } x \in \mathcal{H}, q \in \mathbb{H}.$$

Next, we recall the notion of the *S*-resolvent operator and the *S*- resolvent equation which plays a vital role in establishing quaternionic functional calculus.

Definition 9 [5, Definition 4.8.3] Let $T \in \mathcal{B}(\mathcal{H})$ and $s \in \rho_S(T)$. Then the left S-resolvent operator is defined by

$$S_L^{-1}(s, T) := -\Delta_s(T)^{-1}(T - \bar{s}I) = \sum_{n=0}^{\infty} T^n s^{-1-n},$$

and the right *S*-resolvent operator by

$$S_R^{-1}(s, T) := -(T - \bar{s}I)\Delta_s(T)^{-1} = \sum_{n=0}^{\infty} s^{-1-n} T^n,$$

for $\|T\| < |s|$.

See [2, 5] for detailed discussion on *S*-resolvent operators.

Note 3 Let A be a bounded linear operator on some complex Hilbert space $\mathcal{K}$ and $\lambda \in \rho(A)$, the resolvent set of A. Then $(\lambda I - A)$ is invertible and its inverse is given by the following power series,

$$(\lambda I - A)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n, \text{ for } \|A\| < \lambda.$$

Moreover, if $\lambda, \mu \in \rho(A)$, then we have the following relation known as resolvent equation:

$$(\lambda I - A)^{-1} - (\mu I - A)^{-1} = (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1}. \tag{11}$$

One of the crucial observation in estabilshing the quaternionic functional calculus is the quaternionic analogue of Eq. (11). It is called the *S*-resolvent equation. We recall the result here.

Theorem 5 [2, *Theorem* 3.8] *Let* $T \in \mathcal{B}(\mathcal{H})$ *and let* $s, p \in \rho_S(T)$. *Then the S- resolvent equation is given by*

$$\begin{aligned}
S_R^{-1}&(s, T)S_L^{-1}(p, T) \\
&= \left[\left(S_R^{-1}(s, T) - S_L^{-1}(p, T)\right)p - \bar{s}\left(S_R^{-1}(s, T) - S_L^{-1}(p, T)\right) \right] \\
&\quad (p^2 - 2\mathrm{re}(s)p + |s|^2)^{-1}.
\end{aligned}$$

Equivalently,

$$
\begin{aligned}
S_R^{-1}&(s, T)S_L^{-1}(p, T) \\
&= (s^2 - 2\mathrm{re}(p)s + |p|^2)^{-1}\Big[\big(S_R^{-1}(s, T) - S_L^{-1}(p, T)\big)p \\
&\quad - \bar{s}\big(S_R^{-1}(s, T) - S_L^{-1}(p, T)\big)\Big].
\end{aligned}
$$

Definition 10 Let $T \in \mathcal{B}(\mathcal{H})$, $W \subseteq \mathbb{H}$ be open and $U \subseteq \mathbb{H}$ be a domain in $\mathbb{H}$. Then

1. U is said to be a *T-admissible open set*, if U is axially symmetric *s*-domain that contains the *S*-spectrum $\sigma_S(T)$ such that the boundary $\partial(U \cap \mathbb{C}_m)$ is the union of a finite number of continuously differentiable Jordan curves, for every $m \in \mathbb{S}$.
2. A function $f \in \mathcal{R}^L(W)$ is said to be locally *left regular function* on $\sigma_S(T)$, if there is *T*-admissible domain U in $\mathbb{H}$ such that $\overline{U} \subseteq W$.
3. A function $f \in \mathcal{R}^R(W)$ is said to be *locally right regular function* on $\sigma_S(T)$, if there is *T*-admissible domain U in $\mathbb{H}$ such that $\overline{U} \subseteq W$.

The class of all locally left and locally right regular functions on $\sigma_S(T)$ are denoted by $\mathcal{R}^L_{\sigma_S(T)}$ and $\mathcal{R}^R_{\sigma_S(T)}$ respectively.

Using quaternionic versions of Cauchy Integral formula and Hahn Banach theorem [5, Theorem 4.1.10], the definition of quaternionic functional calculus is obtained as below.

Definition 11 [5*Definition 4.10.4*](quaternionic functional calculus) Let $T \in \mathcal{B}(\mathcal{H})$ and $U \subset \mathbb{H}$ be a *T*-admissible domain. Then

$$
f(T) = \frac{1}{2\pi} \int\limits_{\partial(U \cap \mathbb{C}_m)} S_L^{-1}(s, T)ds_m f(s), \ \text{for all } f \in \mathcal{R}^L_{\sigma_S(T)} \tag{12}
$$

and

$$
f(T) = \frac{1}{2\pi} \int\limits_{\partial(U \cap \mathbb{C}_m)} f(s)ds_m S_R^{-1}(s, T), \ \text{for all } f \in \mathcal{R}^R_{\sigma_S(T)}, \tag{13}
$$

where $ds_m = -ds \cdot m$. Note that the integrals that appear in Eqs. (12), (13) are independent of the choice of imaginary unit $m \in \mathbb{S}$ and *T*-admissible domain U.

4.2 Riesz decomposition theorem

Before proving our result, let us discuss the adjoint of the operator $f(T)$ defined as in Definition 11.

Remark 3 Let $T \in \mathcal{B}(\mathcal{H})$ and W be an axially symmetric open set in $\mathbb{H}$. For every $f : W \to \mathbb{H}$, we define $\hat{f} : W \to \mathbb{H}$ by

$$\hat{f}(q) = \overline{f(\bar{q})}, \text{ for all } q \in W.$$

Let $f \in \mathcal{R}^L(W)$. Then for every $m \in \mathbb{S}$ and $x, y \in \mathbb{R}$, we see that

$$\frac{\partial}{\partial x}\hat{f}(x + my) + m\frac{\partial}{\partial y}\hat{f}(x + my)$$

$$= \frac{\partial}{\partial x}\overline{f(x - my)} + m\frac{\partial}{\partial y}\overline{f(x - my)}$$

$$= \overline{\frac{\partial}{\partial x}f(x - my) - m\frac{\partial}{\partial y}f(x - my)}$$

$$= \overline{\frac{\partial}{\partial x}f(x + \overline{m}y) + \overline{m}\frac{\partial}{\partial y}f(x + \overline{m}y)}$$

$$= 0.$$

This show that $\hat{f} \in \mathcal{R}^L(W)$. Further, if we assume that f is locally left regular function that is, $f \in \mathcal{R}^L_{\sigma_S(T)}$ then by Definition 10, there is a T-admissible domain U such that $\overline{U} \subseteq W$. Since $\sigma_S(T) = \sigma_S(T^*)$ and by the above arguments, we conclude that $\hat{f} \in \mathcal{R}^L_{\sigma_S(T)}$. Now we compute the adjoint of $f(T)$, whenever $f \in \mathcal{R}^L_{\sigma_S(T)}$, as follows:

$$\langle x, f(T)y \rangle = \frac{1}{2\pi}\int_{\partial(U \cap \mathbb{C}_m)} \langle x, S_L^{-1}(s, T)y \rangle \, ds_m \, f(s)$$

$$= \frac{1}{2\pi}\int_{\partial(U \cap \mathbb{C}_m)} \langle S_L^{-1}(s, T)^* x, y \rangle \, ds_m \, f(s)$$

$$= \frac{1}{2\pi}\int_{\partial(U \cap \mathbb{C}_m)} \langle S_R^{-1}(\bar{s}, T^*)x, y \rangle \, ds_m \, f(s).$$

If we put $s = \bar{t}$, then $ds = d\bar{t}$ and $ds_m = -d\bar{t}\, m$. Since the integration over the domain $\partial(U \cap \mathbb{C}_m)$ which is symmetric about the real line, we see that $\overline{d\bar{t}_m} = dt_m$. Thus above integral can be modified as,

$$\langle x, f(T)y \rangle = \frac{1}{2\pi}\int_{\partial(U \cap \mathbb{C}_m)} \langle S_R^{-1}(t, T^*)x, y \rangle d\bar{t}_m f(\bar{t})$$

$$= \frac{1}{2\pi}\overline{\int_{\partial(U \cap \mathbb{C}_m)} \overline{f(\bar{t})} \, \overline{d\bar{t}_m}\langle y, S_R^{-1}(t, T^*)x \rangle}$$

$$= \frac{1}{2\pi}\langle y, \int_{\partial(U \cap \mathbb{C}_m)} \hat{f}(t) \, dt_m \, S_R^{-1}(t, T^*)x \rangle$$

$$= \langle \hat{f}(T^*)x, y \rangle,$$

for all $x, y \in \mathbb{H}$. Therefore, $f(T)^* = \hat{f}(T^*)$ for all $f \in \mathcal{R}^L_{\sigma_S(T)}$. Similarly, the result holds true for $\mathcal{R}^R_{\sigma_S(T)}$. For further details about algebraic properties of quaternionic functional calculus, we refer the reader to [5, Proposition 4.11.1].

In the following lemma, we show that for any compact set in $\mathbb{H}$, there is an axially symmetric s-domain such that its intersection with $\mathbb{C}_m$ is a Cauchy domain in $\mathbb{C}_m$, for every $m \in \mathbb{S}$.

Lemma 2 *Let K be an axially symmetric compact subset of $\mathbb{H}$ and W be an axially symmetric s- domain containing K. Then there is an axially symmetric s- domain U with the boundary $\partial(U \cap \mathbb{C}_m)$ is the union of a finite number of continuously differentiable Jordan curves (for every $m \in \mathbb{S}$) such that $K \subset U$ and $\overline{U} \subset W$.*

Proof This result follows in a standard way of the classical proof given in [16]. For every $m \in \mathbb{S}$, we define $K_m := K \cap \mathbb{C}_m$ and $W_m := W \cap \mathbb{C}_m$. Since K_m is a compact subset of the open set W_m, by [16, Lemma 2.3], there exists a Cauchy domain U_m such that $K_m \subset U_m$ and $\overline{U}_m \subset W$ for every $m \in \mathbb{S}$. Let us take $U = \Omega_{U_m}$. Then U is an axially symmetric s-domain containing K and the boundary $\partial(U \cap \mathbb{C}_m)$ is the union of finite number of continuously differentiable Jordan curves. The closure of U follows from Eq. (3) as,

$$\overline{U} = \overline{\Omega}_{U_m} = \Omega_{\overline{U}_m} \subset \Omega_{W_m} = W.$$

Hence the result.   □

Corollary 1 *Let $T \in \mathcal{B}(\mathcal{H})$ and let $W \subseteq \mathbb{H}$ be an axially symmetric s-domain containing the S-spectrum $\sigma_S(T)$. Then there is a T-admissible domain U such that $\overline{U} \subseteq W$.*

Proof Since $\sigma_S(T)$ is an axially symmetric compact subset of $\mathbb{H}$, the result follows from Lemma 2.   □

Theorem 6 *Let $T \in \mathcal{B}(\mathcal{H})$ and let $\sigma_S(T) = \sigma \cup \tau$, where σ and τ are disjoint non-empty axially symmetric closed subsets of $\sigma_S(T)$. Then there exist a pair $\{\mathcal{M}_\sigma, \mathcal{M}_\tau\}$ of non-trivial invariant subspaces of T such that*

$$\sigma = \sigma_S(T|_{\mathcal{M}_\sigma}) \quad and \quad \tau = \sigma_S(T|_{\mathcal{M}_\tau}).$$

Proof Let $m \in \mathbb{S}$. It is clear from the hypothesis that $\sigma \cap \mathbb{C}_m$ and $\tau \cap \mathbb{C}_m$ are disjoint non-empty compact subsets of the Hausdorff space $\mathbb{C}_m$. Then there is a pair of disjoint open sets, say $\mathcal{O}^{(m)}_\sigma$ and $\mathcal{O}^{(m)}_\tau$ of $\mathbb{C}_m$ such that $\sigma \cap \mathbb{C}_m \subset \mathcal{O}^{(m)}_\sigma$ and $\tau \cap \mathbb{C}_m \subset \mathcal{O}^{(m)}_\tau$. By axially symmetric property of σ and τ, we can write

$$\sigma = \Omega_{\sigma \cap \mathbb{C}_m} \subset \Omega_{\mathcal{O}^{(m)}_\sigma} \text{ and } \tau = \Omega_{\tau \cap \mathbb{C}_m} \subseteq \Omega_{\mathcal{O}^{(m)}_\tau}.$$

Note that $\Omega_{\sigma \cap \mathbb{C}_m}$ and $\Omega_{\tau \cap \mathbb{C}_m}$ are nonempty disjoint s-domains in $\mathbb{H}$. By Lemma 2, there exist a pair of axially symmetric s-domains, denote them by U_σ and U_τ,

containing compact sets σ and τ respectively. Also, the boundaries $\partial(U_\sigma \cap \mathbb{C}_m)$ and $\partial(U_\tau \cap \mathbb{C}_m)$ are the union of finite number of continuously differentiable Jordan curves satisfying,

$$\overline{U}_\sigma \subseteq \Omega_{\mathcal{O}_\sigma^{(m)}} \text{ and } \overline{U}_\tau \subseteq \Omega_{\mathcal{O}_\tau^{(m)}}.$$

Now we define quaternionic operators corresponding to σ and τ as follows:

$$P_\sigma = \frac{1}{2\pi} \int\limits_{\partial(U_\sigma \cap \mathbb{C}_m)} ds_m \, S_R^{-1}(s, T) \tag{14}$$

and

$$P_\tau = \frac{1}{2\pi} \int\limits_{\partial(U_\tau \cap \mathbb{C}_m)} ds_m \, S_R^{-1}(s, T), \tag{15}$$

where $ds_m = -ds \cdot m$. Note that $P_\sigma = \chi_\sigma(T)$ and $P_\tau = \chi_\tau(T)$, where χ_σ and χ_τ are characteristic functions on sets σ and τ respectively. The fact that P_σ and P_τ are orthogonal projections is proved in [2] and these are called Riesz projectors. If we take $\mathcal{M}_\sigma = R(P_\sigma)$ and $\mathcal{M}_\tau = R(P_\tau)$, then it follows that $\mathcal{H} = \mathcal{M}_\sigma \oplus \mathcal{M}_\tau$. From the computations as shown in [2] and [5], one obtains that P_σ and P_τ commutes with T and hence M_σ and M_τ are invariant subspaces of T.

Finally, we show that $\sigma = \sigma_S(T|_{\mathcal{M}_\sigma})$ and $\tau = \sigma_S(T|_{\mathcal{M}_\tau})$. Suppose that $q \notin \sigma$, then $[q] \notin \sigma$ since σ is axially symmetric. With out loss of generality, we assume that there is an axially symmetric s-domain U_σ containing σ such that $\partial(U_\sigma \cap \mathbb{C}_m)$ is the union of a finite number of continuously differentiable Jordan curves for every $m \in \mathbb{S}$. Let us fix $m \in \mathbb{S}$. Define the operator

$$\mathcal{Q}_\sigma^{(q)} := \frac{1}{2\pi} \int\limits_{\partial(U_\sigma \cap \mathbb{C}_m)} S_L^{-1}(t, T) \, dt_m \big(t^2 - 2\mathrm{re}(q)t + |q|^2\big)^{-1},$$

where $dt_m = -dt \cdot m$. We claim that $\mathcal{M}_\sigma$ is invariant subspace of $\mathcal{Q}_\sigma^{(q)}$. For any $p \in \mathbb{C}_m$, if we define a map $\xi_p(t) = \big(t^2 - 2\mathrm{re}(q)t + |q|^2\big)^{-1} S_L^{-1}(p, t)$, for all $t \in U_\sigma$. It is clear that

$$\big(t^2 - 2\mathrm{re}(q)t + |q|^2\big)^{-1} \in \mathbb{C}_m, \quad \text{whenever } t \in \mathbb{C}_m$$

and $S_L^{-1}(p, t)$ is a left s-regular function in variable t. Thus $\xi_p \in R^L(U_\sigma)$ by [5, Proposition 4.11.5]. So, by the quaternionic functional calculus, we deduce that

$$\begin{aligned}
\xi_p(T) &= \mathcal{Q}_\sigma^{(q)} S_L^{-1}(p, T) \\
&= \frac{1}{2\pi} \int\limits_{\partial(U_\sigma \cap \mathbb{C}_m)} S_L^{-1}(t, T) \, dt_m \big(t^2 - 2\mathrm{re}(q)t + |q|^2\big)^{-1} S_L^{-1}(p, t). \tag{16}
\end{aligned}$$

This implies the following:

$$\mathcal{Q}_\sigma^{(q)} P_\sigma$$

$$= \frac{1}{2\pi} \int\limits_{\partial(U_\sigma' \cap \mathbb{C}_m)} \mathcal{Q}_\sigma^{(q)} S_L^{-1}(p, T) dp_m$$

$$= \frac{1}{4\pi^2} \int\limits_{\partial(U_\sigma' \cap \mathbb{C}_m)} \int\limits_{\partial(U_\sigma \cap \mathbb{C}_m)} S_L^{-1}(t, T) \, dt_m \big(t^2 - 2\mathrm{re}(q)t + |q|^2 \big)^{-1} S_L^{-1}(p, t) dp_m,$$

by Equation (16)

$$= \frac{1}{2\pi} \int\limits_{\partial(U_\sigma \cap \mathbb{C}_m)} S_L^{-1}(t, T) \, dt_m \big(t^2 - 2\mathrm{re}(q)t + |q|^2 \big)^{-1} \bigg(\frac{1}{2\pi} \int\limits_{\partial(U_\sigma' \cap \mathbb{C}_m)} S_L^{-1}(p, t) dp_m \bigg)$$

by Fubini's theorem

$$= \frac{1}{2\pi} \int\limits_{\partial(U_\sigma \cap \mathbb{C}_m)} S_L^{-1}(t, T) \, dt_m \big(t^2 - 2\mathrm{re}(q)t + |q|^2 \big)^{-1},$$

since $\dfrac{1}{2\pi} \displaystyle\int\limits_{\partial(U_\sigma' \cap \mathbb{C}_m)} S_L^{-1}(p, t) \, dp_m = 1$

$$= \mathcal{Q}_\sigma^{(q)}.$$

Similarly, $P_\sigma \mathcal{Q}_\sigma^{(q)} = \mathcal{Q}_\sigma^{(q)}$. Equivalently, $\mathcal{Q}_\sigma^{(q)}|_{\mathcal{M}_\sigma} \in \mathcal{B}(\mathcal{M}_\sigma)$. Next, we show that $\Delta_q(T)|_{\mathcal{M}_\sigma}$ is invertible. For this, let us define a map φ_q by

$$\varphi_q(t) = t^2 - 2\mathrm{re}(q)t + |q|^2, \text{ for every } t.$$

Then one can verify that φ_q is locally s-regular function on $\sigma_S(T)$. Moreover, by following similar arguments, we express that

$$\Delta_q(T)|_{\mathcal{M}_\sigma} P_\sigma = \varphi_q(T)|_{\mathcal{M}_\sigma} P_\sigma$$

$$= \frac{1}{2\pi} \int\limits_{\partial(U_\sigma' \cap \mathbb{C}_m)} S_L^{-1}(p, T) \, dp_m (p^2 - 2\mathrm{re}(q)p + |q|^2), \qquad (17)$$

where $dp_m = -dp \cdot m$.

Also we know that $\varphi_q(p) S_L^{-1}(t, p)$ is left s-regular function in the variable p, for every t. By [5, Proposition 4.11.5], it follows that

$$\Delta_q(T)|_{\mathcal{M}_\sigma} S_L^{-1}(t, T) = \frac{1}{2\pi} \int\limits_{\partial(U_\sigma' \cap \mathbb{C}_m)} S_L^{-1}(p, T) \, dp_m \, \varphi_q(p) S_L^{-1}(t, p). \qquad (18)$$

From Eqs. (17), (18) we compute $\Delta_q(T)|_{\mathcal{M}_\sigma} \mathcal{Q}_\sigma^{(q)}$ as follows:

$$\Delta_q(T)|_{\mathcal{M}_\sigma}\mathcal{Q}_\sigma^{(q)}$$

$$= \frac{1}{2\pi}\int\limits_{\partial(U_\sigma\cap\mathbb{C}_m)} \Delta_q(T)|_{\mathcal{M}_\sigma}S_L^{-1}(t,T)\,dt_m\,\big(t^2 - 2\mathrm{re}(q)t + |q|^2\big)^{-1}$$

$$= \frac{1}{4\pi^2}\int\limits_{\partial(U_\sigma\cap\mathbb{C}_m)}\int\limits_{\partial(U_\sigma'\cap\mathbb{C}_m)} S_L^{-1}(p,T)\,dp_m\,\varphi_q(p)S_L^{-1}(t,p)\,dt_m\big(t^2 - 2\mathrm{re}(q)t + |q|^2\big)^{-1}$$

$$= \frac{1}{2\pi}\int\limits_{\partial(U_\sigma'\cap\mathbb{C}_m)} S_L^{-1}(p,T)\,dp_m\,\varphi_q(p)\frac{1}{2\pi}\int\limits_{\partial(U_{\sigma\cap\mathbb{C}_m})} S_L^{-1}(t,p)\,dt_m\big(t^2 - 2\mathrm{re}(q)t + |q|^2\big)^{-1}$$

$$= \frac{1}{2\pi}\int\limits_{\partial(U_\sigma'\cap\mathbb{C}_m)} S_L^{-1}(p,T)\,dp_m\,\varphi_q(p)\big(p^2 - 2\mathrm{re}(q)p + |q|^2\big)^{-1}.$$

Since $\varphi_q(p)\big(p^2 - 2\mathrm{re}(q)p + |q|^2\big)^{-1} = 1$, it will imply that $\Delta_q(T)|_{\mathcal{M}_\sigma}\mathcal{Q}_\sigma^{(q)} = P_\sigma$. Similarly, one can show that $\mathcal{Q}_\sigma^{(q)}\Delta_q(T)|_{\mathcal{M}_\sigma} = P_\sigma$. In other words, for every $x \in \mathcal{M}_\sigma$, we see that

$$\Delta_q(T|_{\mathcal{M}_\sigma})\mathcal{Q}_\sigma^{(q)}x = \mathcal{Q}_\sigma^{(q)}\Delta_p(T|_{\mathcal{M}_\sigma})x = x, \ \text{ for all } x \in \mathcal{M}_\sigma.$$

Thus $\Delta_q(T|_{\mathcal{M}_\sigma})$ is invertible and hence $q \in \rho_S(T|_{\mathcal{M}_\sigma})$. It follows that

$$\sigma_S(T|_{\mathcal{M}_\sigma}) \subseteq \sigma. \tag{19}$$

By the similar arguments, we achieve that

$$\sigma_S(T|_{\mathcal{M}_\tau}) \subseteq \tau. \tag{20}$$

Now we prove reverse inclusions. Suppose that $q \notin \sigma_S(T|_{M_\sigma}) \cup \sigma_S(T|_{M_\tau})$. Then both operators $\Delta_q(T|_{\mathcal{M}_\sigma})$ and $\Delta_q(T|_{\mathcal{M}_\tau})$ are invertible. Hence $\Delta_q(T) \in \mathcal{B}(\mathcal{H})$ is invertible since $\mathcal{H} = \mathcal{M}_\sigma \oplus \mathcal{M}_\tau$. Equivalently, $q \notin \sigma_S(T)$. This shows that

$$\sigma_S(T) \subseteq \sigma_S(T|_{M_\sigma}) \cup \sigma_S(T|_{M_\tau}) \subseteq \sigma \cup \tau = \sigma_S(T).$$

Therefore, by Eqs. (19), (20) and using the fact that σ and τ are disjoint, we conclude that

$$\sigma_S(T|_{M_\sigma}) = \sigma \ \text{ and } \ \sigma_S(T|_{M_\tau}) = \tau.$$

Hence the result. $\square$

Corollary 2 *Let $T \in \mathcal{B}(\mathcal{H})$. If the S-spectrum $\sigma_S(T)$ is disconnected by a pair of disjoint nonempty axially symmetric closed subsets, then T is strongly reducible.*

Proof From the hypothesis, assume that there is a pair $\{\sigma, \tau\}$ of disjoint nonempty axially symmetric closed subsets of $\sigma_S(T)$ satisfying,

$$\sigma_S(T) = \sigma \cup \tau.$$

Then by Theorem 6, there exist a pair of nontrivial mutually orthogonal invariant subspaces M_σ and M_τ of T such that

$$\sigma_S(T|_{M_\sigma}) = \sigma \quad \text{and} \quad \sigma_S(T|_{M_\tau}) = \tau.$$

Equivalently, T commutes with the corresponding projections P_σ and P_τ as shown in step III of Theorem 6. This implies that T is strongly reducible. $\qquad\square$

Compliance with ethical standards

Conflict of interest   The author declares that there is no conflict of interest.

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